Combinatorics and Physics

Chapter 1
Introduction to enumerative combinatorics, ordinary generating functions

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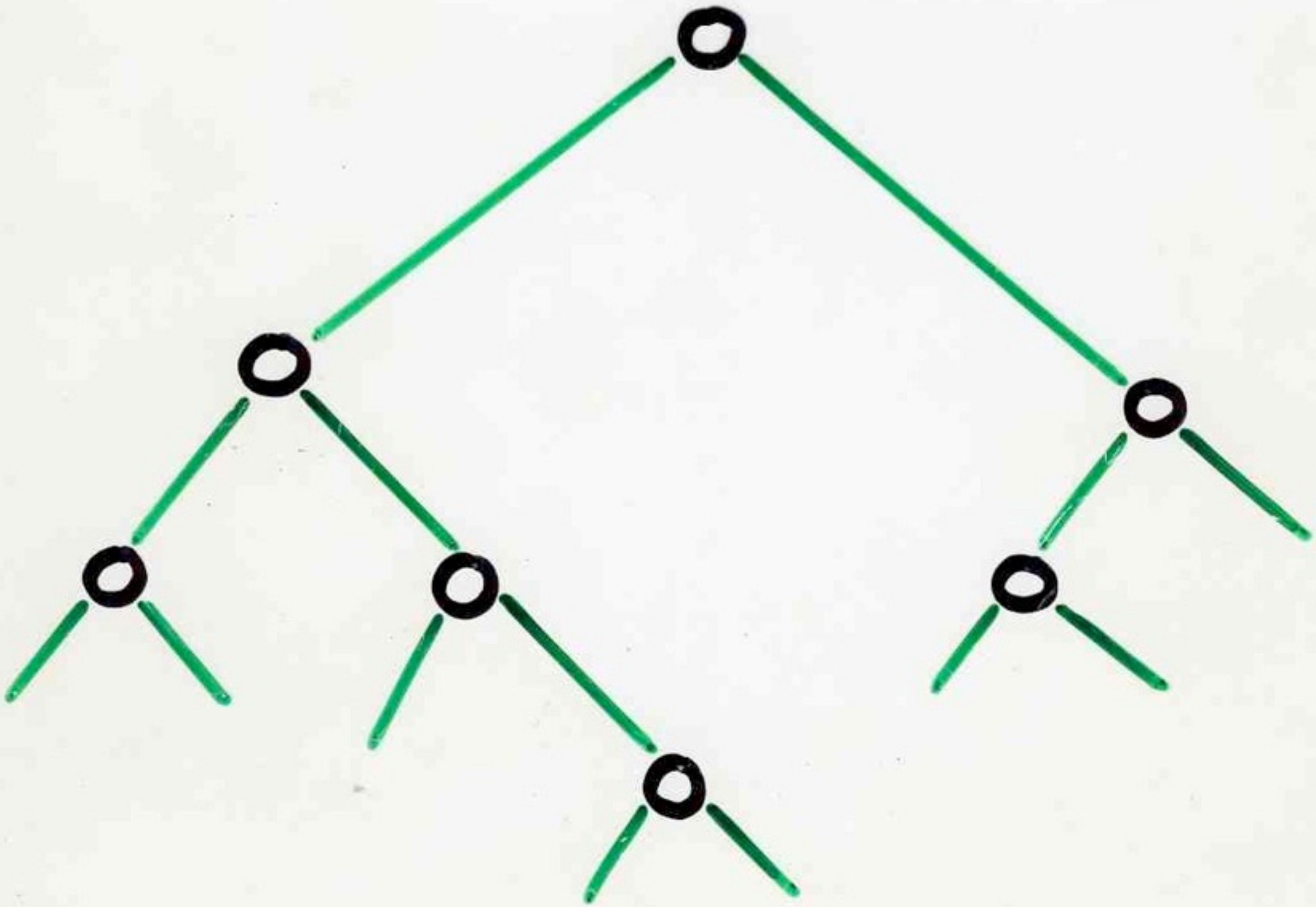
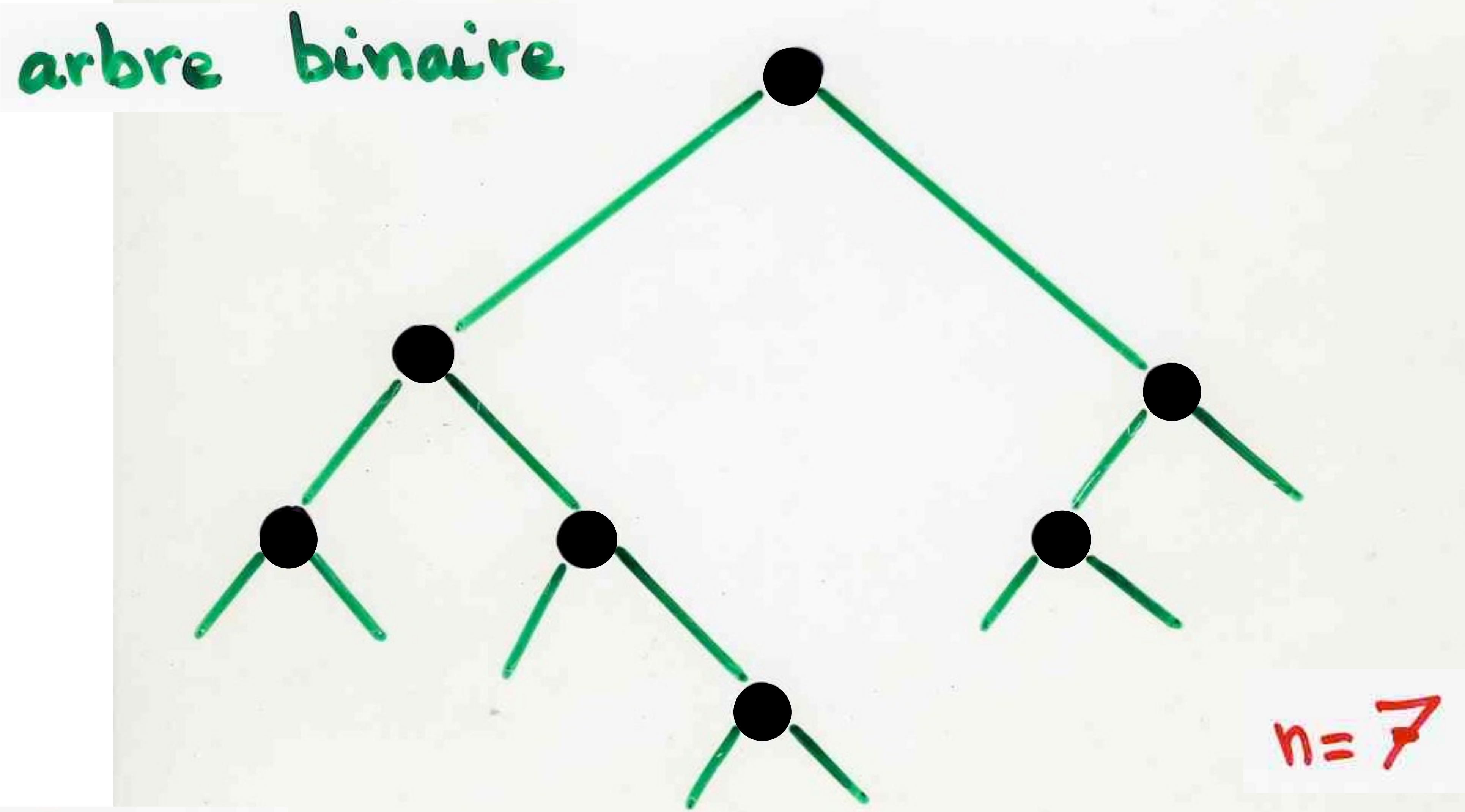
binary tree

