Combinatorics and Physics

Chapter 2c
Dimers, Tilings, Non-crossing paths and determinant

IIT-Madras
2 February 2015

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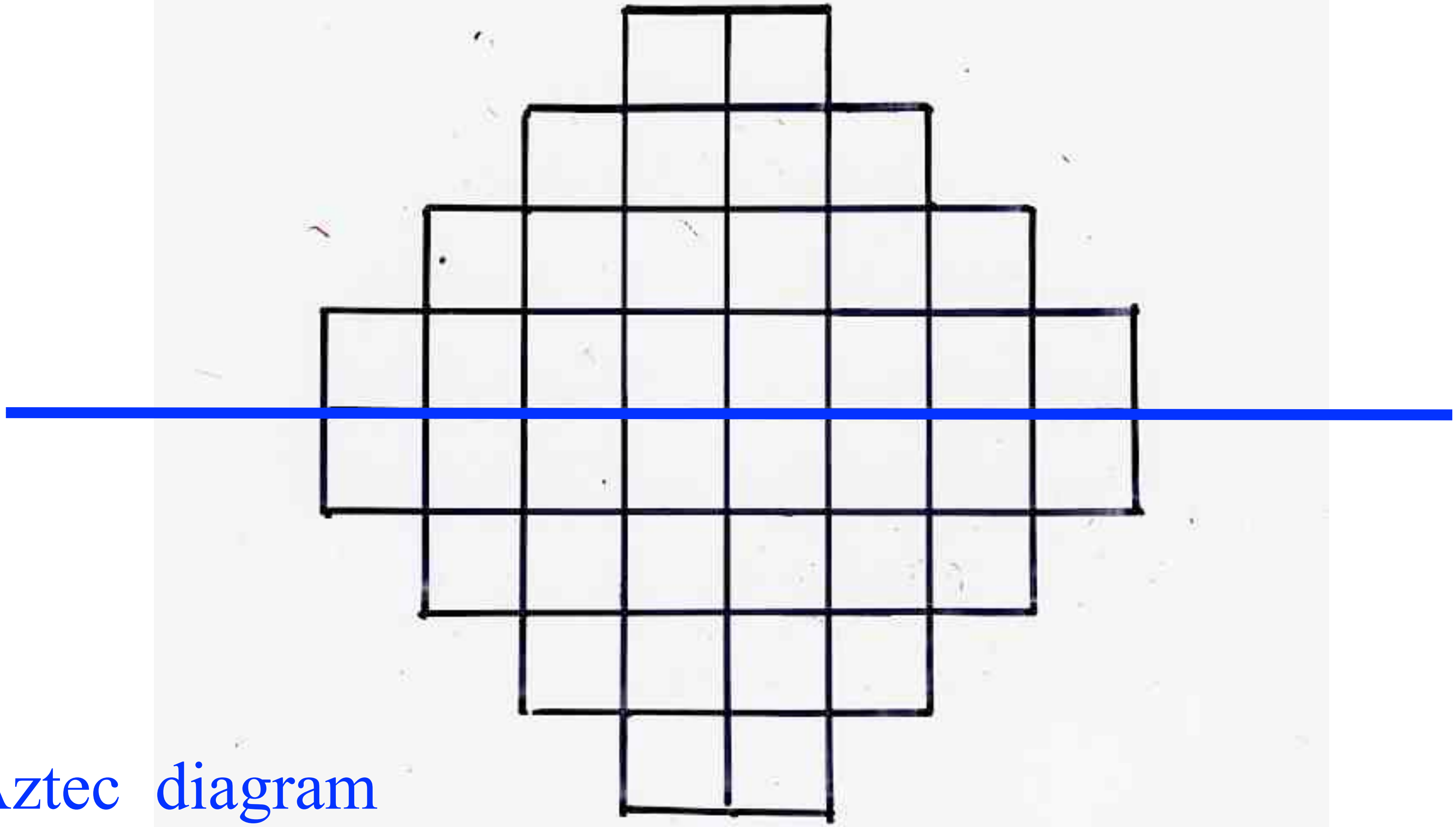
§ 8 Aztec tilings



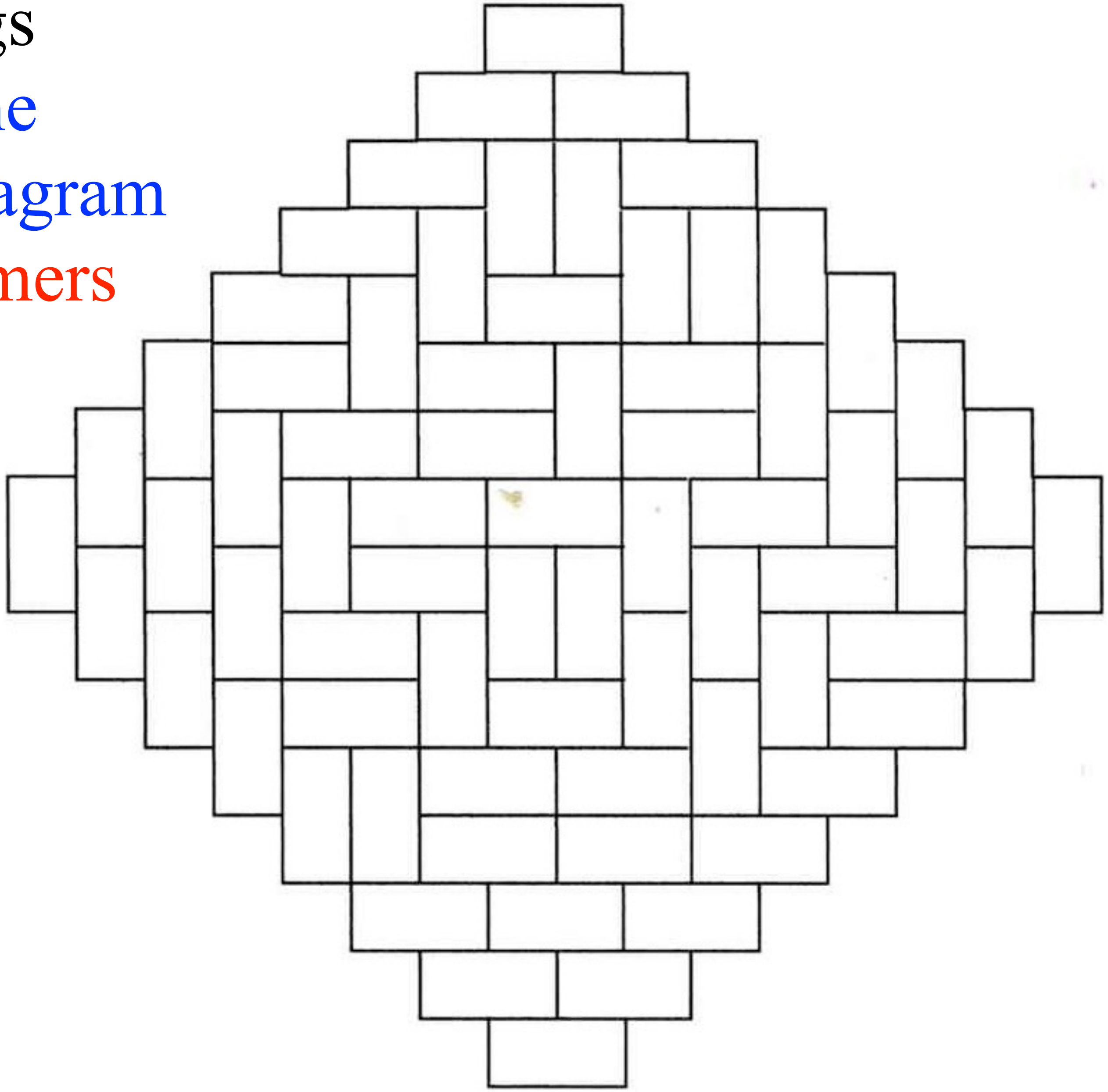
Aztec diagram

tilings

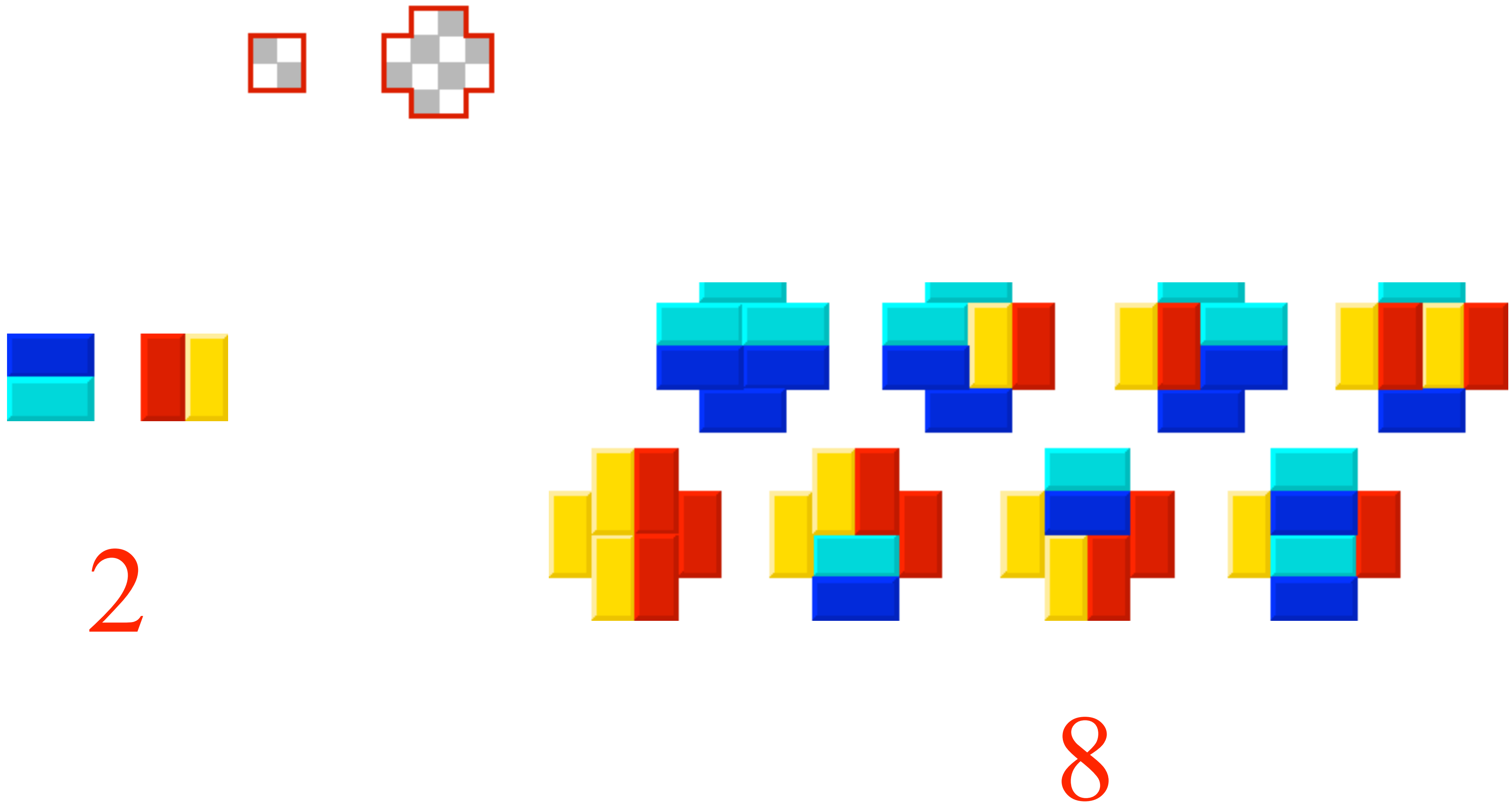
of the

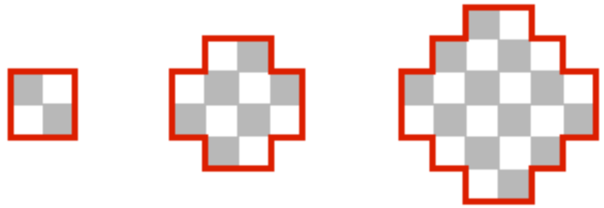
Aztec diagram

with dimers



2


2
8



number of
tilings

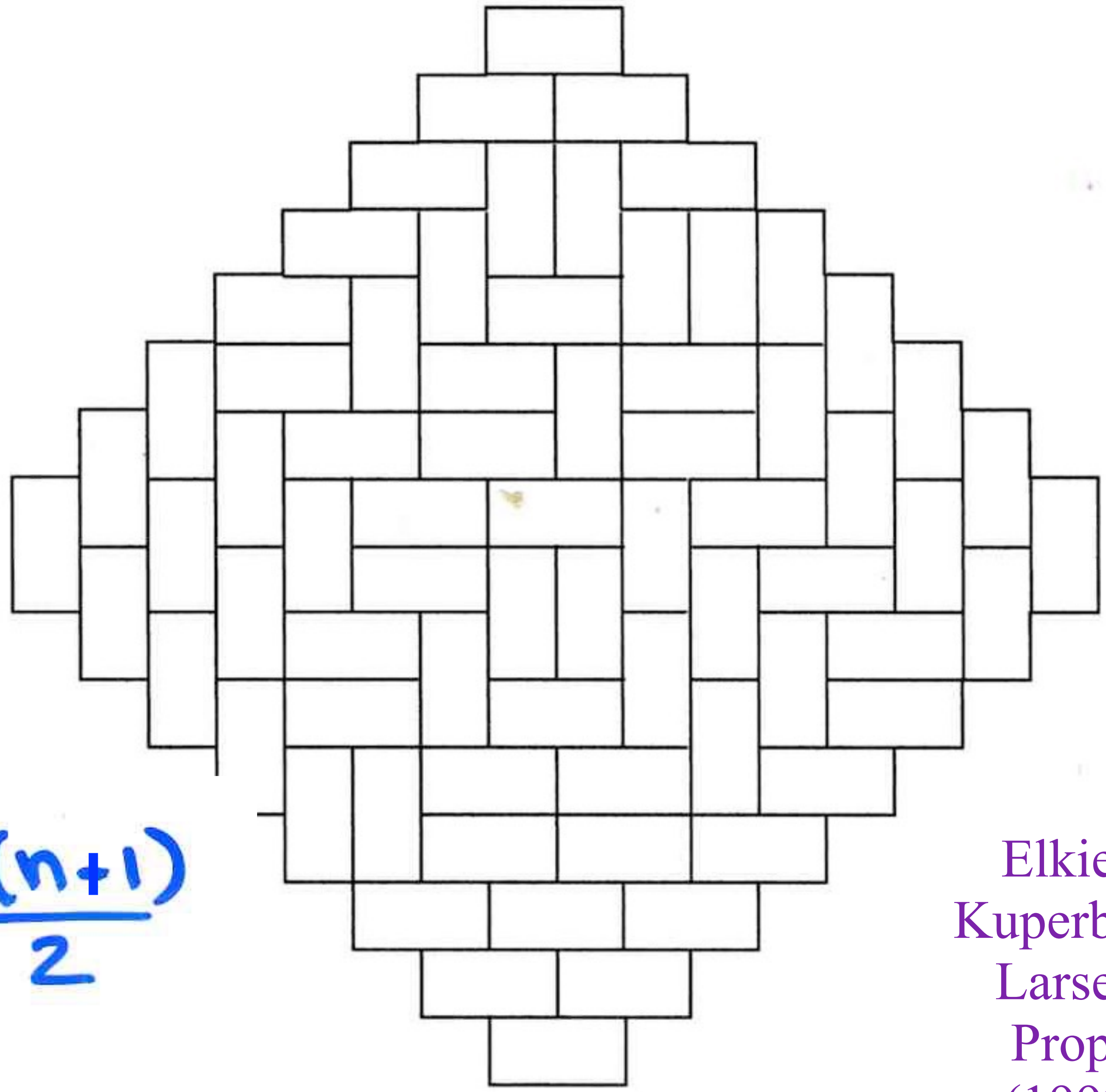
2 8 64 1024

2^1 2^3 2^6 2^{10}

2^1 $2^{(1+2)}$ $2^{(1+2+3)}$ $2^{(1+2+3+4)}$

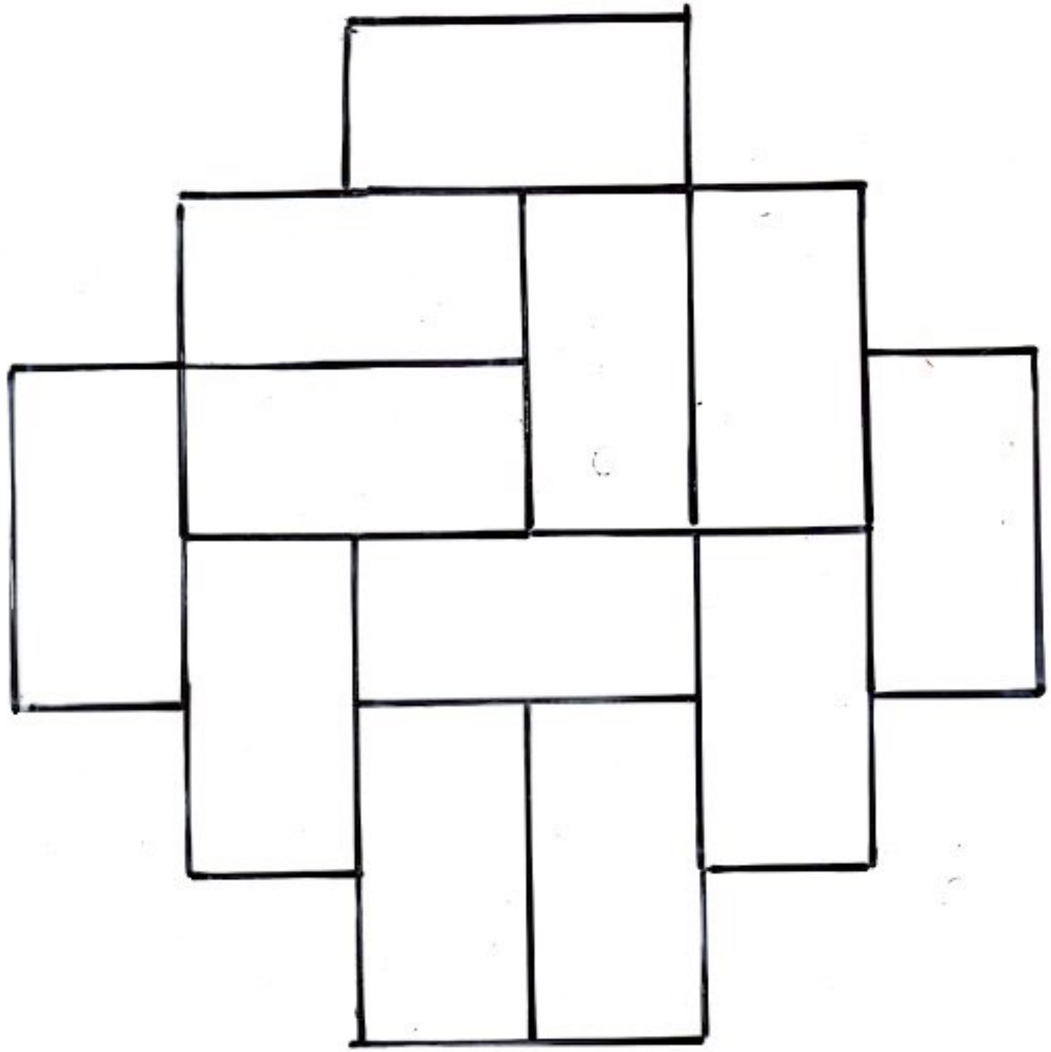
number of tilings of the Aztec diagram with dimers

