Combinatorics and Physics

Chapter 2b
Dimers, Tilings, Non-crossing paths and determinant

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§3 Formulae for binomial determinant

$$\text{det} = \frac{(\text{product})}{(\text{product})}$$

<u>Lemma 1</u>. If $b_1 \neq 0$, then

$$\binom{a_1, \dots, a_k}{b_1, \dots, b_k} = \frac{a_1 \cdots a_k}{b_1 \cdots b_k} \binom{a_1 - 1, \dots, a_k - 1}{b_1 - 1, \dots, b_k - 1}$$

Lemma 1. If $b_1 \neq 0$, then

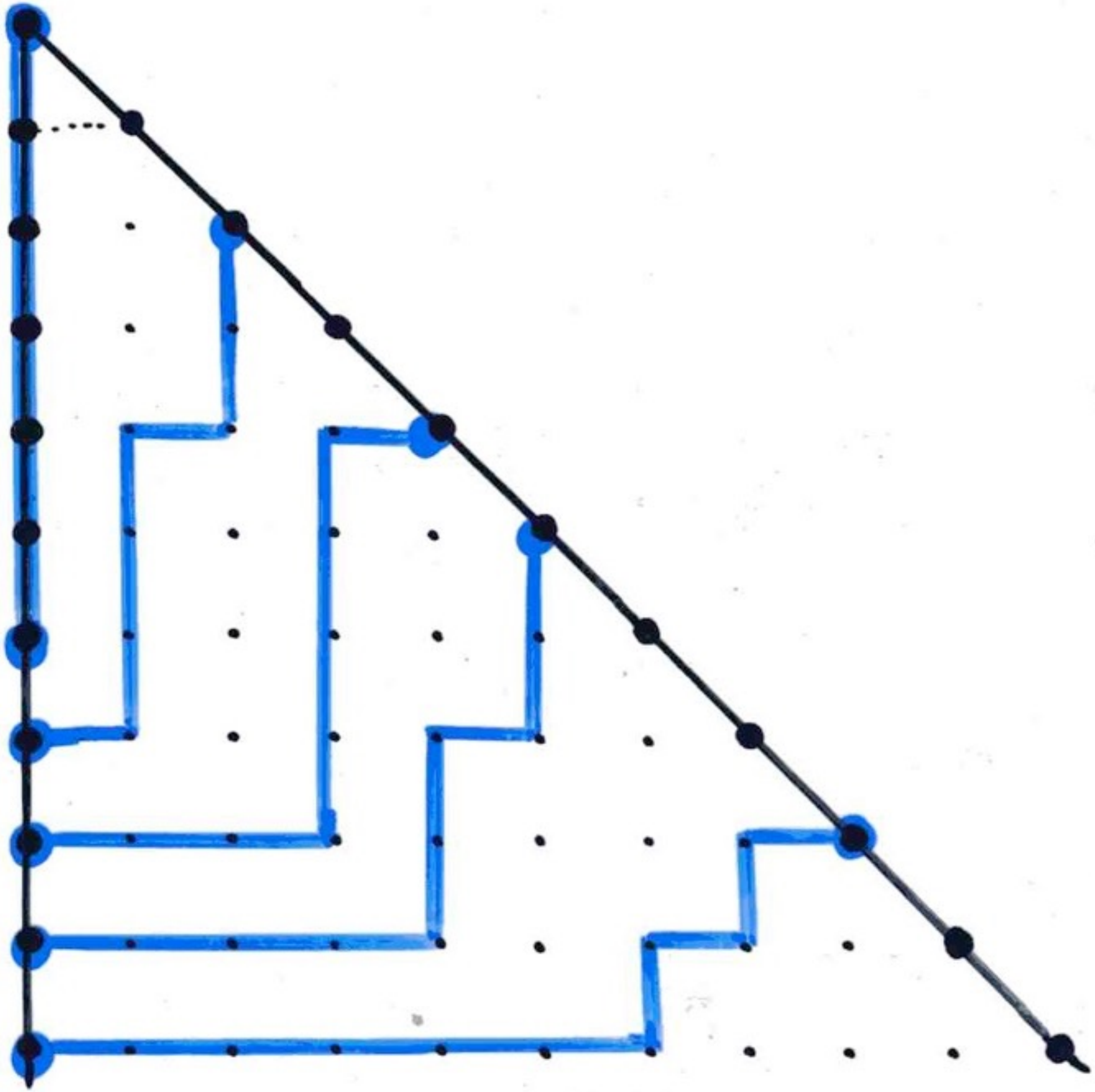
$$\binom{a_1, \dots, a_k}{b_1, \dots, b_k} = \frac{a_1 \cdots a_k}{b_1 \cdots b_k} \binom{a_1 - 1, \dots, a_k - 1}{b_1 - 1, \dots, b_k - 1}$$

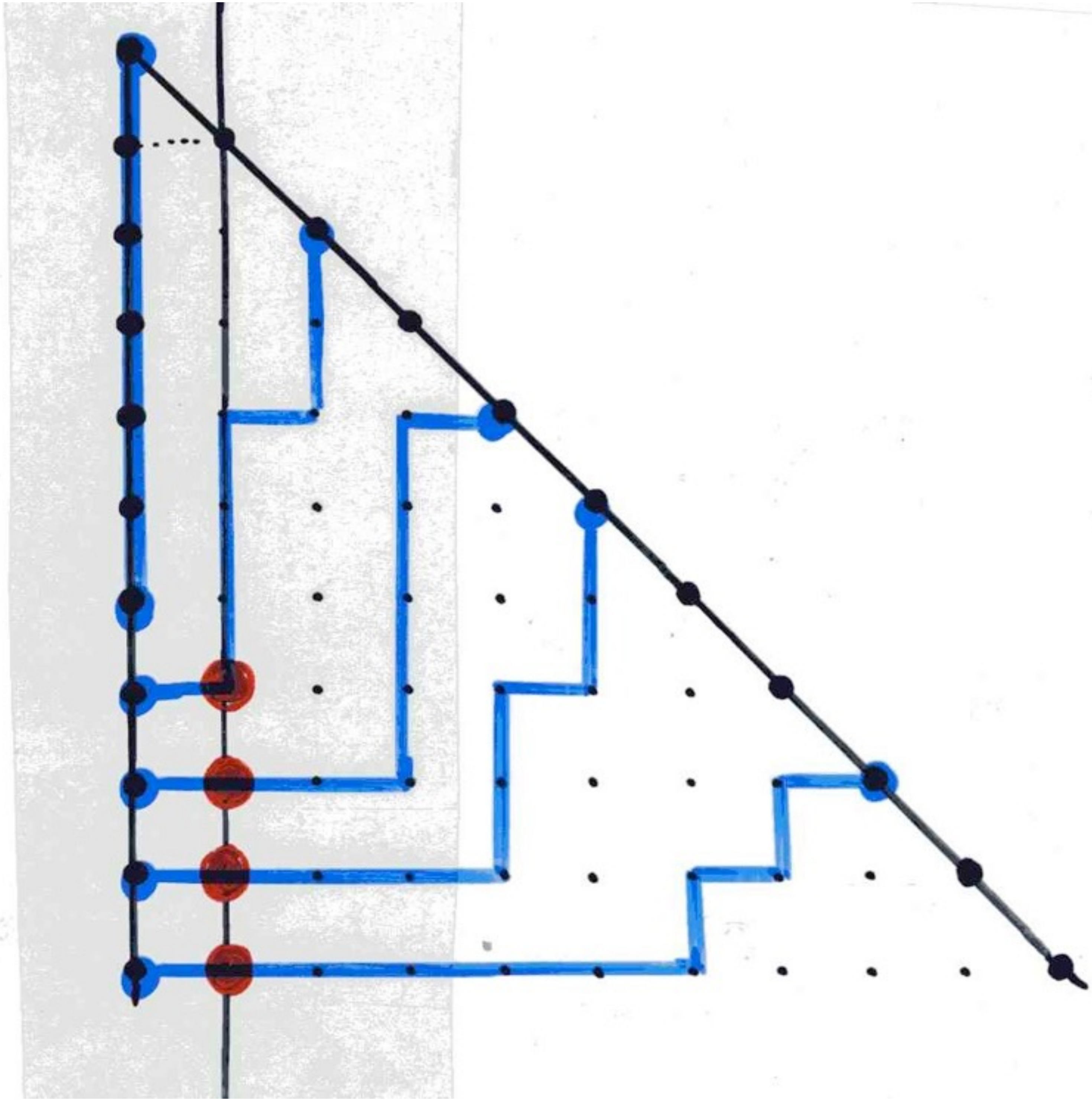
$$\binom{a_1, \dots\dots, a_k}{b_1, \dots, b_k}$$

$$= \det\left(\binom{a_i}{b_j}\right)_{1 \leq i,j \leq k}$$

Lemma 2 -

$$\begin{pmatrix} a, & a+1, & \dots, & a+k-1 \\ 0, & b_2, & \dots, & b_k \end{pmatrix} = \begin{pmatrix} a, & a+1, & \dots, & a+k-2 \\ b_2-1, & b_3-1, & \dots, & b_k-1 \end{pmatrix}$$

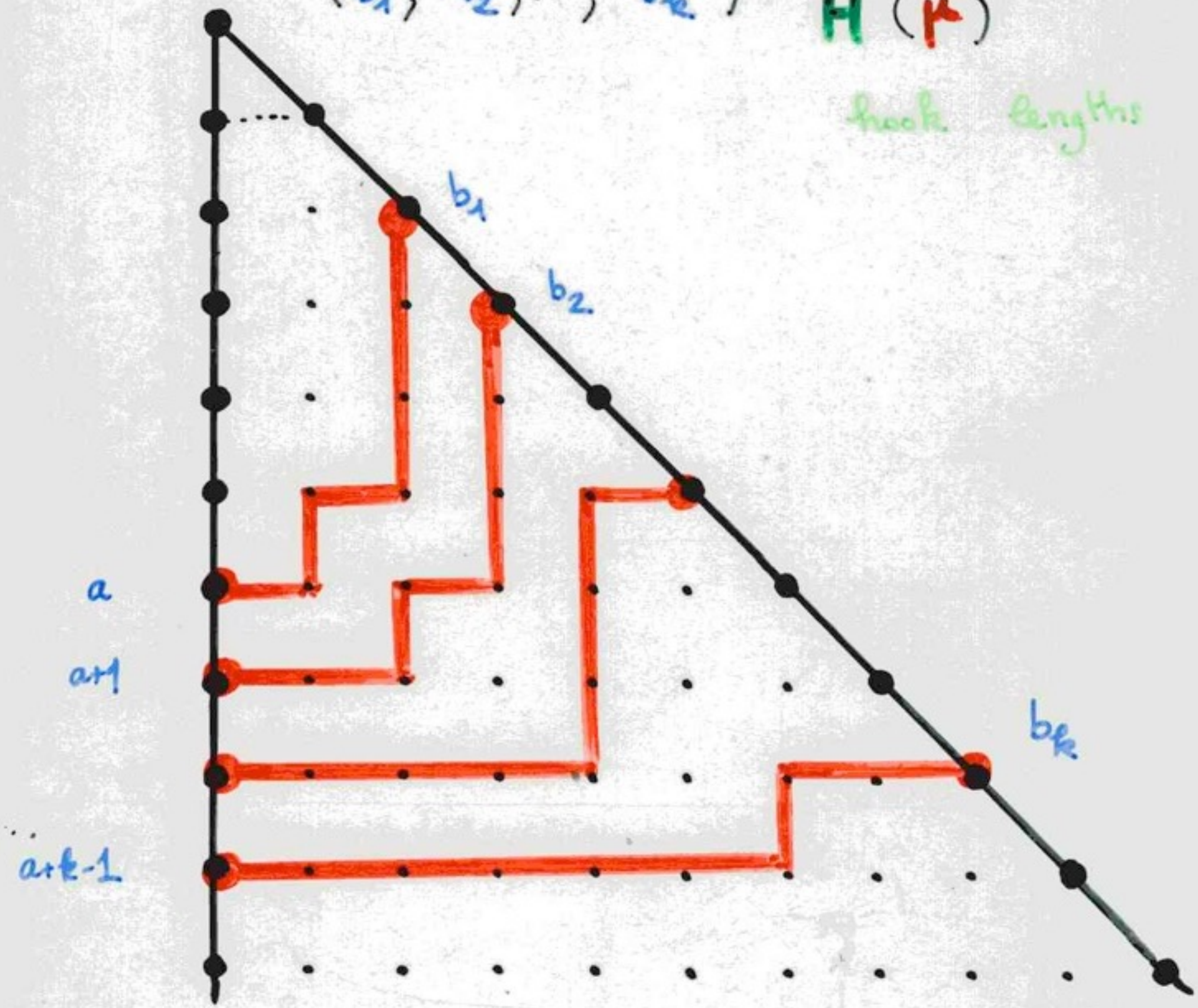




contents

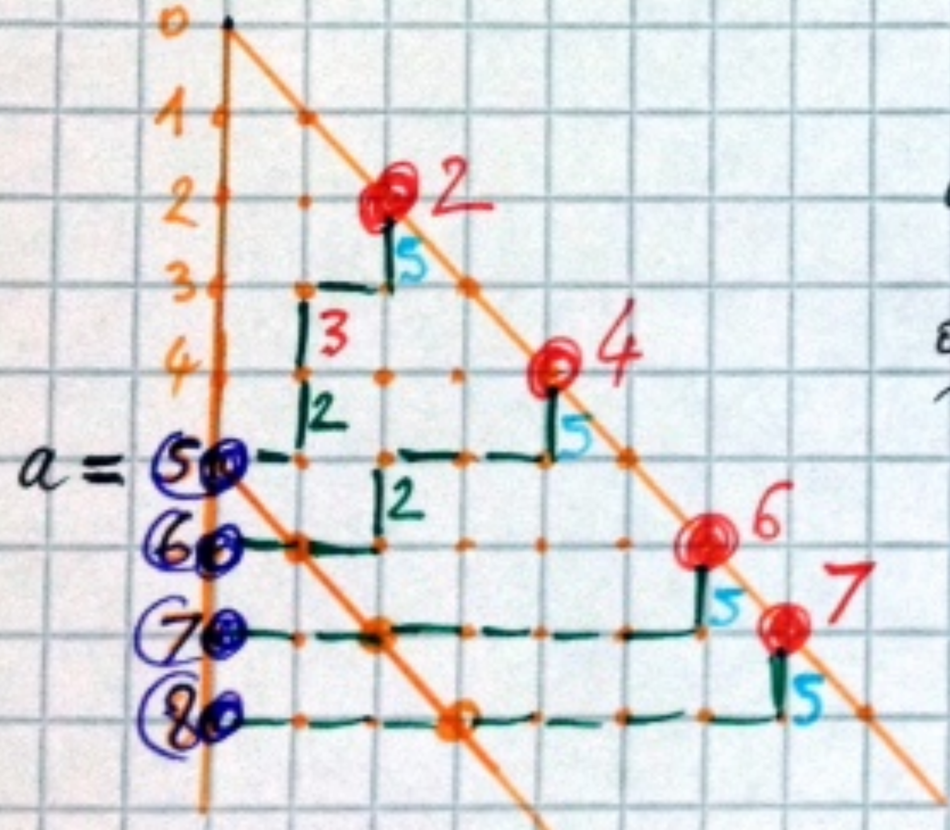
$$\begin{pmatrix} a, & a+1, & \dots, & a+k-1 \\ b_1, & b_2, & \dots, & b_k \end{pmatrix} = \frac{C_a(\mu)}{H(\mu)}$$

hook lengths

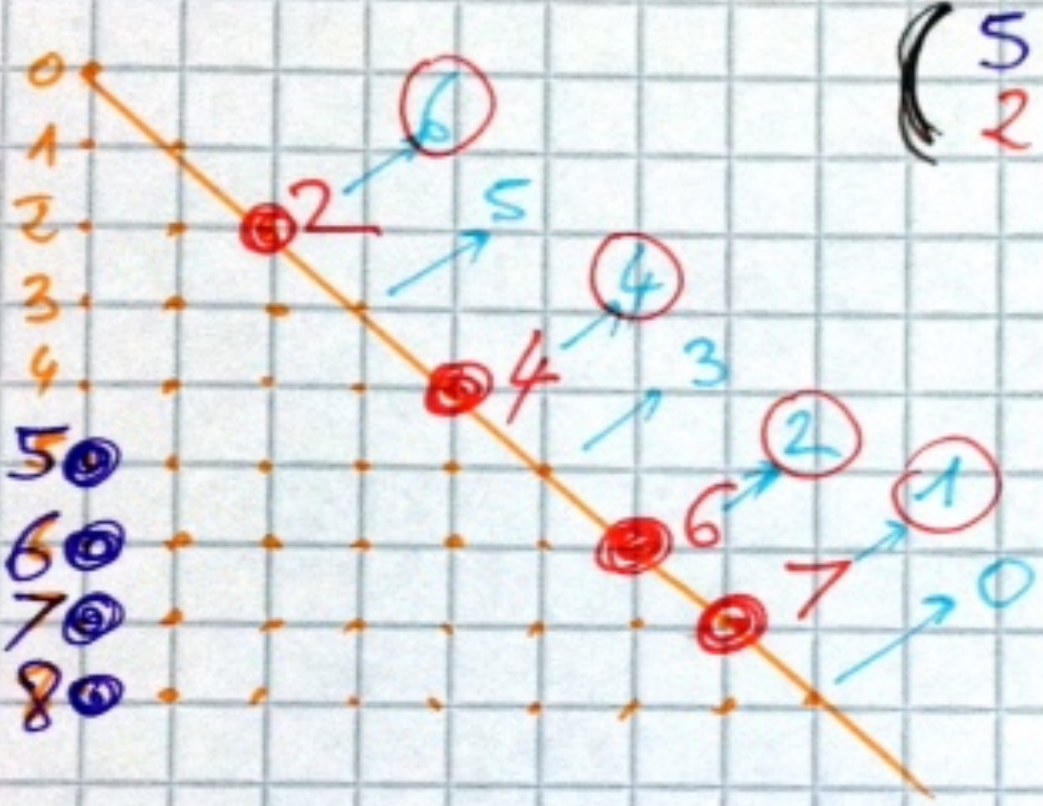


example.

binomial determinant $\begin{pmatrix} 5, 6, 7, 8 \\ 2, 4, 6, 7 \end{pmatrix}$



a configuration of non-crossing paths related to $\begin{pmatrix} 5, 6, 7, 8 \\ 2, 4, 6, 7 \end{pmatrix}$



$$\begin{pmatrix} 5, & 6, & 7, & 8 \\ 2, & 4, & 6, & 7 \end{pmatrix}$$

$$\downarrow$$

$$6 > 4 > 2 > 1$$

$6-3 \quad 4-2 \quad 2-1$

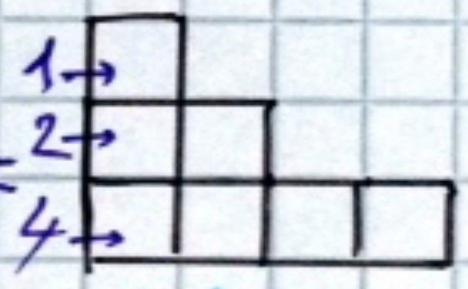
$$\downarrow$$

$$3 \geqslant 2 \geqslant 1 \geqslant 1 = \lambda^*$$

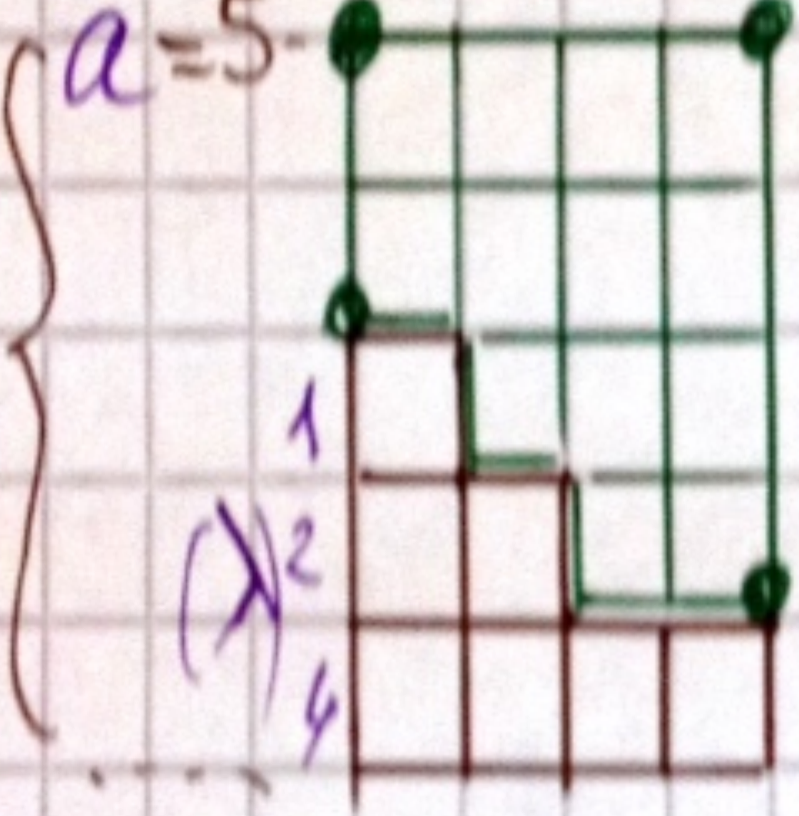
$$\lambda = (4, 2, 1)$$

(transpose)

