

Combinatorics and Physics

Chapter 0

Introduction

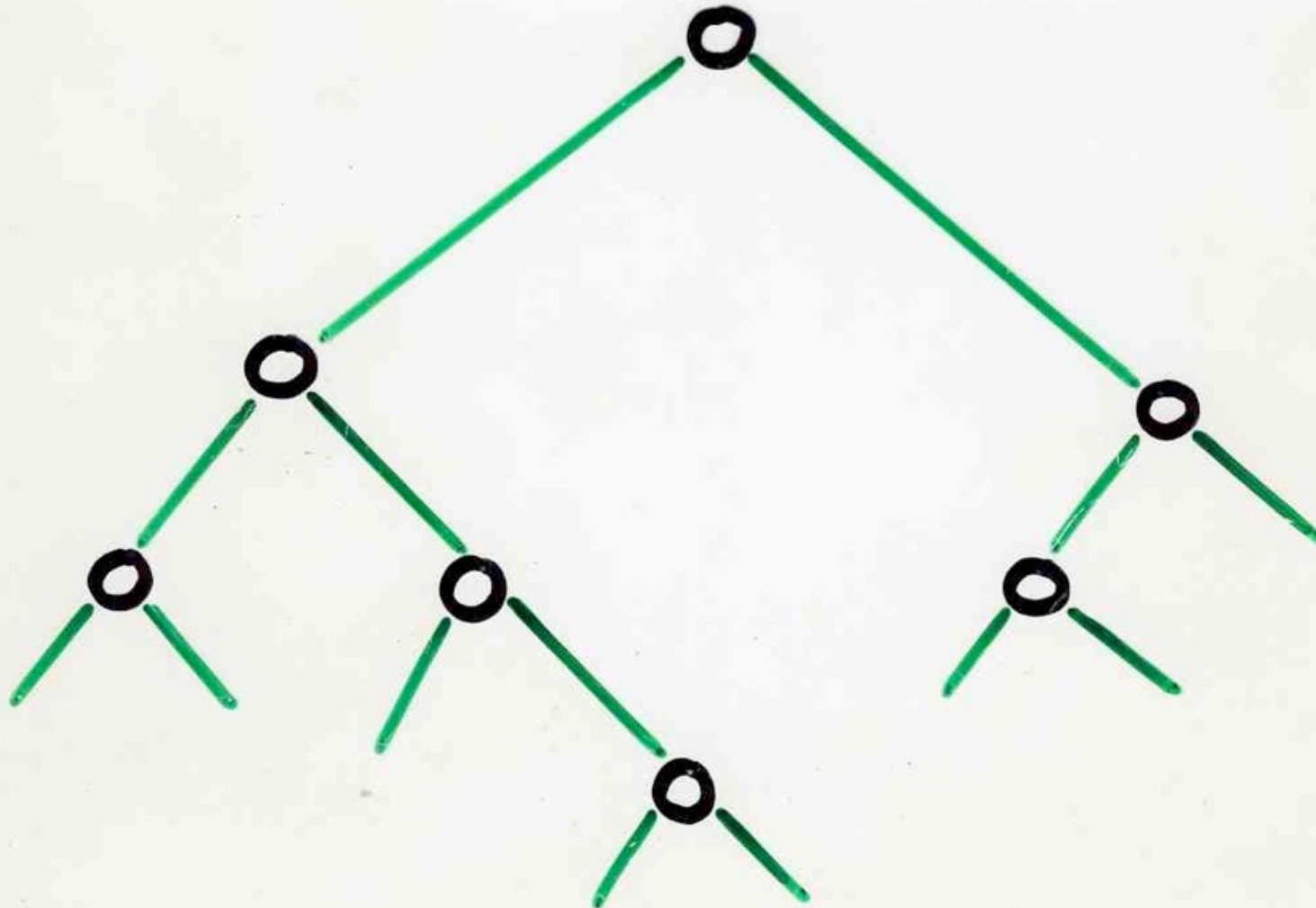
Overview of the course (part 1)