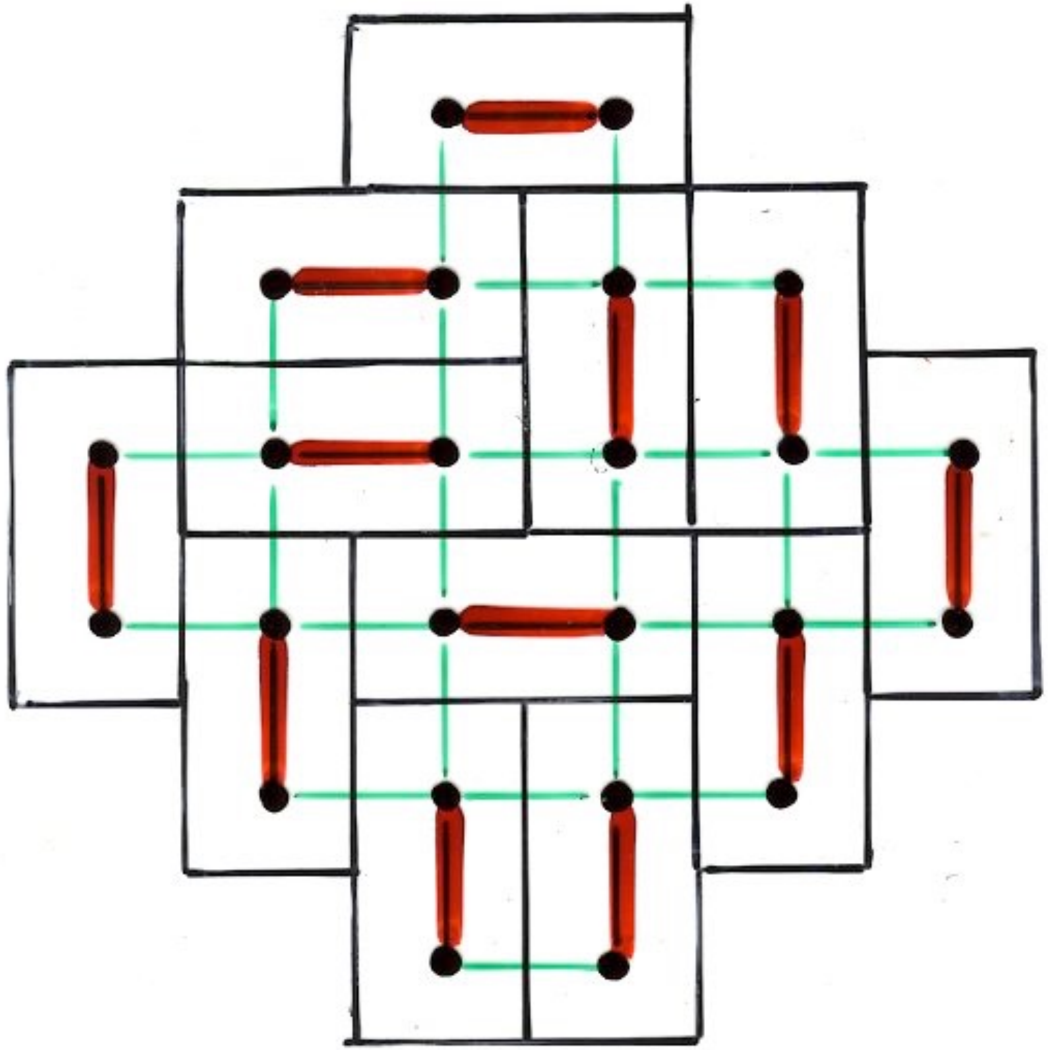
$$2^{(1+2+3+4+....+n)}$$

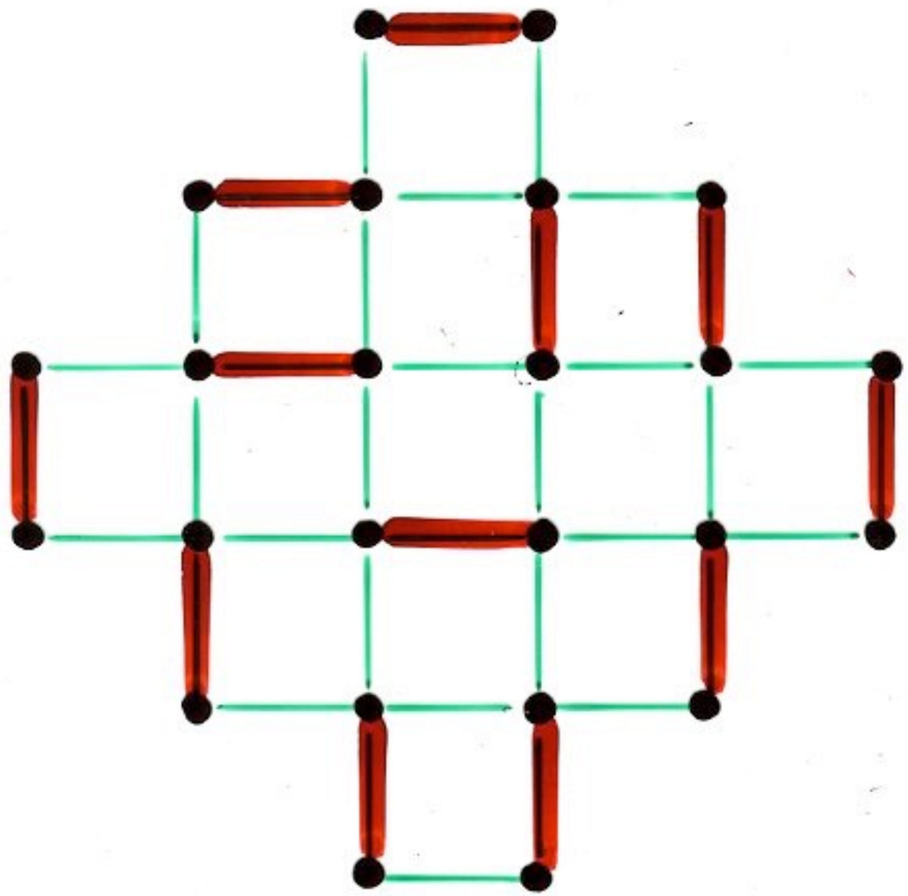


$$2^{\frac{n(n+1)}{2}}$$

Elkies, Kuperberg, Larsen, Propp (1992)