Binary tree

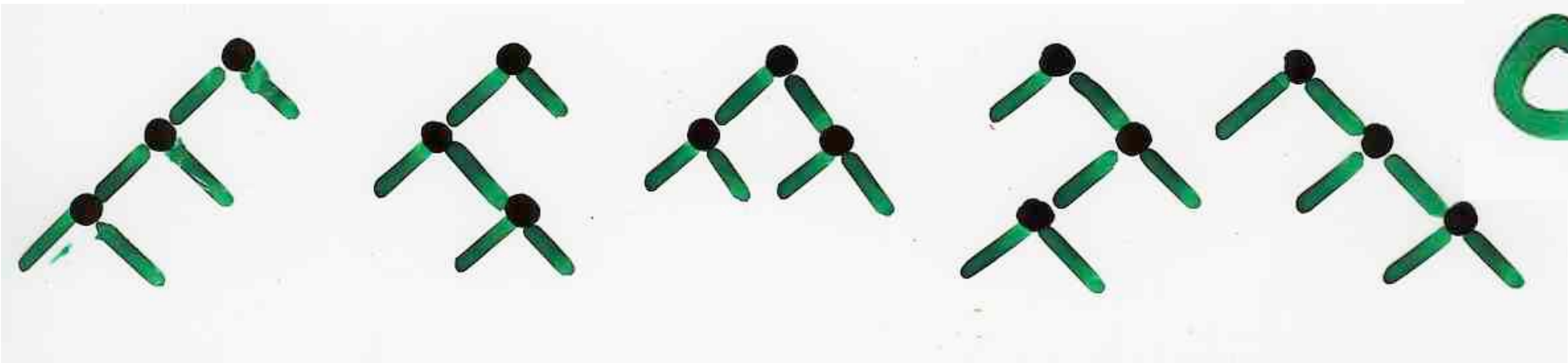




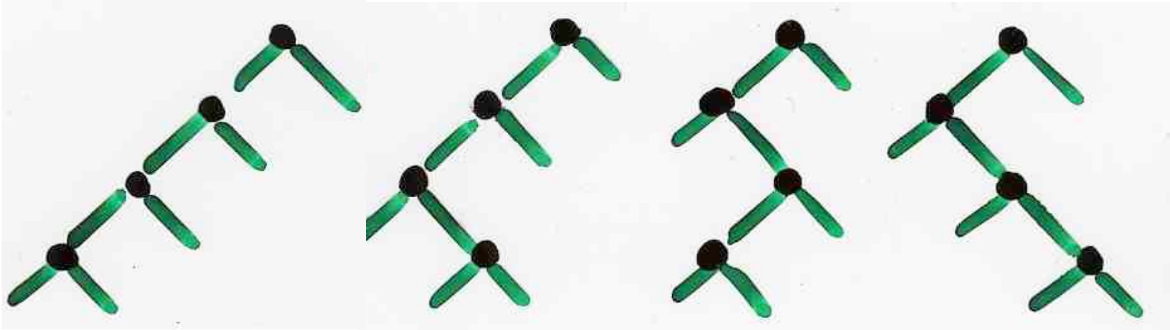
C_n = number of binary trees having n internal vertices ● (or n+1) leaves (external vertices)

$C_1 = 1$

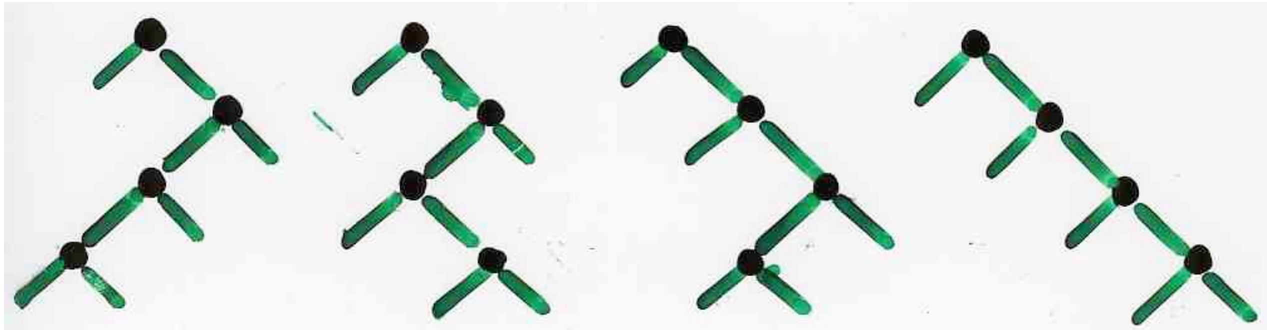
$C_2 = 2$

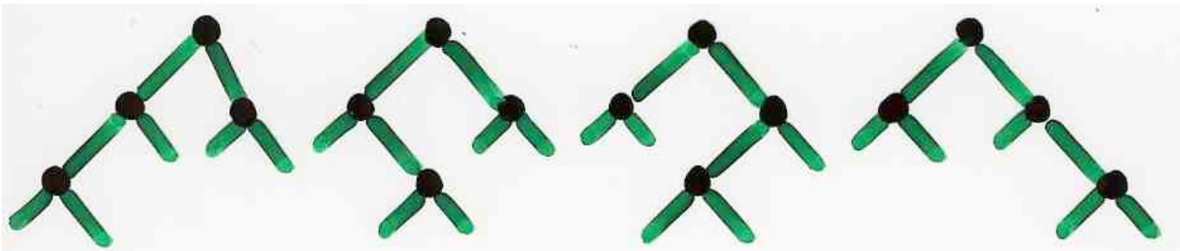


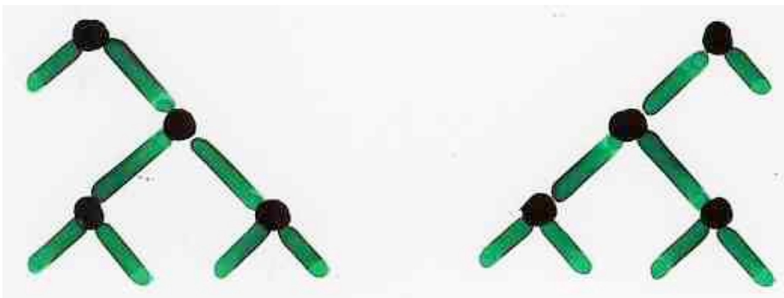
$C_3 = 5$



$C_4 = 14$







recurrence

$$C_{n+1} = \sum_{i+j=n} C_i \, C_j$$

$$C_0 = 1$$

$C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5$

$1, 1, 2, 5, 14, 42, \ldots$

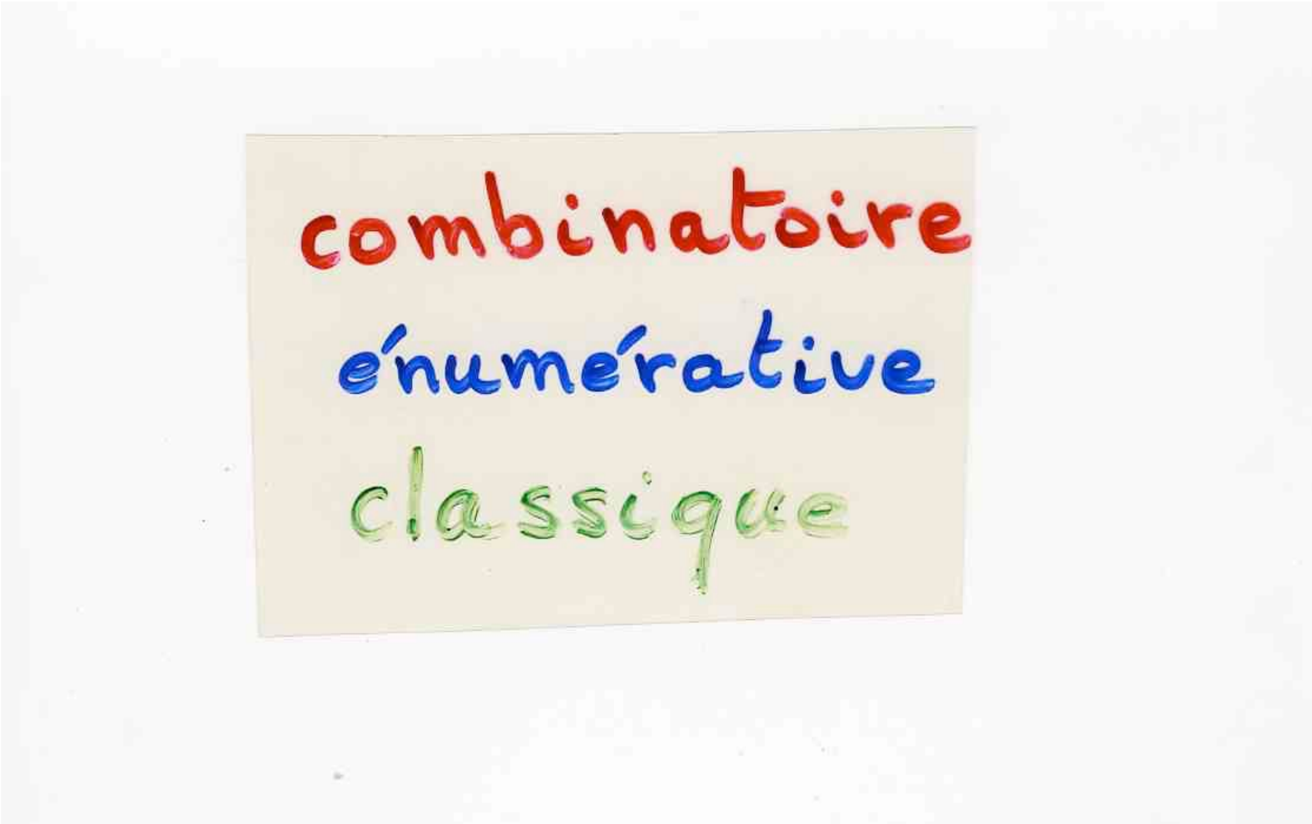
$$C_6 = C_0 C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5 C_0$$

$$132 \qquad 1 \times 42 + 1 \times 14 + 2 \times 5 + 5 \times 2 + 14 \times 1 + 42 \times 1$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$



classical
enumerative
combinatorics

(1838)