IIT-Madras
14 January 2015

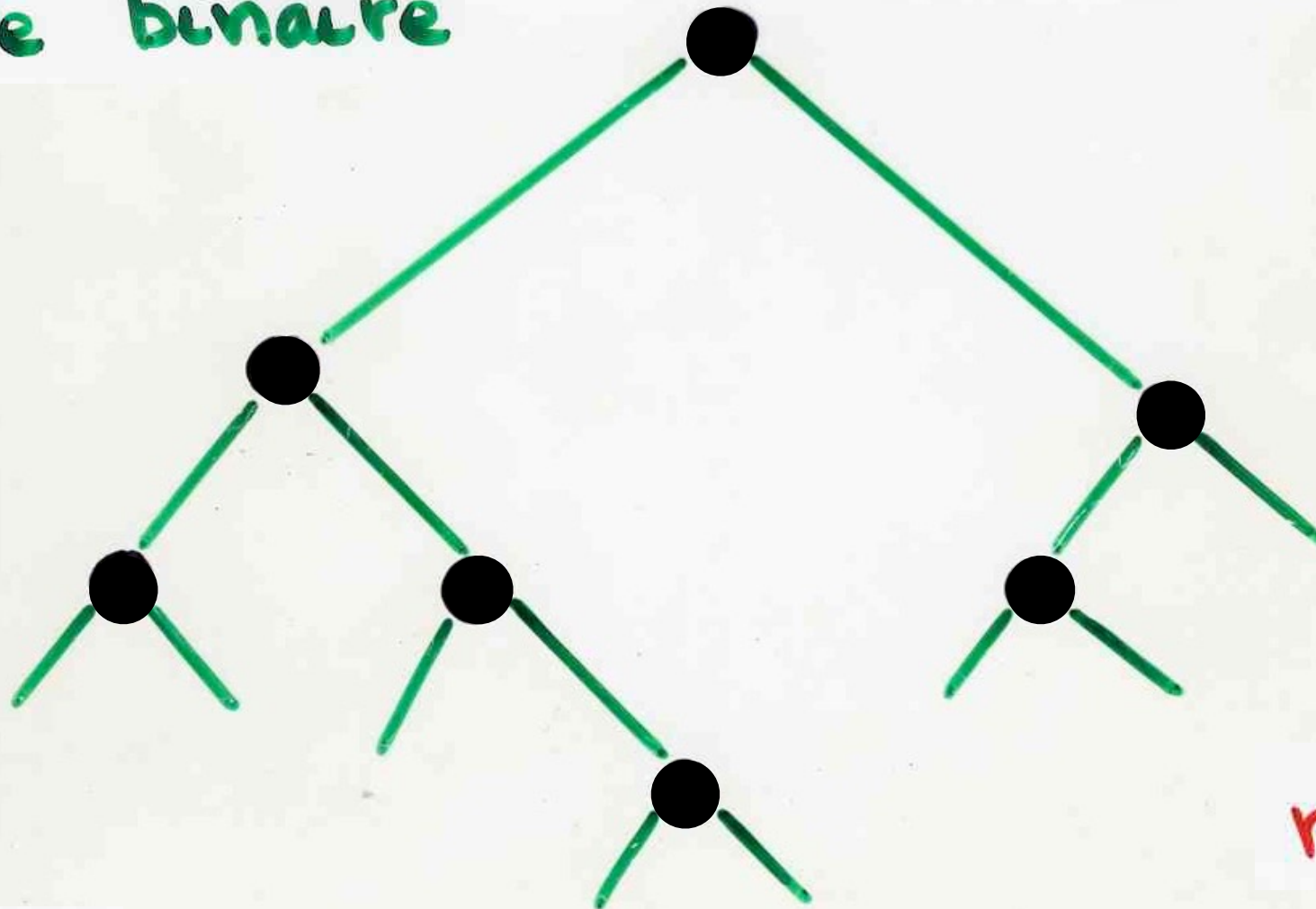
Xavier Viennot
CNRS, LaBRI, Bordeaux

enumerative combinatorics

Binary tree



arbre binaire



$$n = 7$$

$$C_h =$$

number of binary trees
having n internal vertices

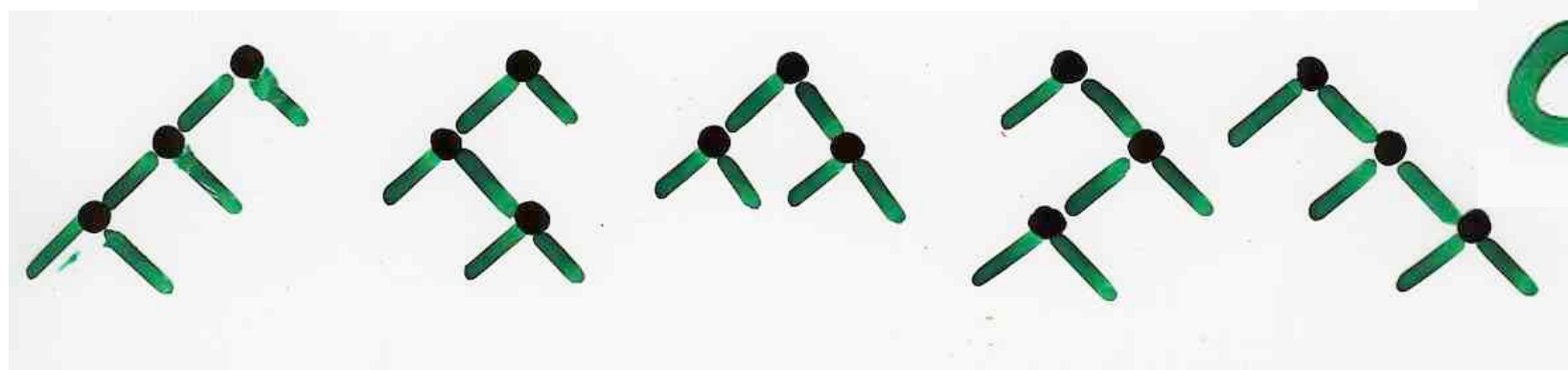
(or $n+1$) leaves (external vertices)