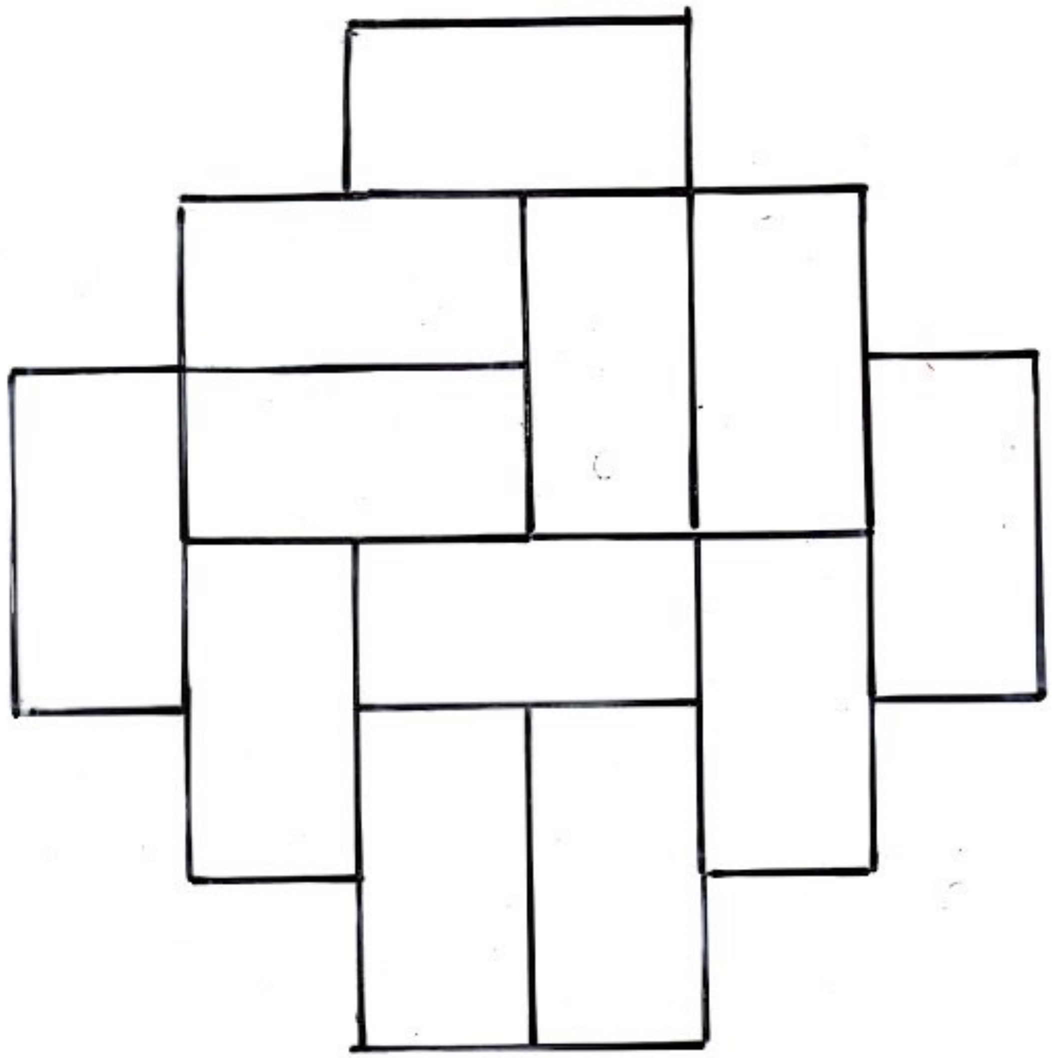


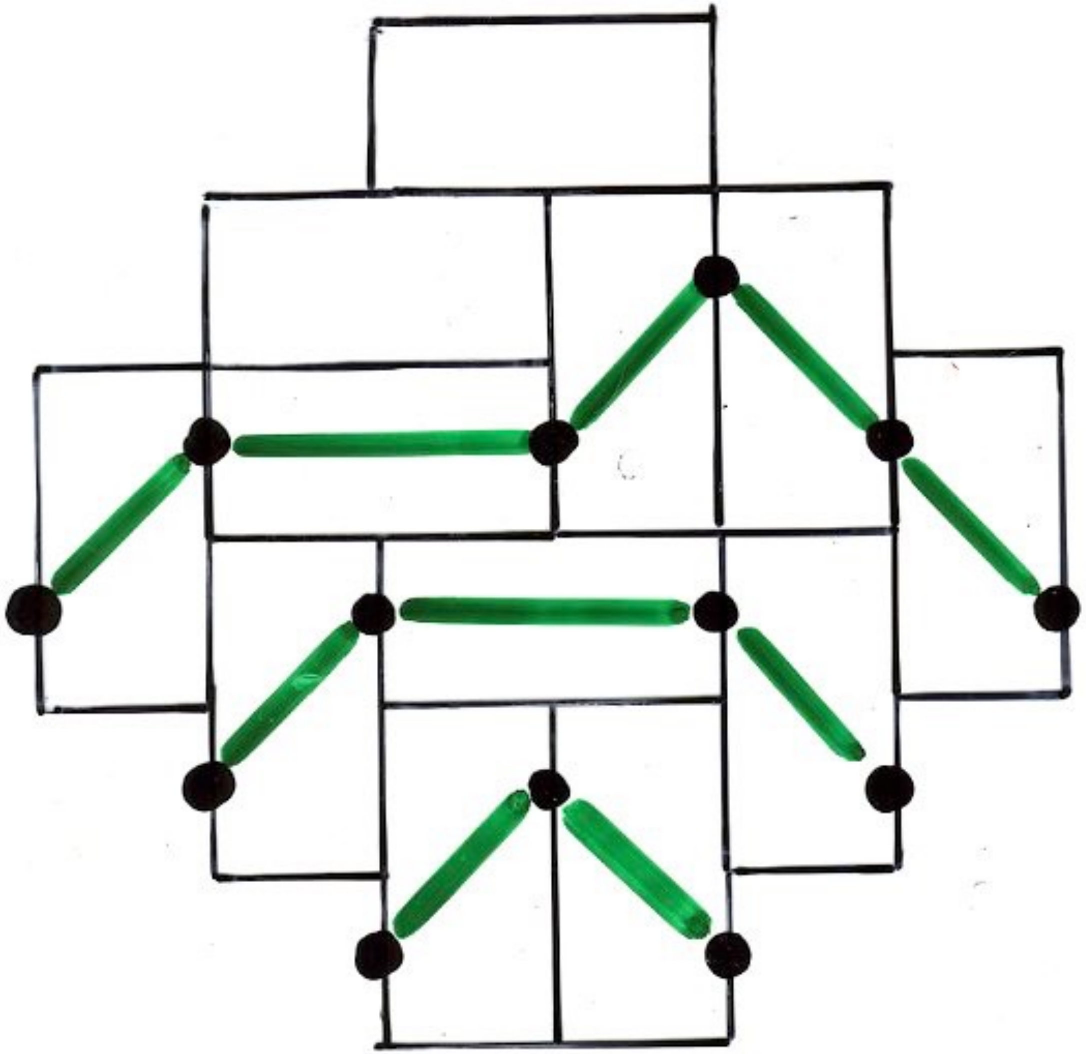
bijection

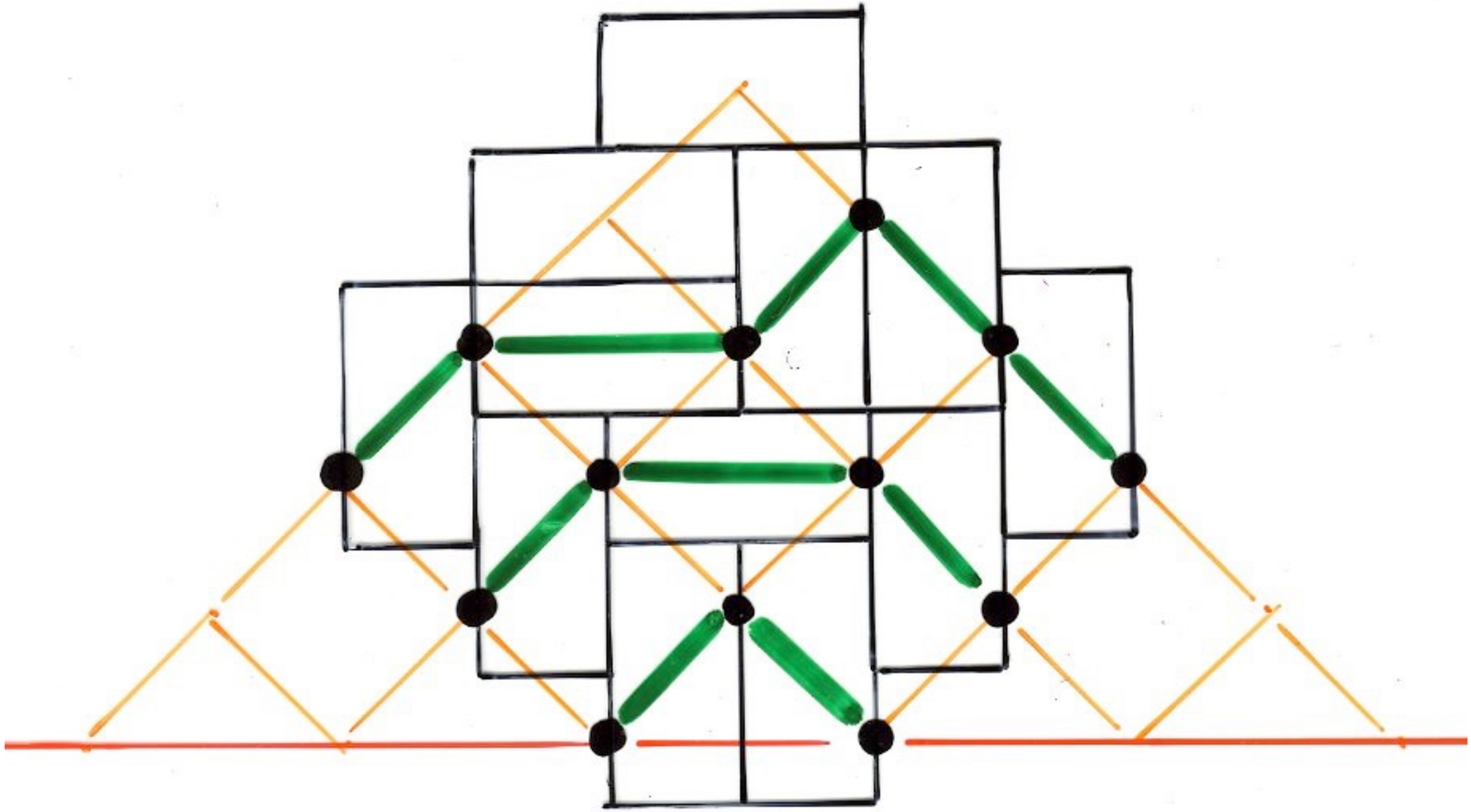
Aztec tilings

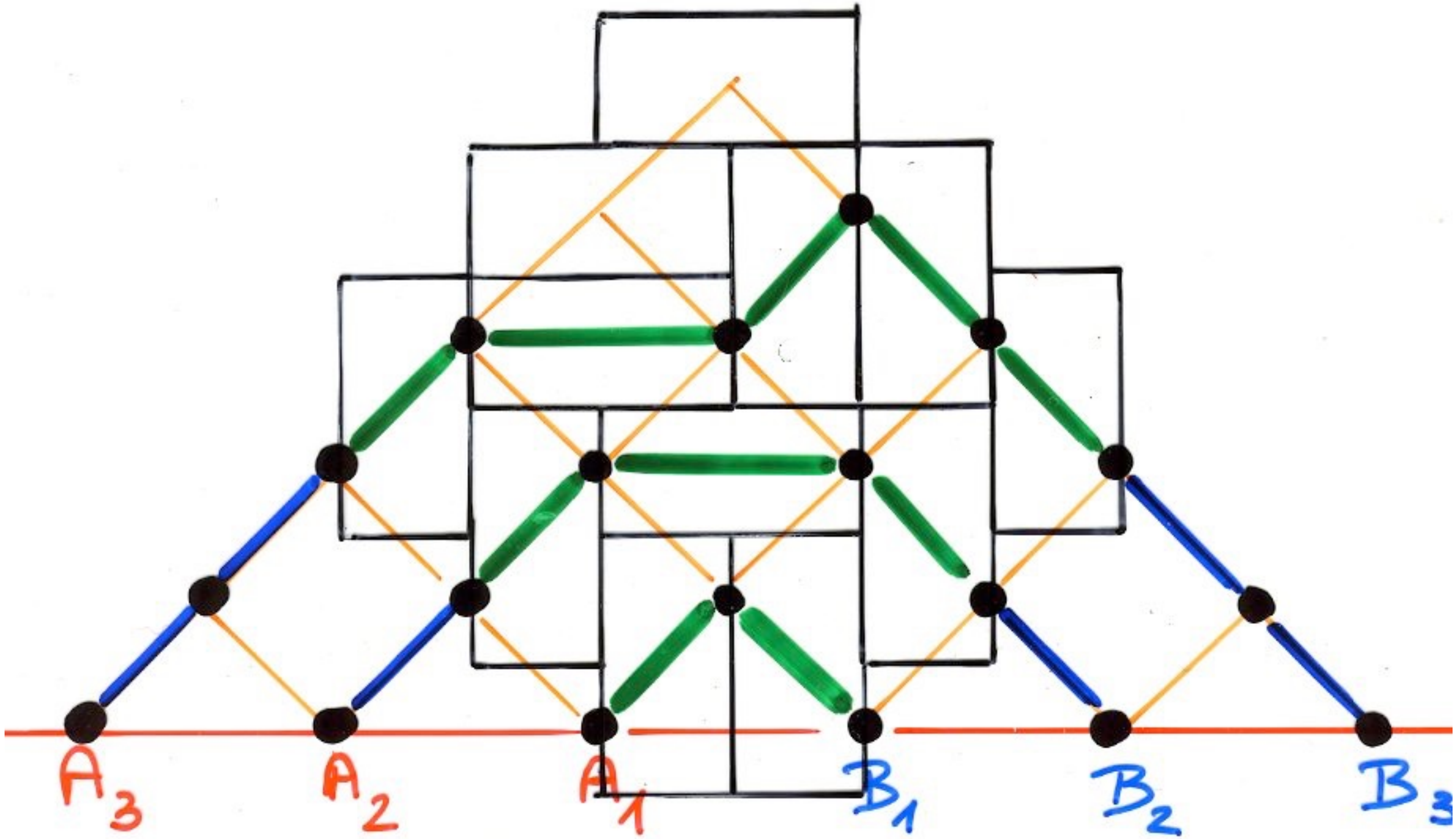
$\updownarrow$

non-intersecting paths
related to a Hankel determinant



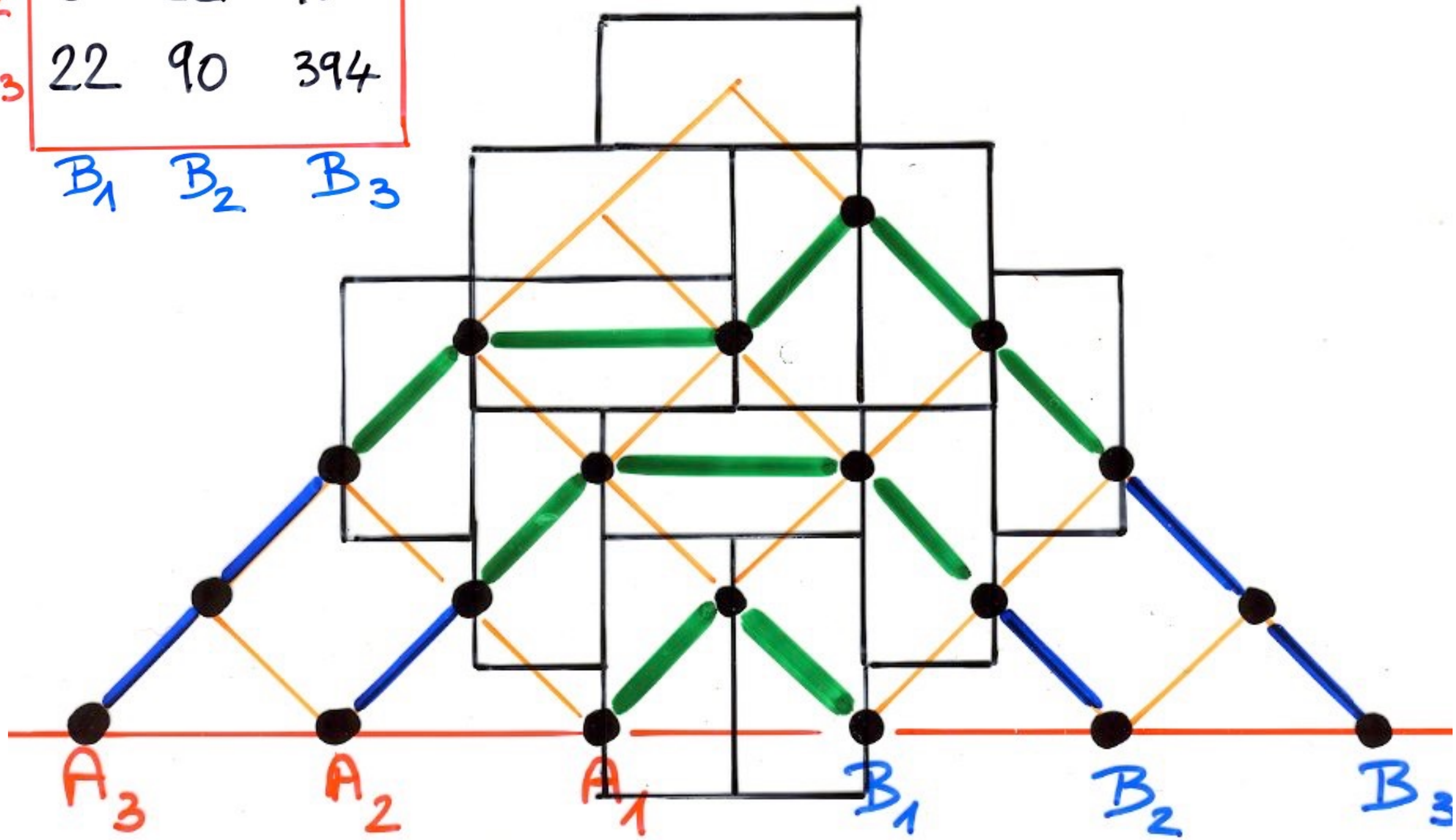






A_3
A_2
A_1
B_1
B_2
B_3

	B_1	B_2	B_3
A_1	2	6	22
A_2	6	22	90
A_3	22	90	394



$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$

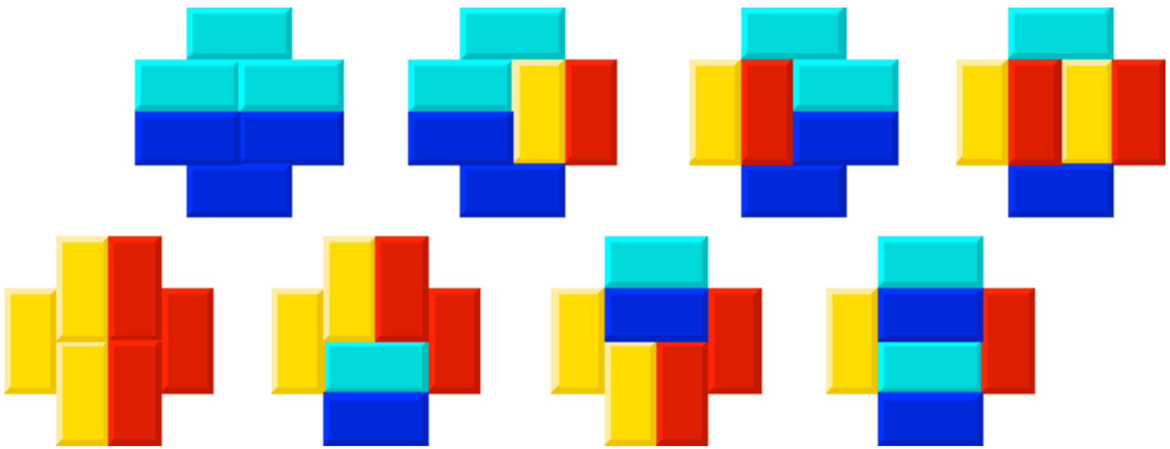
$$= 44 - 36$$

$$\det\begin{pmatrix} 2 & 6 \\ 6 & 22 \end{pmatrix} = (2 \times 22) - (6 \times 6)$$

$$= 44 - 36$$

$$= 8 = 2^3$$

(!)



$$\det\begin{pmatrix} 2 & 6 & 22 \\ 6 & 22 & 90 \\ 22 & 90 & 394 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 22 & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ 6 & \cdot & \cdot \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ \cdot & \cdot & 90 \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$+\,17336 \qquad +\,11880 \qquad +\,11880 \;\rightarrow\; 41096$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & \cdot & 90 \\ \cdot & 90 & \cdot \end{pmatrix} \quad \begin{pmatrix} \cdot & 6 & \cdot \\ 6 & \cdot & \cdot \\ \cdot & \cdot & 394 \end{pmatrix} \quad \begin{pmatrix} \cdot & \cdot & 22 \\ \cdot & 22 & \cdot \\ 22 & \cdot & \cdot \end{pmatrix}$$

$$-\,16200 \qquad -\,14184 \qquad -\,10648 \;\rightarrow\; -41032$$

$$64$$

$$= 2^6 \quad (!!)$$

Narayana numbers

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

Dyck paths by number of peaks

$$\sum_{k\geq 1} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^k$$

Schröder numbers

$$S_n = \sum_{k\geq 1} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} 2^k$$

Narayana numbers

	k=1	2	3	4	5	6	- -	Catalan
n=1	1							→ 1
2	1	1						→ 2
3	1	3	1					→ 5
4	1	6	6	1				→ 14
5	1	10	20	10	1			→ 42
6	1	15	50	50	15	1		→ 132
⋮	- - -							

Schröder numbers

					1
1×2					→ 2
1×2	1×4				→ 6
1×2	3×4	1×8			→ 22
1×2	6×4	6×8	1×16		→ 90
1×2	10×4	20×8	10×16	1×32	→ 394

Schröder numbers $S_n = \sum_{k \geq 1} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} 2^k$

also $S_n = \sum_{i=0}^{n} \binom{2n-i}{i} C_{n-i}$

Catalan

Schröder numbers

(1)

$1\times 2 \longrightarrow$ (2)

$1\times 2 \quad 1\times 4 \longrightarrow$ (6)

$1\times 2 \quad 3\times 4 \quad 1\times 8 \longrightarrow$ (22)

$1\times 2 \quad 6\times 4 \quad 6\times 8 \quad 1\times 16 \longrightarrow$ (90)

$1\times 2 \quad 10\times 4 \quad 20\times 8 \quad 10\times 16 \quad 1\times 32 \longrightarrow$ (394)

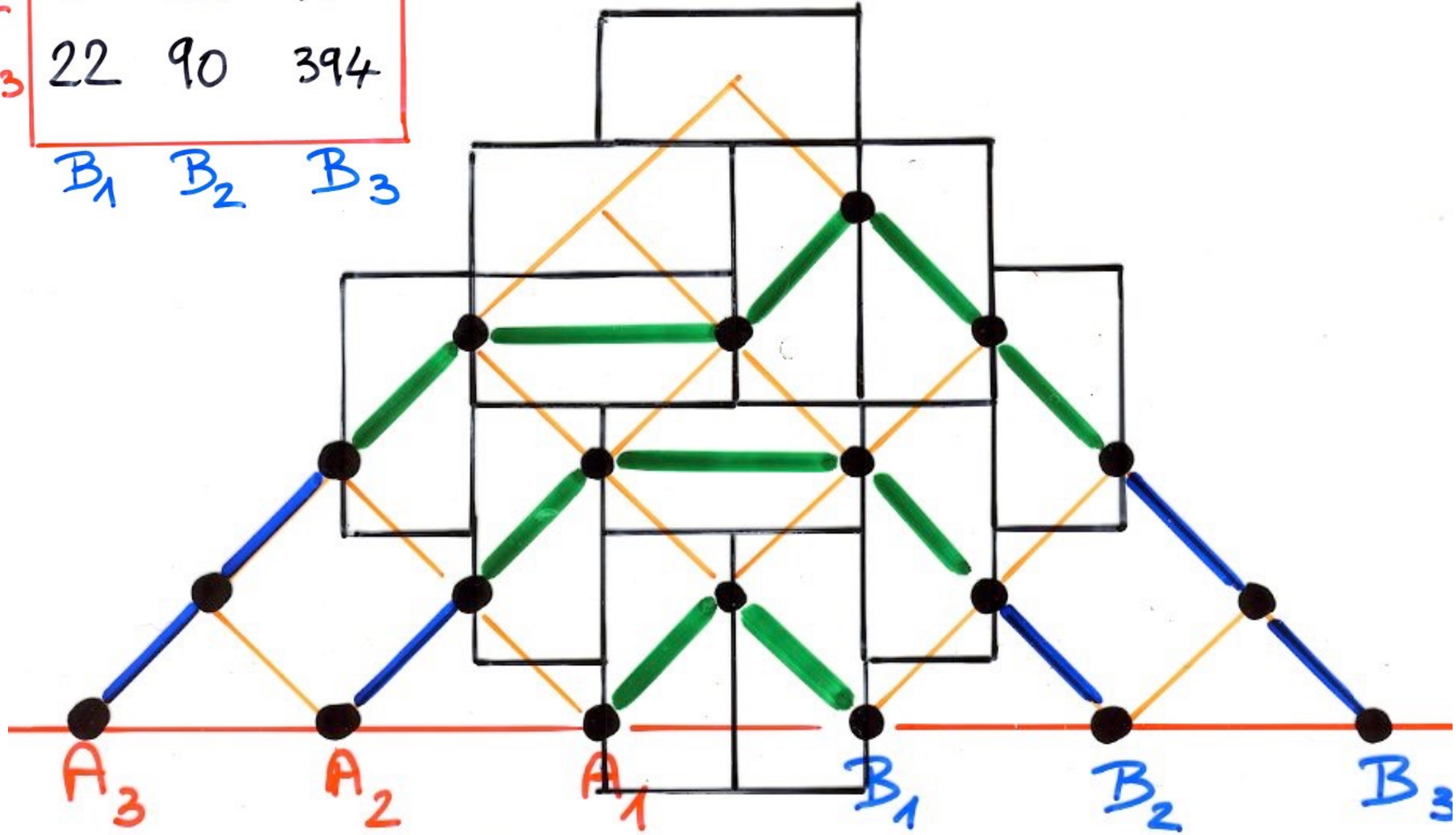
(litte) Schröder = $\frac{1}{2} S_n$

1, 1, 3, 11, 45, 197, …, 103 049

Hipparchus number

§9 computing the Hankel determinant of Schröder numbers giving the number of tilings of the Aztec diagram

	B_1	B_2	B_3
A_1	2	6	22
A_2	6	22	90
A_3	22	90	394

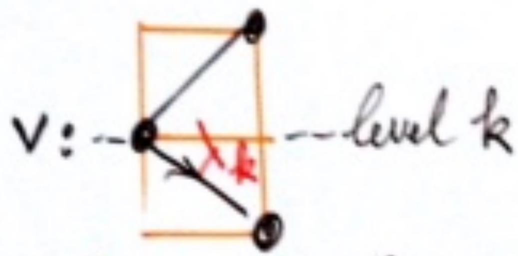


Schröder numbers

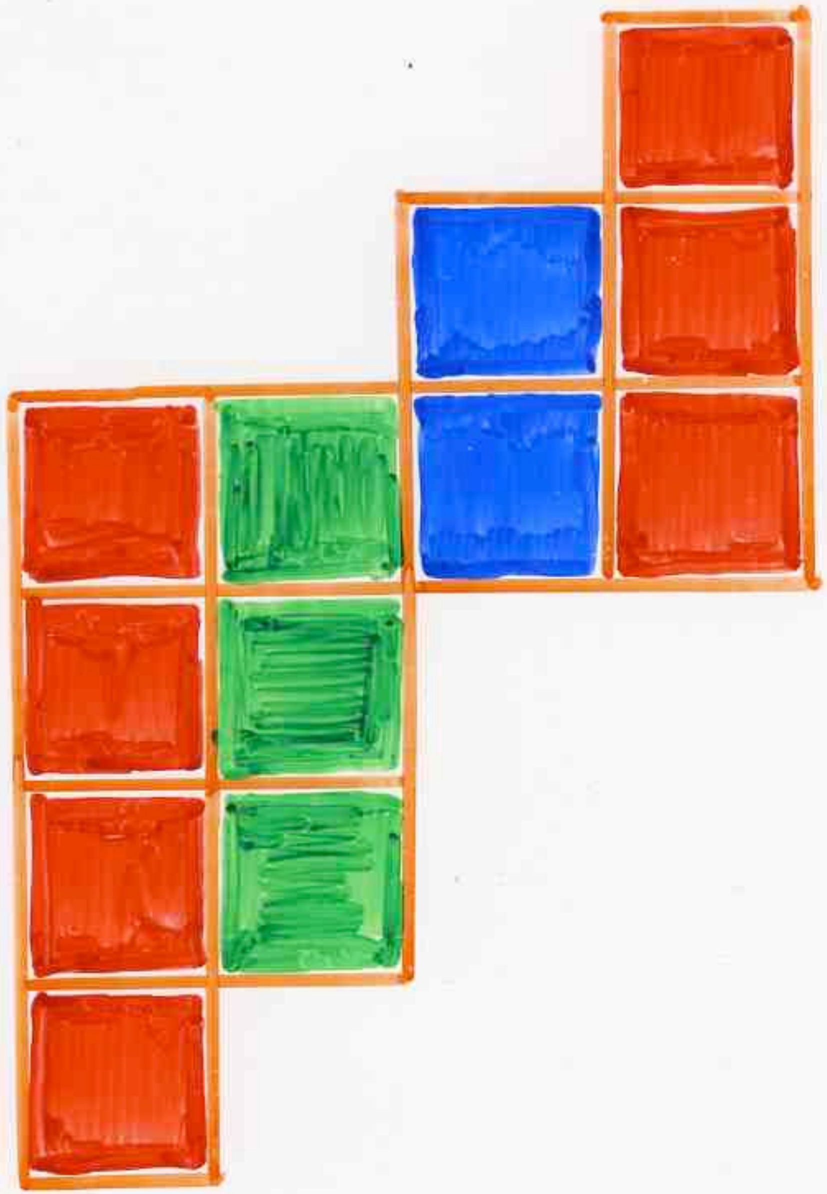
$$S_n = \sum_{k \geq 1} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} 2^k$$