Note sur une Équation aux différences finies;

Par E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_4P_{n-2} + P_3P_{n-1} + P_n, \quad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \qquad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3} \cdot \frac{10}{4} \cdot \frac{14}{5} \cdots\cdots \frac{4n-6}{n} P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\ldots(4n-6)}{2.3.\ 4.\ 5\ldots\ n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\ldots(4n-6) = 2^{n-1}.1.3.5.7\ldots(2n-3)$$
$$= \frac{2^{n-1}.1.2.3.4.5\ldots(2n-2)}{2.4.6.8\ldots(2n-2)} = \frac{1.2.3.4\ldots(2n-2)}{1.2.3\ldots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\ldots(2n-2)}{2.3.4\ldots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1} C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned} \frac{1}{n+1} C_{2n,n} = \frac{1}{n} C_{2n-2,n-1} + \frac{1}{n-1} C_{2n-4,n-2} \times \frac{1}{2} C_{2,1} \\ + \frac{1}{n-2} C_{2n-6,n-3} \times \frac{1}{3} C_{4,2} + \ldots + \frac{1}{n} C_{2n-2,n-1}. \end{aligned}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

1, 2, 6, 20, 70, 252, 924...;

qu'on les divise respectivement par

1, 2, 3, 4, 5, 6, 7...;

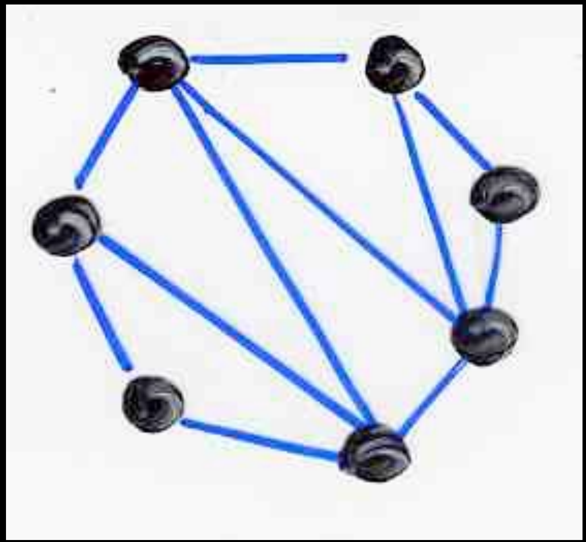
...e nombres,

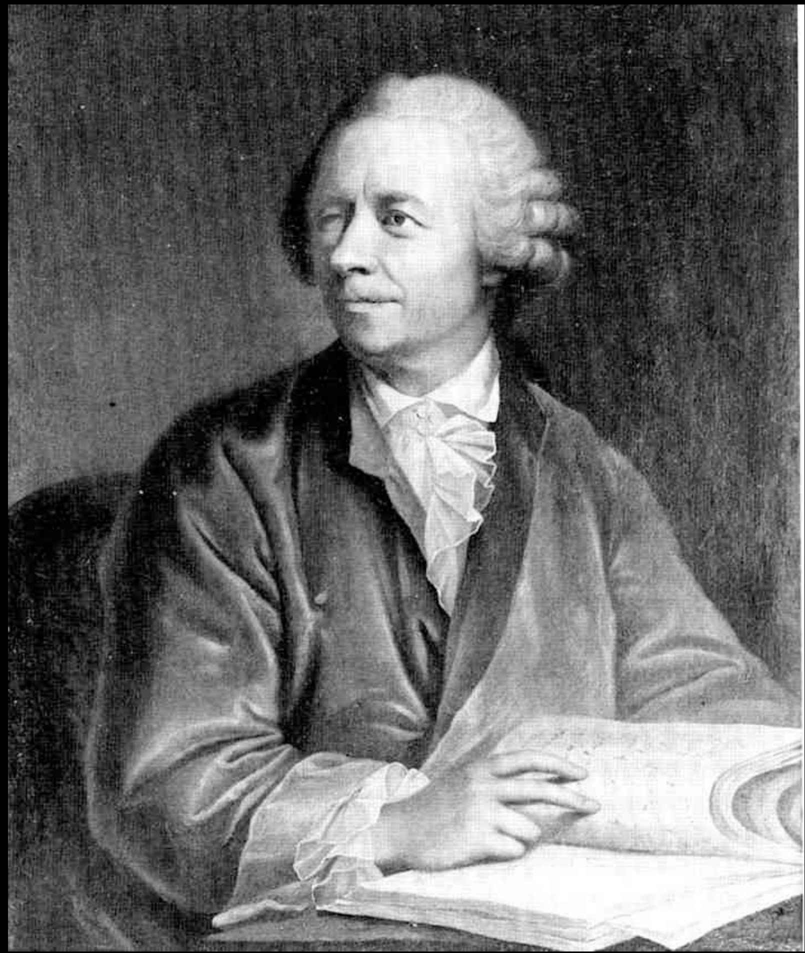
... 14, 42, 132..., (A)

...té :

...ite (A) est égal à la somme des ...ivant au-dessous d'elle-même, et ...s termes précédents, et en multi- ...les deux séries.

...5 + 5.2 + 14.1 + 42.1.





$$2(2n+1)C_n = (n+2)C_{n+1}$$

$$\frac{1}{n+1}\binom{2n}{n}$$

ordinary generating functions

formal power series

Catalan numbers

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

polynomial

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

$$+ \cdots$$

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \ldots + C_n t^n + \ldots$$

Wilf corde à linge

série génératrice

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots\dots$$
$$\dots + a_n t^n + \dots\dots\dots$$

generating function

formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^n + \ldots$$

a little exercise

$$\frac{1}{1-(t+t^2)} = ?$$

$$\frac{1}{1-(t+t^2)} = ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4$$
$$+ 8t^5 + 13t^6 + 21t^7$$
$$+ 34t^8 + 55t^9 + \dots$$

$$\sum_{i \geqslant 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \ldots$$
$$+ (t^5 — — —$$
$$\ddots$$

$$\sum_{i \geqslant 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \dots$$

$$+(t^5 \; - - - -$$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

$$f(t) = \sum_{n \geqslant 0} a_n t^n$$

$$t + t + t + \ldots\ldots t + \ldots$$

$$1 + 1 + 1 + \ldots$$

$$\cancel{t + t + t + \ldots\; t + \ldots}$$

$$\cancel{1 + 1 + 1 + \ldots}$$

formal power series algebra

formalisation

Formal power series

Formal power series algebra in one variable

$\mathbb{K}$ commutative ring

$\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \quad \mathbb{Z}[\alpha, \beta, \ldots]$

$\mathbb{K}[t]$ polynomials algebra

$\deg(P)$ degree

$\mathbb{K}[[t]]$ formal power series algebra

(in one variable t and coefficients in $\mathbb{K}$)

$$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$$a_0 + a_1 t + a_2 t + \dots + a_n t^n + \dots$$

generating power series

of the coefficients (numbers) a_n

$$\sum_{n \geq 0} a_n t^n = f(t)$$

s.g. ordinaire

exponential $\sum a_n \frac{t^n}{n!}$

convergence

- formal
- real (complex)

ultrametric topology

algebra

- sum $\quad f + g = h, \quad a_n + b_n = c_n$
- product $\quad fg = h, \quad c_n = \sum_{\substack{p+q=n \\ p,q \geq 0}} a_p b_q$
- product (by a scalar) $\quad \lambda f = h, \quad c_n = \lambda a_n$

$$f = \sum_{n \geq 0} a_n t^n, \quad g = \sum_{n \geq 0} b_n t^n, \quad h = \sum_{n \geq 0} c_n t^n$$

summable family

$$\sum_{i \in I} f_i^{(t)}$$

infinite product

$$\prod_{i \in I} \left(1 + g_i^{(t)}\right)$$

other operations

- substitution

$$f(t)=\sum_{n\geq 0} a_n t^n \,, \qquad g(t)=\sum_{n\geq 0} b_n t^n$$

$$b_0=0$$

$$f\circ g(t) \; ; \; f(g(t)) = \sum_{n\geq 0} a_n (g(t))^n$$

- Inverse

$$\frac{1}{1-f} = 1+f+f^2+\dots+f^n+\dots$$

(si ord(f) ≥ 1)

- derivative

$$f' \qquad \frac{df}{dt} = \sum_{n\geq 1} n\, a_n t^{n-1}$$

exponential

logarithm

binomial power series

$$\exp(t) = \sum_{n\geq 0} \frac{t^n}{n!}$$

$$\log(1-t)^{-1} = \sum_{n\geq 1} \frac{t^n}{n}$$

$$(1+t)^{\alpha} = \sum_{n\geq 0} \binom{\alpha}{n} t^n$$

$$= \sum_{n\geq 0} \alpha(\alpha-1)\cdots(\alpha-n+1)\frac{t^n}{n!}$$

$\mathrm{ord}(f)\geq 1$ $\quad \exp(f) \quad \log(1+f) \quad (1+f)^{\alpha}$

formal power series in several variables

$$f(t_1, t_2, \dots, t_p) = \sum_{n_1, \dots, n_p} a_{n_1, \dots, n_p} t_1^{n_1} t_2^{n_2} \cdots t_p^{n_p}$$