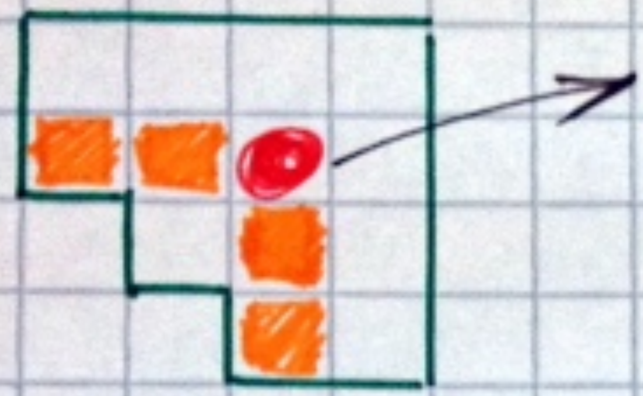
$\lambda =$



$(3\ 2\ 1\ 1) = \lambda^*$



$= \mu$ complement of λ according to a



hook
length
= 5

hook lengths

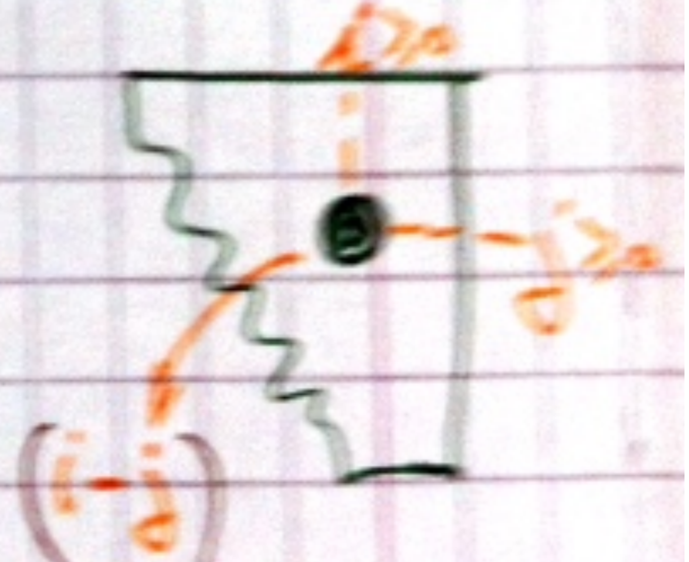
2	4	6	7
1	3	5	6
	1	3	4
		1	2

8	7	6	5
7	6	5	4
	5	4	3
		3	2

contents + a

3	2	1	0
2	1	0	-1
	0	-1	-2
		-2	-3

contents



$$\begin{pmatrix}5,6,7,8\\2,4,6,7\end{pmatrix} = \frac{5.6.7.8}{2.4.6.7}\times\underbrace{\begin{pmatrix}4,5,6,7\\1,3,5,6\end{pmatrix}}_{=}$$

$$\frac{4.5.6.7}{1.3.5.6}\times\underbrace{\begin{pmatrix}3,4,5,6\\0,2,4,5\end{pmatrix}}_{=}$$

$$\begin{pmatrix}3,4,5\\1,3,4\end{pmatrix}$$

$$\begin{pmatrix}3 & 4 & 5\\1 & 3 & 4\end{pmatrix} = \frac{3\times4\times5}{1\times3\times4}\times\underbrace{\begin{pmatrix}2,3,4\\0,2,3\end{pmatrix}}_{=}$$

$$\begin{pmatrix}2,3\\1,2\end{pmatrix} = \frac{2\times3}{1\times2}\begin{pmatrix}1,2\\0,1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = 1$$

hook lengths

$$\begin{array}{|cccc|} 2 & 4 & 6 & 7 \\ 1 & 3 & 5 & 6 \\ & 1 & 3 & 4 \\ & & 1 & 2 \end{array} \qquad \begin{array}{|cccc|} 8 & 7 & 6 & 5 \\ 7 & 6 & 5 & 4 \\ & 5 & 4 & 3 \\ & & 3 & 2 \end{array}$$

contents + a

$$\begin{pmatrix} 5,6,7,8 \\ 2,4,6,7 \end{pmatrix} = \frac{5.6.7.8}{2.4.6.7} \times \underbrace{\begin{pmatrix} 4,5,6,7 \\ 1,3,5,6 \end{pmatrix}}_{= \frac{4.5.6.7}{1.3.5.6} \times \underbrace{\begin{pmatrix} 3,4,5,6 \\ 0,2,4,5 \end{pmatrix}}_{= \begin{pmatrix} 3,4,5 \\ 1,3,4 \end{pmatrix}}}$$

$$\begin{pmatrix} 3\ 4\ 5 \\ 1\ 3\ 4 \end{pmatrix} = \frac{3 \times 4 \times 5}{1 \times 3 \times 4} \times \underbrace{\begin{pmatrix} 2,3,4 \\ 0,2,3 \end{pmatrix}}_{=}$$

$$\begin{pmatrix} 2,3 \\ 1,2 \end{pmatrix} = \frac{2 \times 3}{1 \times 2} \begin{pmatrix} 1,2 \\ 0,1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\begin{pmatrix}4, & 5, & 6, & 7\\ 1, & 3, & 5, & 6\end{pmatrix} = \frac{\text{product}\begin{pmatrix}8 & 7 & 6 & 5\\ 7 & 6 & 5 & 4\\ & 5 & 4 & 3\\ & & 3 & 2\end{pmatrix}}{\text{product}\begin{pmatrix}2 & 4 & 6 & 7\\ 1 & 3 & 5 & 6\\ & 1 & 3 & 4\\ & & 1 & 2\end{pmatrix}}$$

$$\begin{pmatrix}8 & 7 & 6 & 5\\ 7 & 6 & 5 & 4\\ & 5 & 4 & 3\\ & & 3 & 2\end{pmatrix} \to \mathbb{C}_a(\mu)$$

$$\begin{pmatrix}2 & 4 & 6 & 7\\ 1 & 3 & 5 & 6\\ & 1 & 3 & 4\\ & & 1 & 2\end{pmatrix} \to H(\mu)$$

$$\begin{pmatrix} 4, & 5, & 6, & 7 \\ 1, & 3, & 5, & 6 \end{pmatrix} = 2^2 \times 5^2 \times 7 = 700$$

exercise

another formula

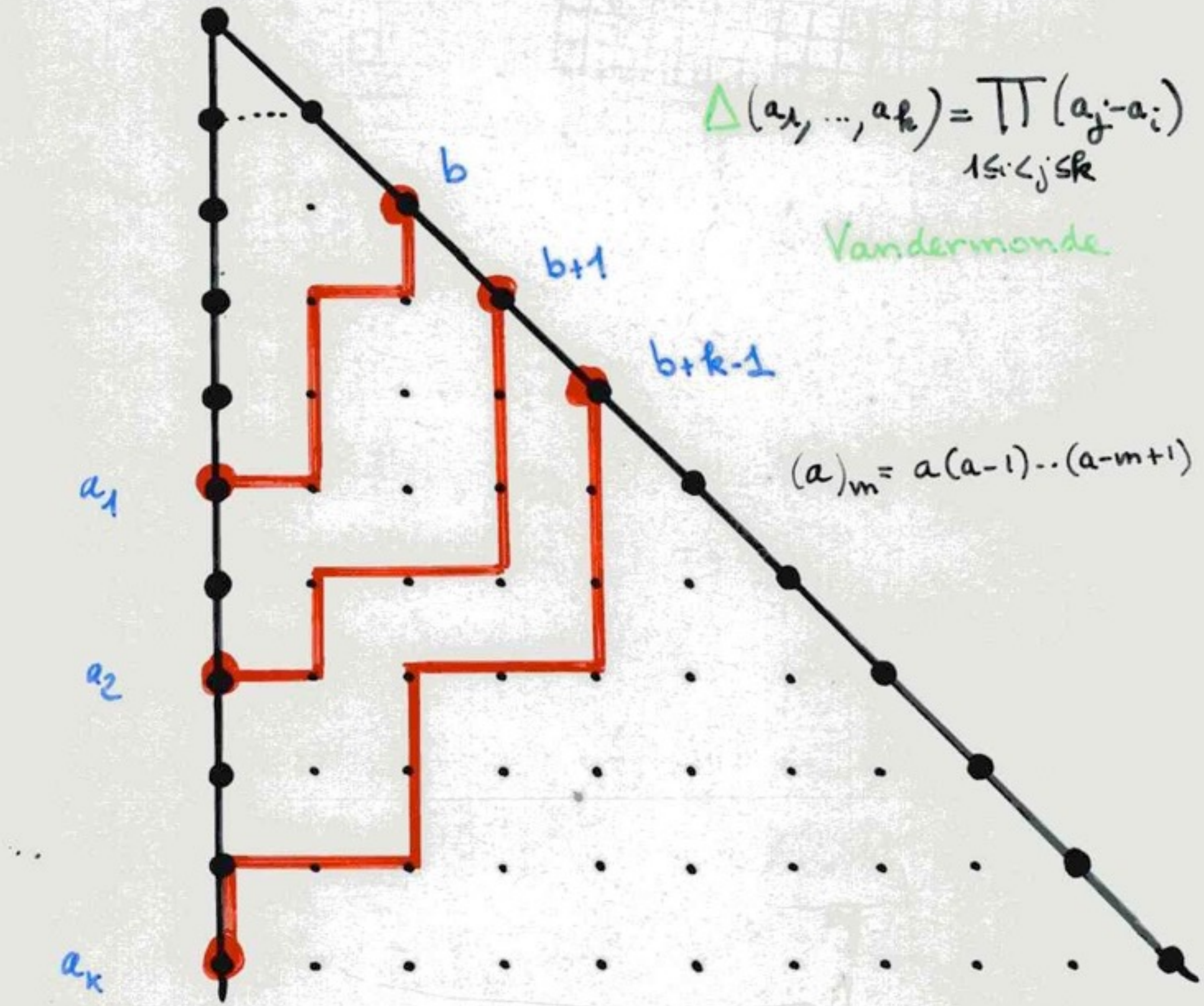
for binomial determinant

$$\binom{a_1,\dots,a_k}{b,\ b+1,\dots,b+k-1} = \frac{(a_1)_b \cdots (a_k)_b}{b! \cdots (b+k-1)!}\,\Delta(a_1,\dots,a_k)$$

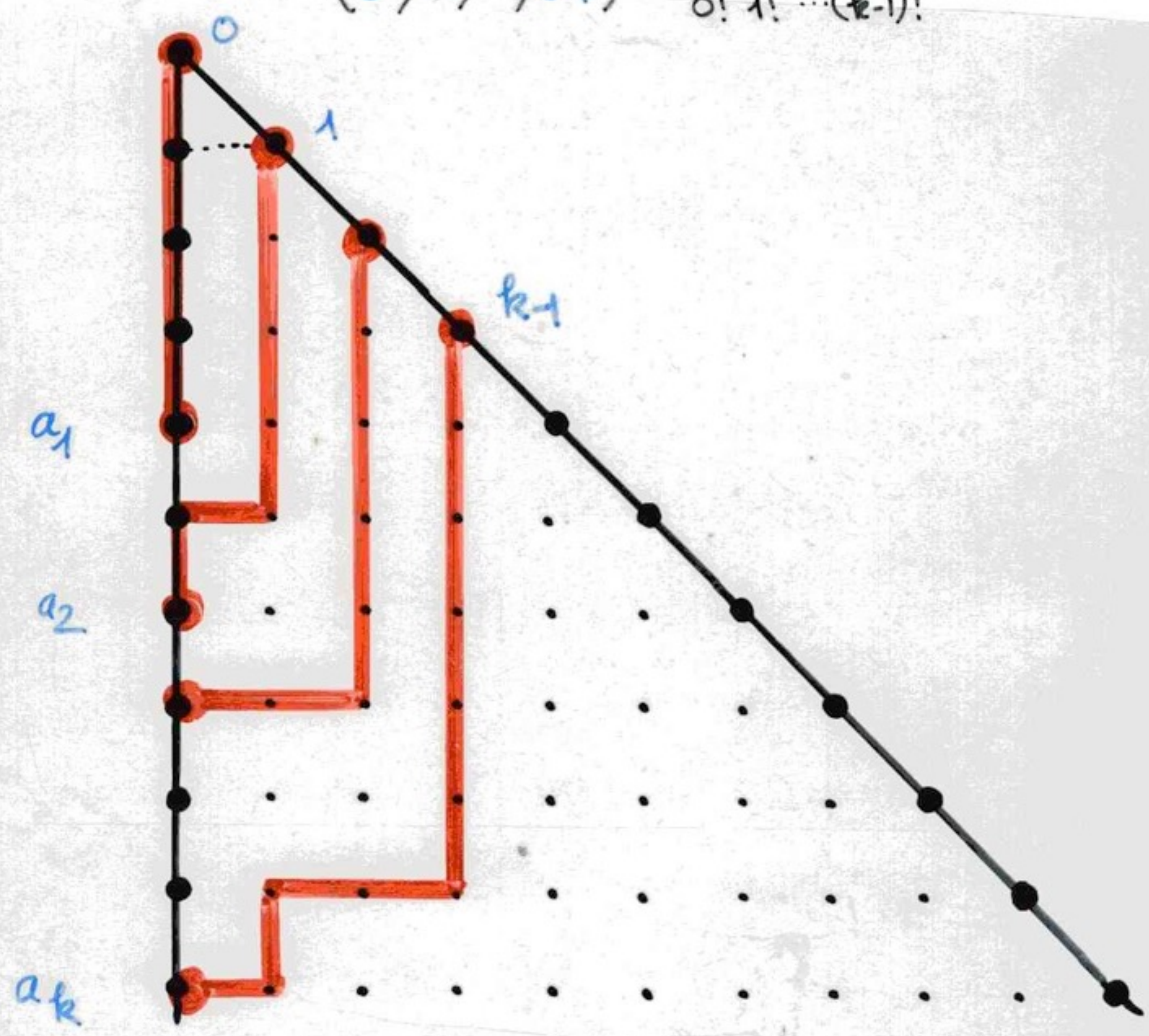
$$\Delta(a_1,\dots,a_k) = \prod_{1\le i<j\le k}(a_j - a_i)$$

Vandermonde

$$(a)_m = a(a-1)\cdots(a-m+1)$$



$$\binom{a_1, a_2, \dots, a_k}{0, 1, \dots, k-1} = \frac{\Delta(a_1, \dots, a_k)}{0!\, 1! \cdots (k-1)!}$$