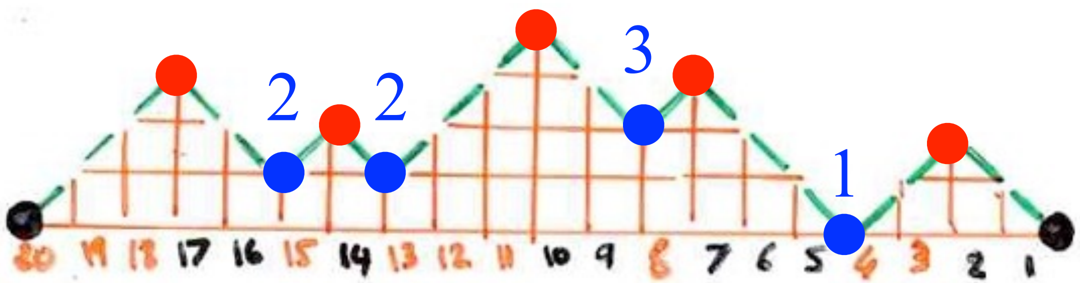
$$S_n = \sum_{\omega} v(\omega)$$

ω Dyck paths $|\omega| = 2n$

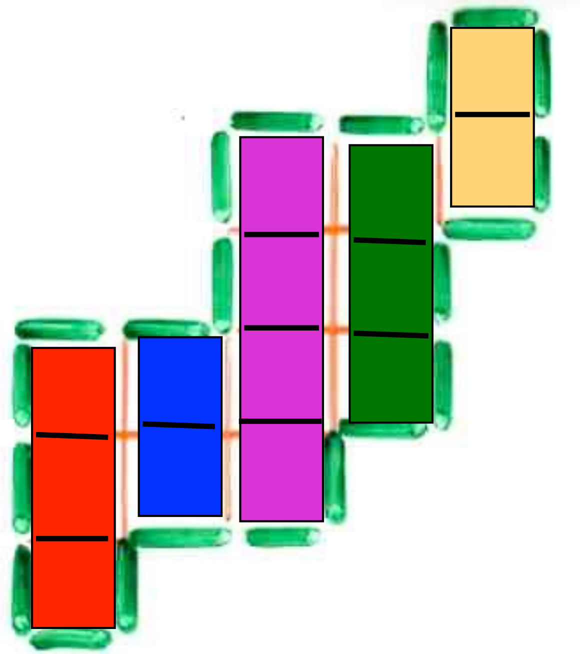


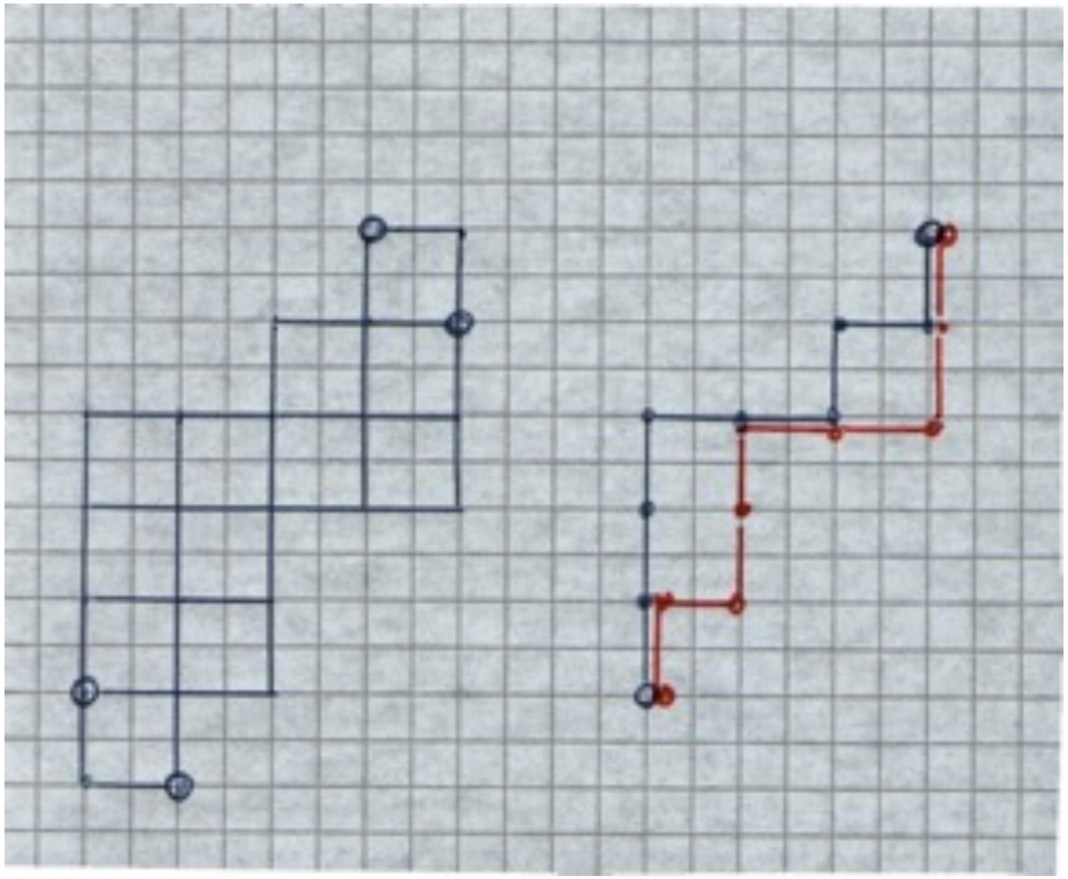
$$\lambda_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$$