$\mathbb{K}[t_1, \dots, t_p]$

$\mathbb{K}[[t_1, \dots, t_p]]$

algebras

operations

$\partial / \partial t_i$

rational power series $\qquad \sum_{n \geq 0} a_n t^n = \frac{N(t)}{D(t)}$

$\cap$

algebraic power series $\qquad P(y,t) = 0$

$\cap$

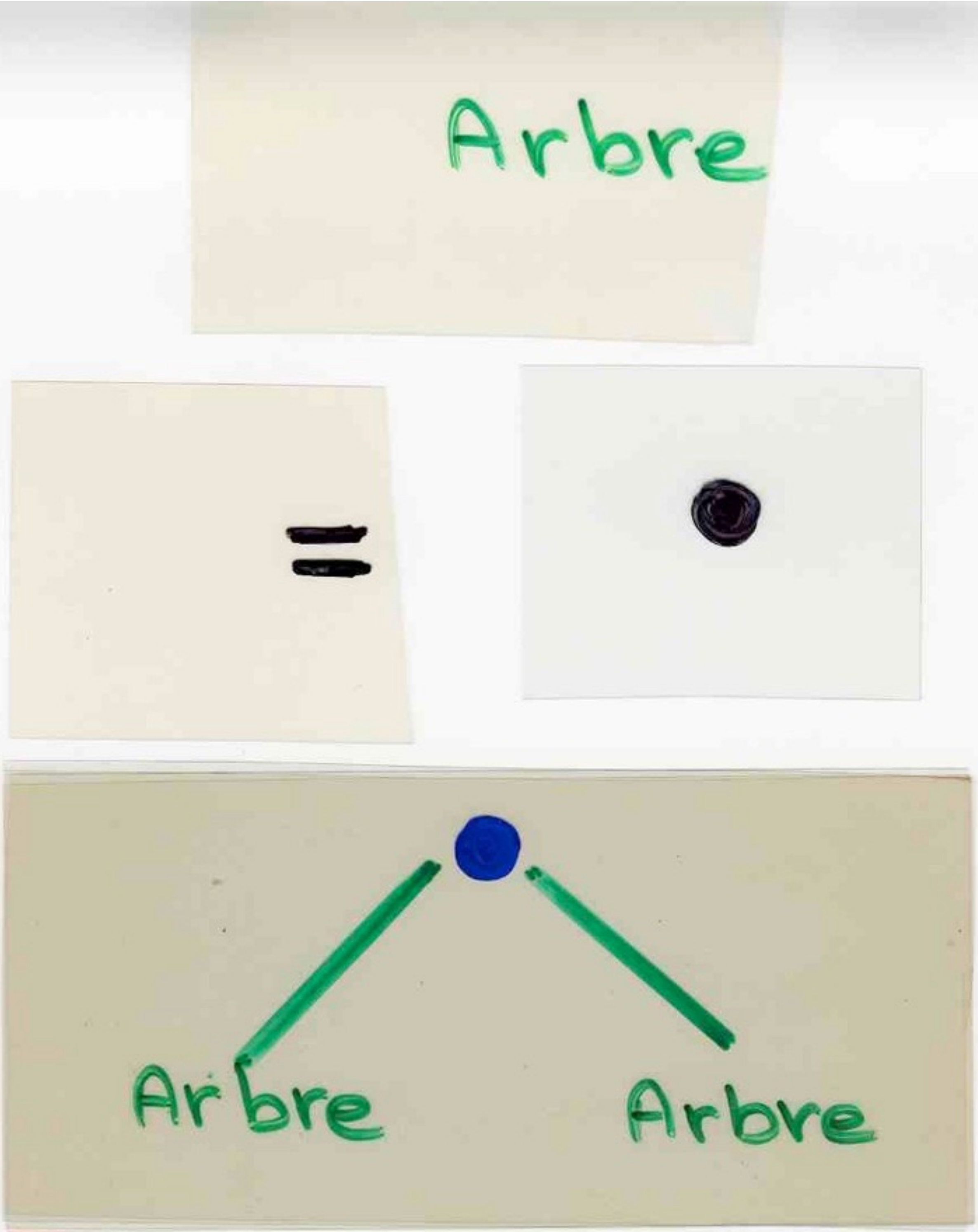
P-recursive (D-finite) power series

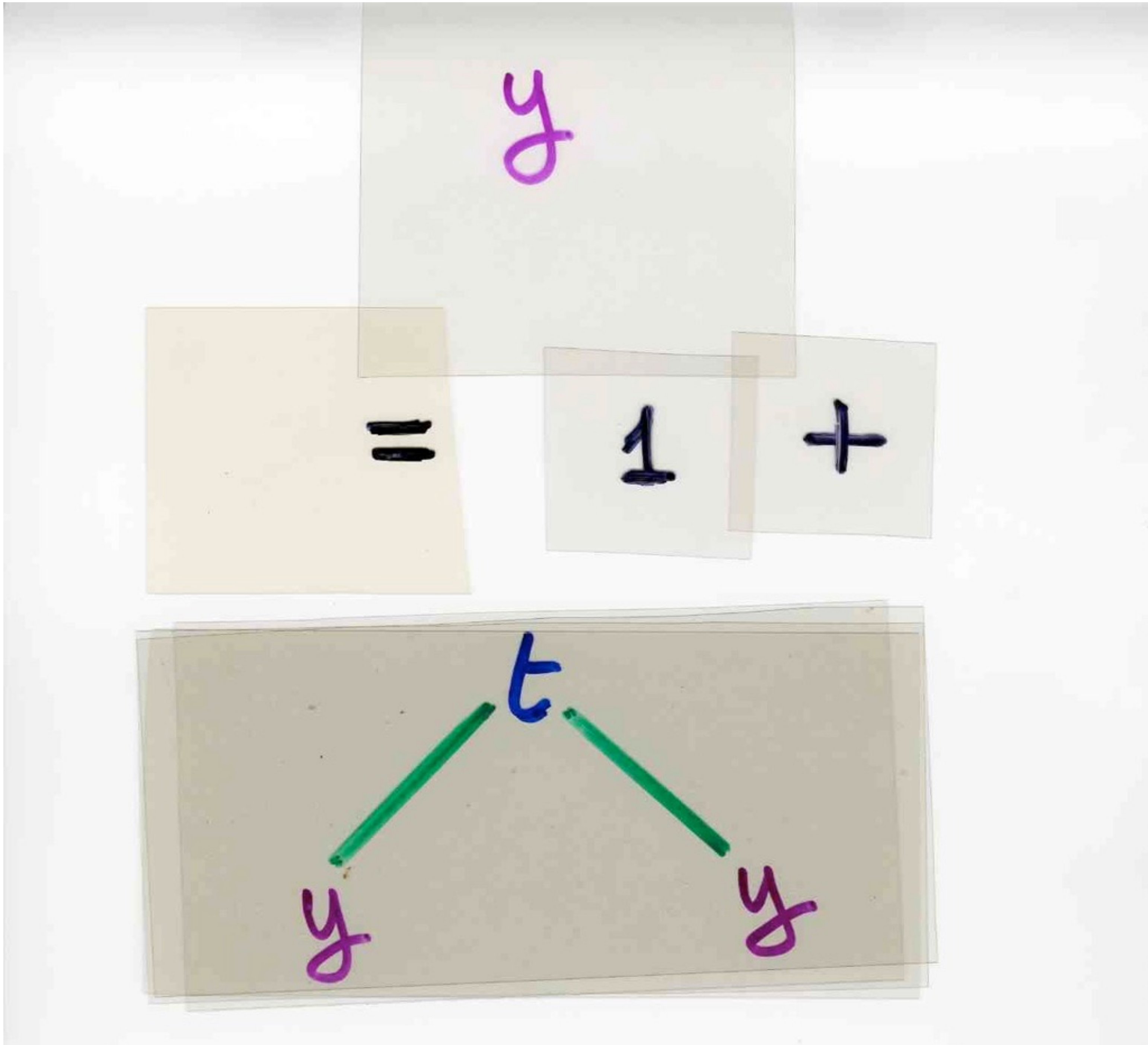
$$P_k(n) a_{n+k} + P_{k-1}(n) a_{n+k-1} + \cdots + P_0(n) a_n = 0$$

operations on combinatorial objects

example: binary trees

Binary Tree




y
=
1
+
t
y
y

$$y = 1 + t(y)^2$$

algebraic equation

$$y = 1 + t\,y^2$$

équation algébrique

$$y = \frac{1-(1-4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \dots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

operations on combinatorial objects

formalisation

Operations on combinatorial objects

Def- class of valued combinatorial objects

$\mathcal{A} = (A, v)$ A finite or enumerable set

$v : A \to \mathbb{K}[x]$

valuation

(*) for w monomial of $\mathbb{K}[x]$,

let $A_w = \{ \alpha \in A$, coeff of w in $v(\alpha)$ is $\neq 0 \}$.