Vandermonde determinant

exercise

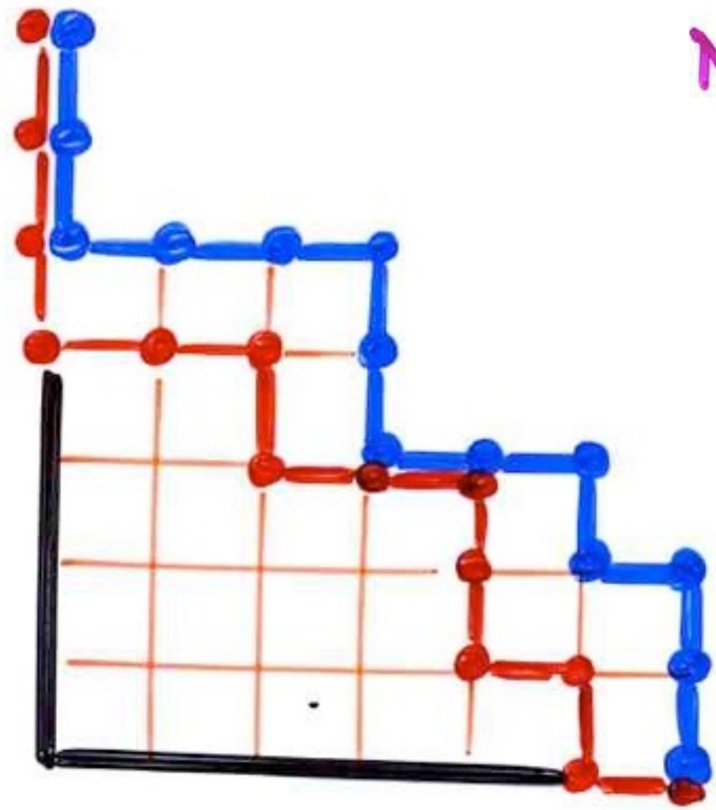
MacMahon- Narayana determinant

MacMahon

déterminant

Krewevas

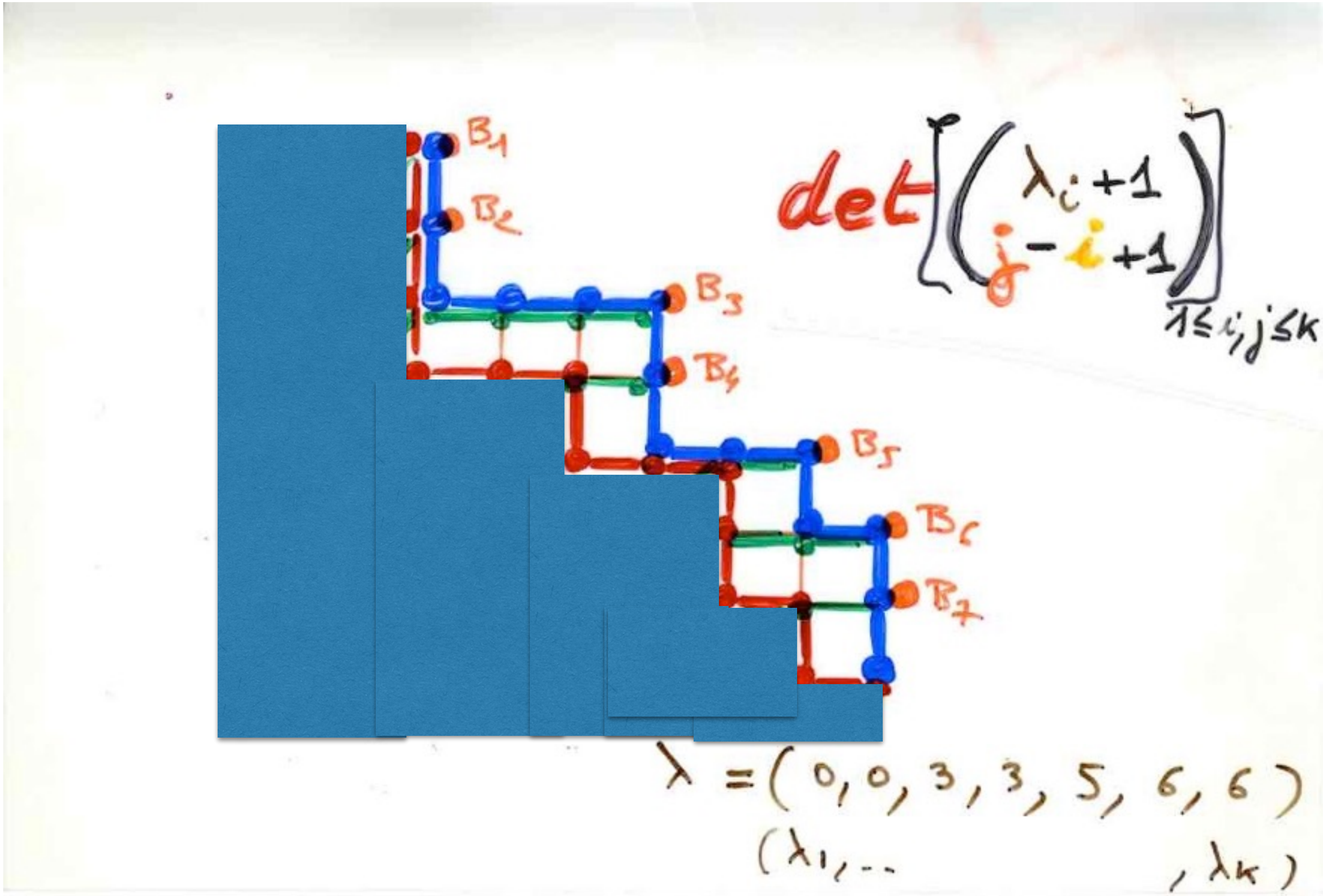
Narayana



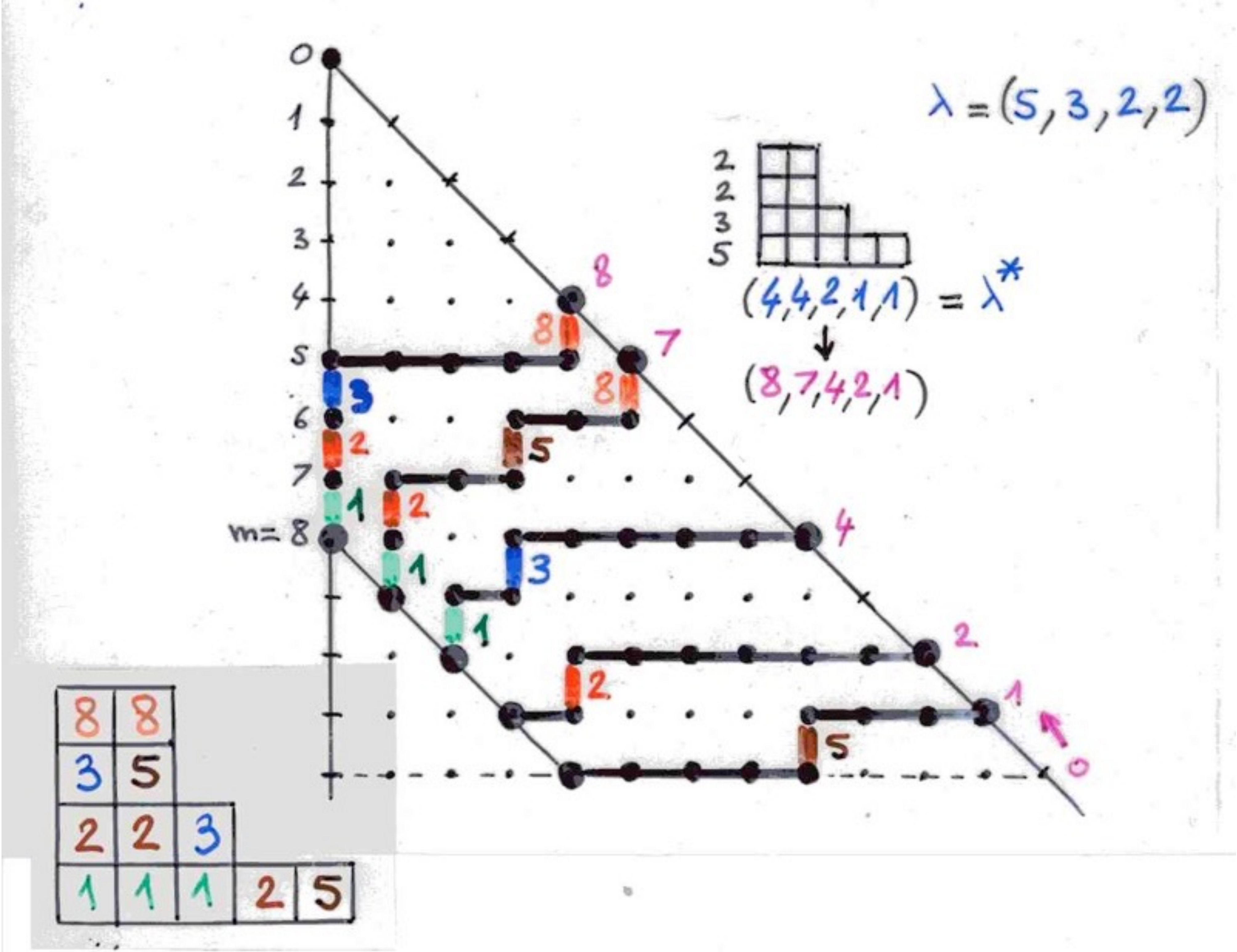
$\lambda = (0, 0, 3, 3, 5, 6, 6)$

TASEP model

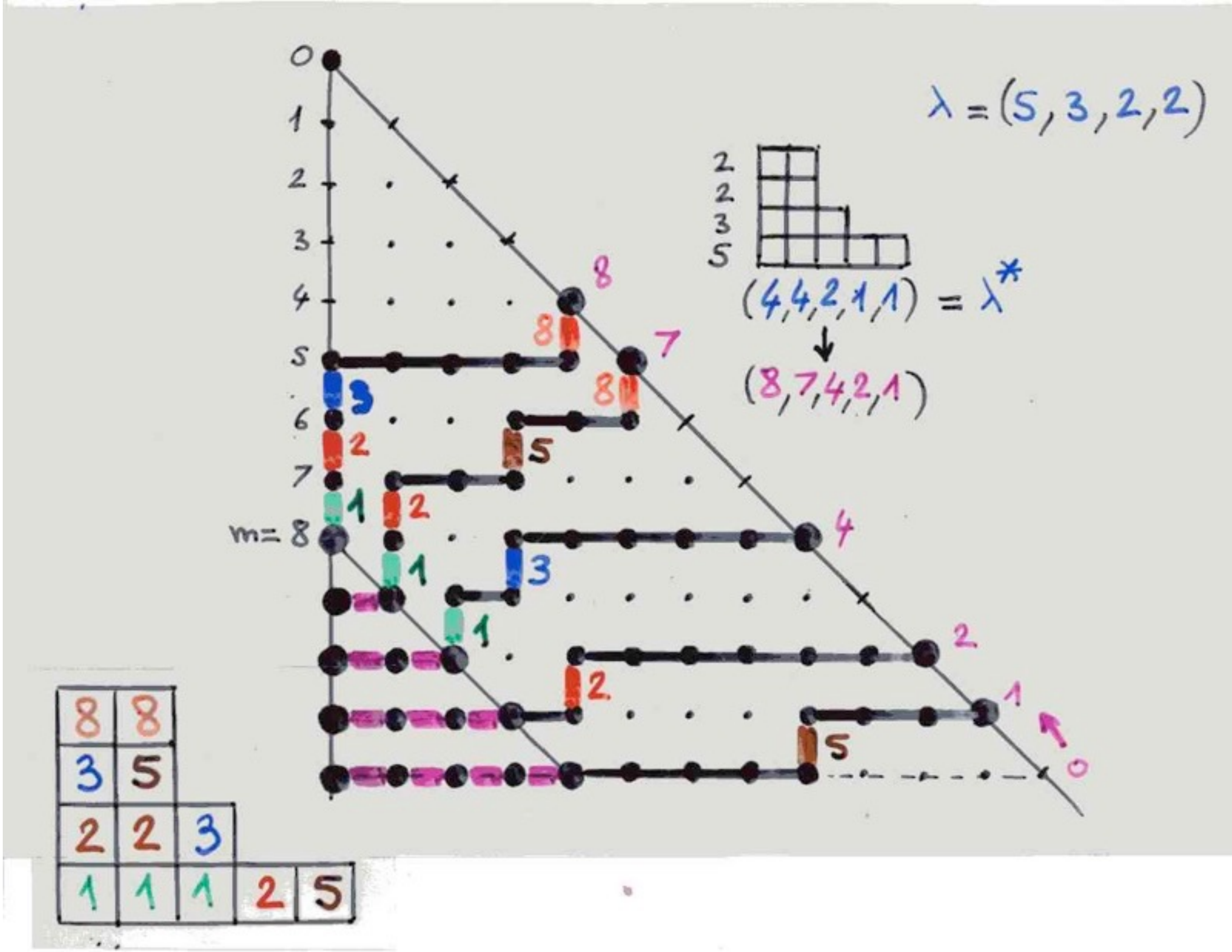
find a configuration of paths
giving a bijective proof of Narayana determinant



§4 (semi-standard) Young tableaux



λ = (5, 3, 2, 2)
(4,4,2,1,1) = λ*
(8,7,4,2,1)
m= 8



λ = (5, 3, 2, 2)
(4,4,2,1,1) = λ*
(8,7,4,2,1)
m= 8

Prop (P. Hall)

the number of Young tableaux of shape λ with entries in $\{1, 2, .., a\}$

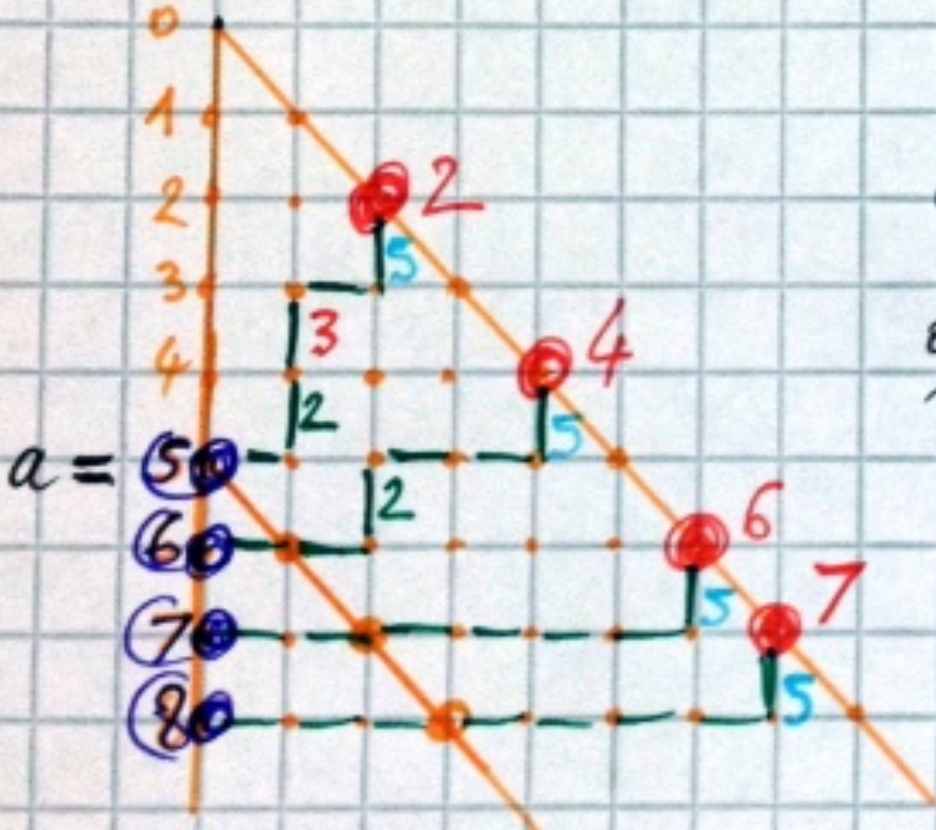
$$= \frac{C_a(\mu)}{H(\mu)}$$

μ « complementary » partition of λ with respect to a

example.

binomial
determinant

$$\begin{pmatrix} 5, 6, 7, 8 \\ 2, 4, 6, 7 \end{pmatrix}$$



a configuration of non-crossing paths related to $\begin{pmatrix} 5, 6, 7, 8 \\ 2, 4, 6, 7 \end{pmatrix}$

The associated

semi-standard Young tableau

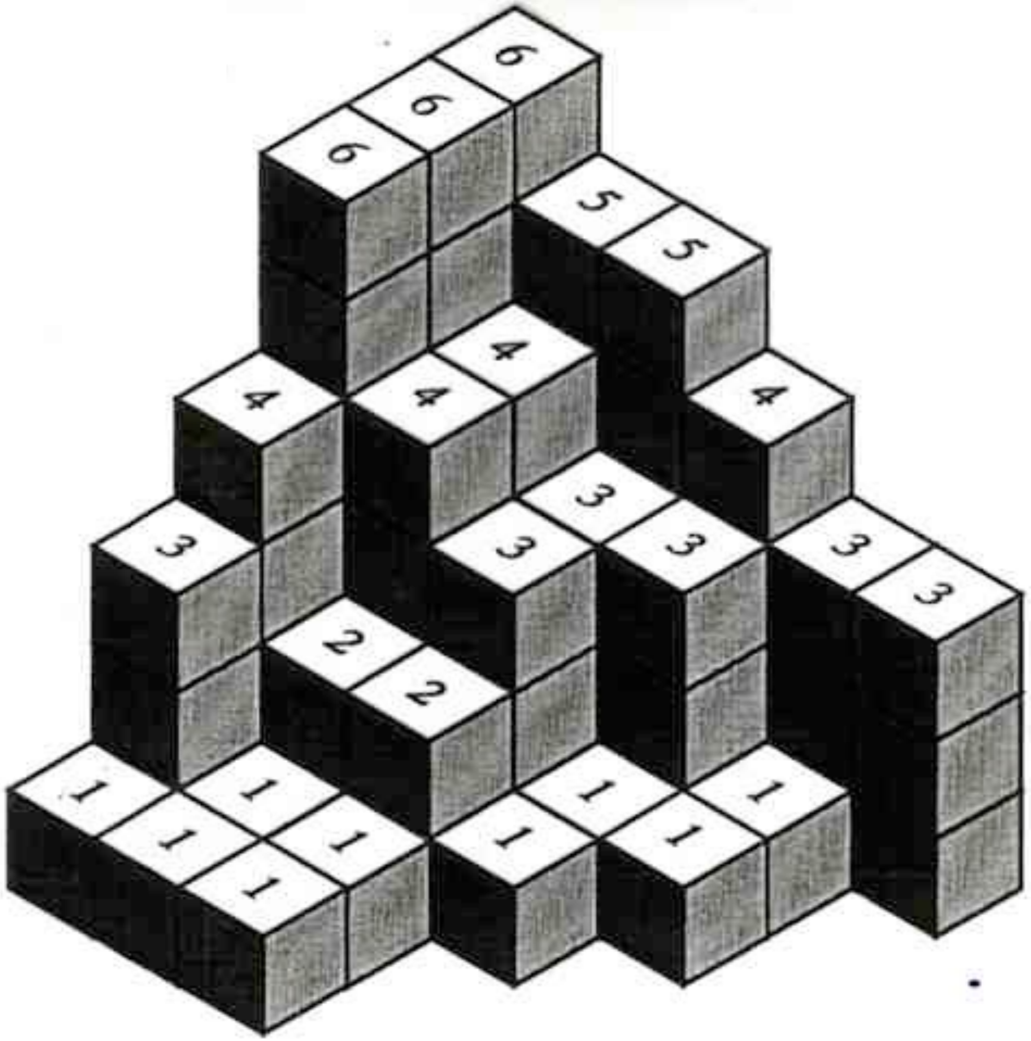
5			
3	5		
2	2	5	5

product

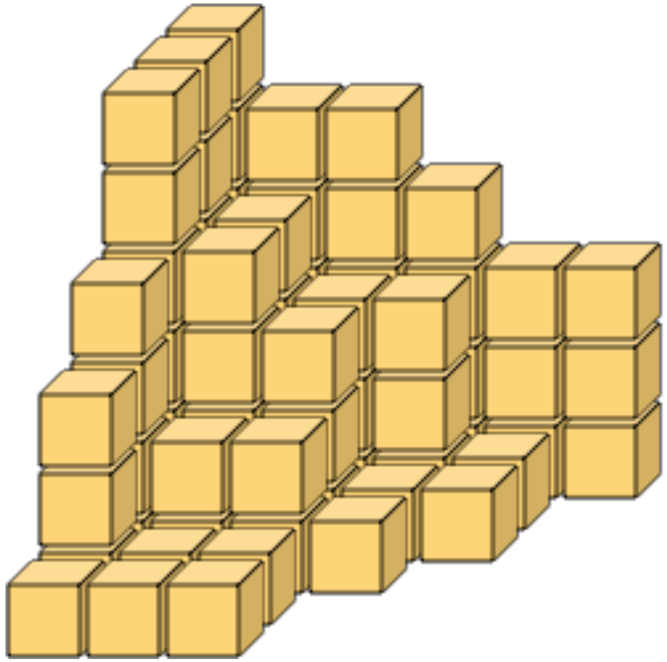
$$\begin{pmatrix}4, & 5, & 6, & 7\\ 1, & 3, & 5, & 6\end{pmatrix} = \frac{\begin{pmatrix}8&7&6&5\\7&6&5&4\\&5&4&3\\&&3&2\end{pmatrix}}{\begin{pmatrix}2&4&6&7\\1&3&5&6\\&1&3&4\\&&1&2\end{pmatrix}} \qquad \begin{matrix}\to \mathbb{C}_a(\mu)\\[2em] \to H(\mu)\end{matrix}$$

product

§5 Planes partitions



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



Partitions planes bornées

diagramme 3D $\quad F \subseteq \mathcal{B}(r,s,t)$

$$\mathcal{B}(a,b,c) = \left\{ (i,j,k) \in \mathbb{N}^3, \begin{array}{l} 1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c \end{array} \right\}$$