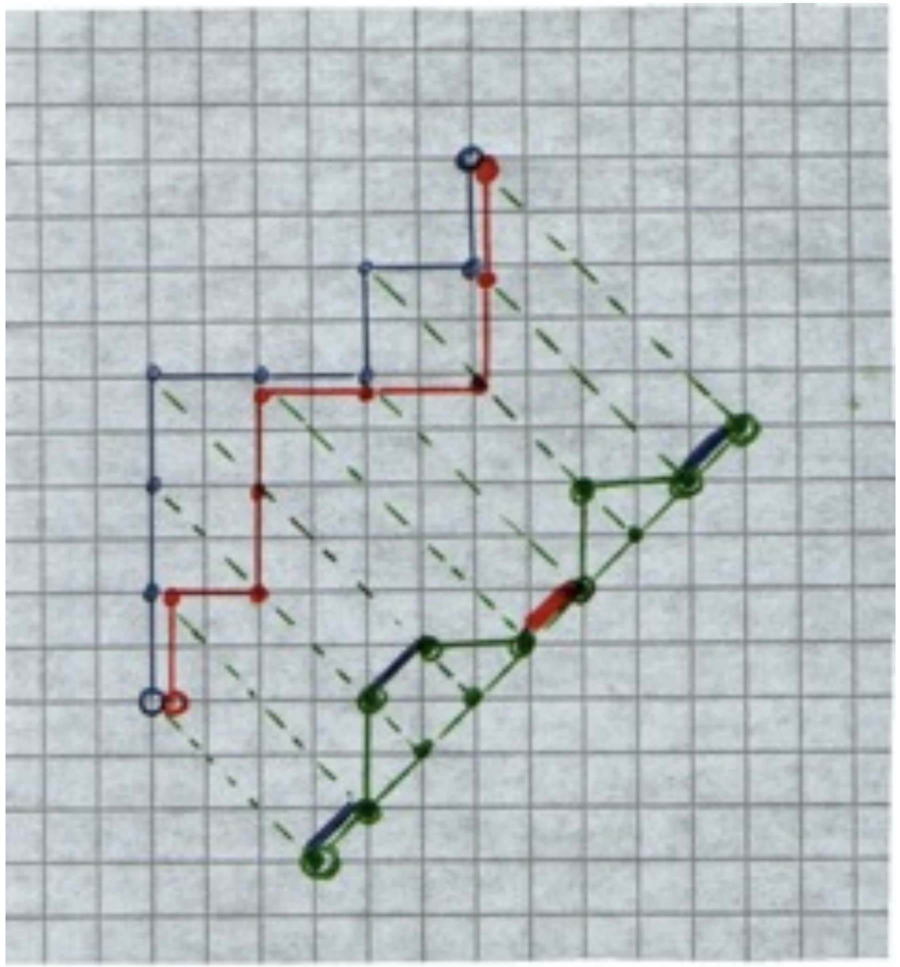


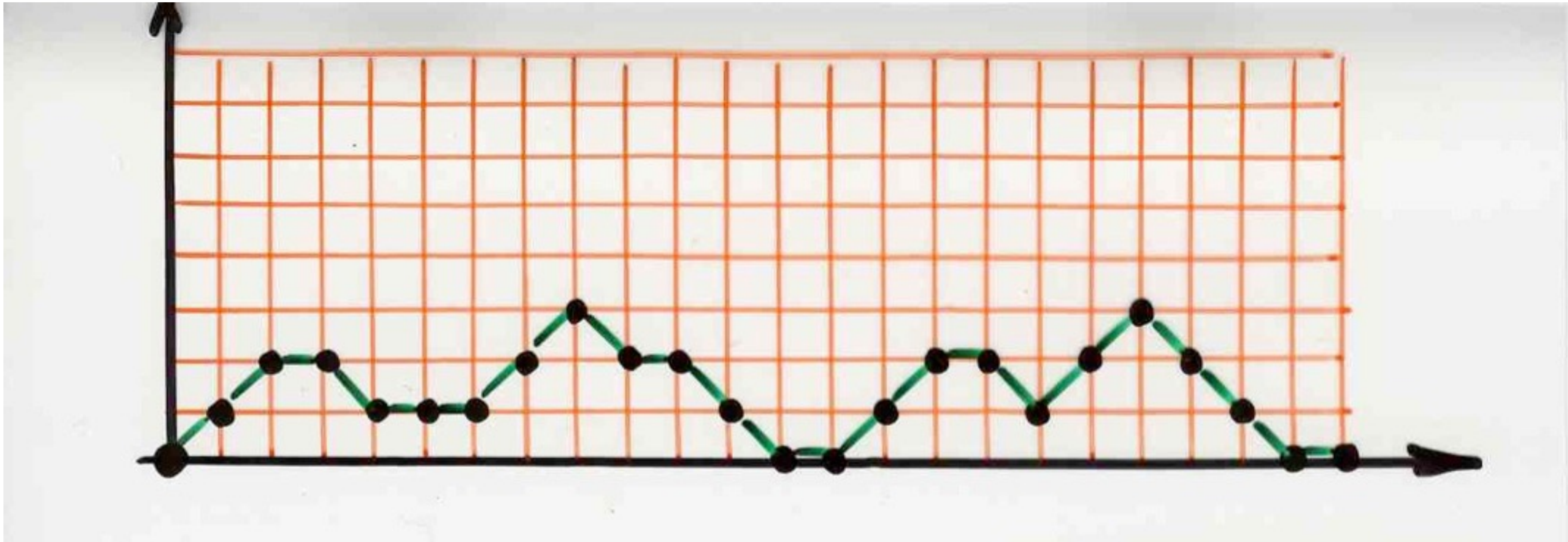


2
2
3
1
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



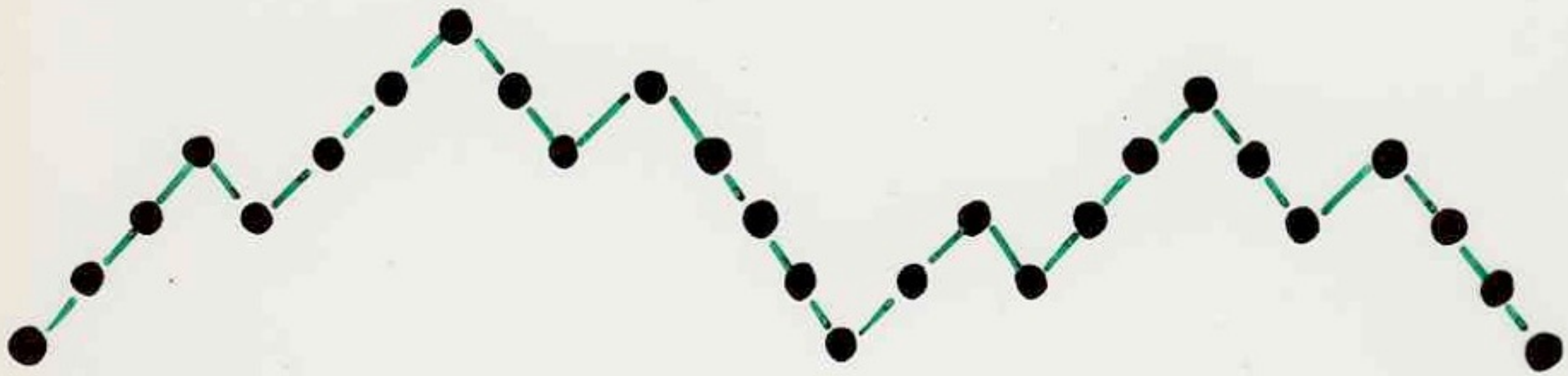






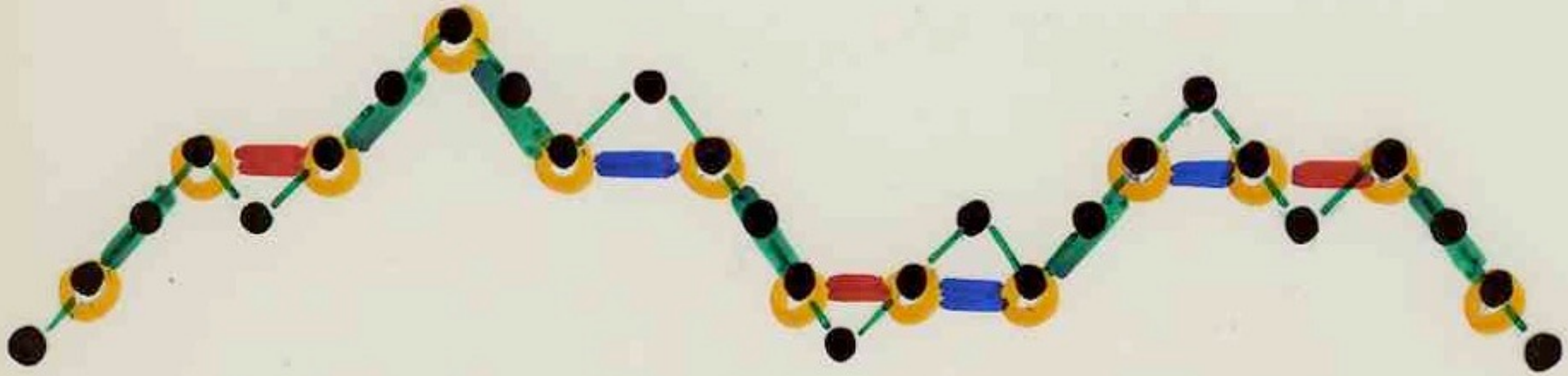
Motzkin path

bijection Dyck path length 2n+2

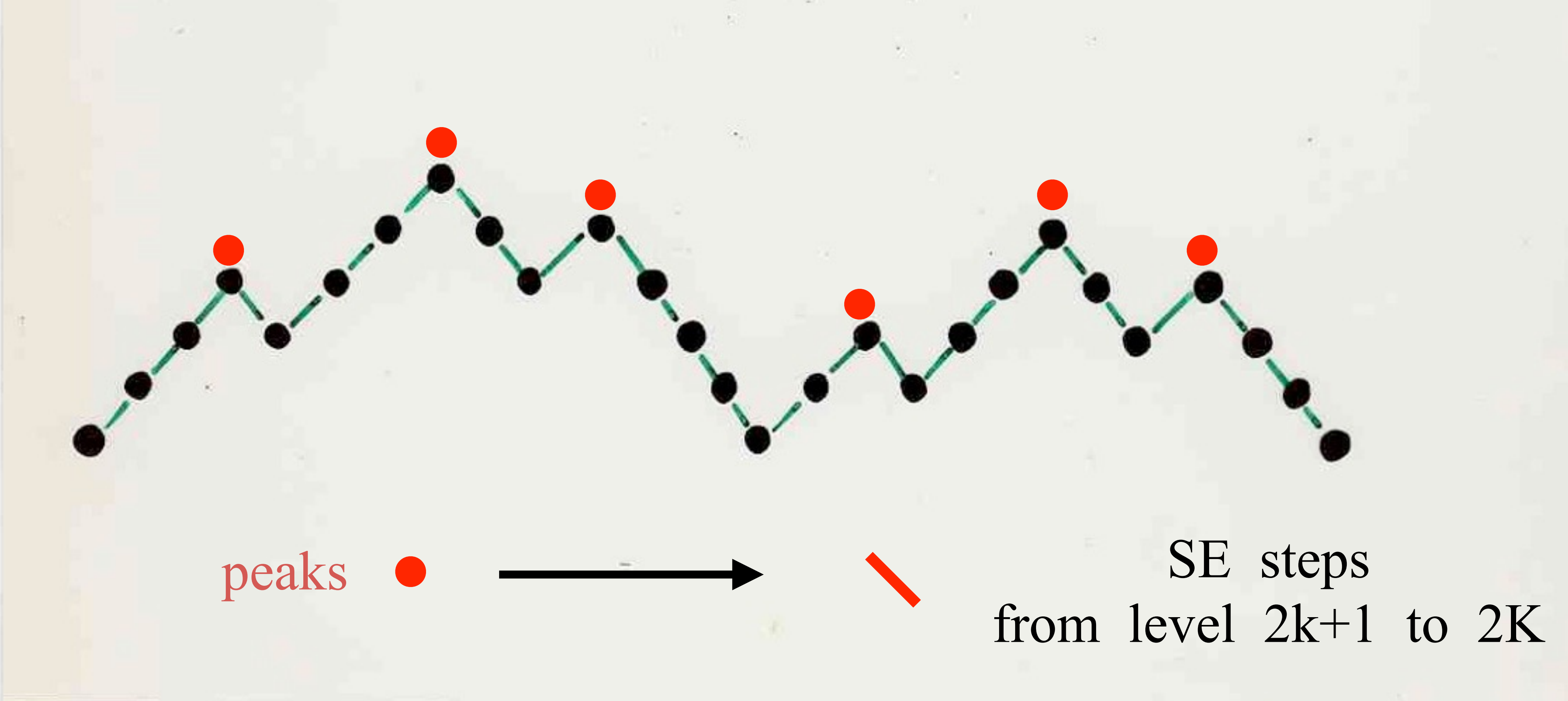


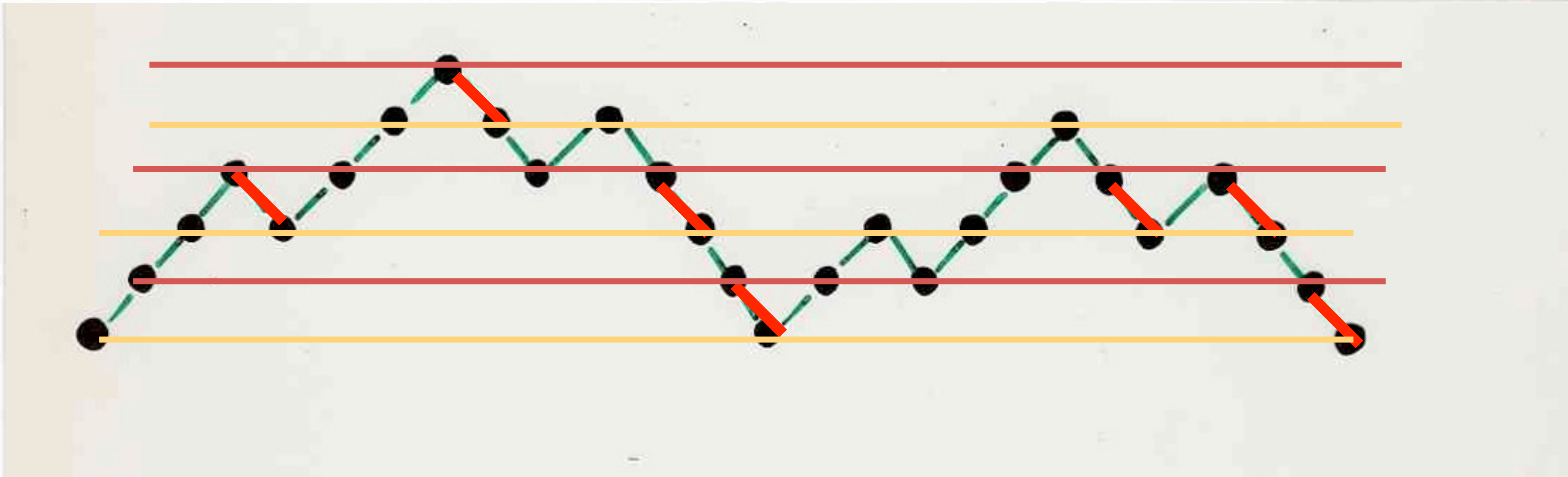
2-colored Motzkin path length n

bijection Dyck path length 2n+2



2-colored Motzkin path length n


peaks
SE steps
from level 2k+1 to 2K



§10 Hankel determinants
and
continued fractions

The fundamental Flajolet Lemma

combinatorial interpretation of a
continued fraction with weighted paths

Discrete Mathematics 32 (1980) 125–161

Discrete Maths (1980)

COMBINATORIAL ASPECTS OF CONTINUED FRACTIONS

P. FLAJOLET

IRIA, 78150 Rocquencourt, France

Received 23 March 1979
Revised 11 February 1980

We show that the universal continued fraction of the Stieltjes-Jacobi type is equivalent to the characteristic series of labelled paths in the plane. The equivalence holds in the set of series in non-commutative indeterminates. Using it, we derive direct combinatorial proofs of continued fraction expansions for series involving known combinatorial quantities: the Catalan numbers, the Bell and Stirling numbers, the tangent and secant numbers, the Euler and Eulerian numbers.... We also show combinatorial interpretations for the coefficients of the elliptic functions, the coefficients of inverses of the Tchebycheff, Charlier, Hermite, Laguerre and Meixner polynomials. Other applications include cycles of binomial coefficients and inversion formulae. Most of the proofs follow from direct geometrical correspondences between objects.

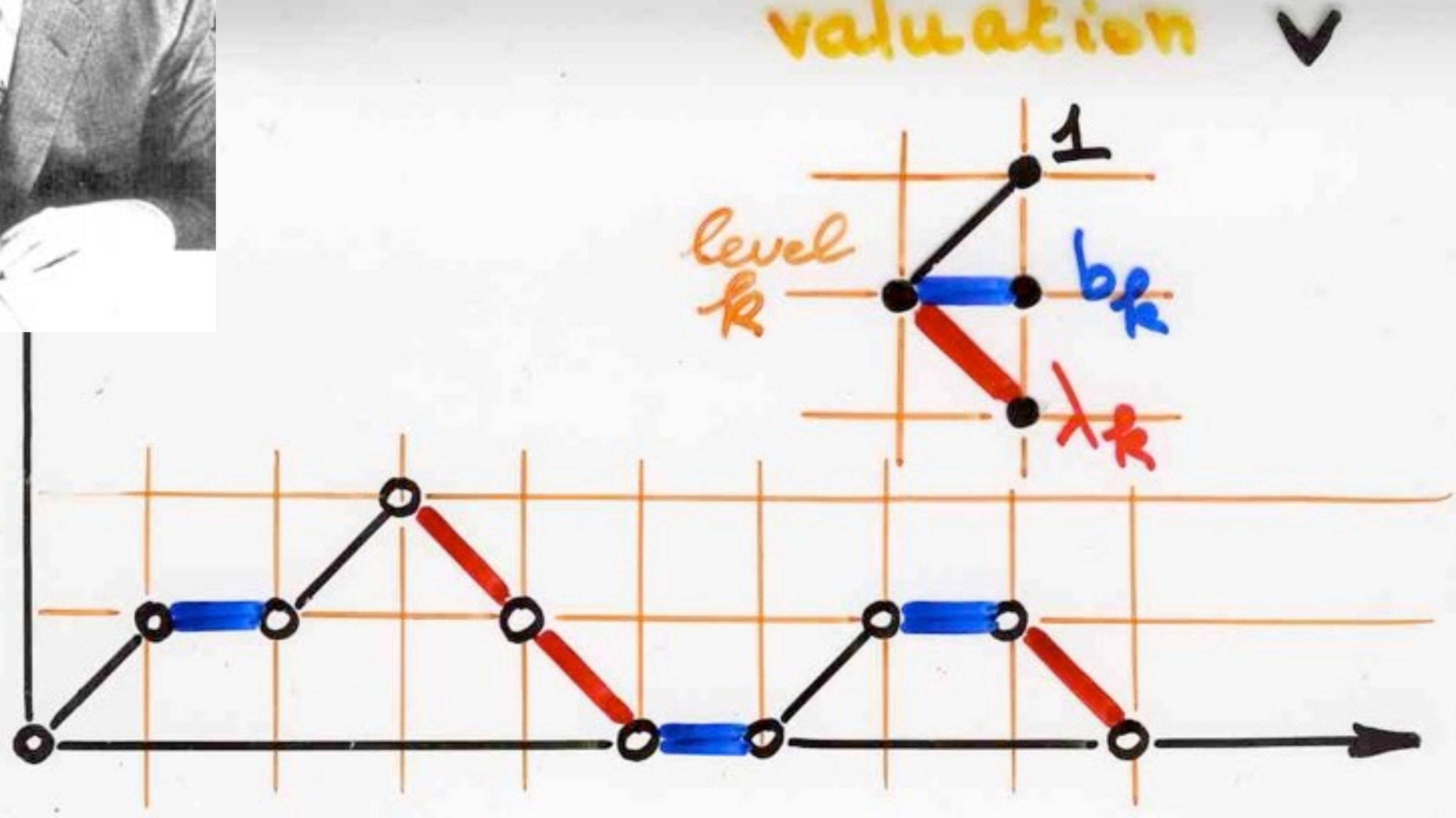
Introduction

In this paper we present a geometrical interpretation of continued fractions together with some of its enumerative consequences. The basis is the equivalence

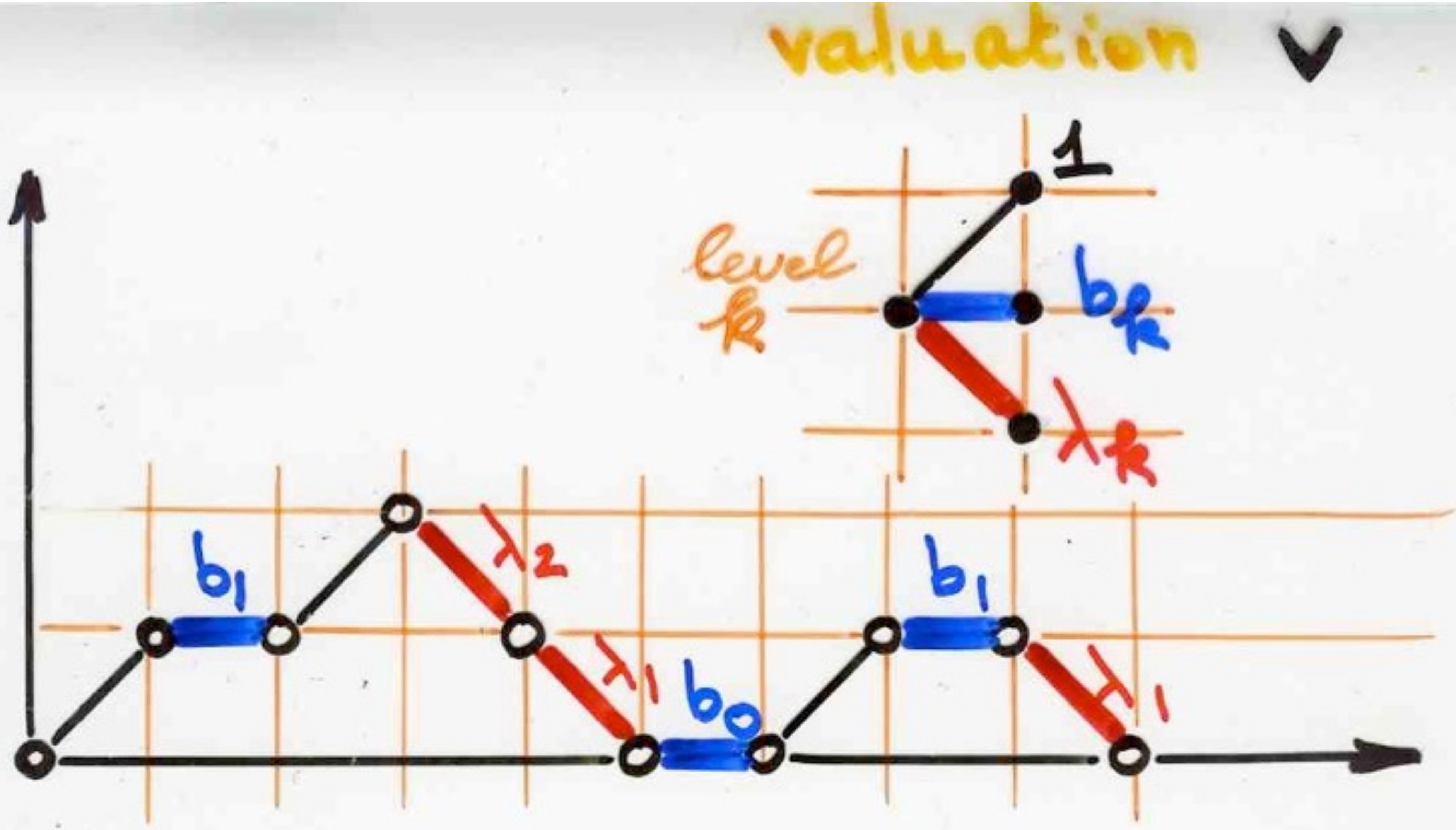
weigthed Dyck paths
and Motzkin paths

$\{b_k\}_{k\geq 0}$ $\{\lambda_k\}_{k\geq 1}$

$b_k, \lambda_k \in \mathbb{K}$ ring



ω Motzkin path



ω Motzkin path

$v(\omega) = b_0 \, b_1^2 \, \lambda_1^2 \, \lambda_2$

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin} \\ |\omega| = n}} v(\omega)$$

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots}}}}}$$

$J(t; b, \lambda)$

Jacobi continued fraction

$b = \{b_k\}_{k\geq 0}$ $\lambda = \{\lambda_k\}_{k\geq 1}$

Jacobi continued fraction

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots}{1 - b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots}}}}}$$

$$\mu_n = \sum_{\substack{\omega \\ \text{Motzkin path} \\ |\omega| = n}} v(\omega)$$

Philippe Flajolet
fundamental
Lemma

continued fractions

$$\sum_{\omega \text{ Dyck path}} v(\omega)\, t^{|\omega|/2} = \cfrac{1}{1-\cfrac{\lambda_1 t}{1-\cfrac{\lambda_2 t}{\cdots\cdots\cfrac{}{1-\cfrac{\lambda_k t}{\cdots\cdots}}}}}$$

$S(t;\lambda)$

Stieltjès continued fraction

combinatorial theory of orthogonal polynomials and continued fractions

Flajolet (1980) Viennot (1983, ...)

- théorie combinatoire des polynômes orthogonaux généraux (x.g.v.)

  Publications du LACIM
  UQAM (Université du Québec à Montréal)
  Notes de conférences (1984) 214 p.
  réédition : n° hors série