then for every monomial w,

A_w is finite

$v(\alpha)$ weight or valuation of α

$\{ v(\alpha) , \alpha \in A \}$ is summable

Def

$$f_A = \sum_{\alpha \in A} v(\alpha)$$

generating power series of objects $\alpha \in A$

weighted by v

ex: objects of size n $\quad A$

$$X = \{t\} \qquad v(\alpha) = t^n$$

n is the size of α, $|\alpha|$

$a_n = A_{t^n}$ (finite)

$=$ number of objects $\alpha \in A$ of size n

$$f_A = \sum a_n t^n$$

ex: more generally

$$X = \{t\} \cup Y \qquad v(\alpha) = w(\alpha)\, t^n$$

in general $a_0 = 1$, only one *empty* object

ε with weight $v(\varepsilon) = 1$

$\alpha = (A, v_A)$ $\beta = (B, v_B)$

- sum $\alpha + \beta = \mathcal{C}$

$A \cap B = \emptyset$ $= (C, v_C)$

- $C = A \cup B$ (disjoint union)
- $v_C/_A = v_A$ $v_C/_B = v_B$

Lemma $f_{\mathcal{C}} = f_\alpha + f_\beta$

- **product** $\alpha \cdot \beta = \mathcal{C}$

$$= (C, v_C)$$

- $C = A \times B$
- $(\alpha, \beta) \in C$

$$v_C(\alpha, \beta) = v_A(\alpha)\, v_B(\beta)$$

ex: size $|(\alpha, \beta)| = |\alpha| + |\beta|$

ex: binary tree

Lemma $f_{\mathcal{C}} = f_{\alpha} \cdot f_{\beta}$

- **sequence**

$\mathcal{A} = (A, v_A)$ $\quad\quad$ $\mathcal{C} = (C, v_C)$

$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \ldots + \mathcal{A}^n + \ldots$$

$$= \mathcal{A}^*$$

Lemma

$$f_{\mathcal{A}^*} = \frac{1}{1 - f_{\mathcal{A}}}$$

ex: Ferrers diagrams

operations on combinatorial objects

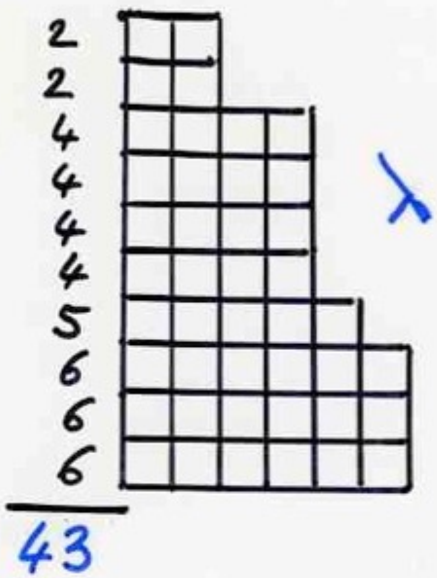
example: integers partitions

q-series

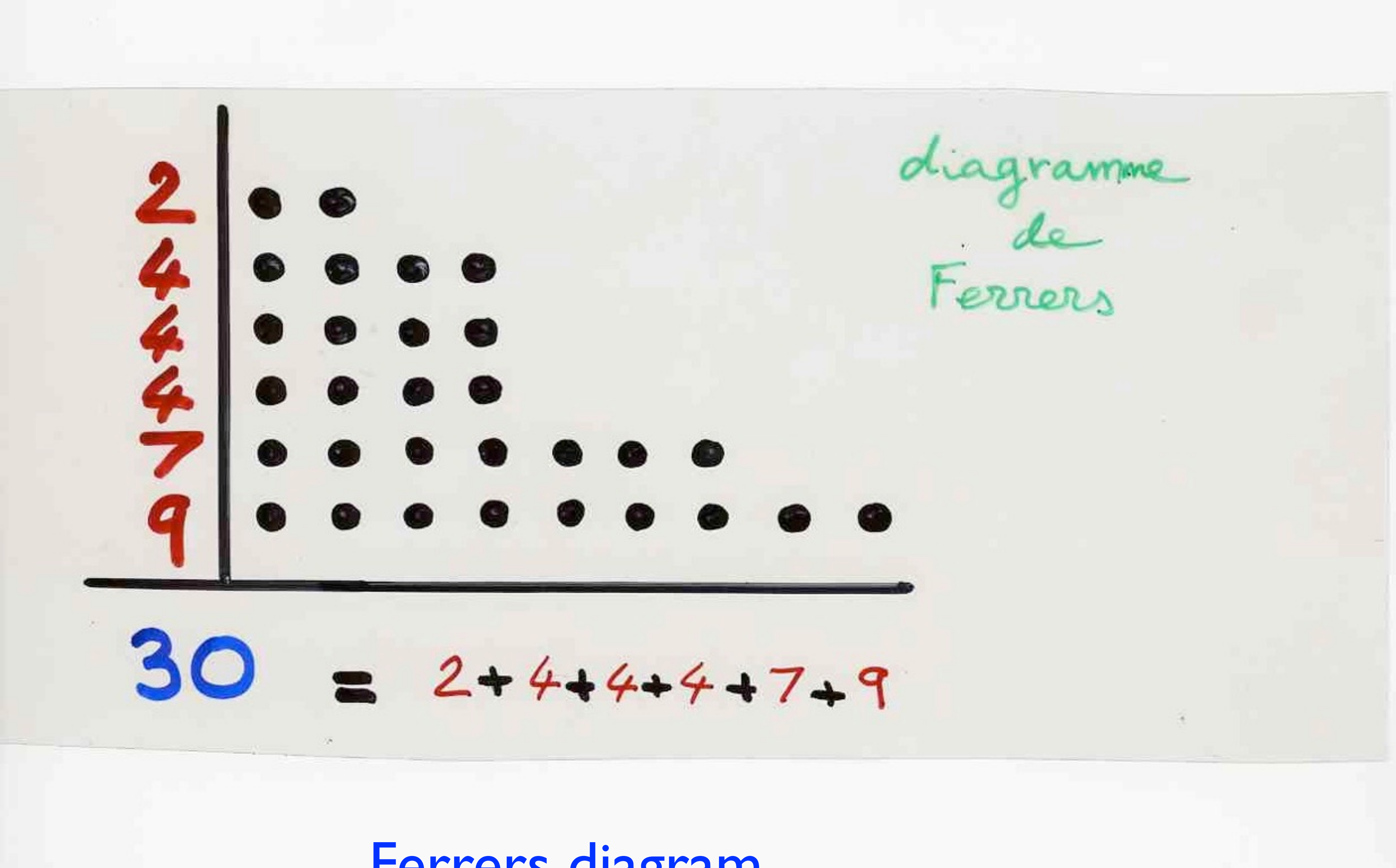
partition of an integer n

$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$



Ferrers diagram



Ferrers diagram

$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots$$

série génératrice

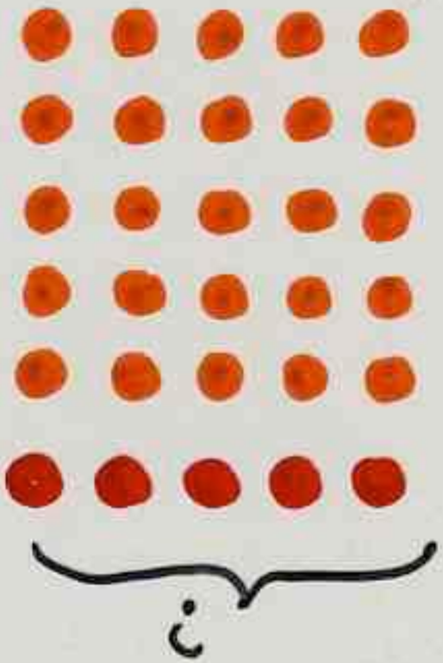
des partitions (d'entiers)