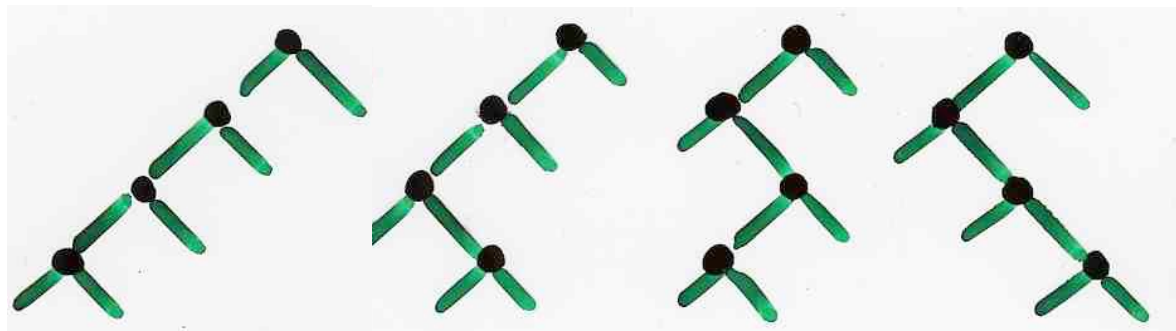
$$C_1 = 1$$



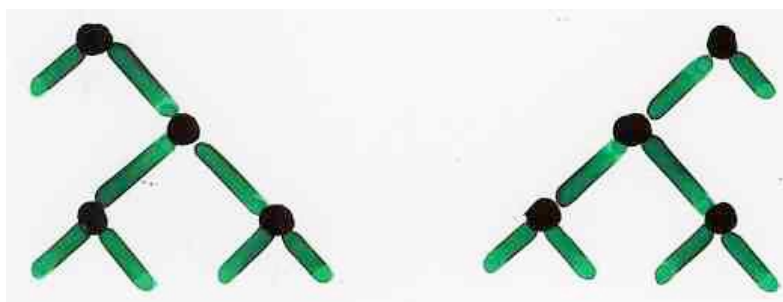
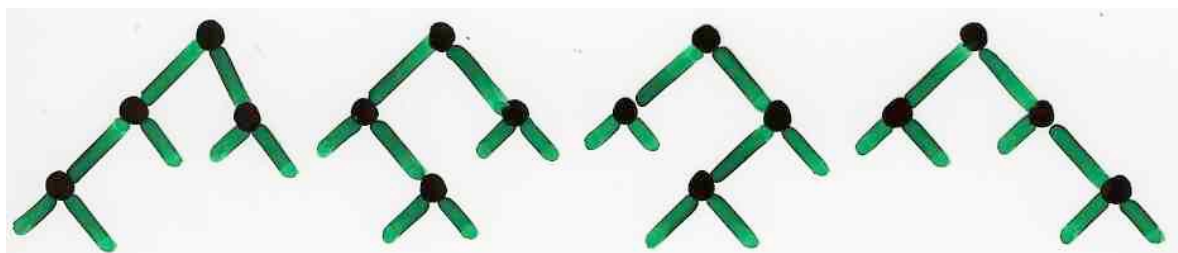
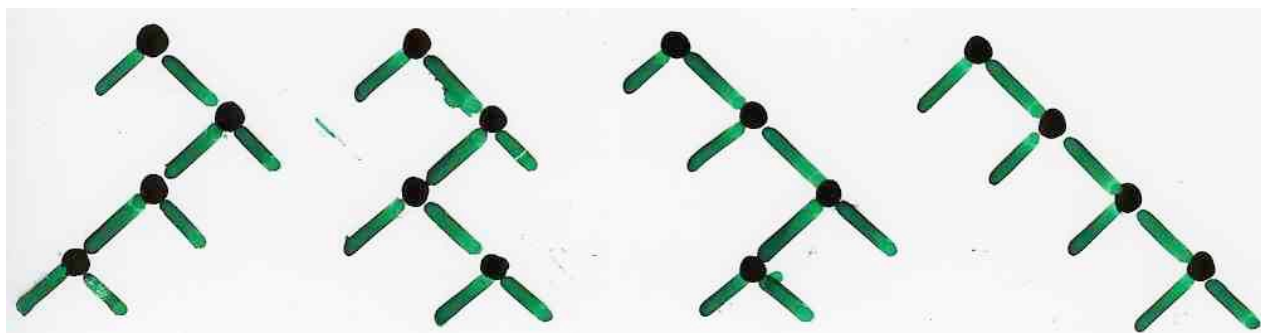
$$C_2 = 2$$



$$C_3 = 5$$



$$C_4 = 14$$



recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)! n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

very classical
enumerative
combinatorics

Note sur une Équation aux différences finies ;

PAR E. CATALAN.

M. Lamé a démontré que l'équation

$P_{n+1} = P_n + P_{n-1}P_2 + P_{n-2}P_3 + \dots + P_4P_{n-4} + P_3P_{n-3} + P_n$, (1)
se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \quad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3} \cdot \frac{10}{4} \cdot \frac{14}{5} \dots \frac{4n-6}{n} P_1;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_1 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2 \cdot 6 \cdot 10 \cdot 14 \dots (4n-6)}{2 \cdot 3 \cdot 4 \cdot 5 \dots n}. \quad (3)$$

Le numérateur

$$\begin{aligned} 2 \cdot 6 \cdot 10 \cdot 14 \dots (4n-6) &= 2^{n-1} \cdot 1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-3) \\ &= \frac{2^{n-1} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots (2n-2)}{2 \cdot 4 \cdot 6 \cdot 8 \dots (2n-2)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n-2)}{1 \cdot 2 \cdot 3 \dots (n-1)}. \end{aligned}$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2) \dots (2n-2)}{2 \cdot 3 \cdot 4 \dots n}. \quad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p ; et si l'on change n en $n+1$, on aura

$$P_{n+1} = \frac{1}{n+1} C_{2n,n}, \quad (5)$$

ou bien

$$P_{n+1} = C_{2n,n} - C_{2n,n-1}. \quad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left. \begin{aligned} \frac{1}{n+1} C_{2n,n} &= \frac{1}{n} C_{2n-2,n-1} + \frac{1}{n-1} C_{2n-4,n-3} \times \frac{1}{2} C_{2,1} \\ &+ \frac{1}{n-2} C_{2n-6,n-3} \times \frac{1}{3} C_{4,2} + \dots + \frac{1}{n} C_{2n-2,n-1}. \end{aligned} \right\} \quad (7)$$

III.