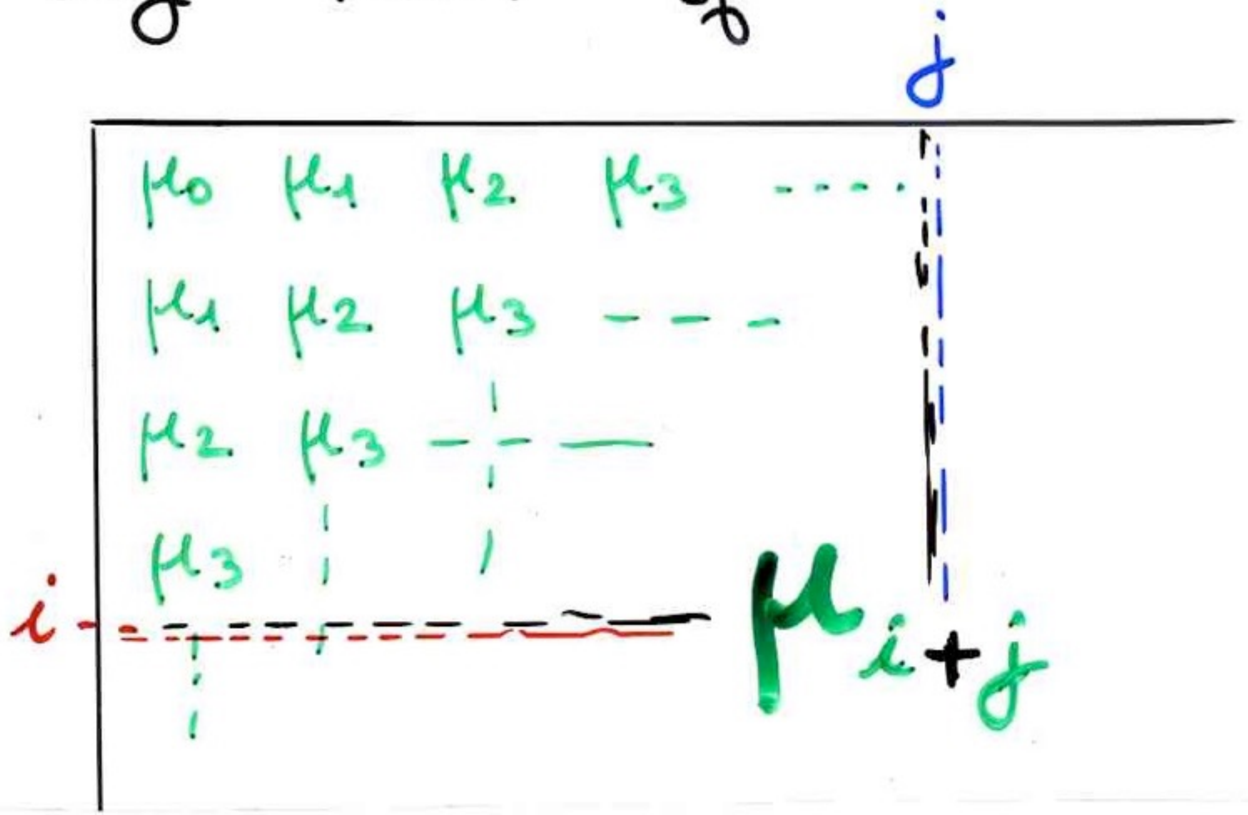
§11 computing the coefficients

λ_k $\quad\quad$ b_k

with Hankel determinants

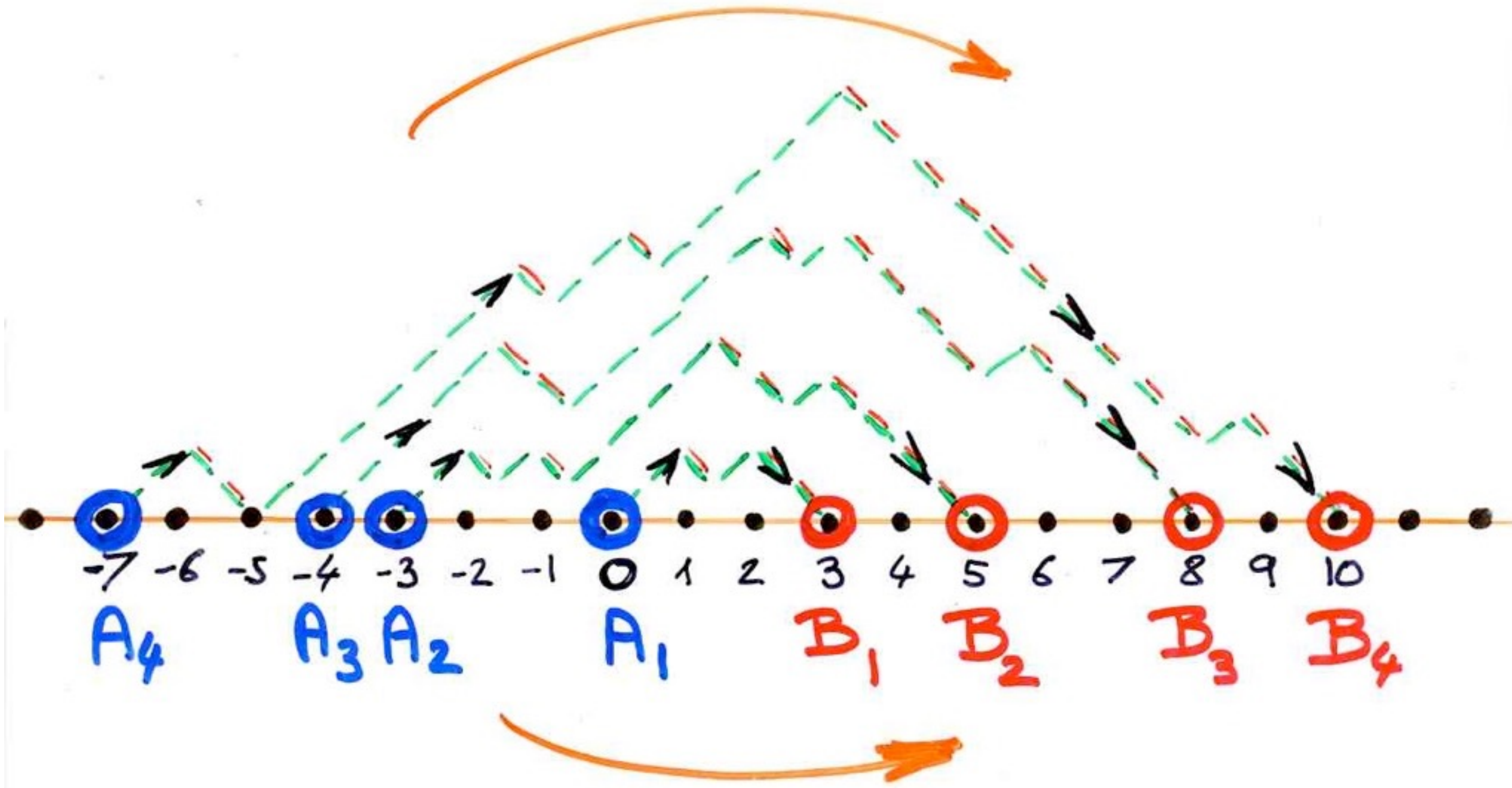
Hankel determinant

any minor of

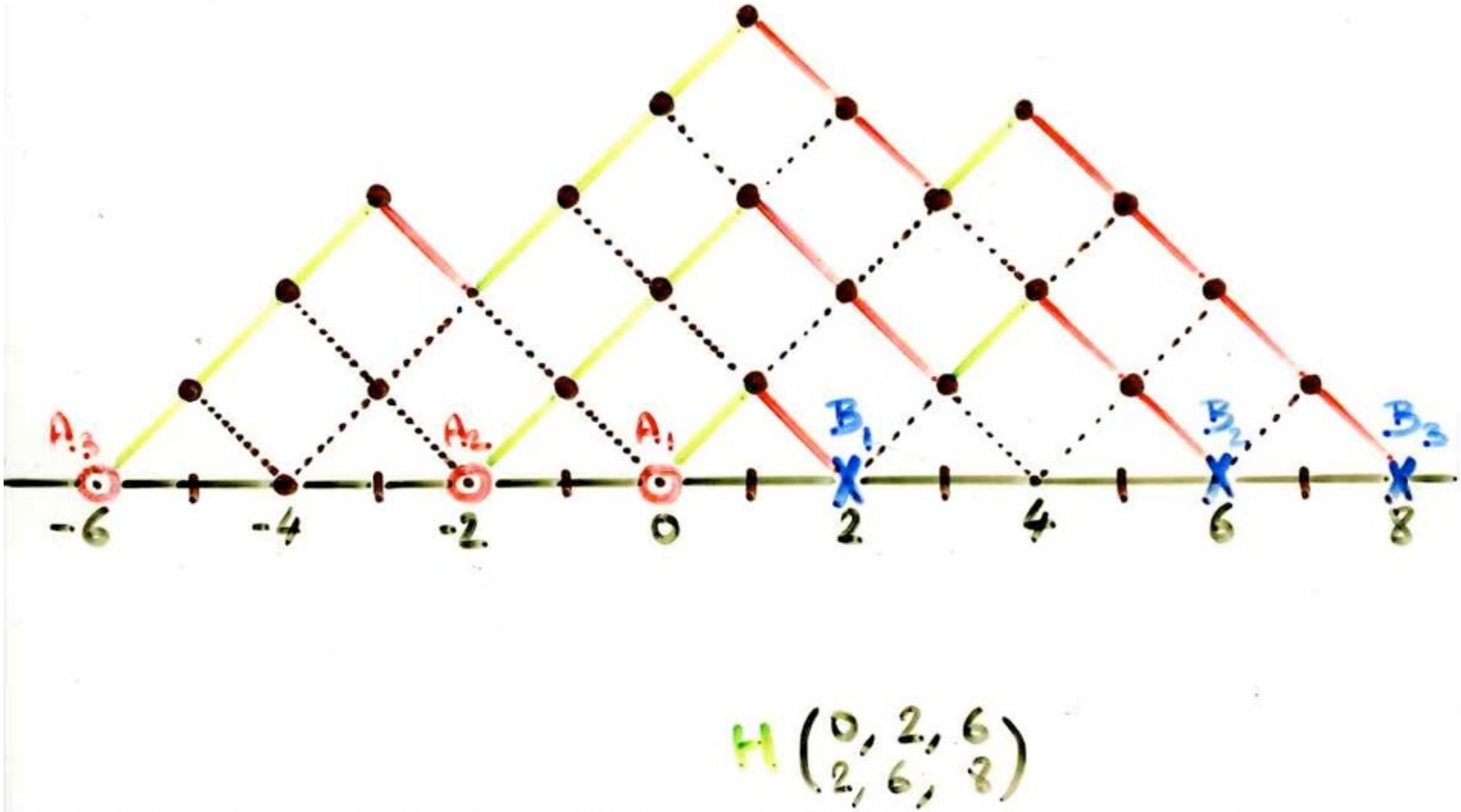


$$\begin{vmatrix} \mu_3 & \mu_5 & \mu_8 & \mu_{10} \\ \mu_6 & \mu_8 & \mu_{11} & \mu_{13} \\ \mu_7 & \mu_9 & \mu_{12} & \mu_{14} \\ \mu_{10} & \mu_{12} & \mu_{15} & \mu_{17} \end{vmatrix}$$

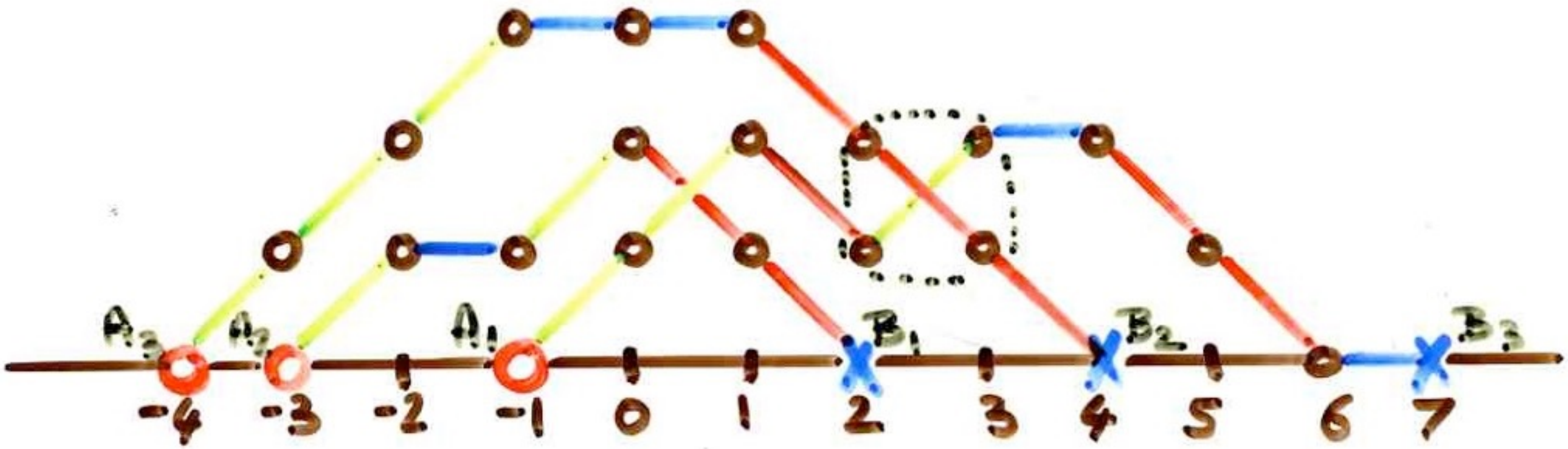
chemins de Dyck



chemins de Dyck

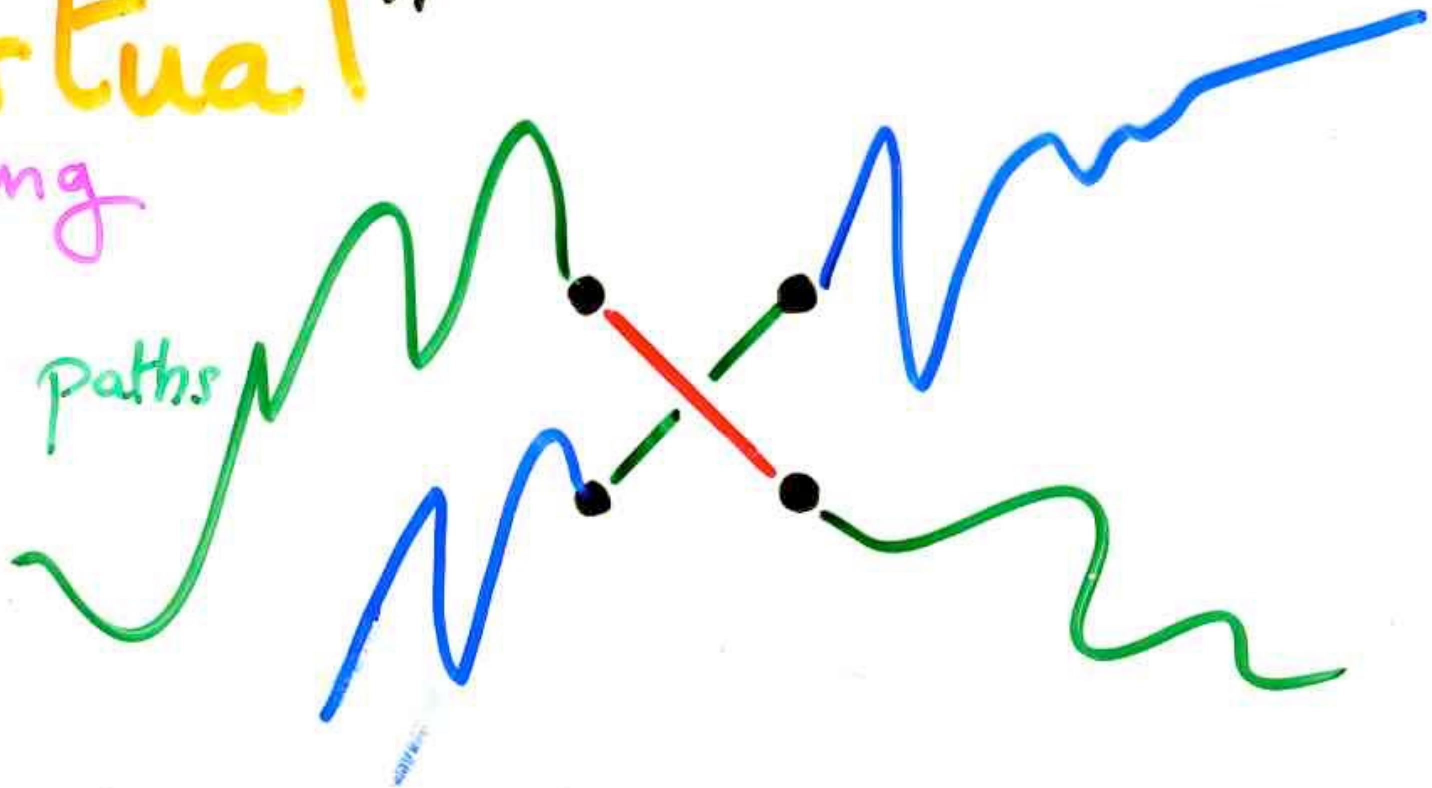


chemins de Motzkin



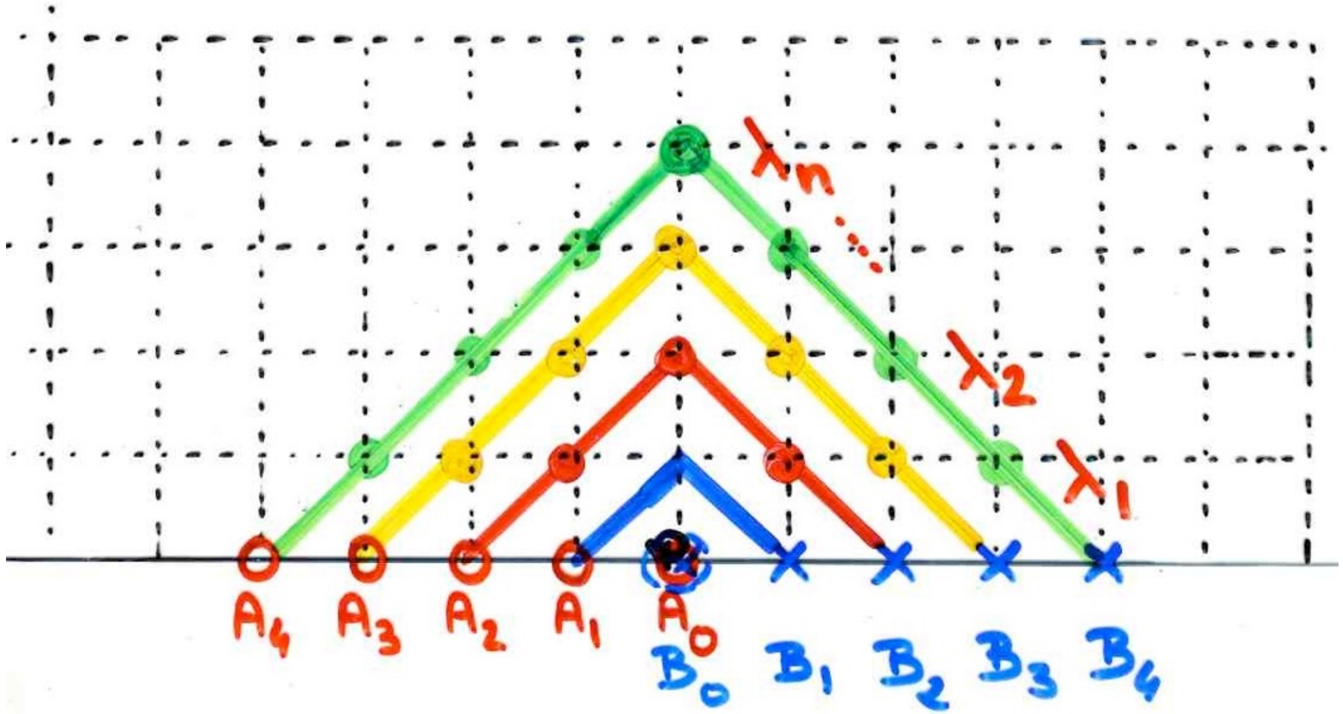
$$H\begin{pmatrix}1, & 3, & 4\\ 2, & 4, & 7\end{pmatrix}$$