$$\sum_{n\geq 0} a_n q^n$$

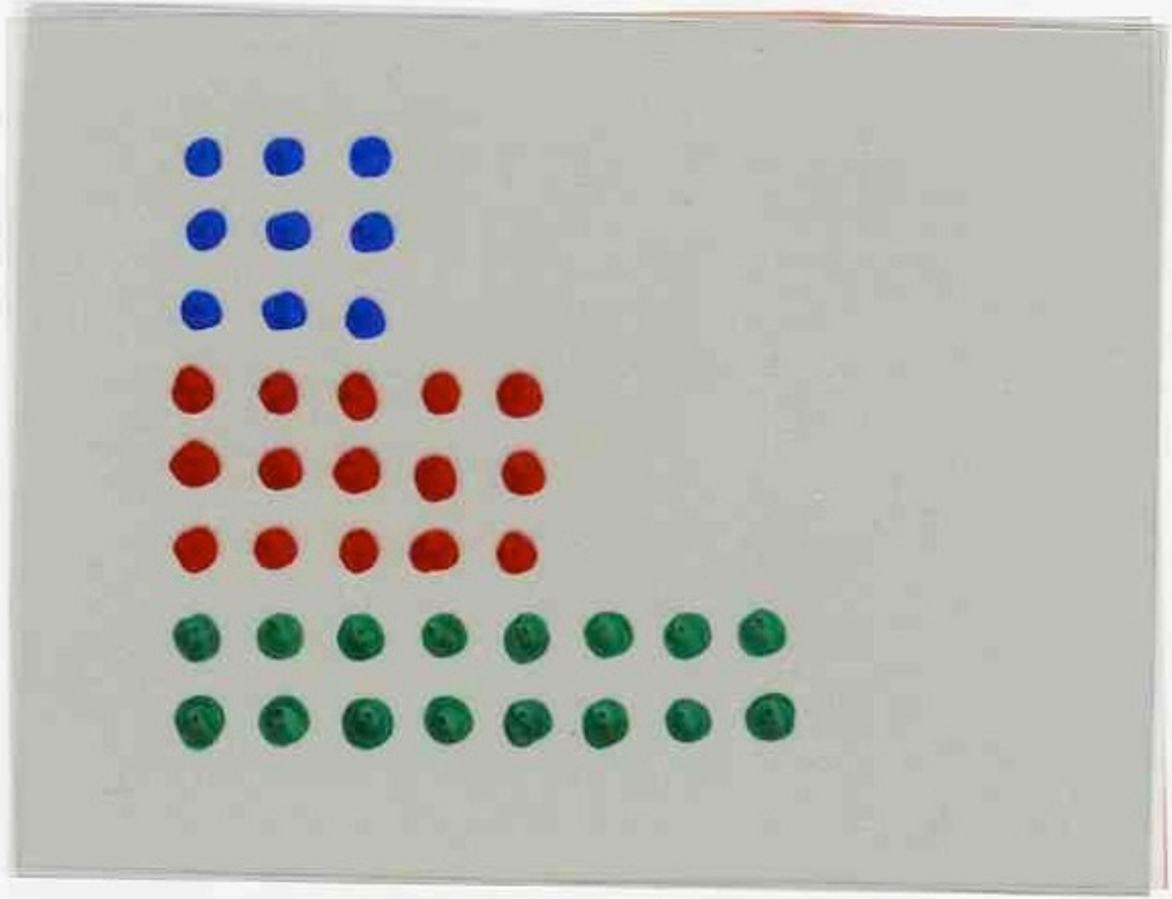
generating function

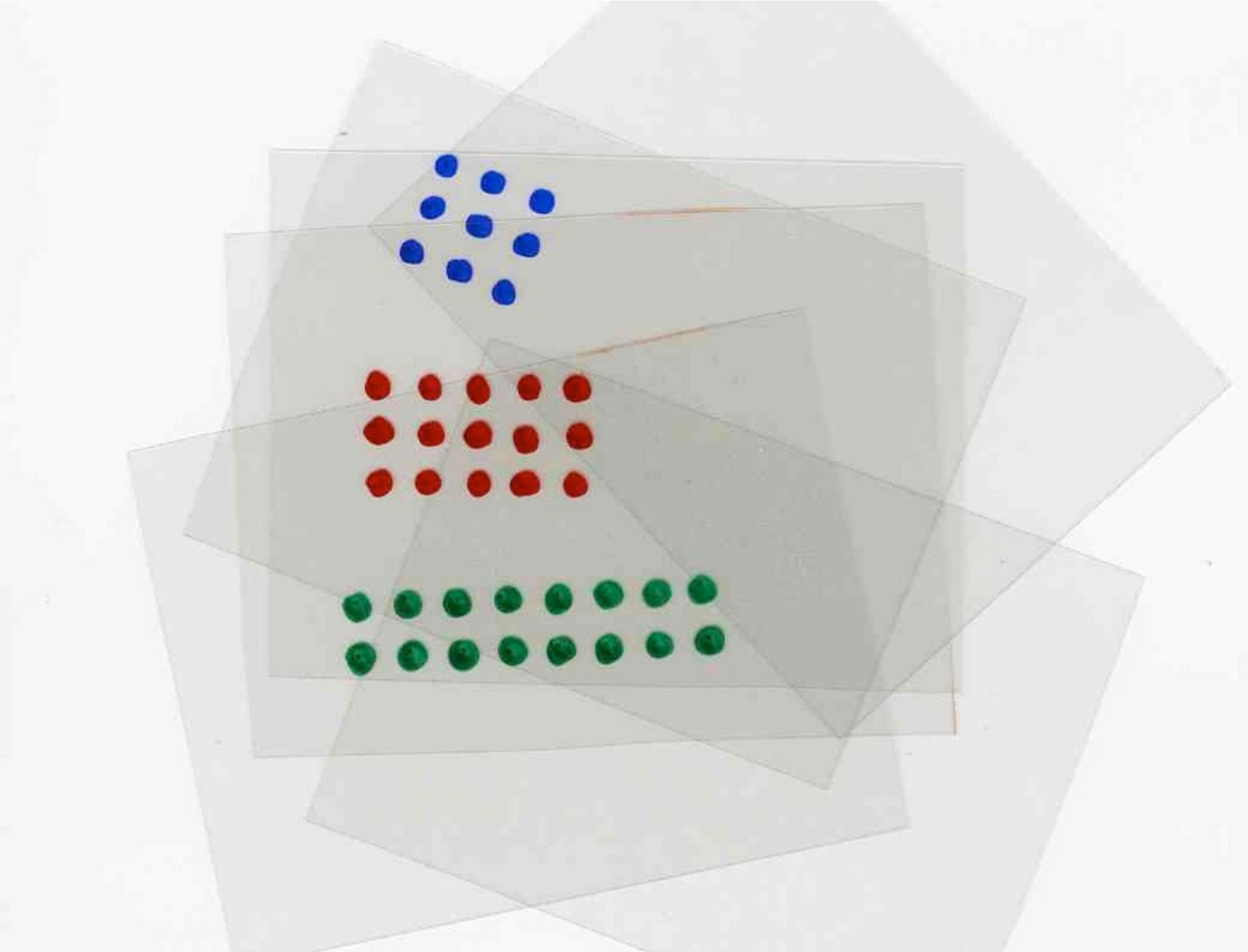
for (integer) partitions

$\underbrace{\bullet \bullet \bullet \bullet \bullet}_{i}$ q^{i}



$$\frac{1}{1-q^{i}}$$





$$\frac{1}{(1-q)(1-q^2)\cdot\cdots\;(1-q^m)}$$

$$\frac{1}{(1-q)(1-q^2)\cdot\cdots\,(1-q^m)}$$

$$\prod_{i\geq 1}\frac{1}{(1-q^i)}$$

exercise 1

$$\sum_{n\geq 0} p(n, I)\, q^n = \prod_{i\in I} \frac{1}{1-q^i}$$

partitions

parts $\lambda_j \in I$

exercise 2

D- partition $\lambda_i - \lambda_{i+1} \geq 2$

$\lambda = (\lambda_1, \dots, \lambda_k)$ $(1 \leq i < k)$

série génératrice des D- partitions

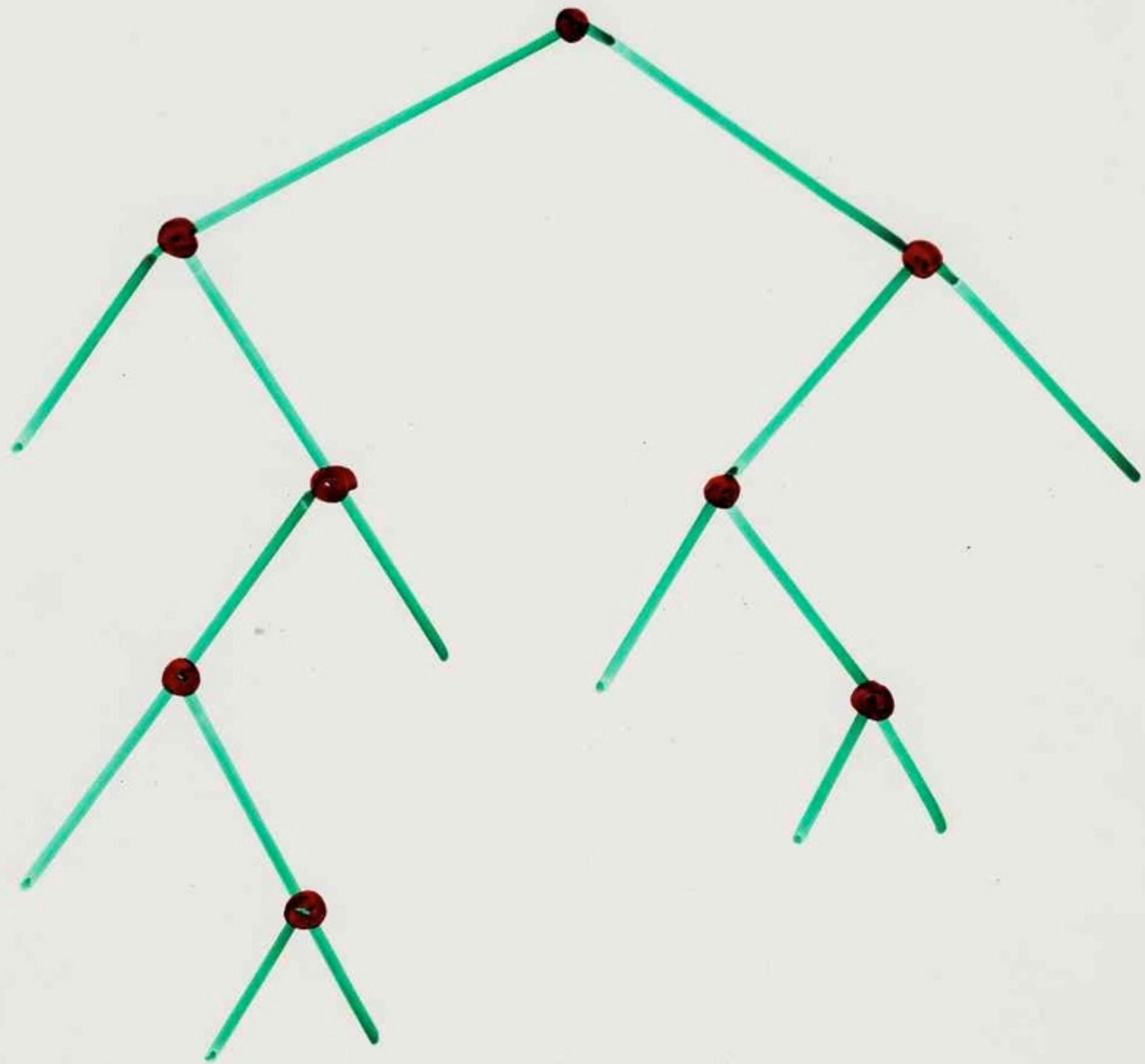
$$\sum_{m\geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