$\mathcal{B}($ a, b, c $)$

at most a rows
at most b columns
parts $\leq$ c

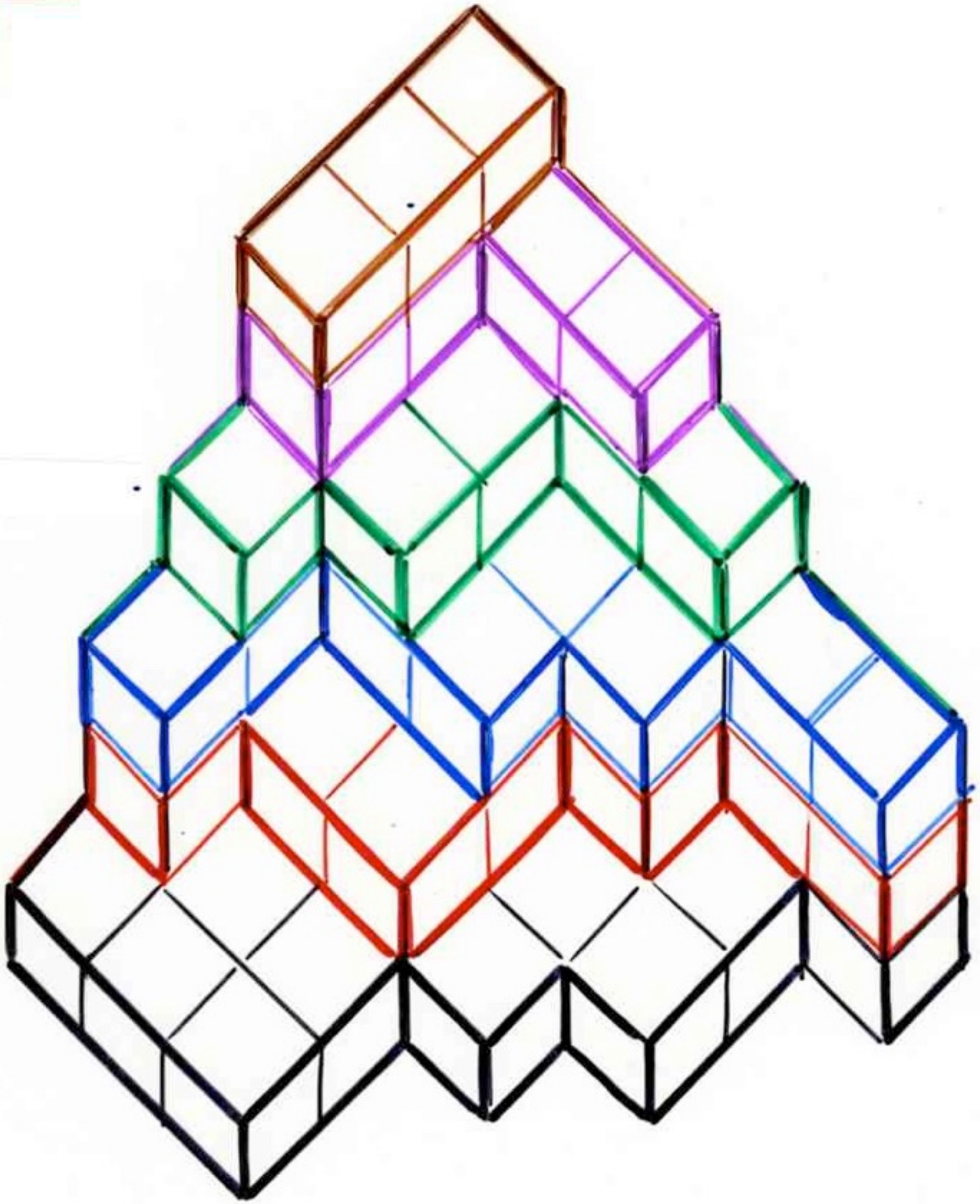
$\mathcal{B}(7,6,6)$

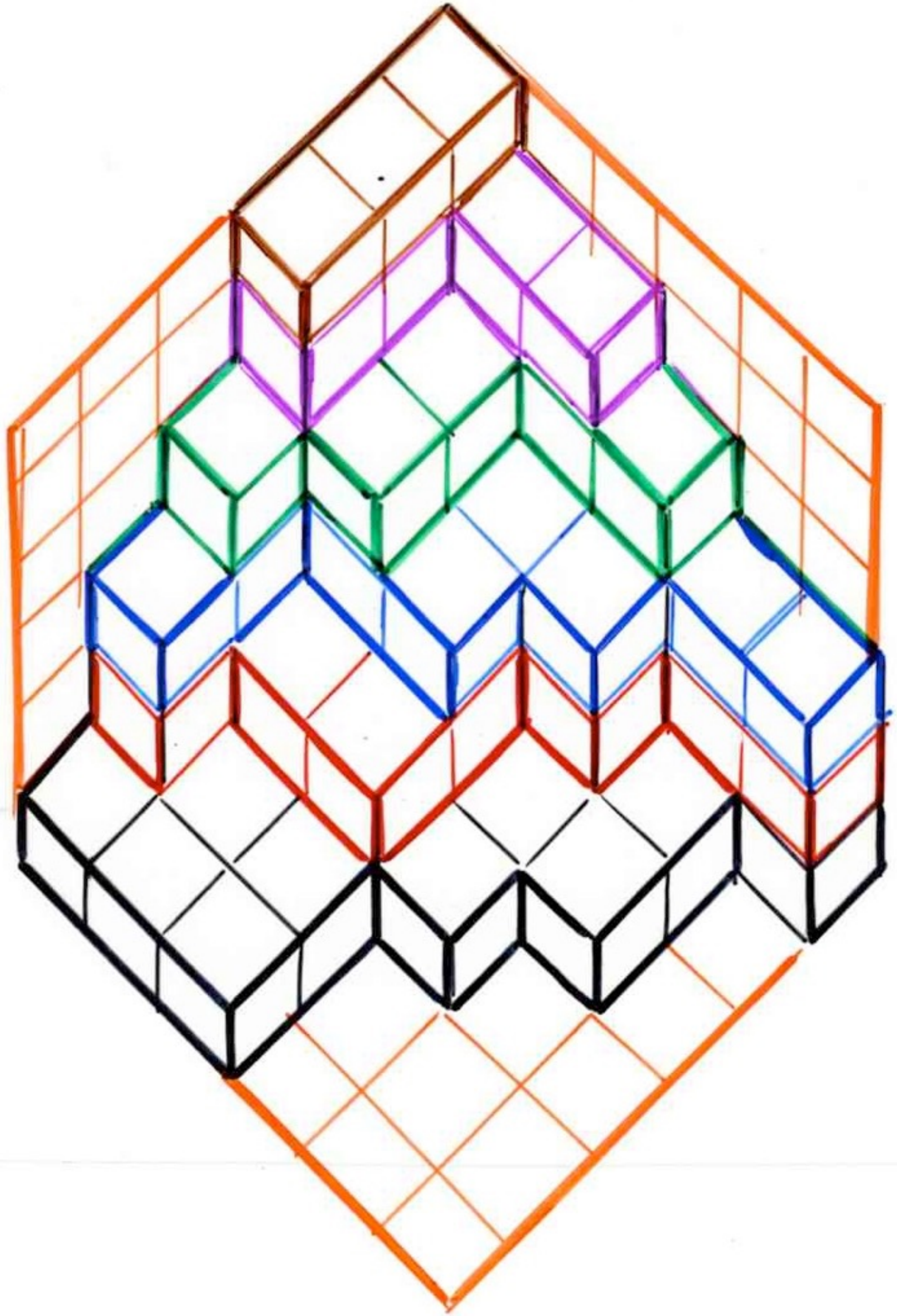
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

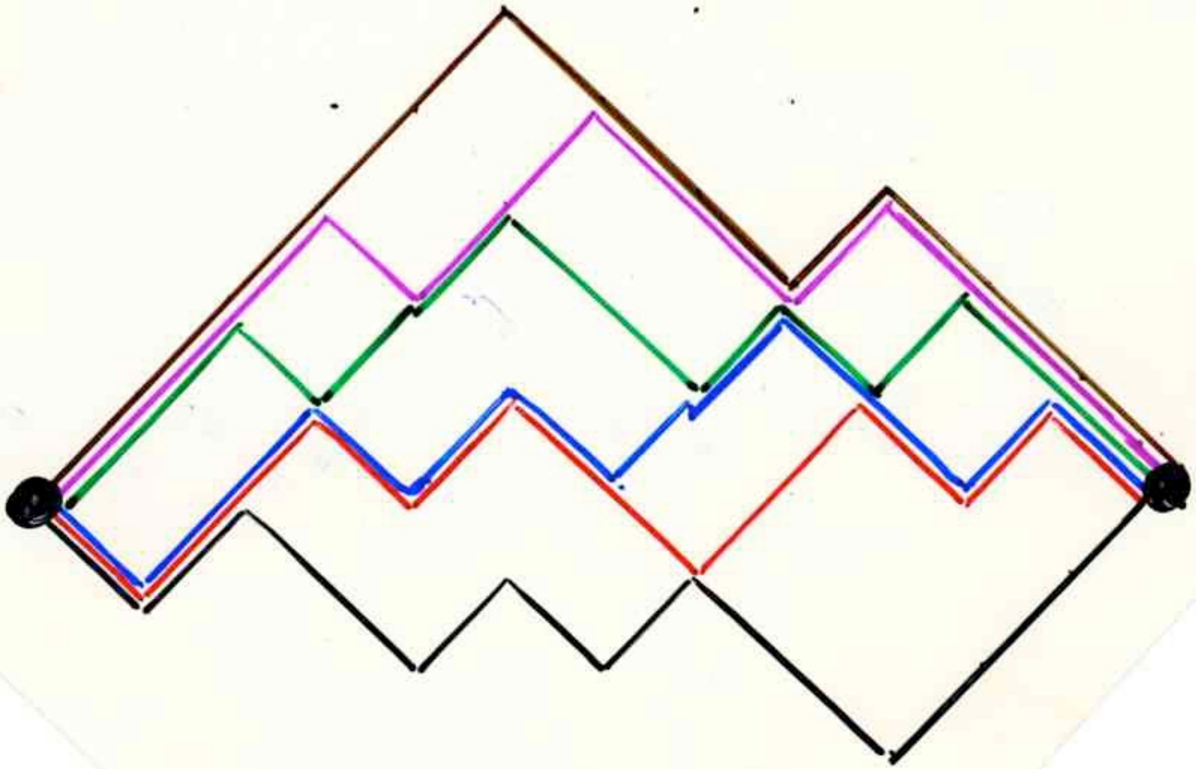
$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

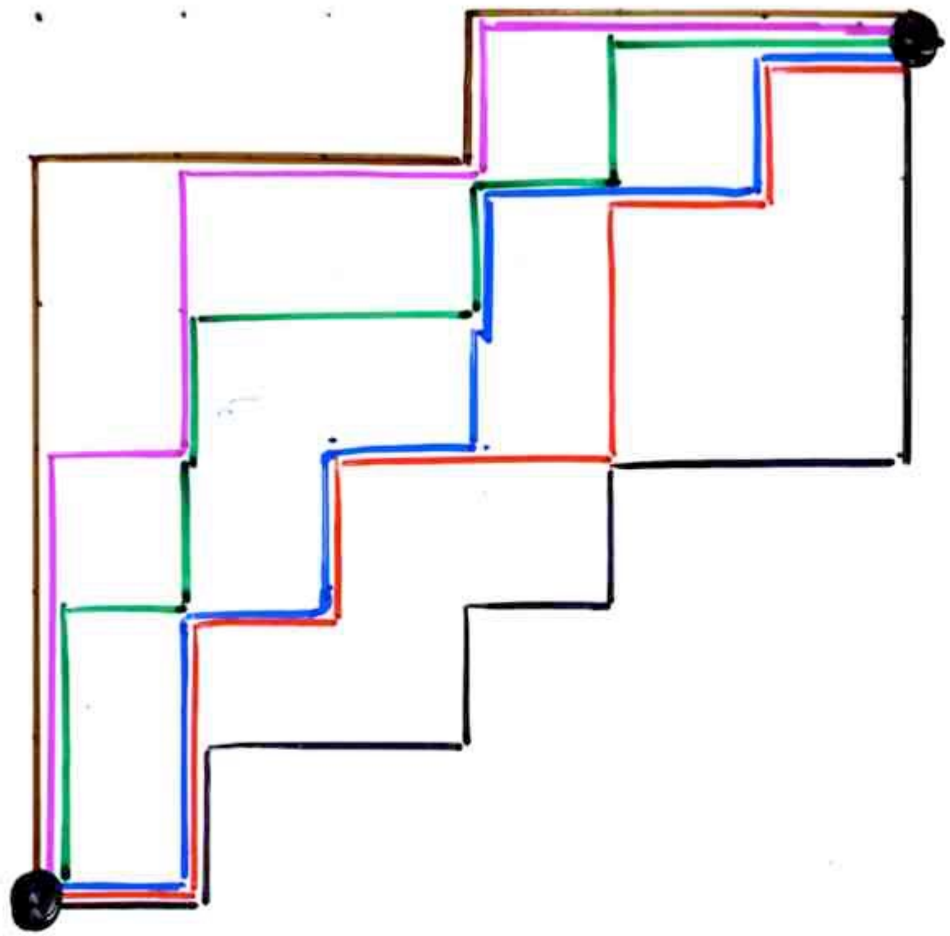
Paths for plane partitions

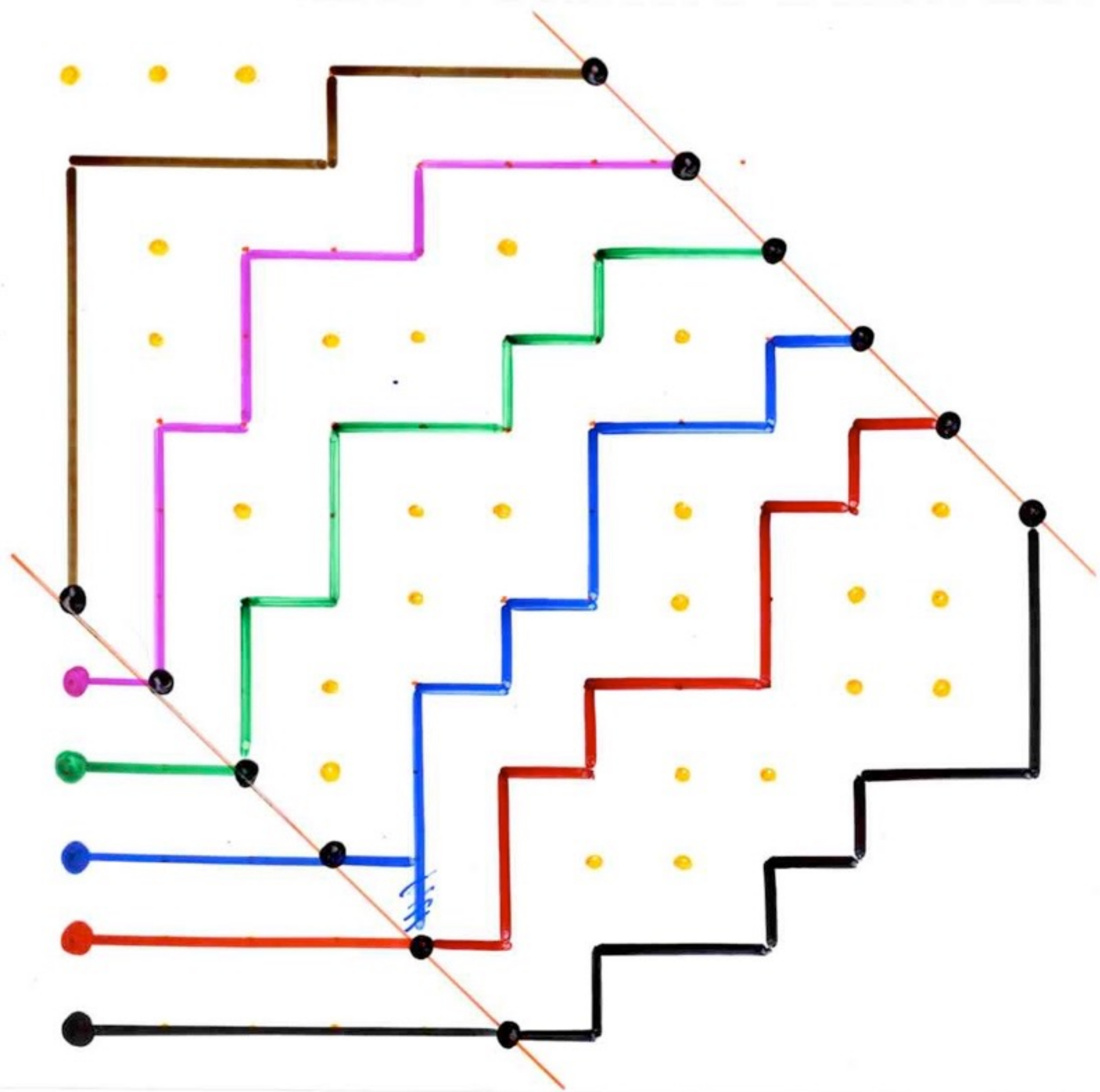
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