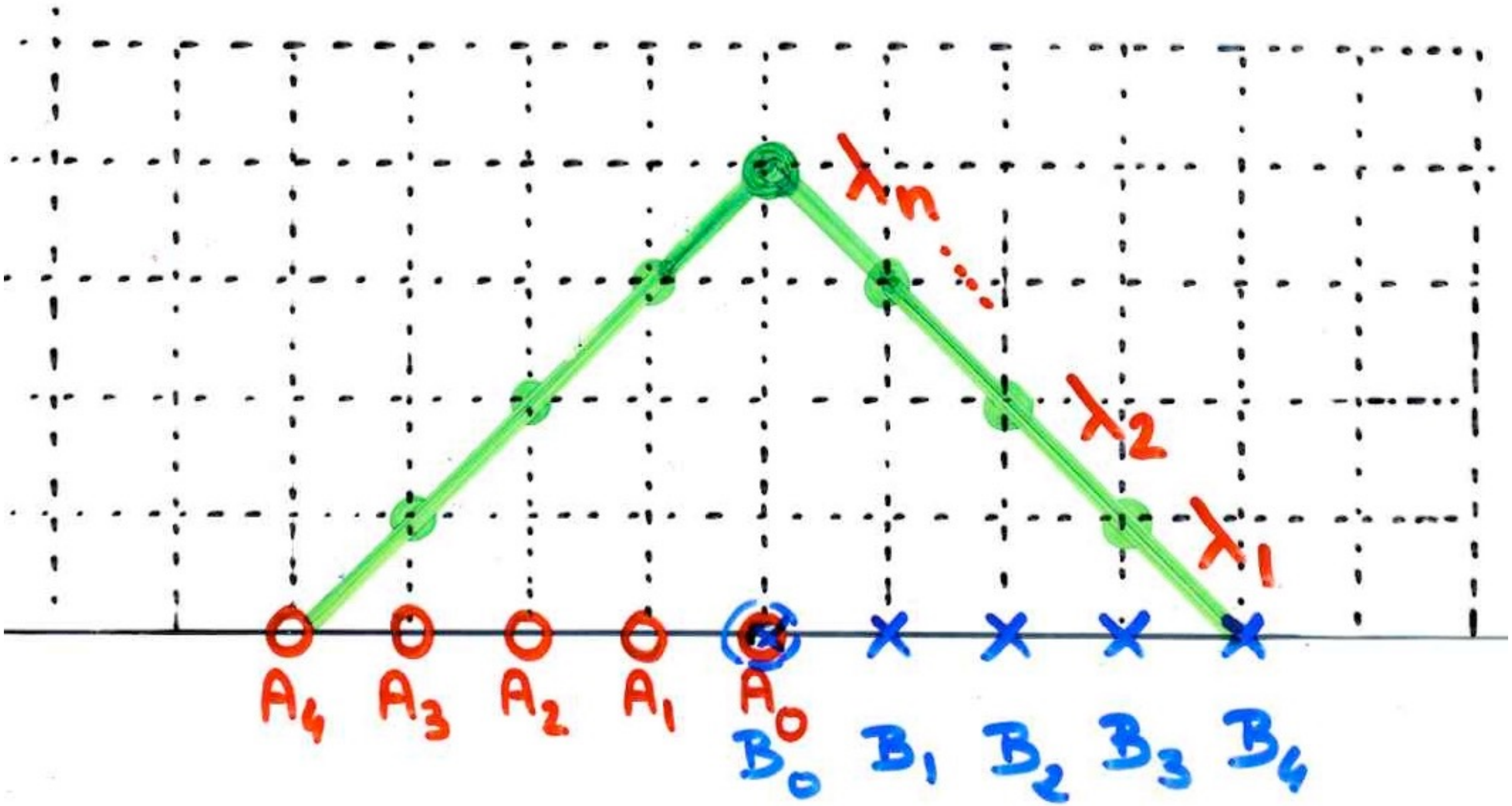
"virtual" crossing of Motzkin paths



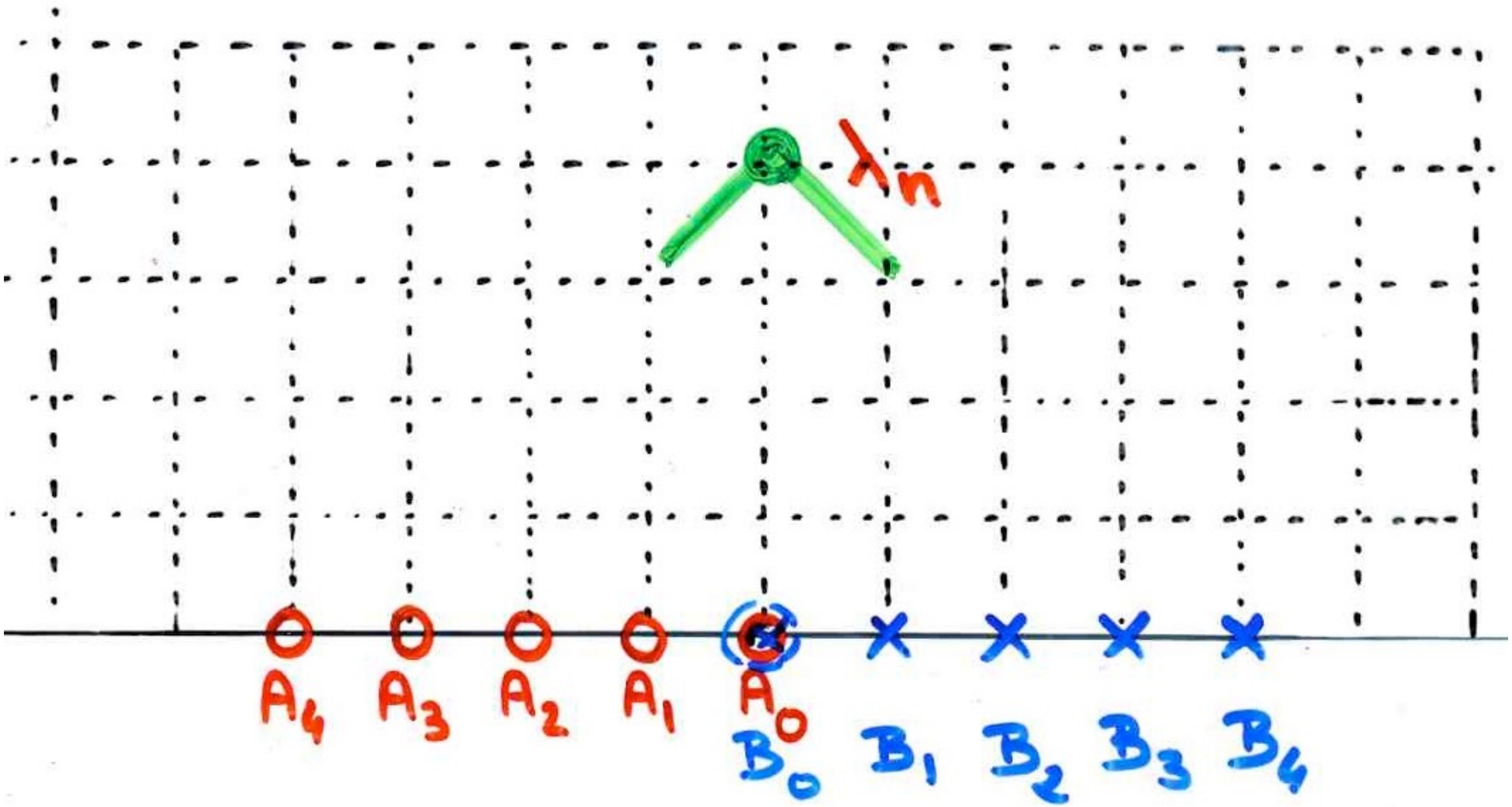
Hankel

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_n & \cdots & \cdots & \mu_{n+n} \end{vmatrix}$$





$$\frac{\Delta_n}{\Delta_{n-1}}$$



$$\frac{\Delta_n}{\Delta_{n-1}} : \frac{\Delta_{n-1}}{\Delta_{n-2}} = \lambda_n$$

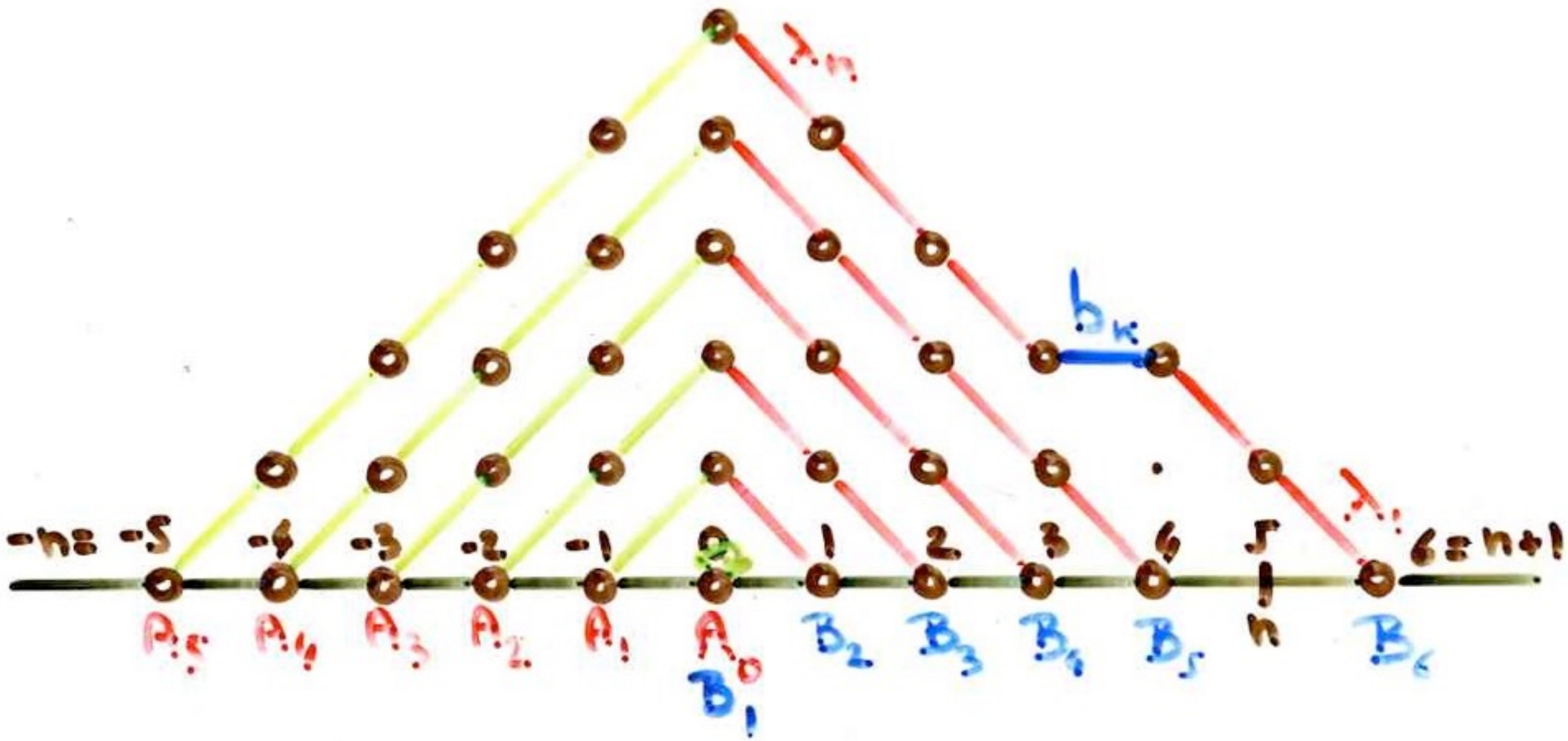
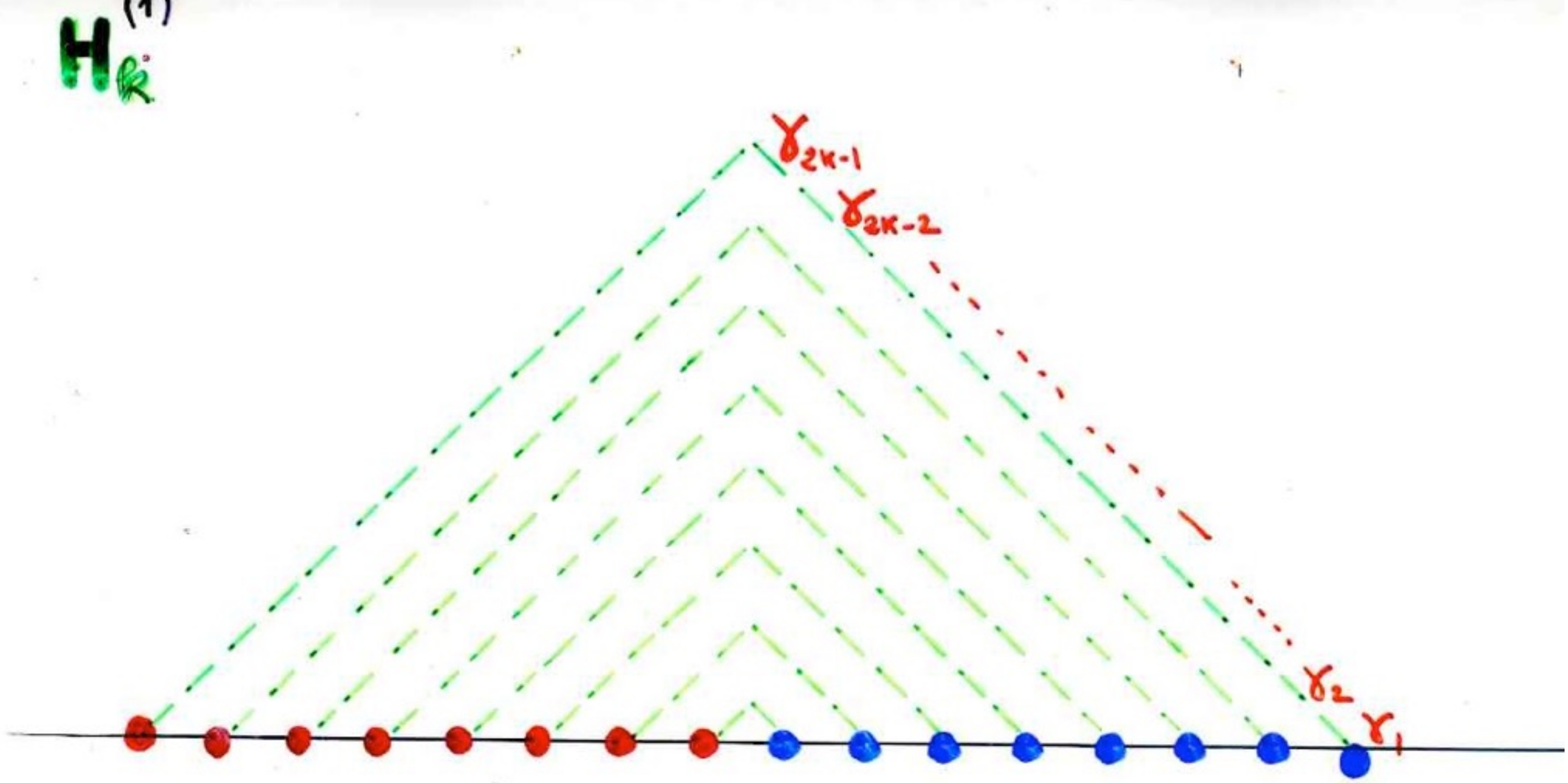


Fig 8. χ_n

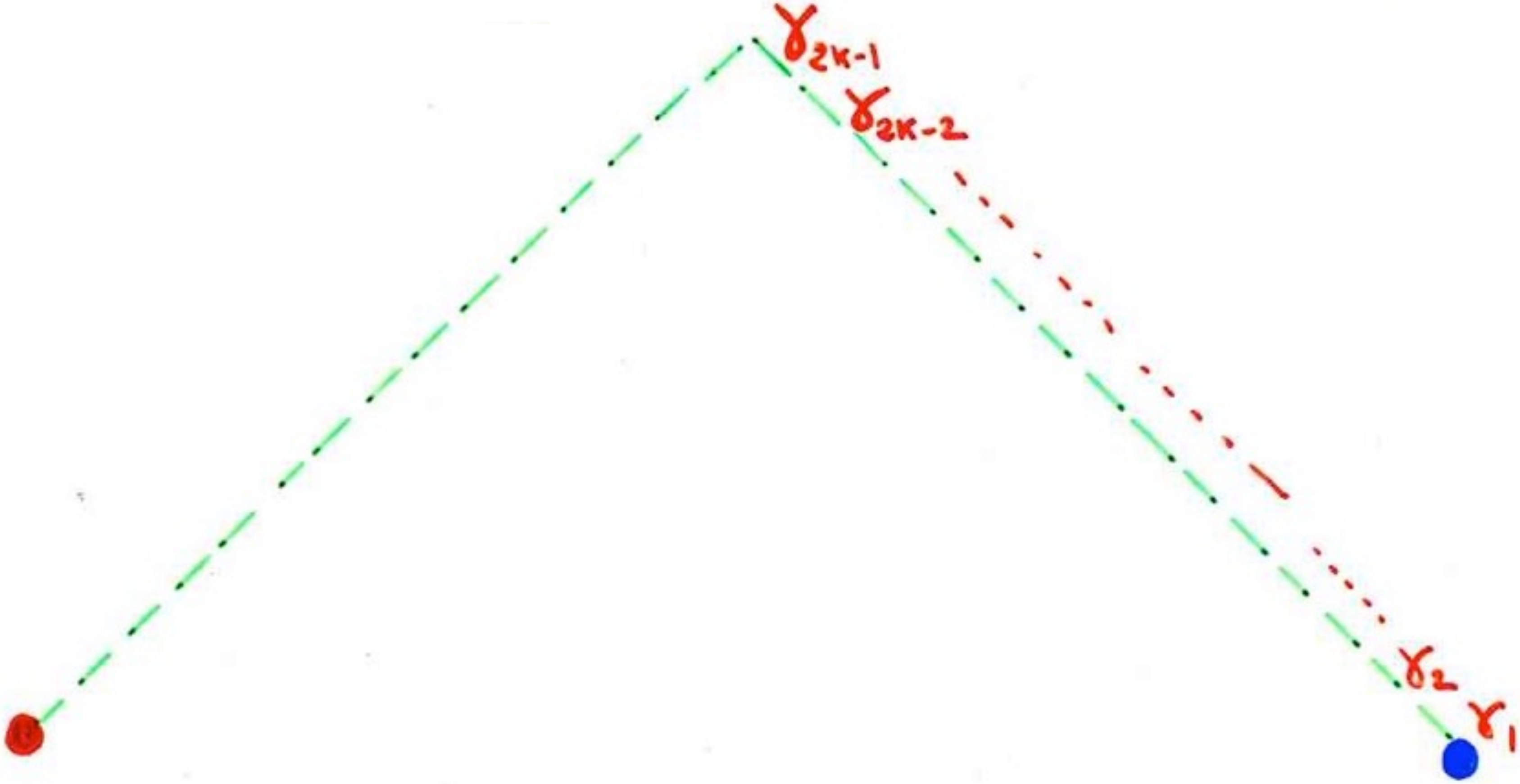
$$\chi_n = \sum_k b_k \Delta_n \qquad b_n = \frac{\chi_n}{\Delta_n} - \frac{\chi_{n-1}}{\Delta_{n-1}}$$

$$H_k^{(1)} = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \cdots & \cdots & \vdots \\ \vdots & & & \vdots \\ \mu_{k+1} & \cdots & \cdots & \mu_{2k+1} \end{vmatrix}$$