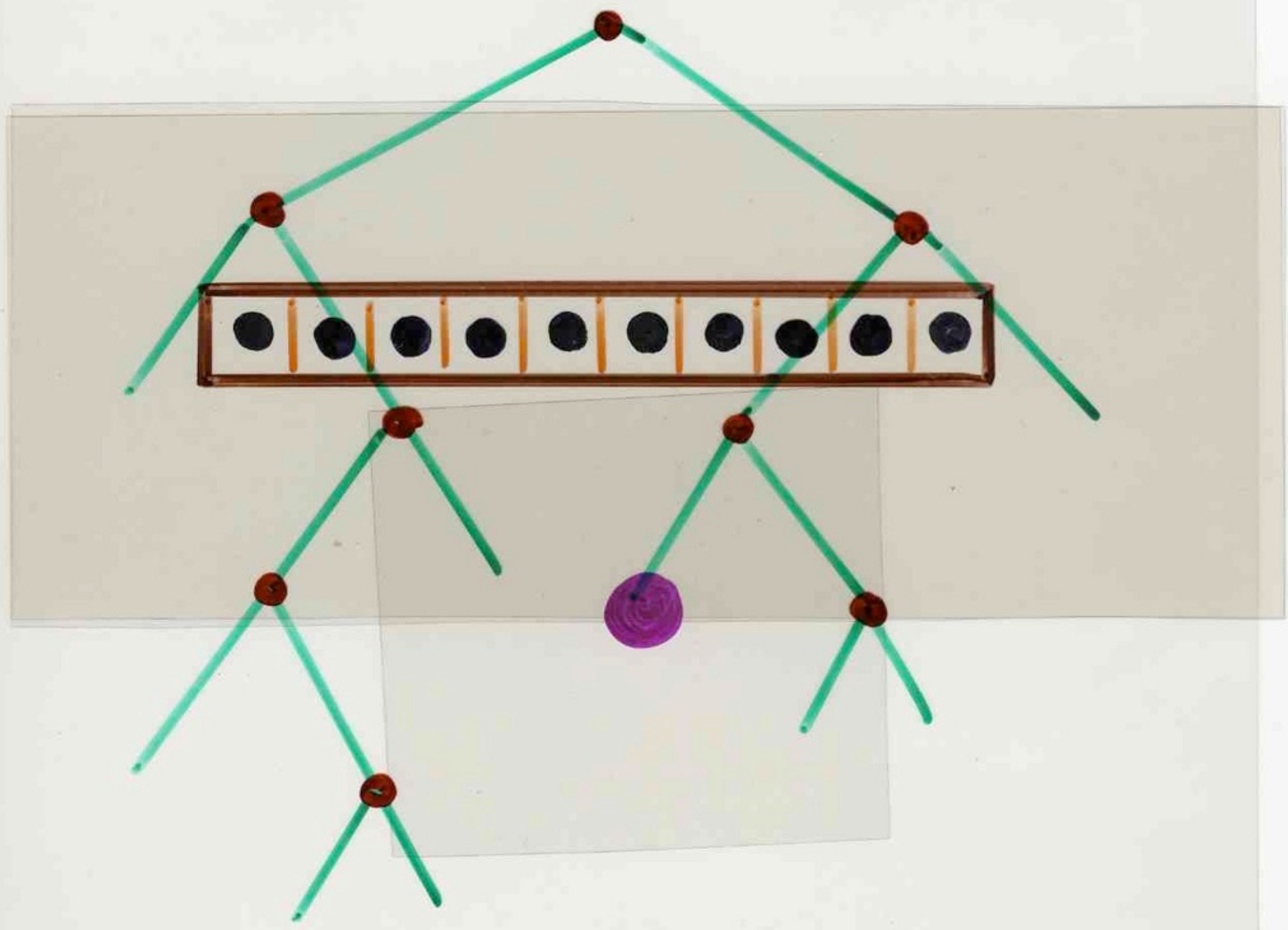
bijective combinatorics

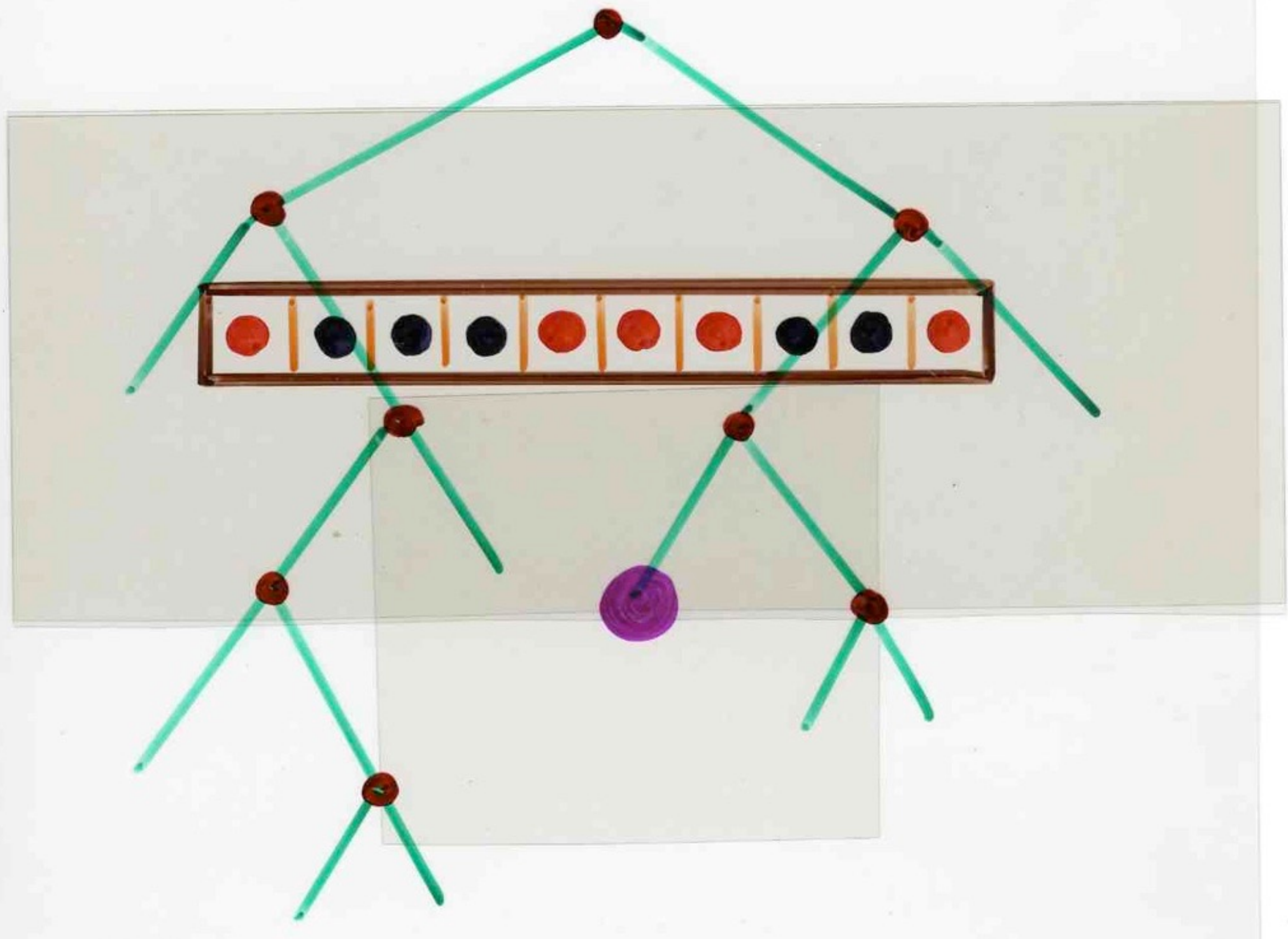
example: Catalan numbers

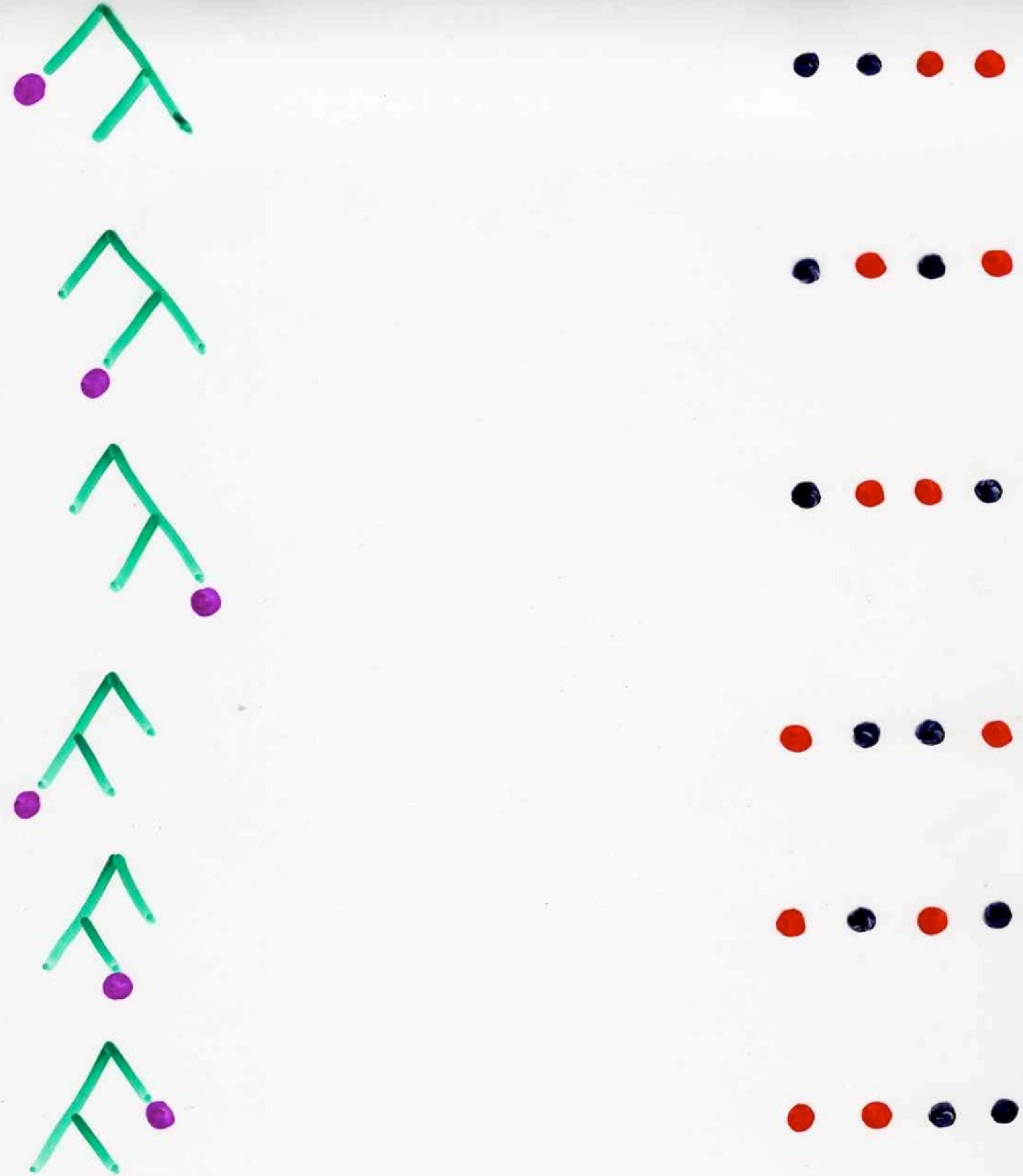
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

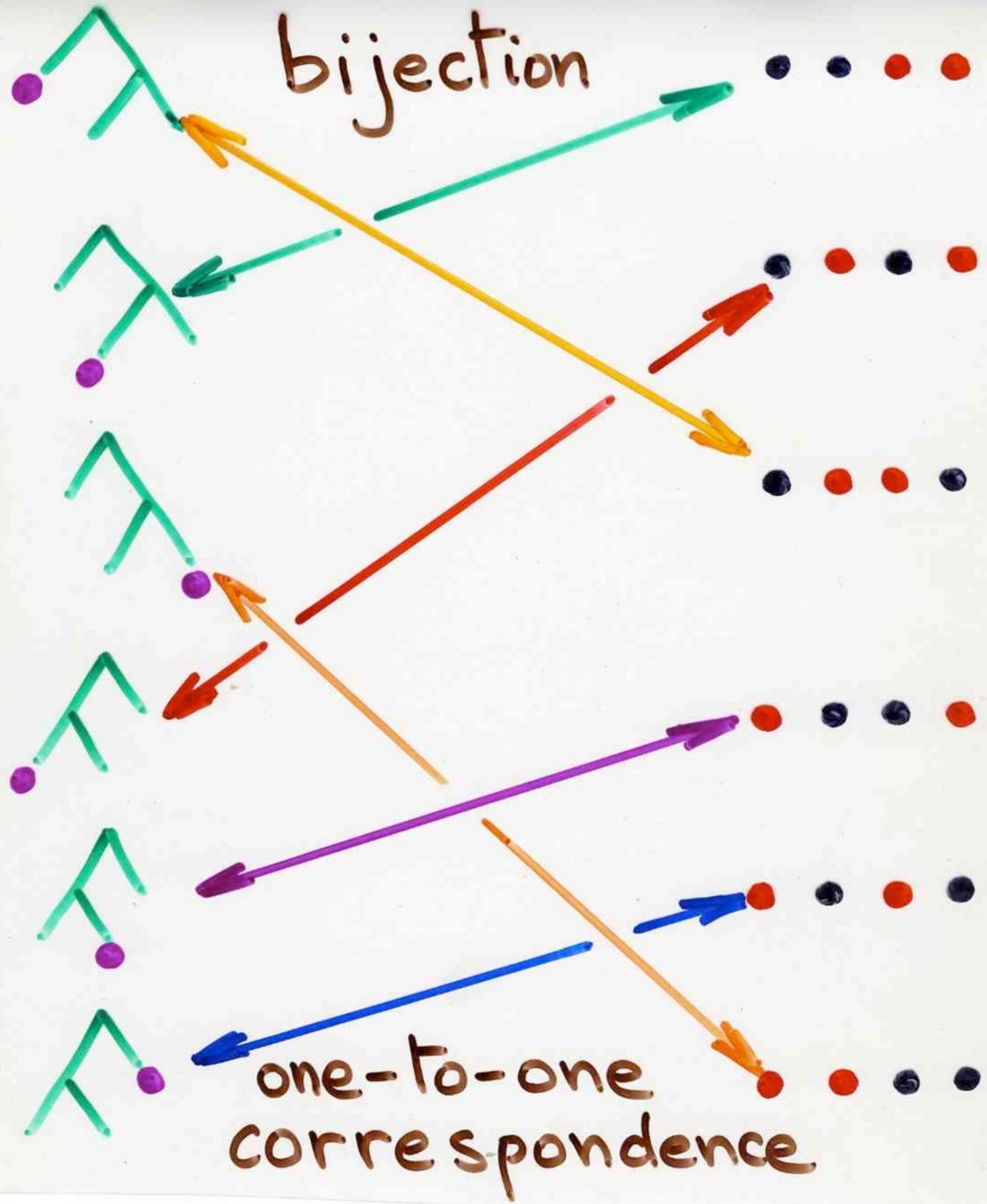
$$(n+1)\,C_n = \binom{2n}{n}$$










bijection
one-to-one
correspondence

exercise 3:

give a bijective proof of the following identity

(multiplicative recurrence relation for Catalan numbers)

$$2(2n+1)C_n = (n+2)C_{n+1}$$

binary trees

generating power series

power series algebra

operations on combinatorial objects

formalisation

example: integers partitions

bijective combinatorics