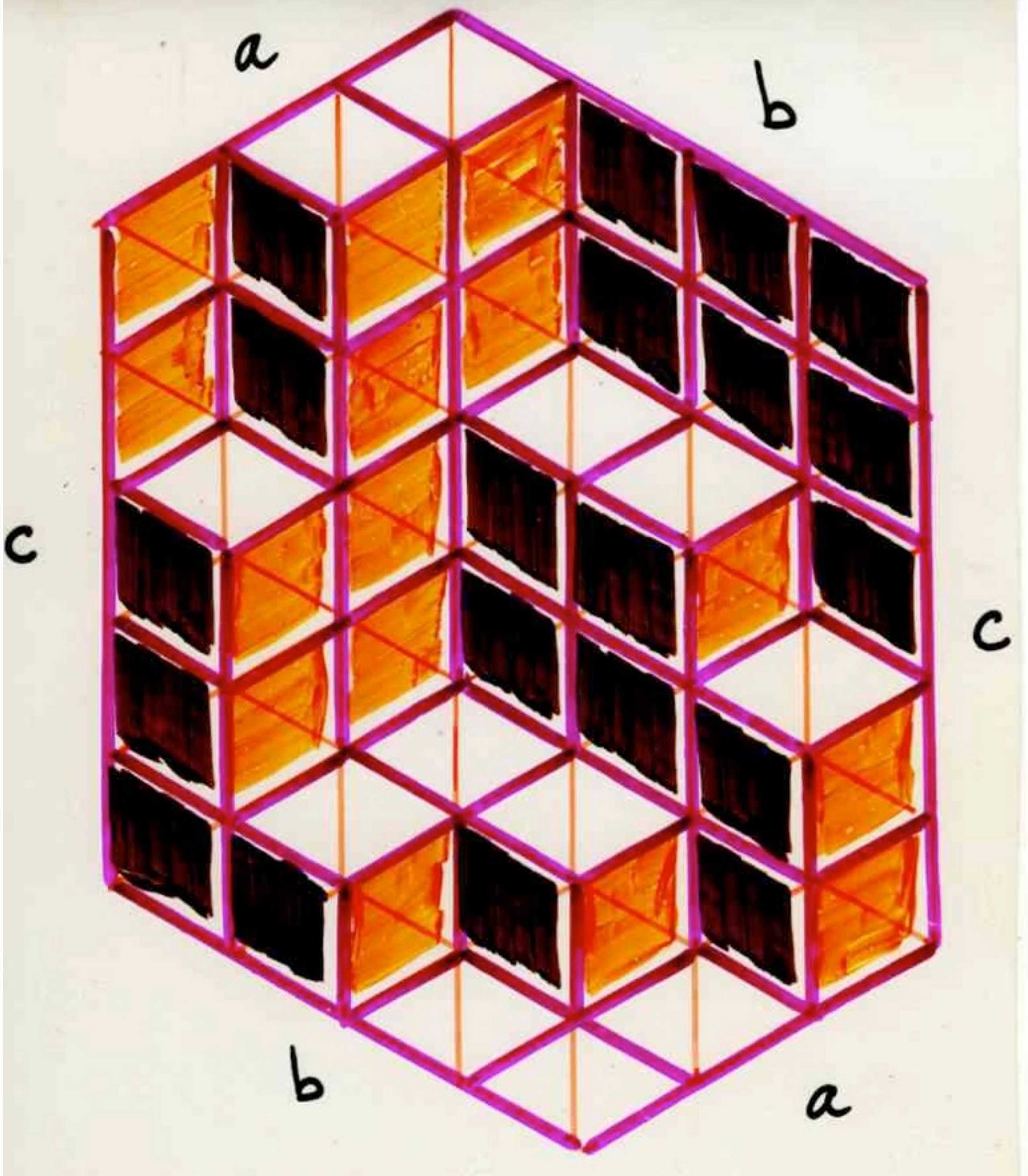




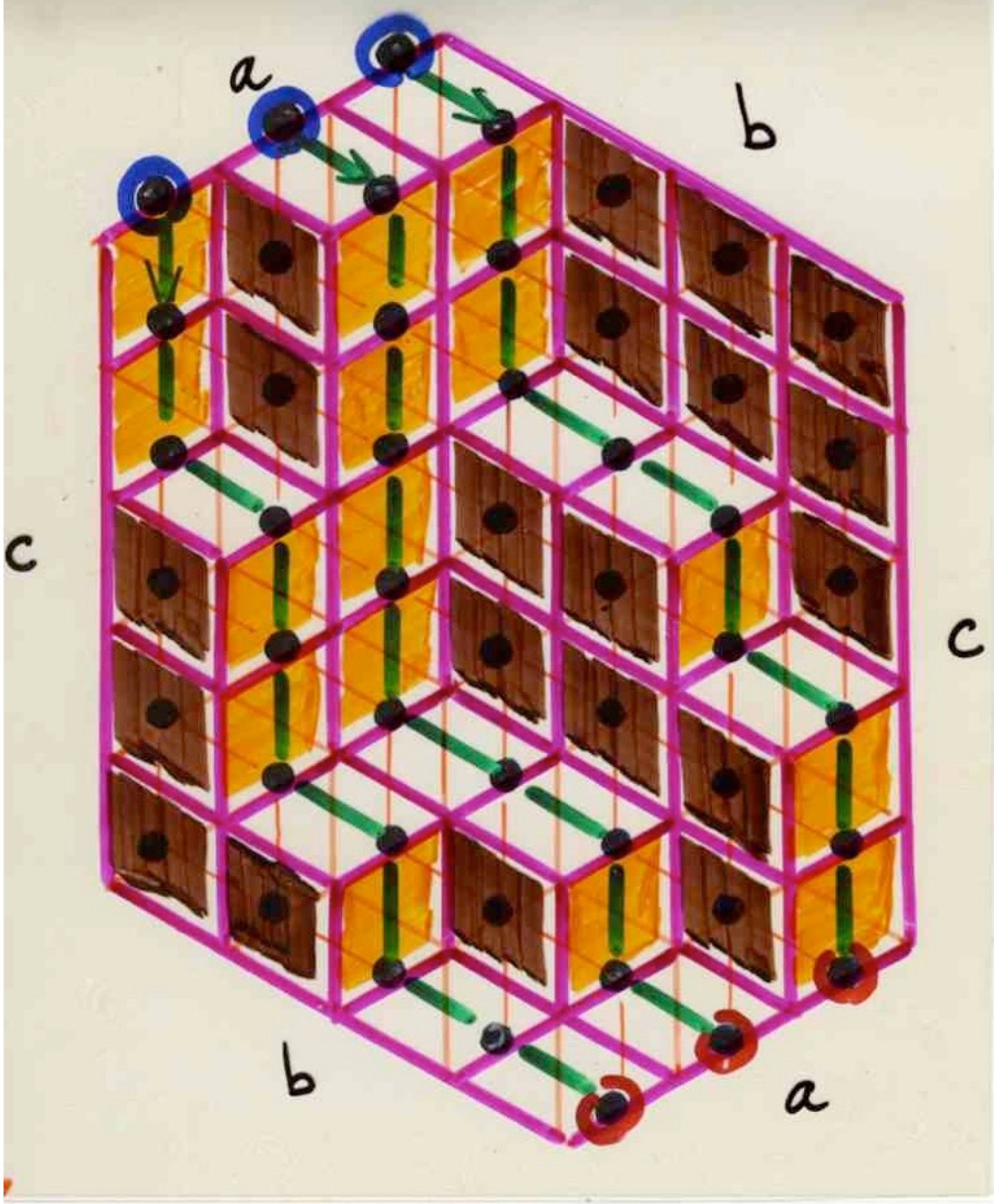


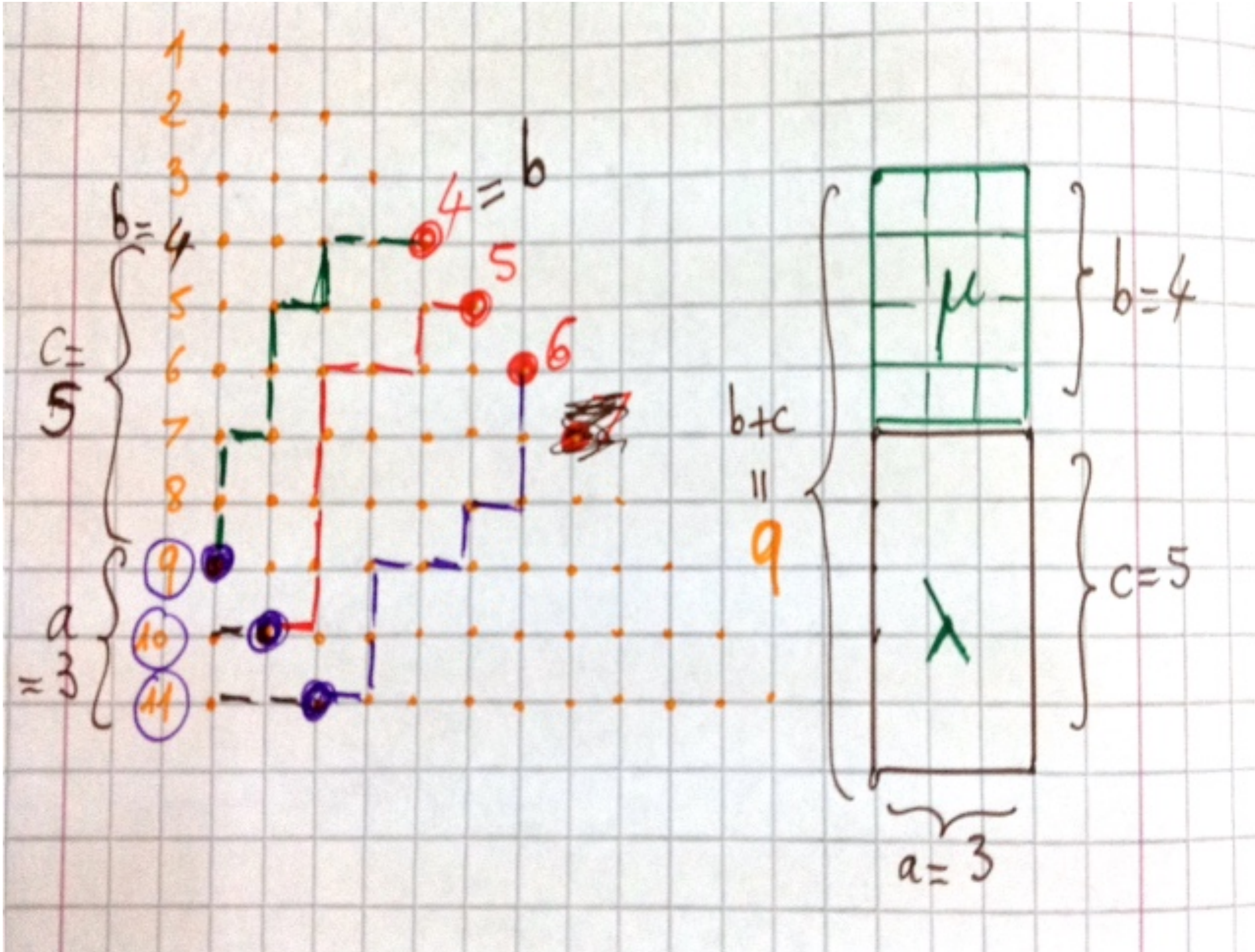




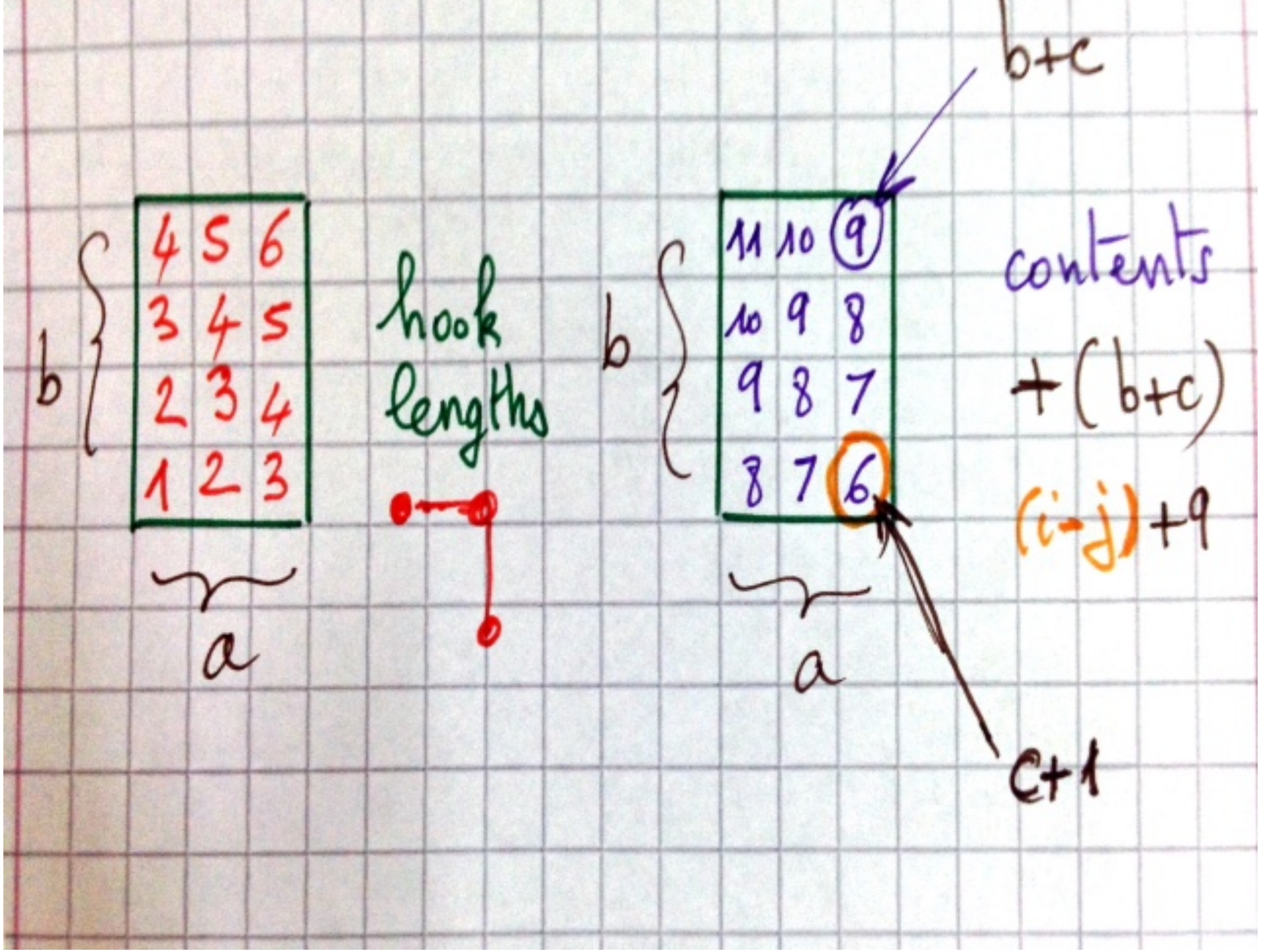

a
b
c
c
b
a

coding of a
plane partition
with N.C.-paths

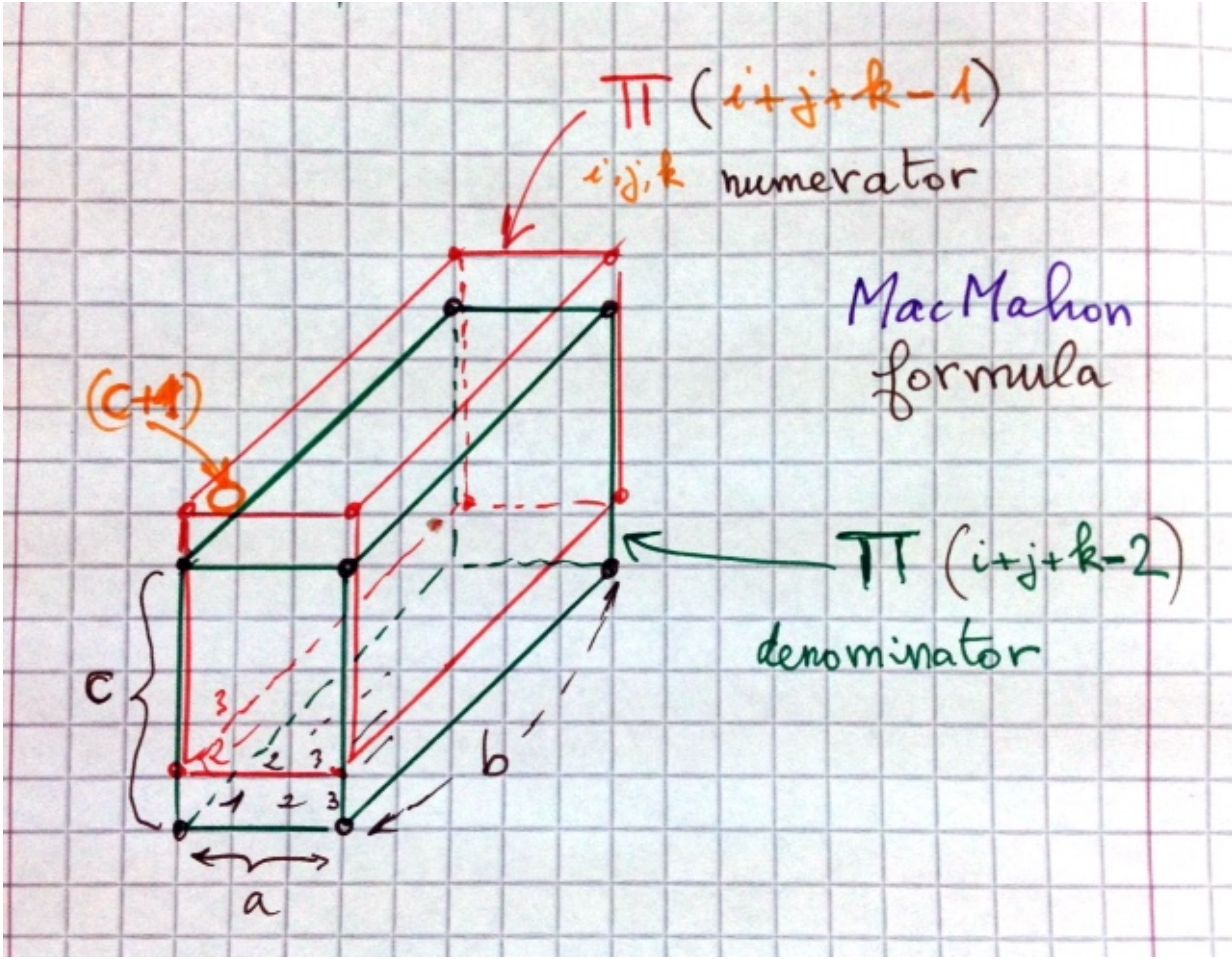

a
b
c
c
b
a



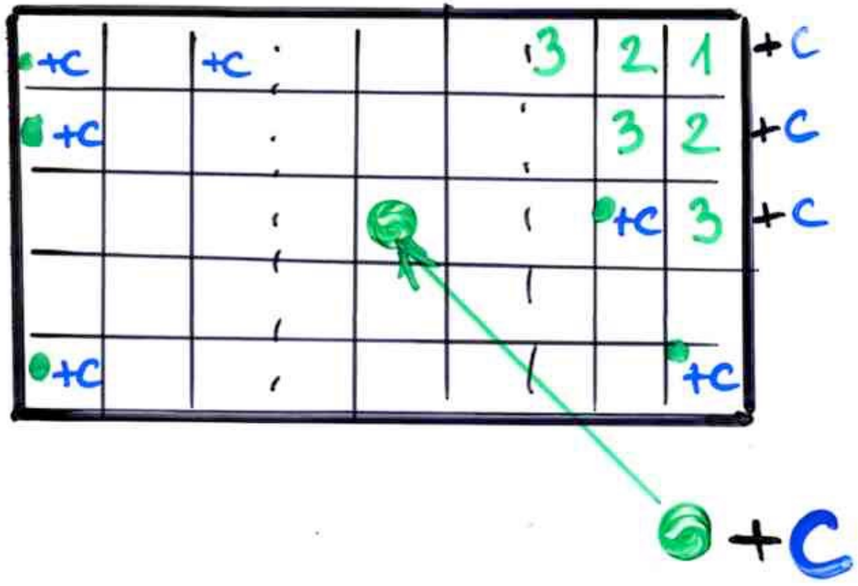
1
2
3
4 = b
b = 4
5
6
c = 5
7
8
9
10
11
a = 3
b+c
=
9
μ
λ
b = 4
c = 5
a = 3


b
4 5 6
3 4 5
2 3 4
1 2 3
a
hook lengths
b+c
b
11 10 9
10 9 8
9 8 7
8 7 6
a
contents
+ (b+c)
(i-j) +9
c+1

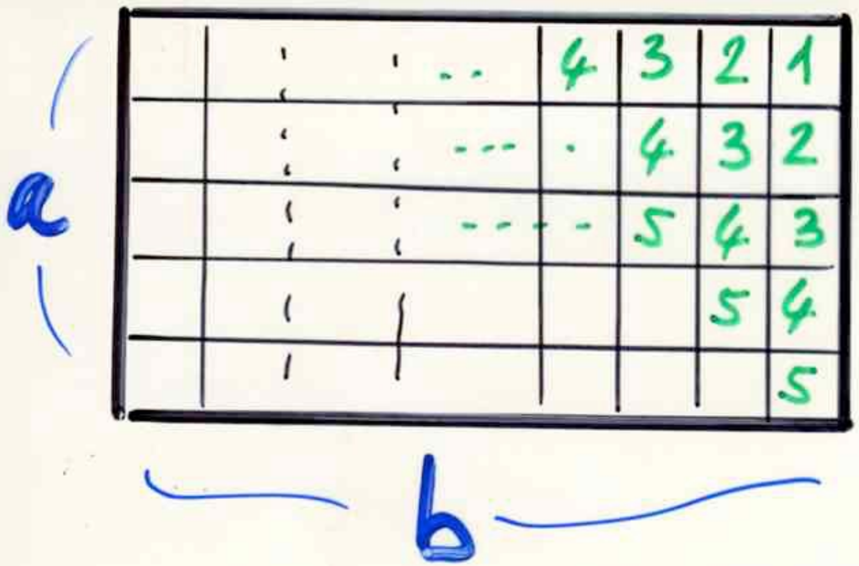
$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$



$\prod_{i,j,k} (i+j+k-1)$
numerator
MacMahon
formula
(c+1)
$\prod (i+j+k-2)$
denominator
c
b
a
1
2
3