On sait que le $(n+1)^{\text{e}}$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

$$1, 2, 6, 20, 70, 252, 924 \dots;$$

qu'on les divise respectivement par

$$1, 2, 3, 4, 5, 6, 7 \dots;$$

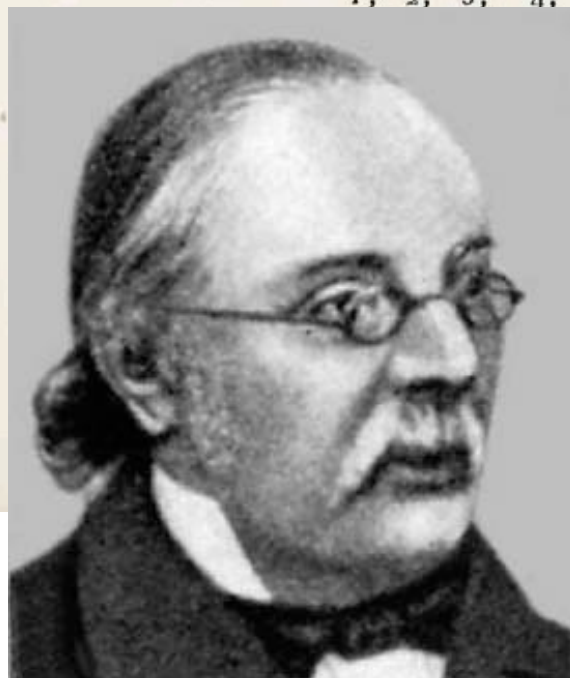
on obtient les nombres,

$$1, 1, 2, 5, 14, 42, 132 \dots, \quad (A)$$

ou

le terme (A) est égal à la somme des termes précédents, et en multipliant par les deux séries.

$$1 + 1 \cdot 1 + 2 \cdot 1 + 5 \cdot 1 + 14 \cdot 1 + 42 \cdot 1 \dots$$



ordinary generating function

série génératrice

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots \\ \dots + a_n t^n + \dots$$

generating function

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \dots + C_n t^n + \dots$$

Will corde a-
linge

Binary Tree

Arbre

=



Arbre

Arbre

y

=

1

+

t

y

y

y

=

1

+

t (y)²

$$y = \frac{1 - (1 - 4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!} u + \frac{m(m-1)}{2!} u^2 + \frac{m(m-1)(m-2)}{3!} u^3 + \dots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)! n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

statistical mechanics

phase transition
critical phenomena

Physics

exactly solved model

Baxter (1982)

Ising model

Onsager (1944)

Potts, ice model

Temperley-Lieb
(1971)

Baxter (1982)