$H_k^{(1)}$
γ_{2k-1}
γ_{2k-2}
γ_2
γ_1

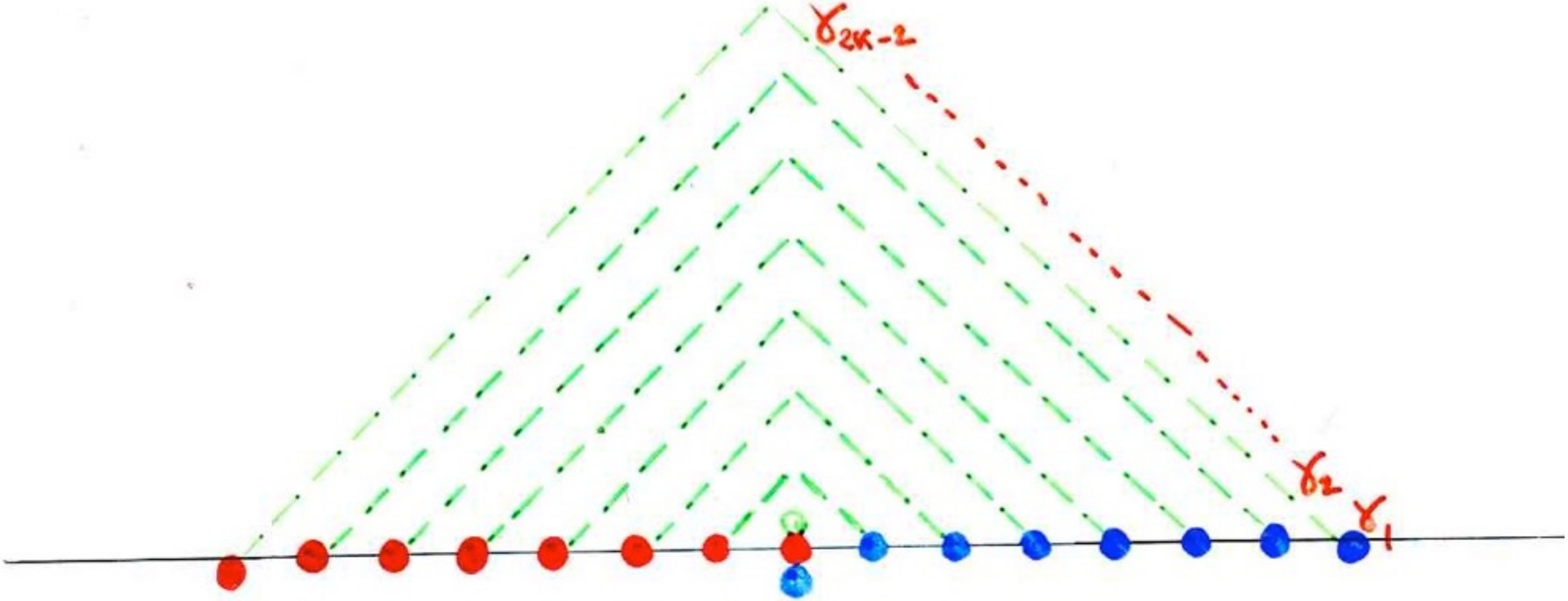
$$\frac{H_k^{(1)}}{H_{k-1}^{(1)}}$$

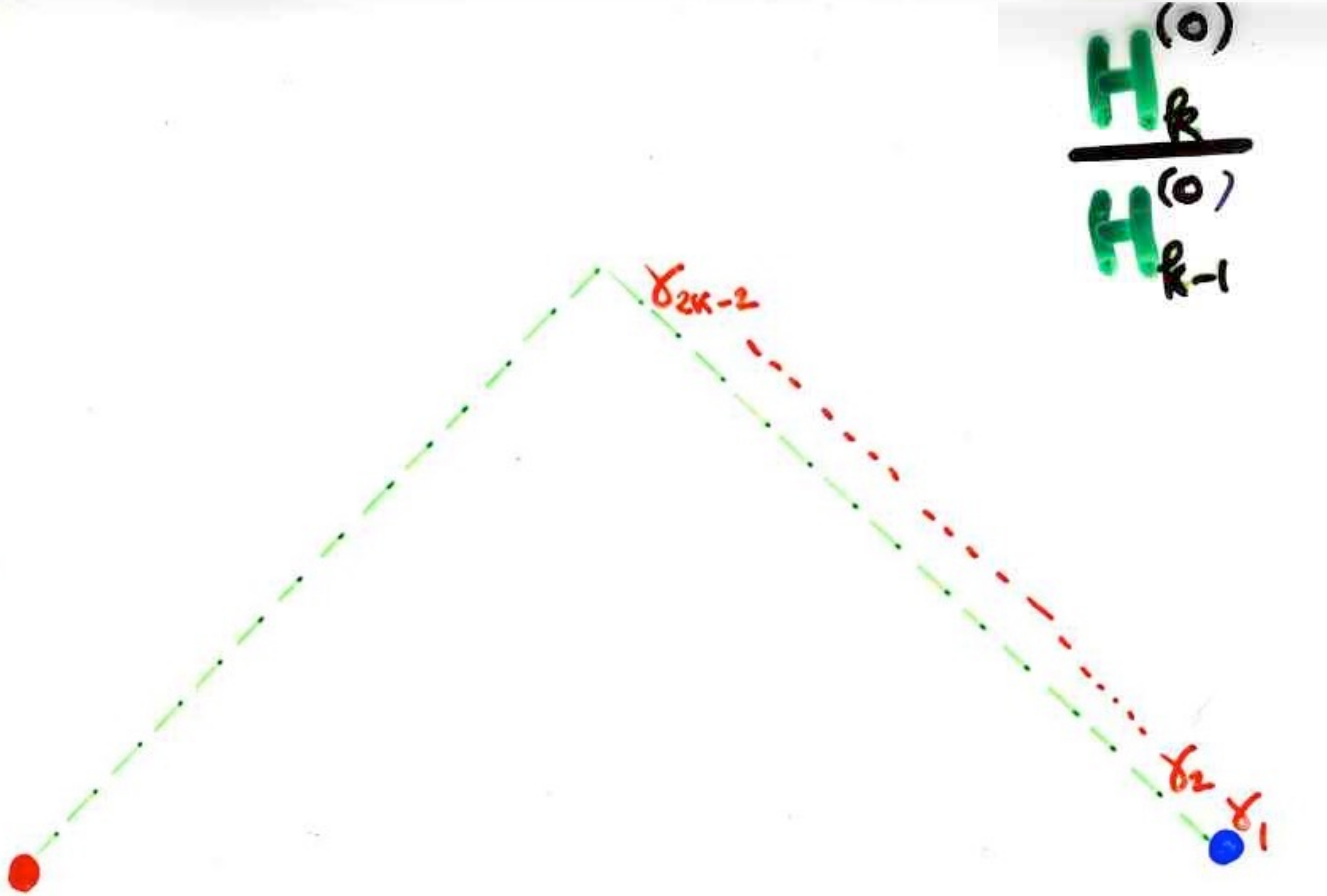


Hankel determinant

$$H_k^{(0)} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \vdots & & & \vdots \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k} \end{vmatrix}$$

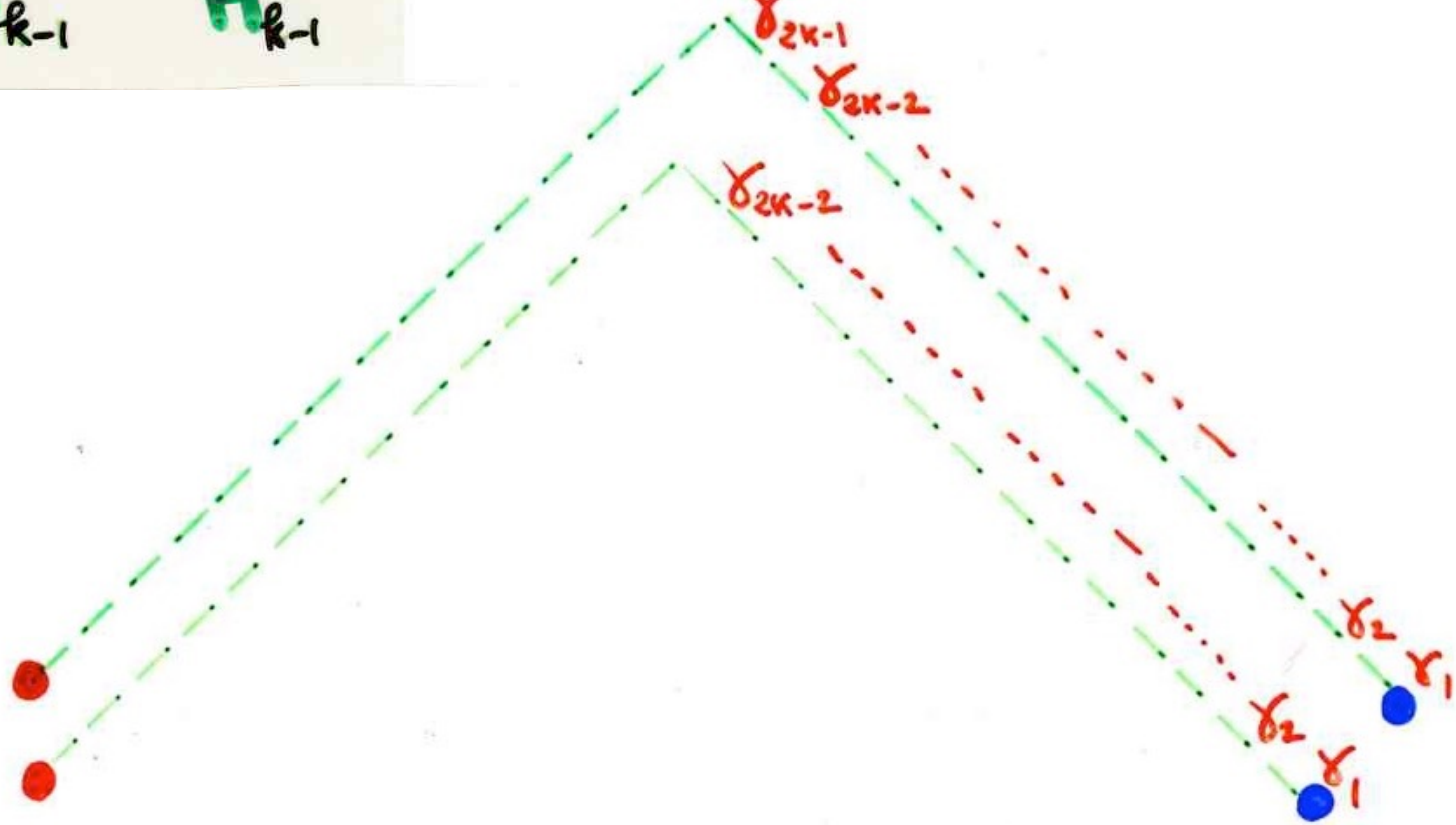
$\mathbf{H}_k^{(0)}$





$\frac{H_k^{(0)}}{H_{k-1}^{(0)}}$
γ_{2k-2}
γ_2
γ_1

$$\frac{H_k^{(1)}}{H_{k-1}^{(1)}} : \frac{H_k^{(0)}}{H_{k-1}^{(0)}} =$$



$$\frac{H_k^{(1)}}{H_{k-1}^{(1)}} : \frac{H_k^{(0)}}{H_{k-1}^{(0)}} = \wedge\gamma_{2k-1}$$

conclusion another example with

tiling,

Ising-like bijection,

non-crossing paths for binomial determinant

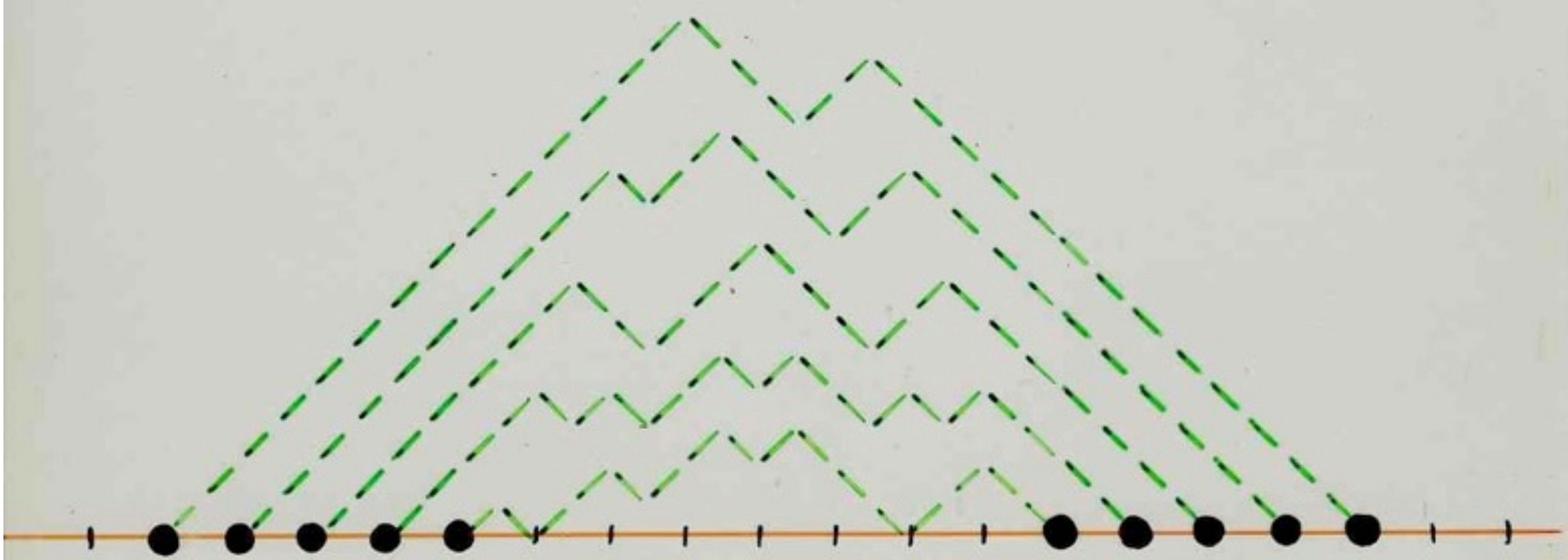
Young tableaux,

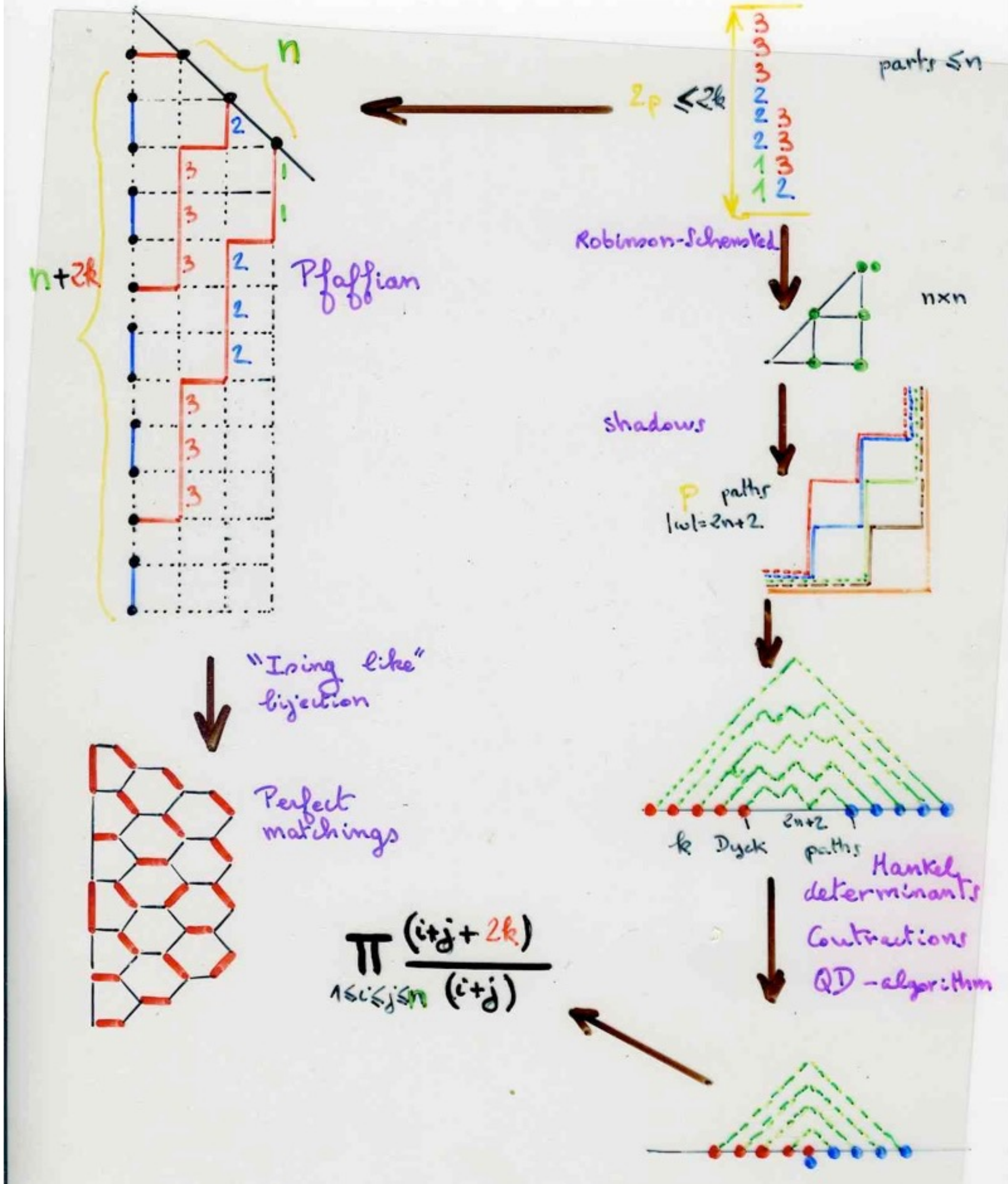
« light and shadow » process,

non-crossing Dyck paths,

Hankel determinant of Catalan numbers,

computation with the « qd-algorithm »





De Sainte-Catherine
and X.V. (1985)