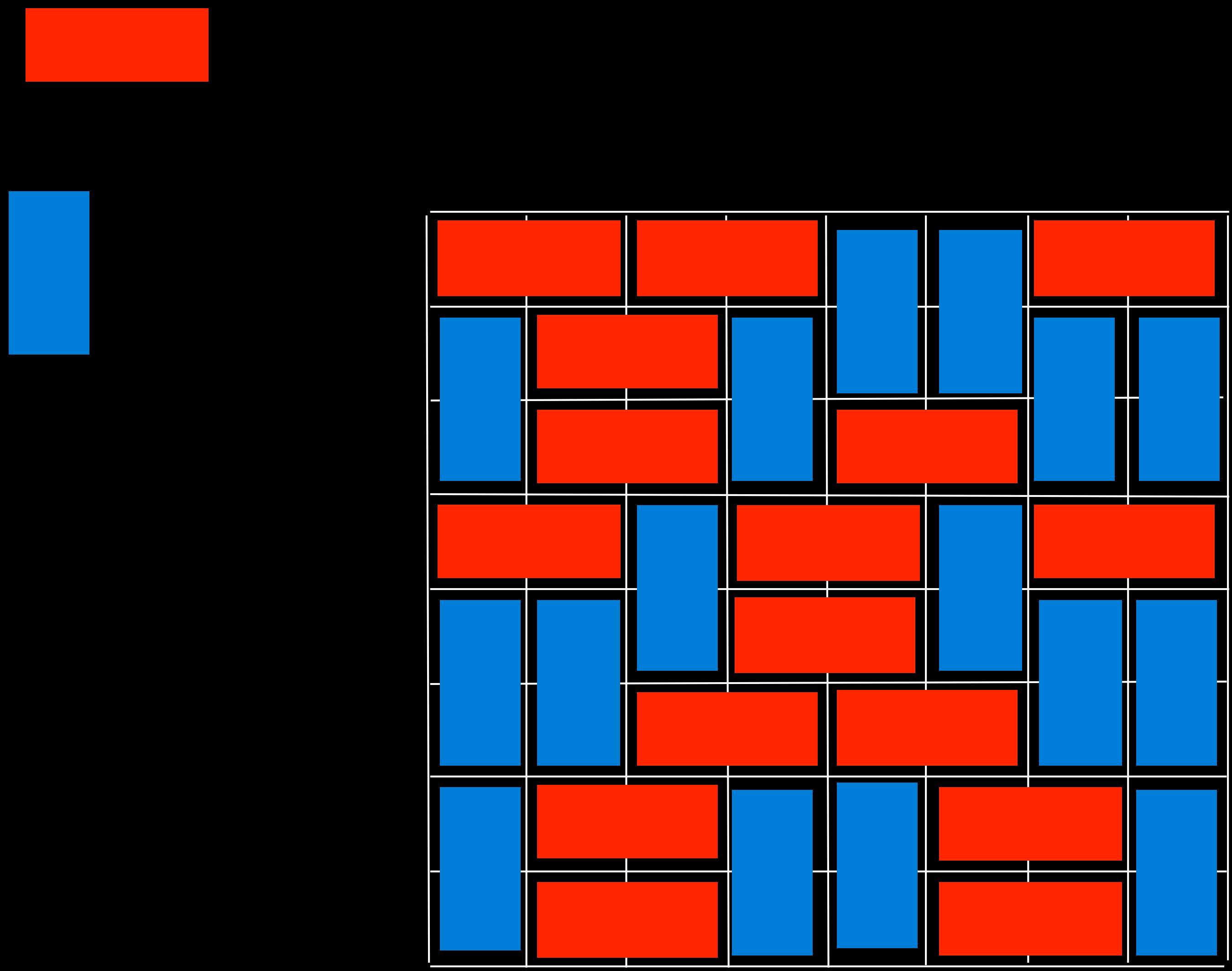
+c
3 2 1
3 2
3
+c



4 3 2 1
4 3 2
5 4 3
5 4
5
a
b

§6 Tilings

tiling in Kuperberg' s bathroom



number of tilings on a 8 x 8 chessboard
= 12 988 816

number of tilings with dimers of a $m \times n$ rectangle

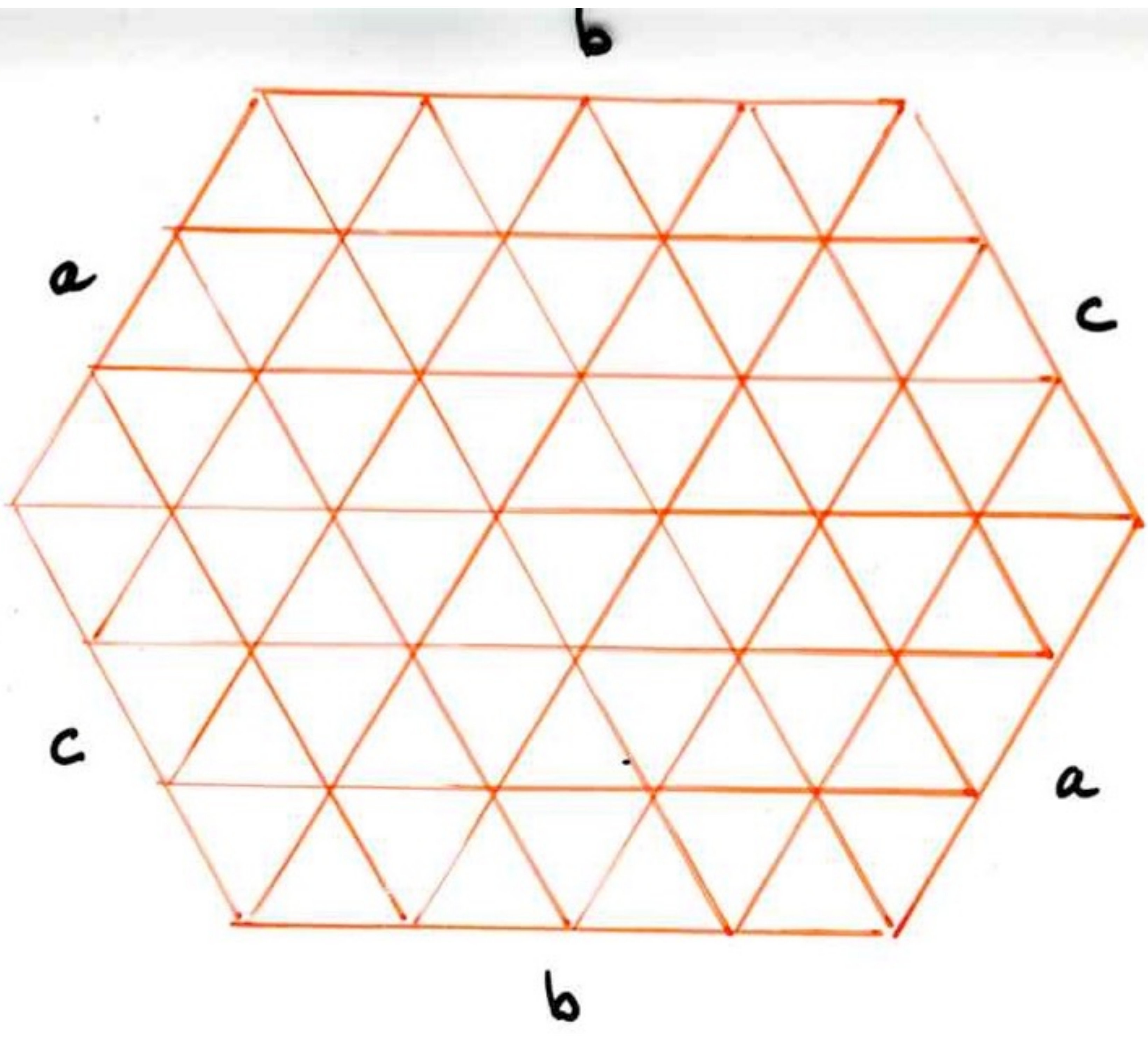
$$4^{mn} \prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

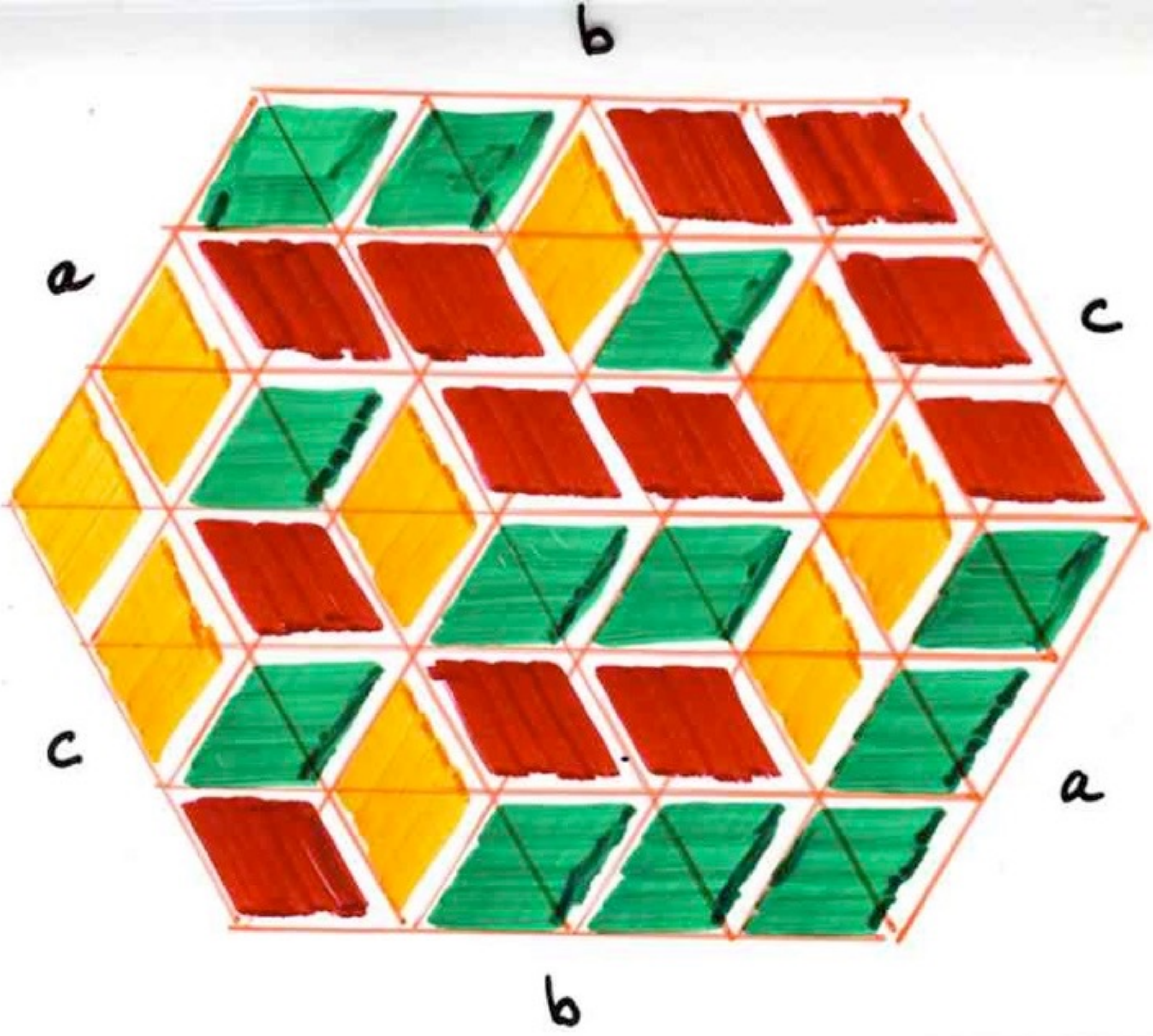
Kasteleyn (1961)

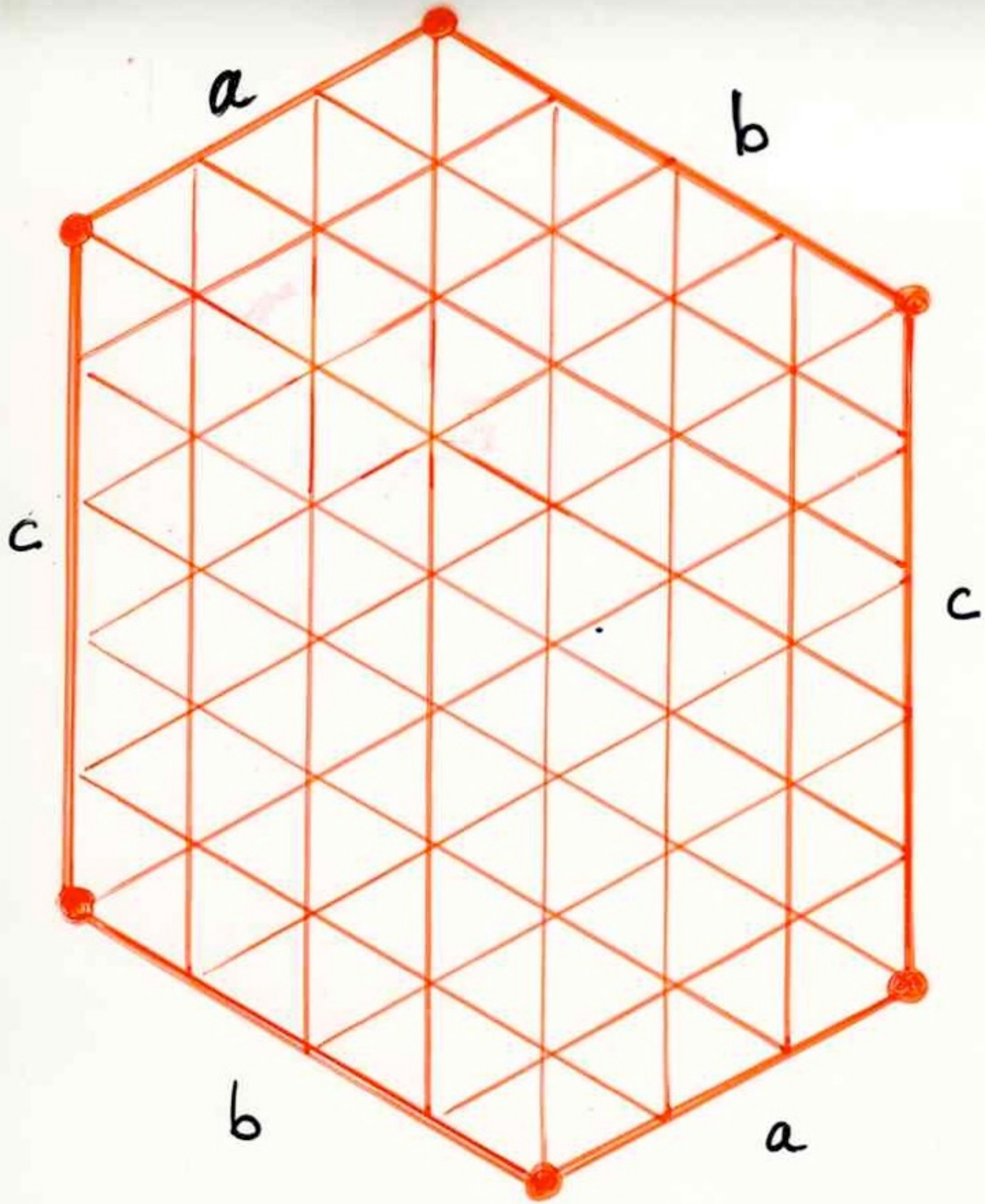
it is an integer !!