exactly solved models

Statistical physics

$F(T)$

$\approx$

$$\frac{1}{(T - T_c)^\alpha}$$

critical
exponent

temperature

critical
temperature

thermodynamic
function

- $F(t) = \sum_{n \geq 0} a_n t^n$

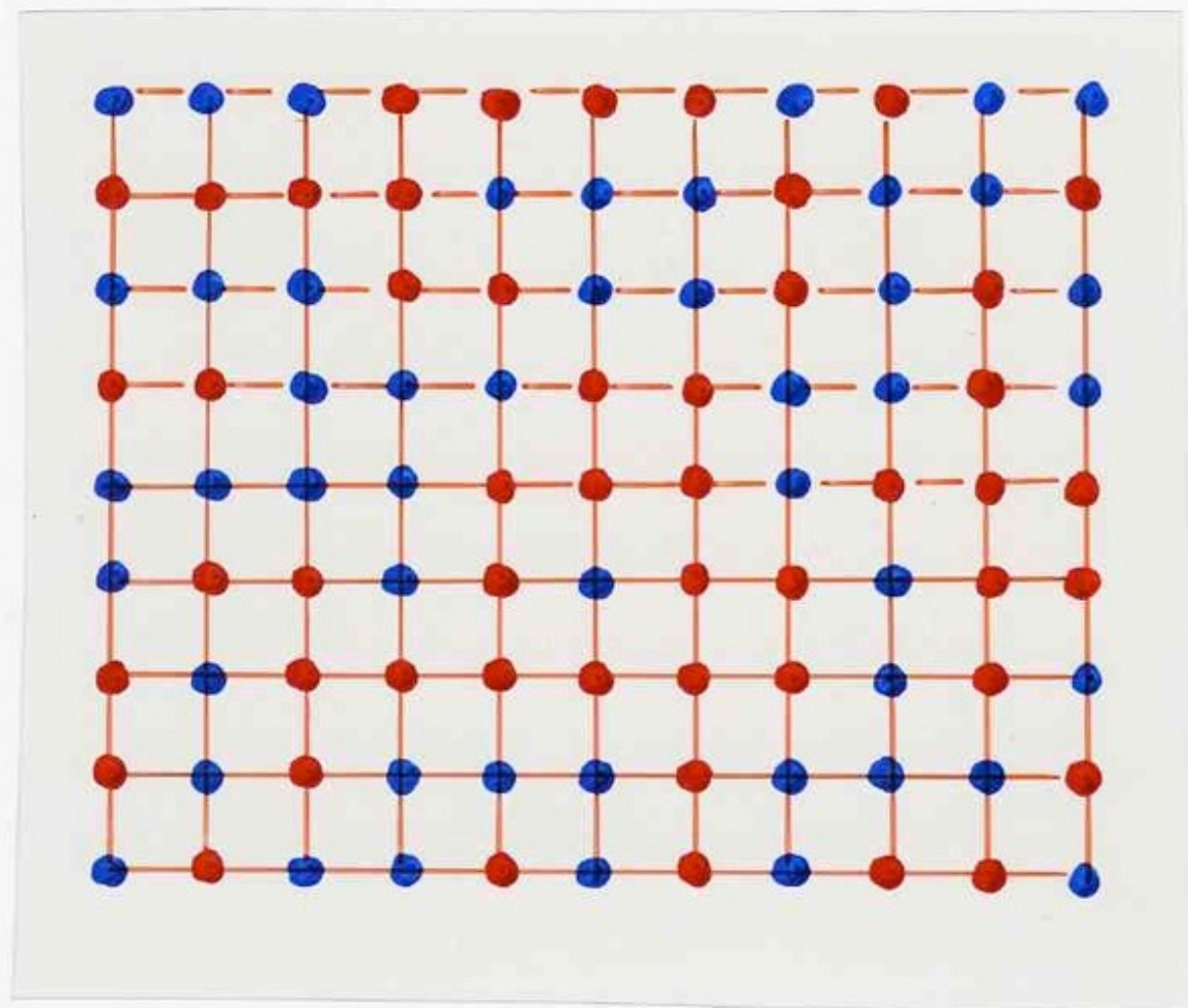
number of

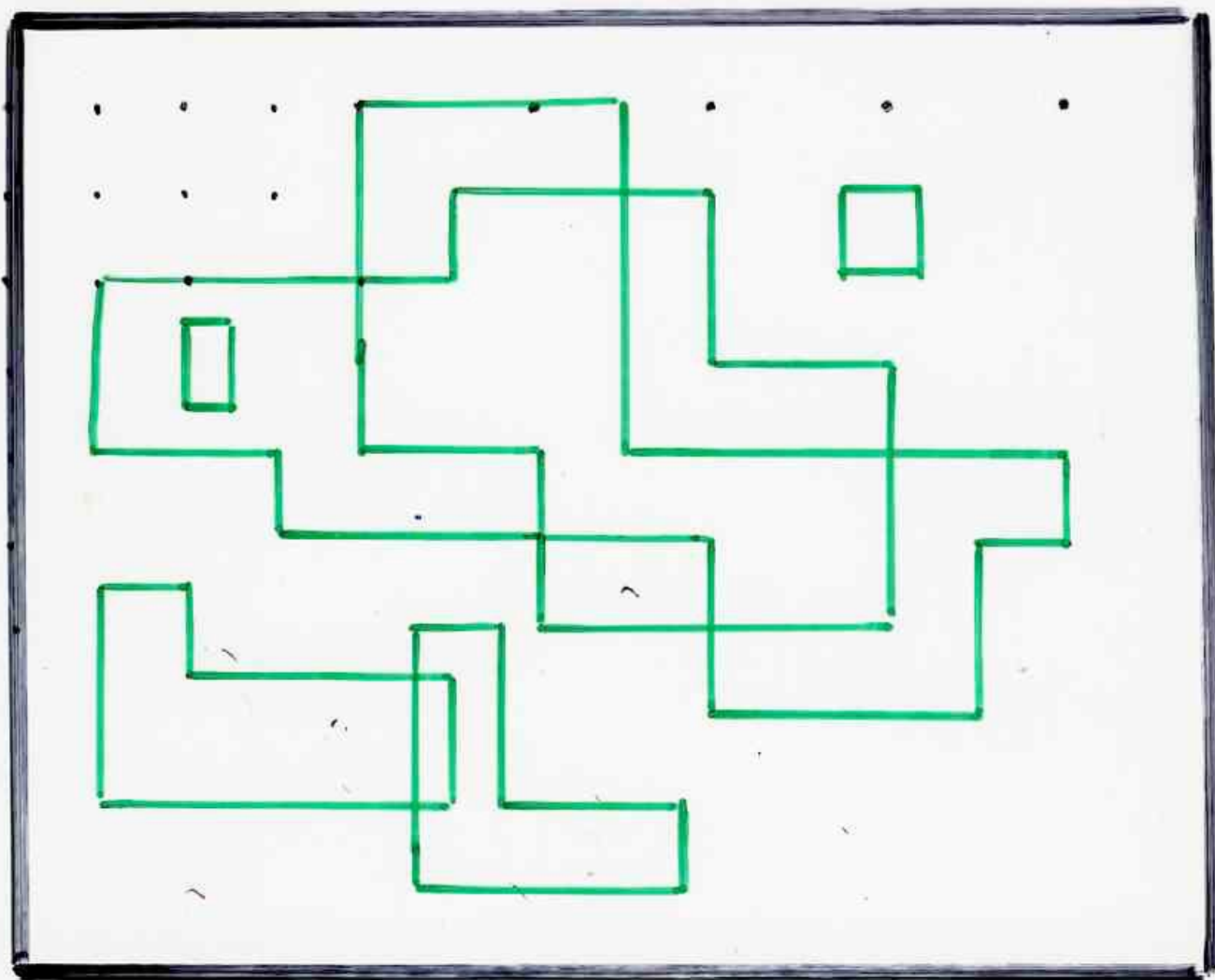
- $a_n \approx \mu^n n^{-\theta}$

connective constant

critical exponent

$$\mu = \frac{1}{t_c}$$





"closed" graph