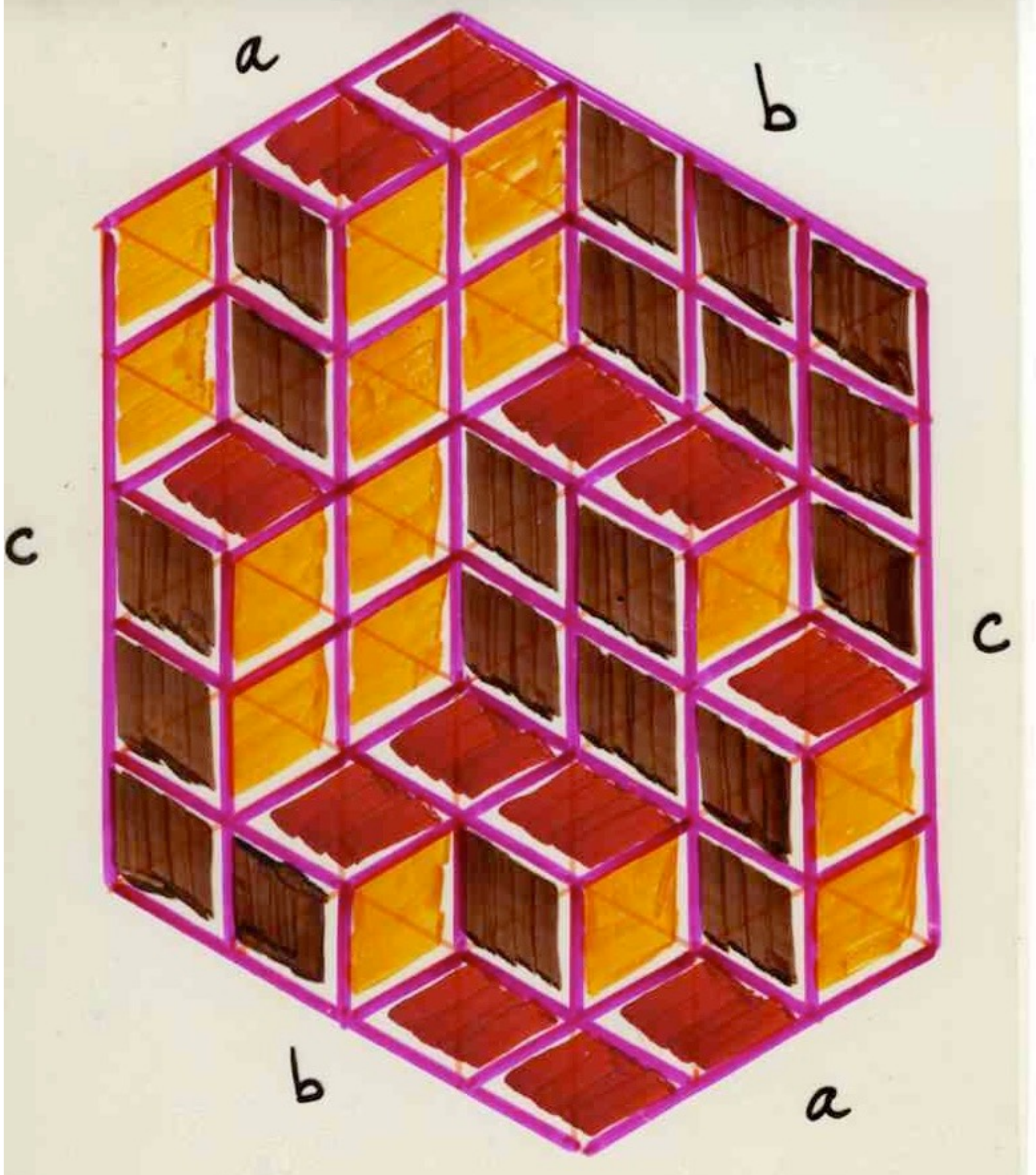
for the chessboard m=8, n=8: 12 988 816

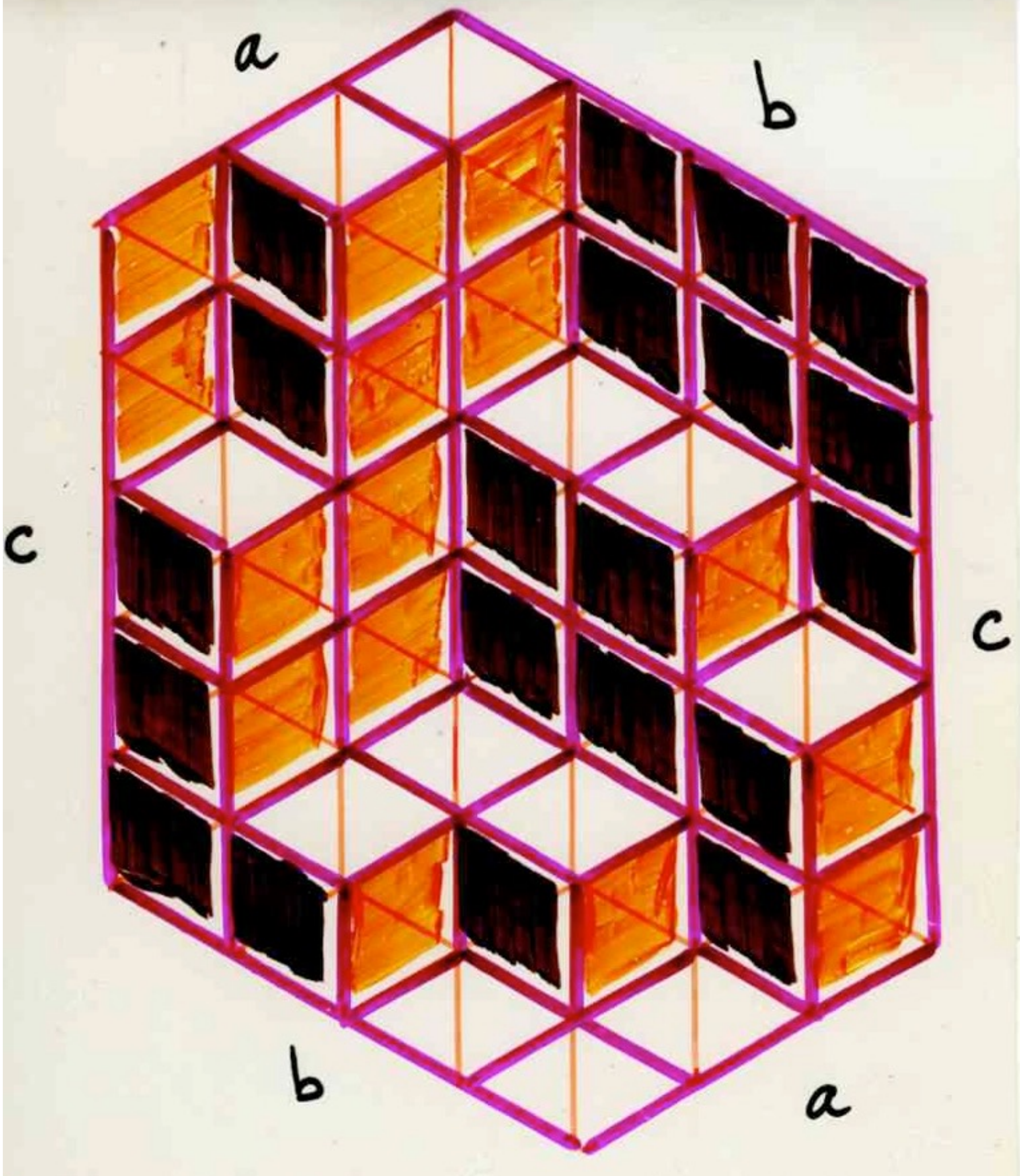
Tilings on triangular lattice

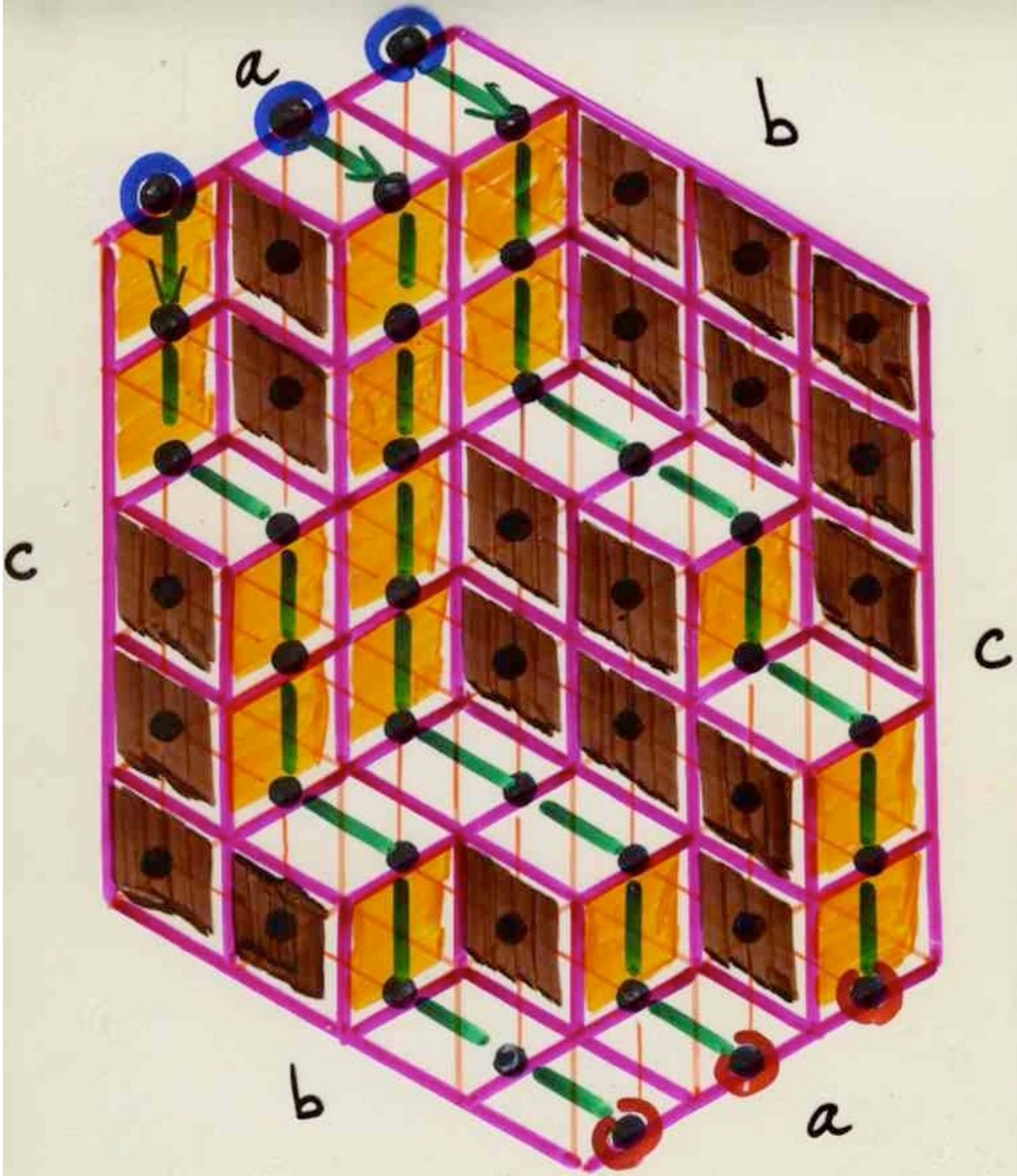

b
a
c
c
a
b


b
a
c
c
a
b


a
b
c
c
b
a


a
b
c
c
b
a

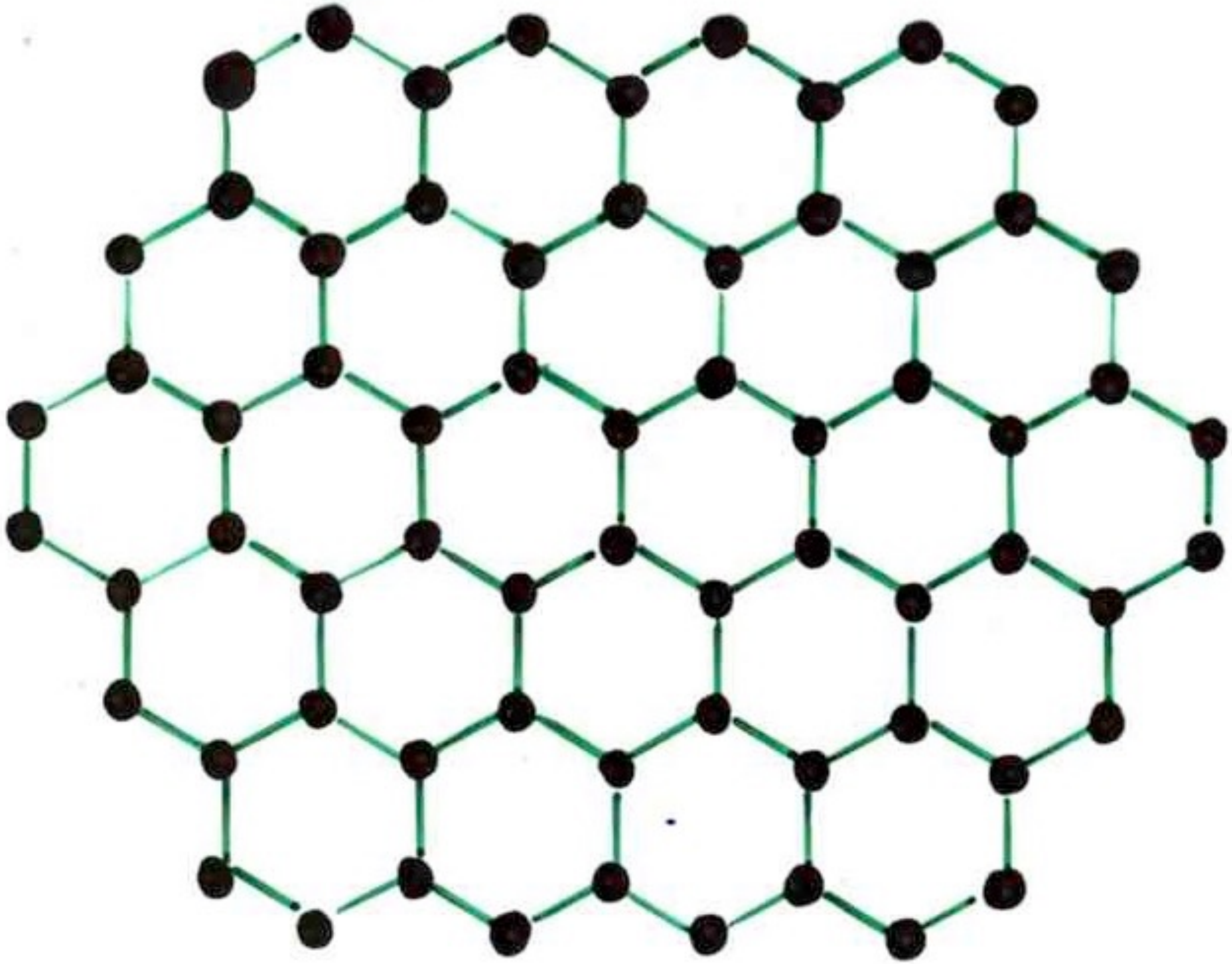

a
b
c
c
b
a

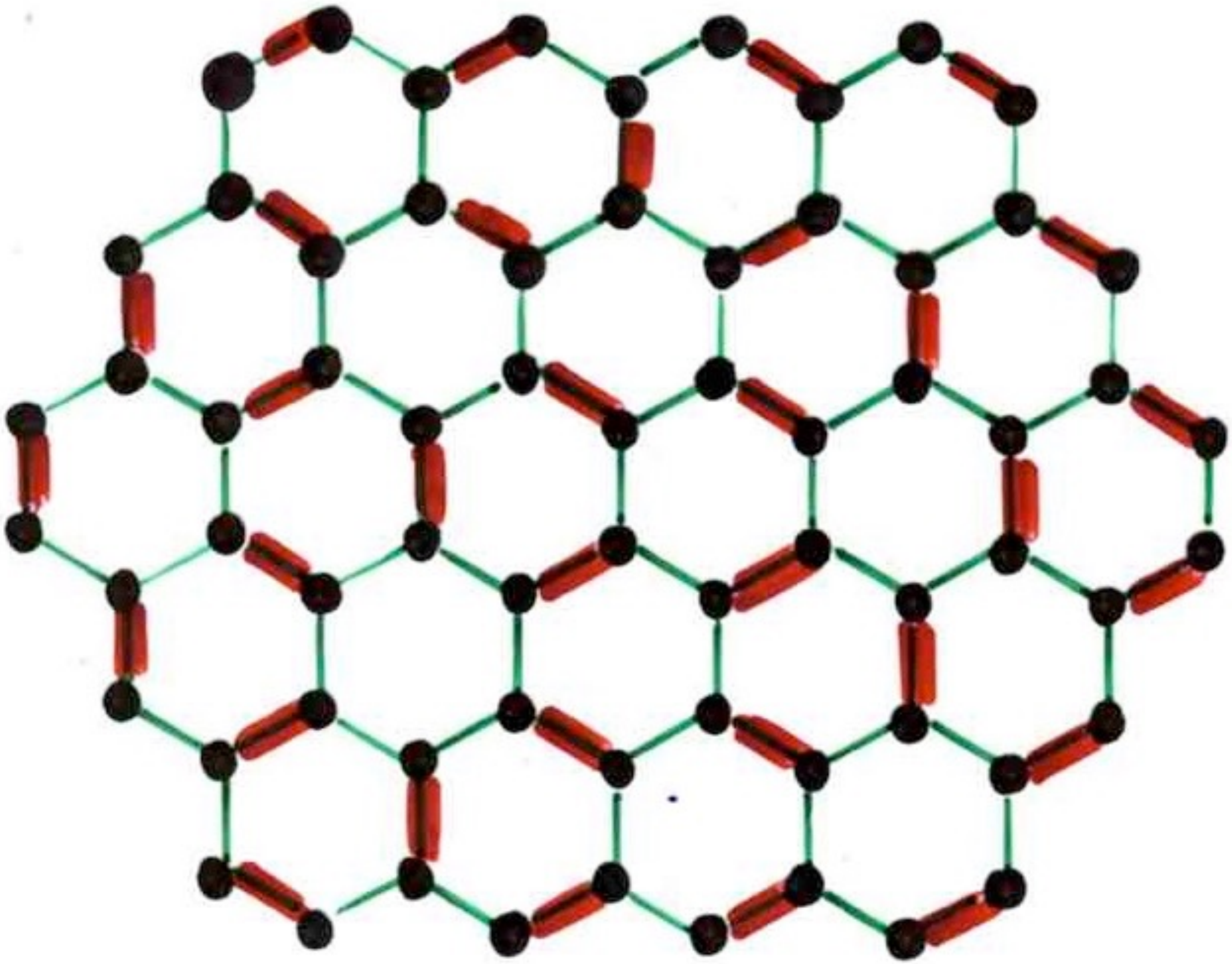


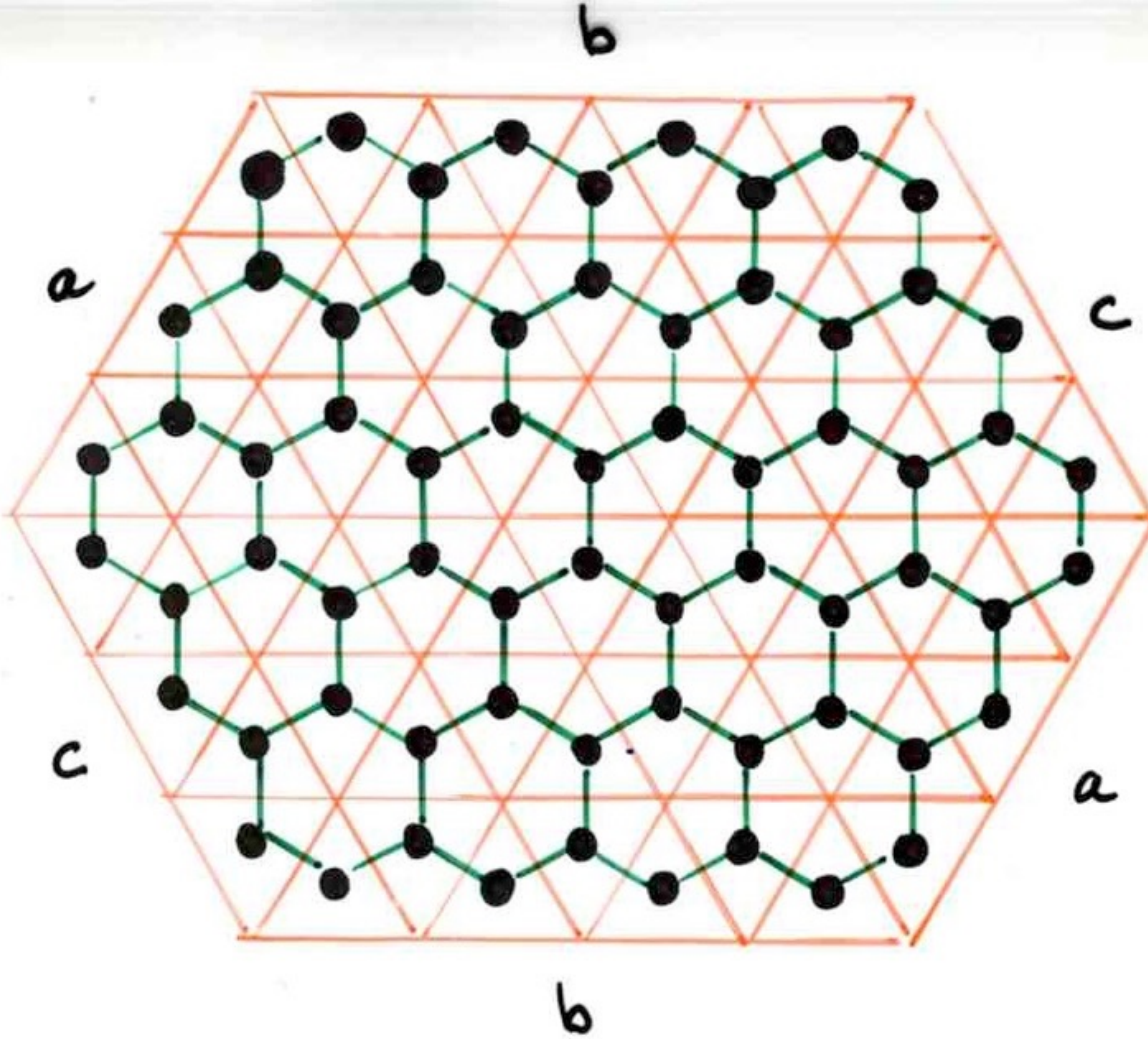
a
b
c
c
b
a

$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

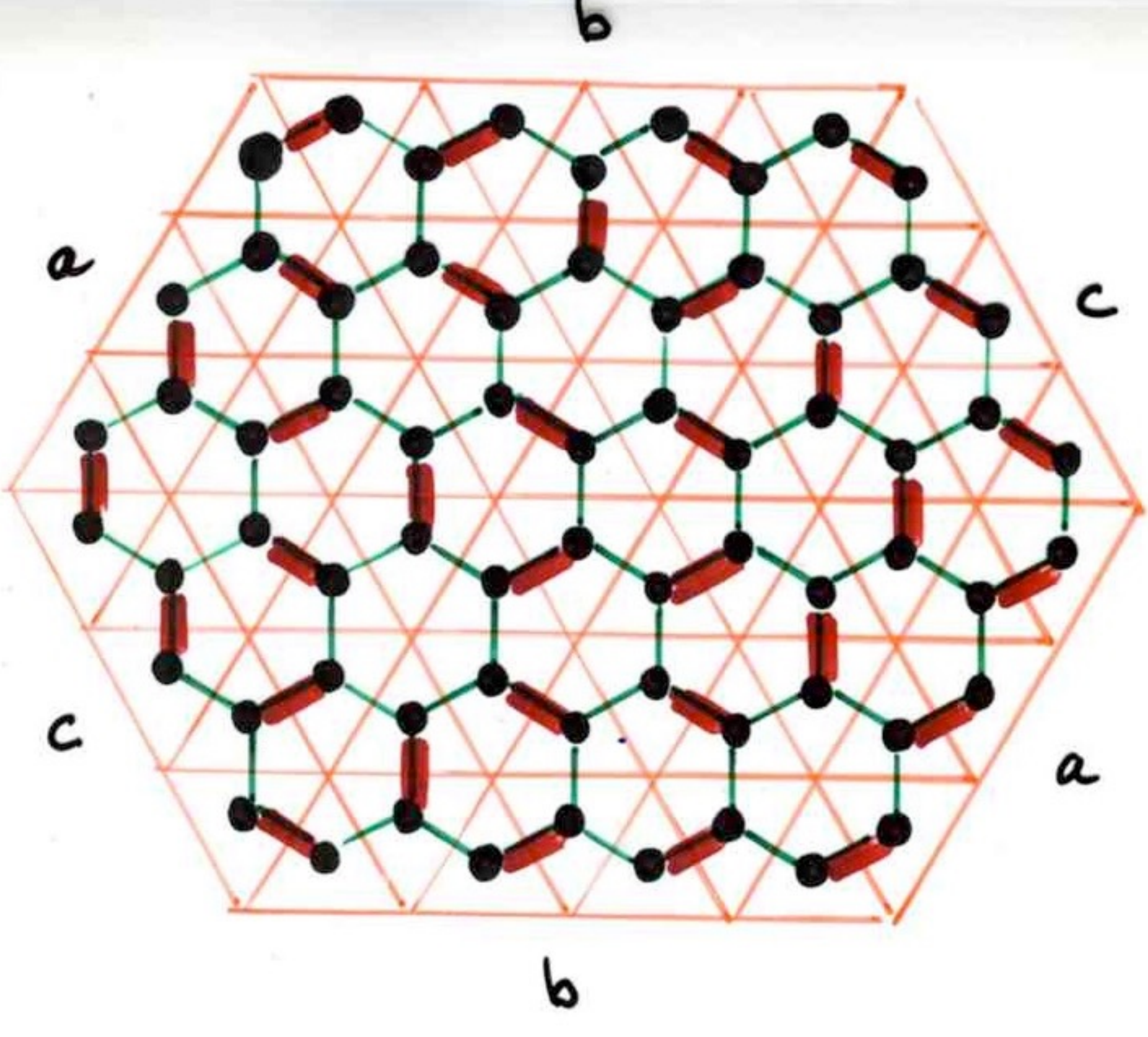
§7 Perfect matchings

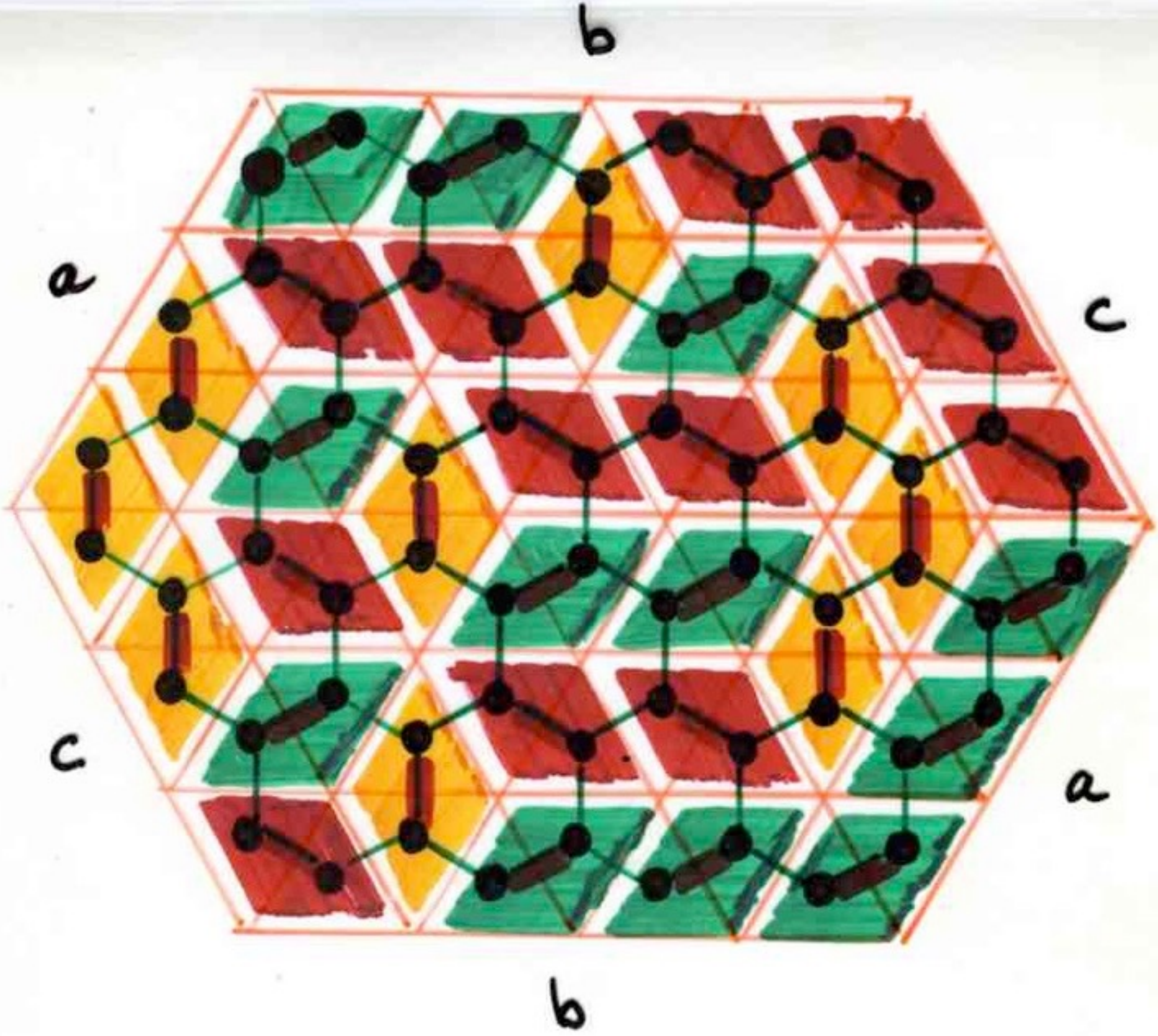






b
a
c
c
a
b


b
a
c
c
a
b


b
a
c
c
a
b

Quantum-chemical theory

resonance theoretic methods

Gutman (1980, ... 1990, ...)