Ising model

- dénombrement de
couplages parfaits

- graphe planaire

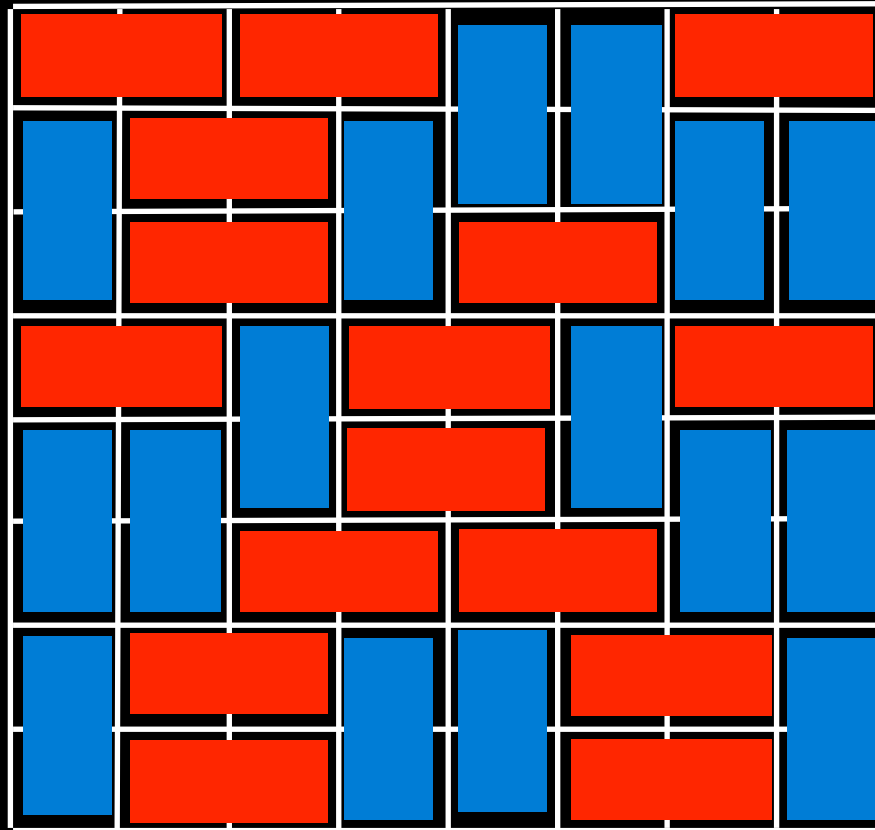
méthode du Pfaffien

- modèle d'Ising (1925)

Kasteleyn, Fisher, Temperley
(1961,)

Onsager (1944)

Tiling



$$4^{mn}$$

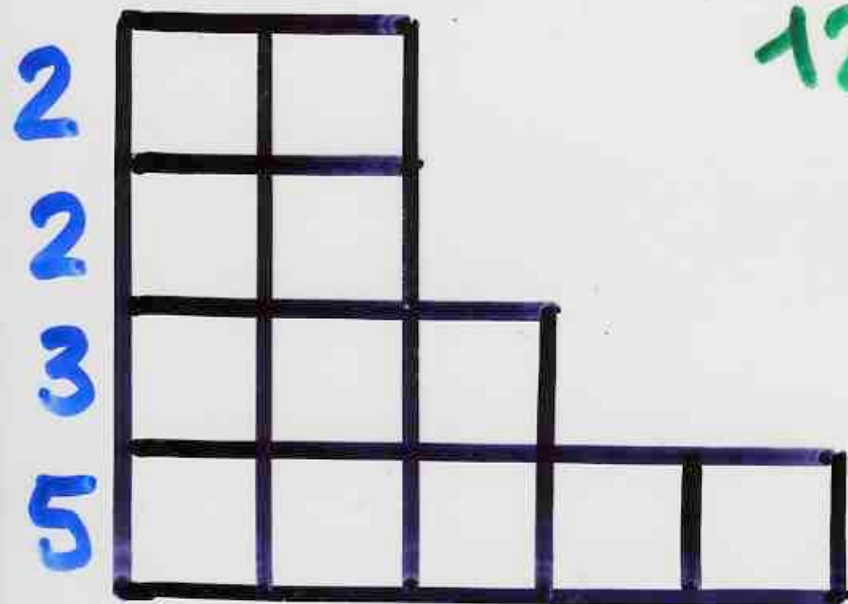
$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kas teley n (1961)

this is an integer !!

more advanced
enumerative combinatorics:

Young tableaux

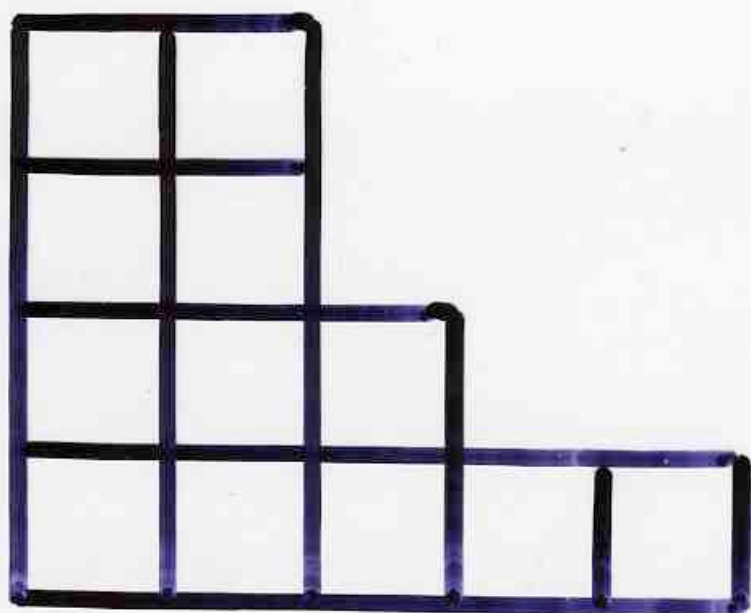


12

$$12 = n = 5 + 3 + 2 + 2$$

Ferrers
diagram

Partition of n



7	12			
6	10			
3	5	9		
1	2	4	8	11

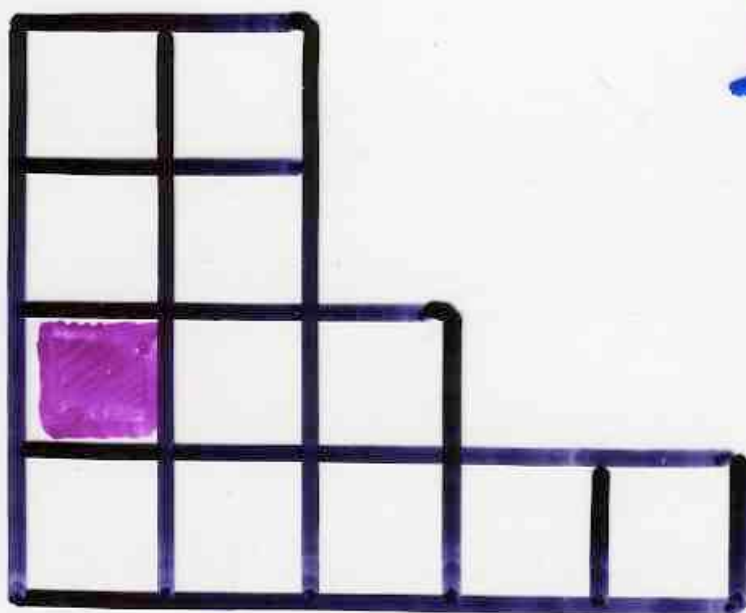
Young
tableau

$f_\lambda = \text{nb of}$
Young
tableaux
shape λ

Hook length formula

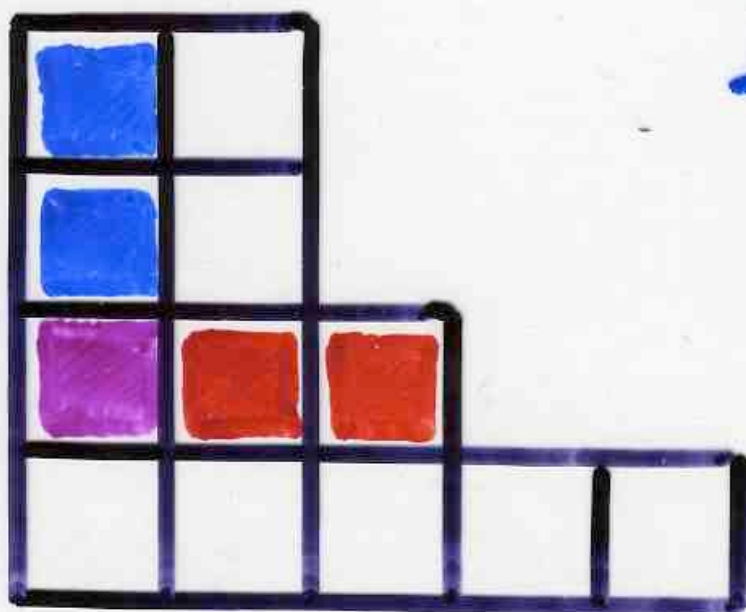
J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954

..... Franzblau-Zeilberger, Remmel, Greene-Wilf, Krattenthaler,
Novelli- Pak-Stoyanovski, ...