Klein, Hite, Seitz, Schmaltz (1986)

Randić, Nikolić, Trinajstić (1988)

... (1990)

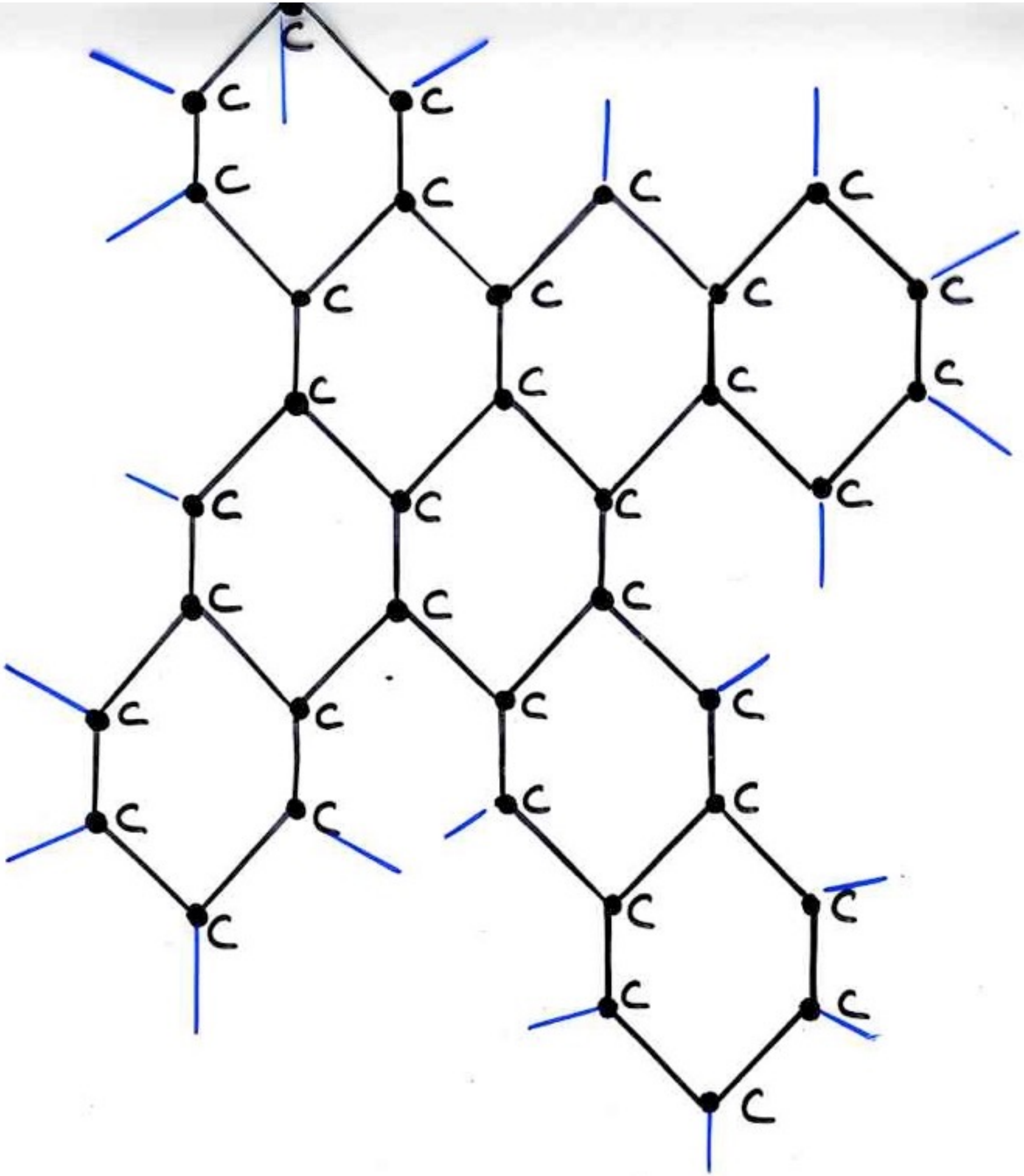
...

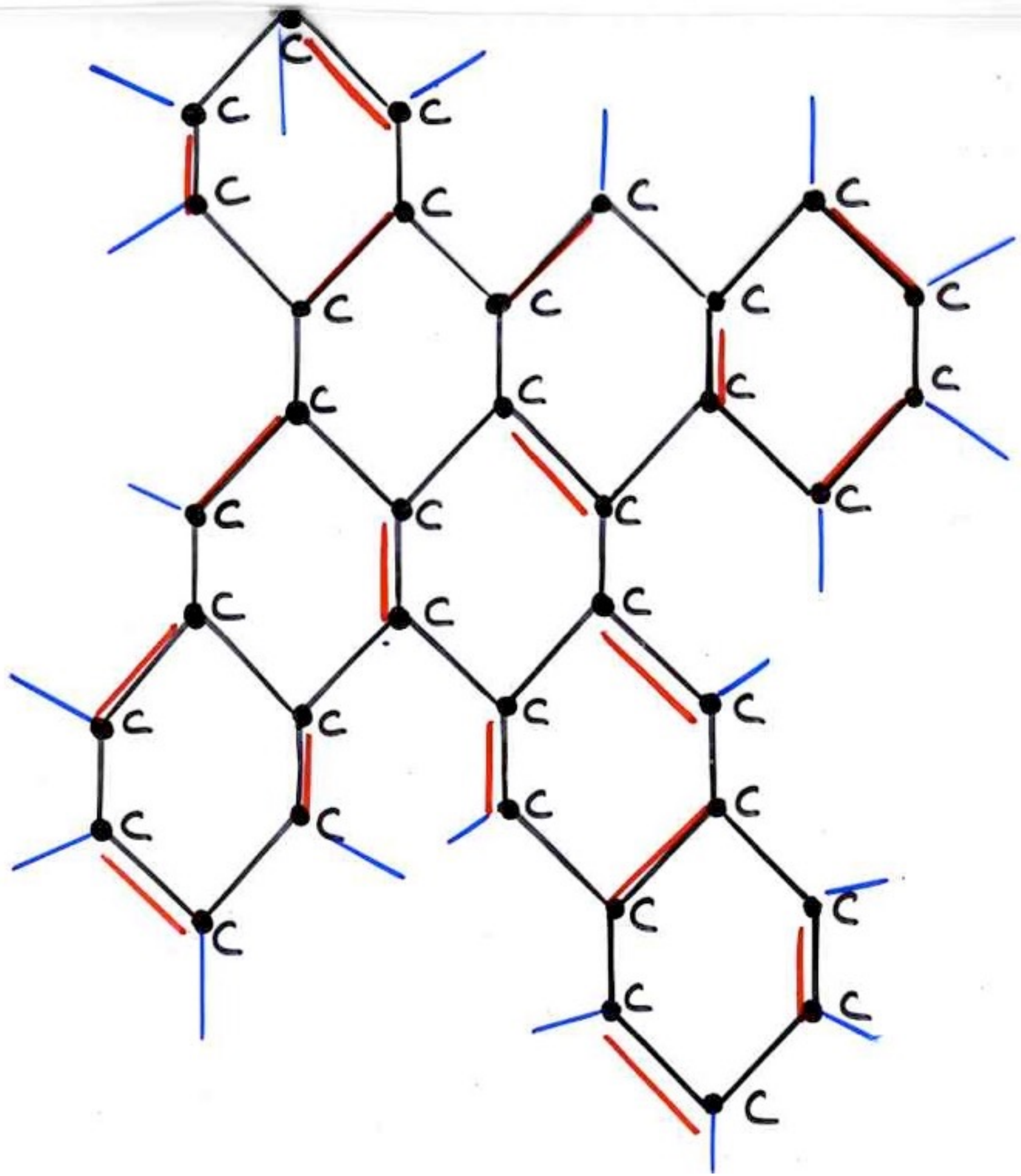
Jerman-Blažič, Živković (1991)

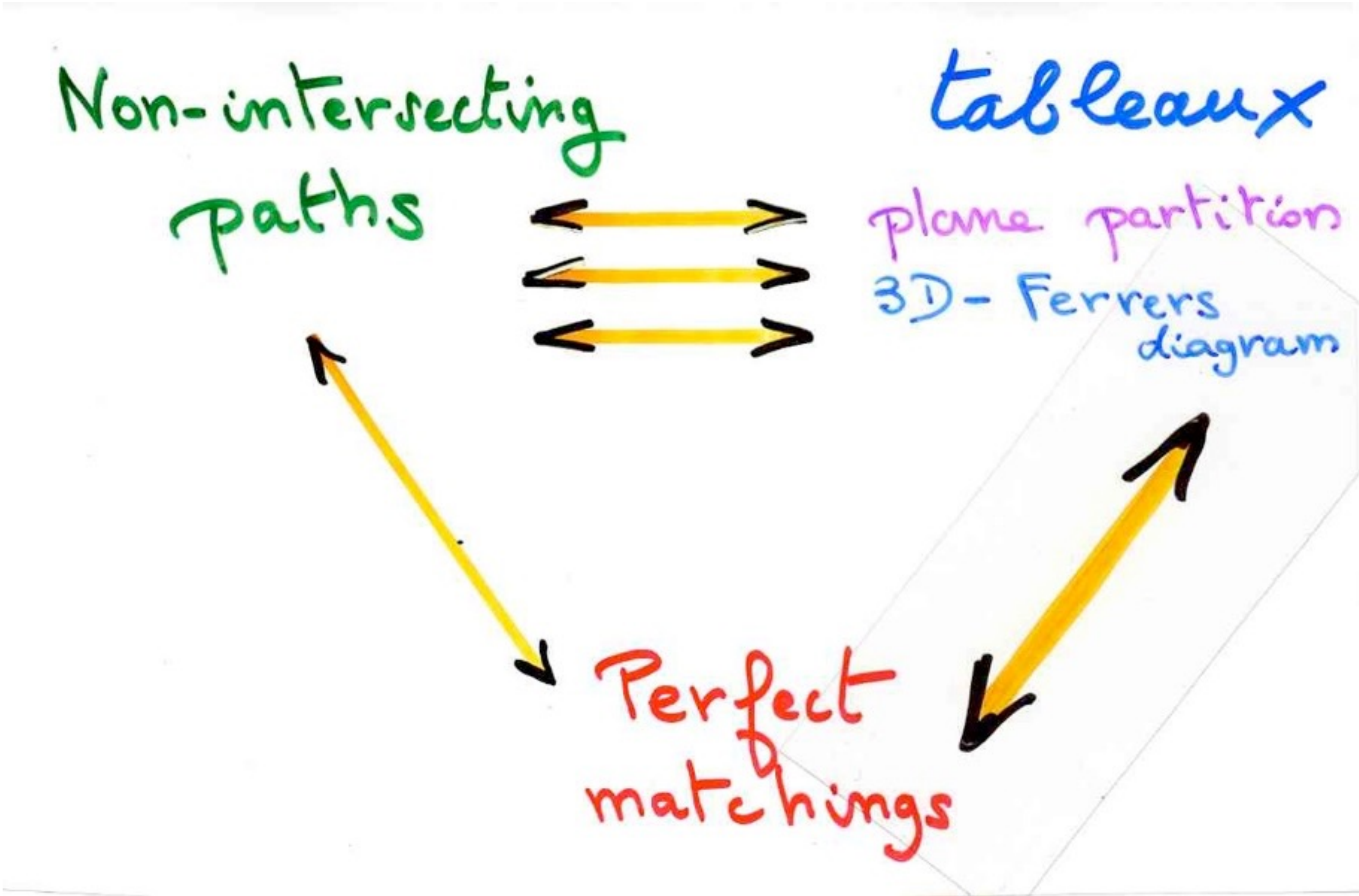
...

Zhang Fuji (1990, ...) Hosaya (1986)

honeycomb graph






Non-intersecting paths
tableaux
plane partition
3D-Ferrers diagram
Perfect matchings

Perfect matchings

Pfaffian methodology

- dénombrement de

  couplages parfaits

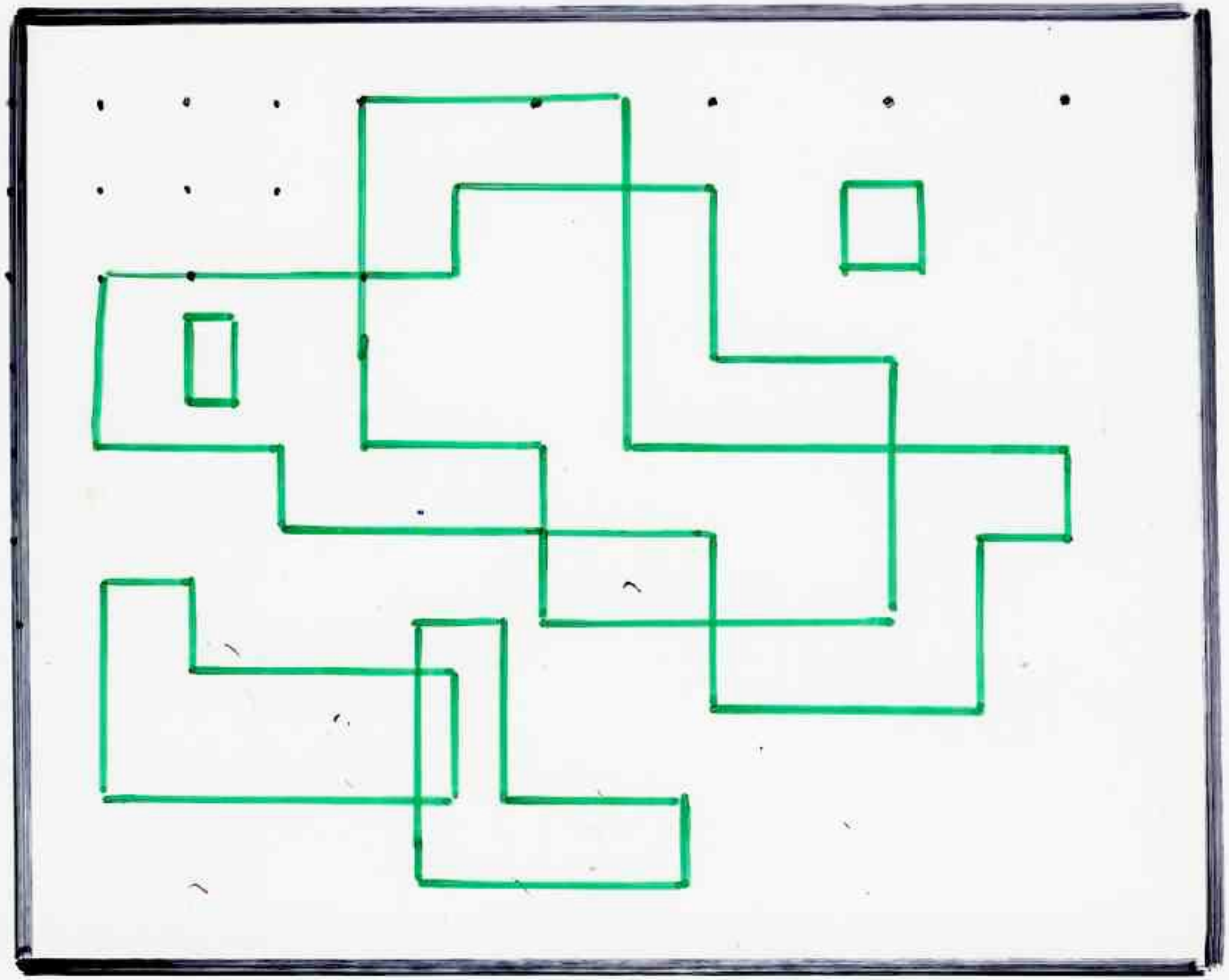
- graphe planaire

  méthode du Pfaffien

- modèle d'Ising (1925)

  Kasteleyn, Fisher, Temperley
  (1961, ...)

  Onsager (1944)



"closed" graph

Ising model

$$w = B^m A^n$$

$$uv = A^n B^m$$

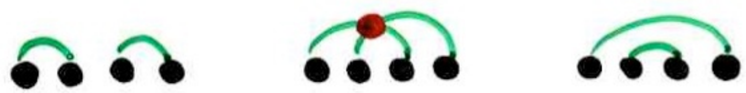
Pfaffian

Pfaf (1815)
Caianiello (1953, 59)
Wick

$T = (a_{ij})$ $\quad 1 \leq i < j \leq 2k$

ex:

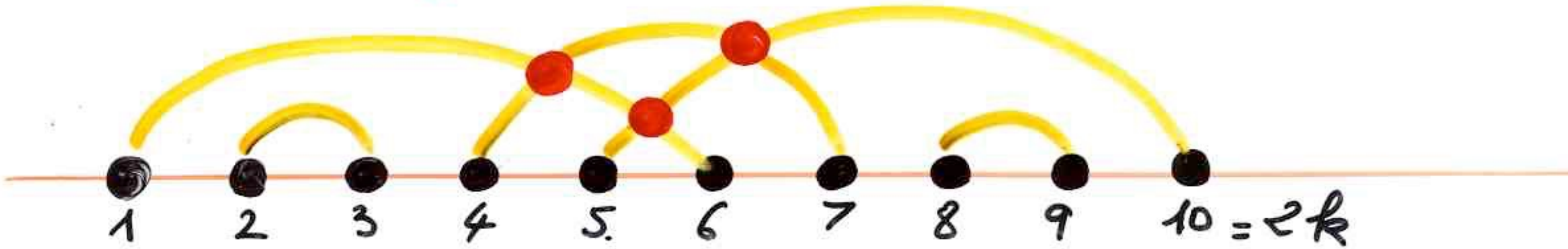
$$\left| \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{array} \right| = a_{12}\,a_{34} - a_{13}\,a_{24} + a_{14}\,a_{23}$$



Involutions α

with no fixed points

crossing number $\quad cr(\alpha) = 3$



skew-symmetric matrix

$$A = (a_{ij})_{1 \leq i,j \leq 2k}$$

$$\begin{cases} a_{ij} = -a_{ji} & i \neq j \\ a_{ii} = 0 & 1 \leq i,j \leq 2k \end{cases}$$

Cayley (1847)

$$\det(A) = (\mathrm{Pf}(T))^2$$

méthode du Pfaffien

graphe G

orientation admissible

$$i \bullet \longrightarrow \bullet j$$

$$a^*_{ij} = \begin{cases} \pm 1 \\ 0 \end{cases}$$

nb de couplages parfaits

$$= \mathrm{Pf}\left(a^*_{ij}\right)_{1 \leq i < j \leq 2k}$$

$$= \left(\mathrm{dét}\left(a^*_{ij}\right)\right)^{1/2}_{1 \leq \substack{i \\ j} \leq 2k}$$

<u>Thm</u> (Kasteleyn, 1967)

Tout graphe planaire

admet

une orientation

admissible