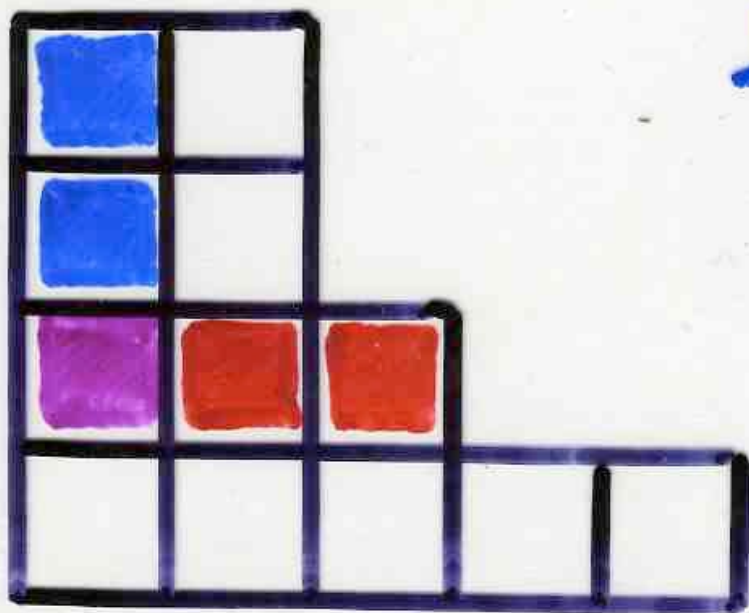
hook





hook





hook



length
5

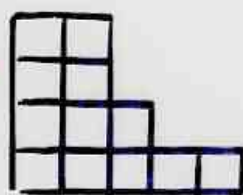
2	1			
3	2			
5	4	1		
8	7	4	2	1

2	1			
3	2			
5	4	1		
8	7	4	2	1

$$f_1 = \frac{n!}{\prod_x h_x}$$

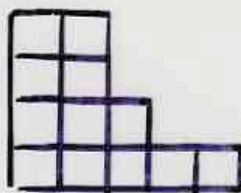
hook
length
formula

ℓ



=

8

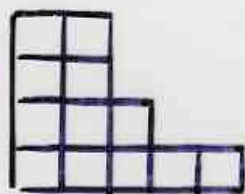


=

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \cdot 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 8}$$

$$1^3 \cdot 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 8$$

8



=

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \cdot 2^3 \cdot 3^2 \cdot 4^2 \cdot 5 \cdot 7 \cdot 8}$$

$$1^3 \cdot 2^3 \cdot 3^2 \cdot 4^2 \cdot 5 \cdot 7 \cdot 8$$

$$= 3^4 \times 5 \times 11 = 4455$$

integer partitions

q-series

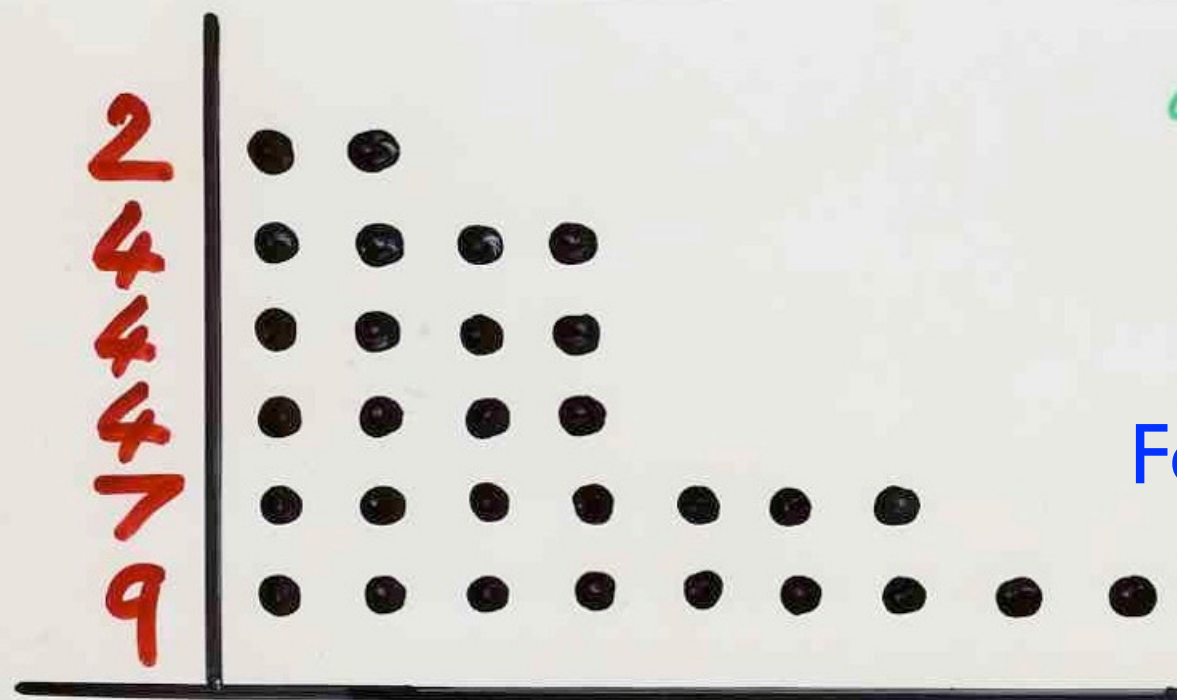


diagramme
de
Ferrers

Ferrers diagram

$$30 = 2 + 4 + 4 + 4 + 7 + 9$$

partition of an integer

série génératrice
des partitions (d'entiers)

$$\sum_{n \geq 0} a_n q^n$$

generating function
for (integer) partitions

$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

$$\prod_{i \geq 1}$$

$$\frac{1}{(1 - q^i)}$$

