Combinatorics and Physics

Chapter 4a
Heaps of pieces and theoretical physics

IIT-Madras
9 February 2015

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CNRS, LaBRI, Bordeaux

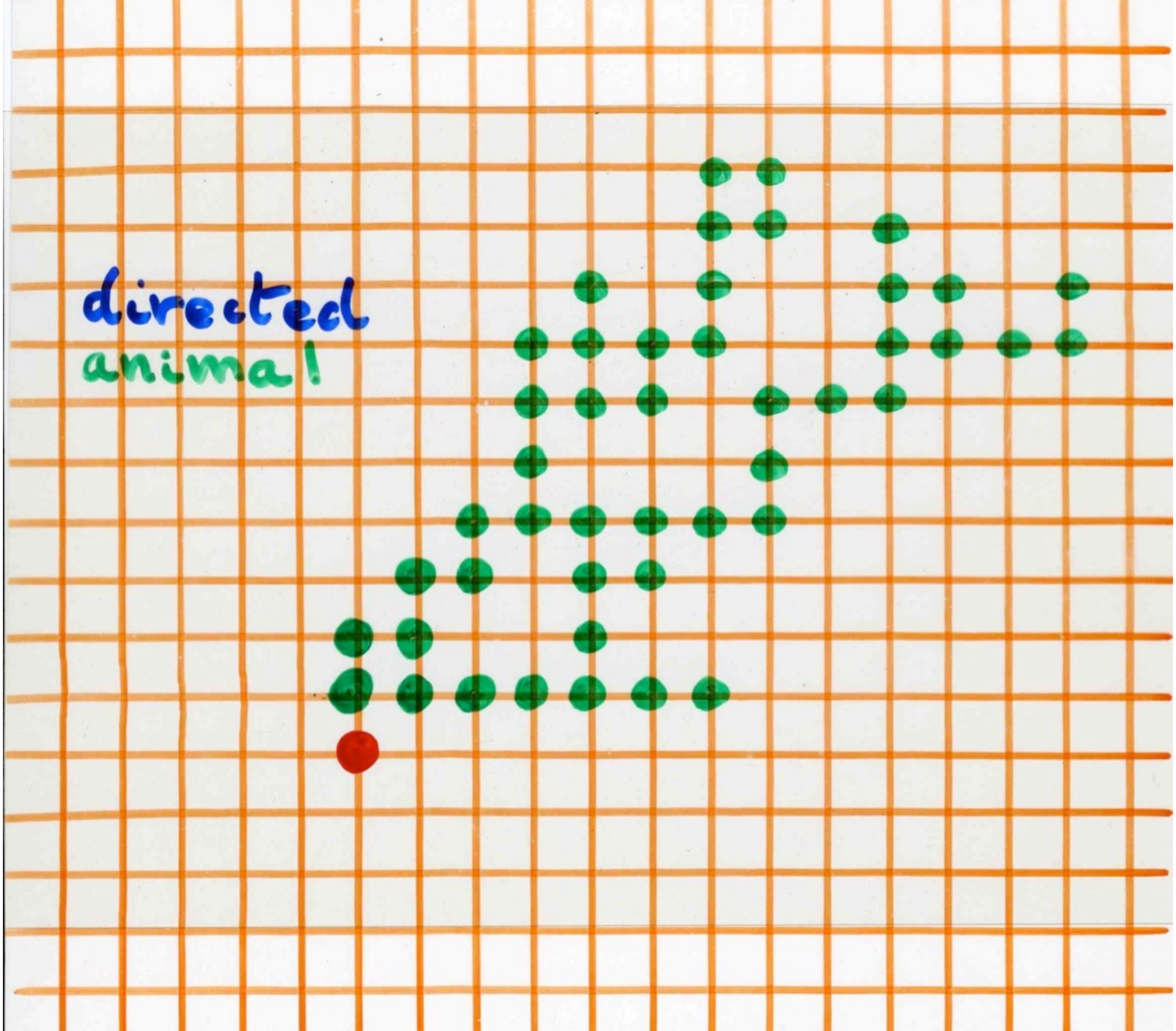
interactions with physics

- the directed animal model
  and hard particule gas model

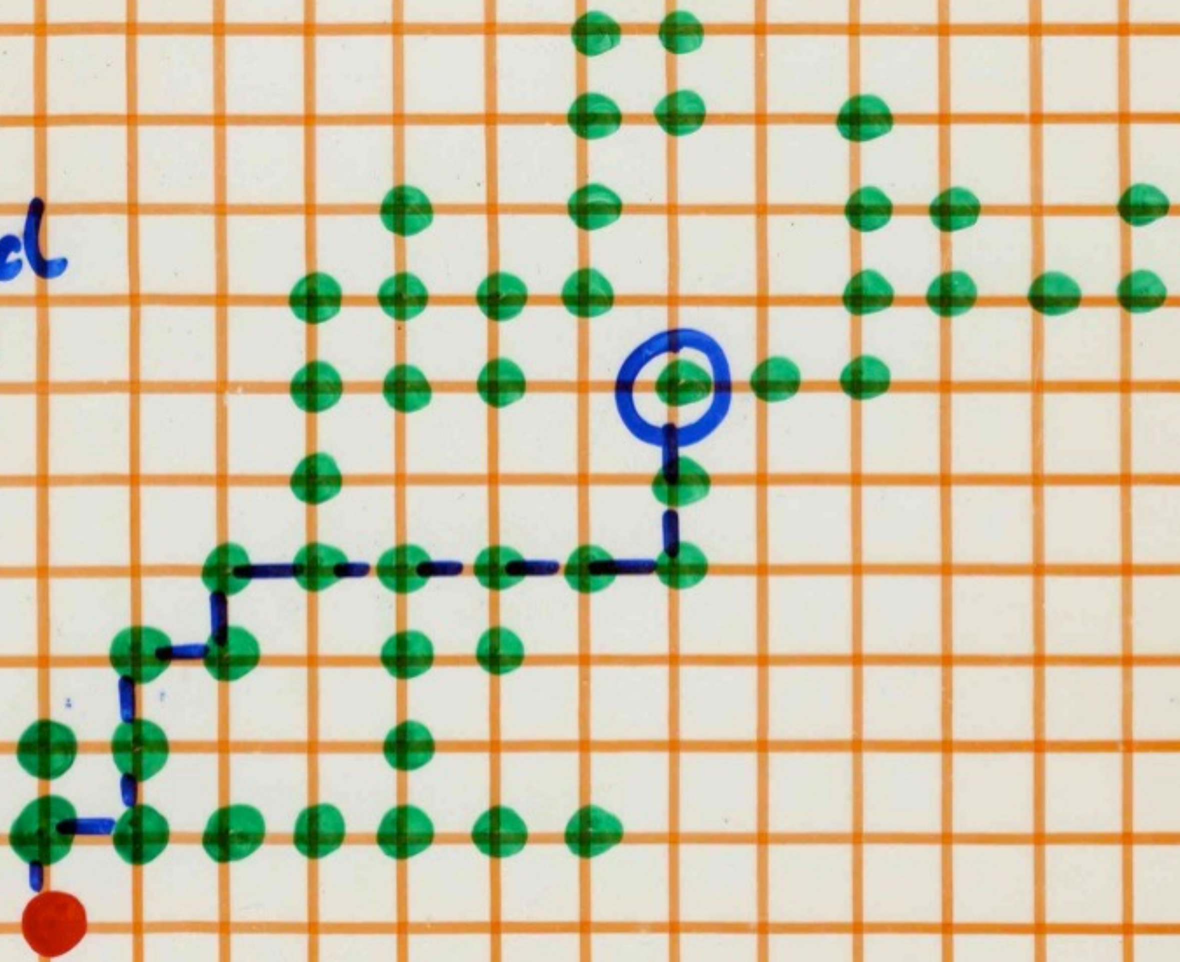
- Lorentzian triangulations in 2D quantum gravity

- q-Bessel functions in statistical physics:
  staircase polygons, SOS model, ...

The directed animal model


directed
animal

directed
animal



Breuer, Janssen (1982)   Cardy (1982)

Day, Lubensky (1982)

Dhar (1982) (1983)   Dhar, Phani, Barma (1982)

Family (1982)

Hakim, Nadal (1983)   Nadal (1983)

Nadal, Derrida, Vannimenus (1982) (1983)

Redner, Coniglio (1982)

Redner, Yang (1982)

Lubensky, Vannimenus (1982)

Stanley, Redner, Yang (1982)

Green, Moore (1982)

Herrmann, Family, Stanley (1983)

Breuer (1984)   Duarte (1985)

animal $d+1$ $\nu_\perp = \nu$

(Parisi, Sourlas) 1981

Lee-Yang edge singularity problem

$(d-1)+1$

$\theta = (d-1)\nu_\perp$ $d \le 7$

$\frac{\nu_\parallel}{\nu_\perp} = z$

(Breuer, Janssen 1982)

(Day, Lubensky 1982)

(Square) Directed Animals problem d

Directed Percolation Problem

(Dhar 1983)

Crystal growth model

average density d

Stochastic lattice-gas $d-1$

(Dhar 1982)

$d=2$ Verhagen

(Hard-square) Baxter gas model

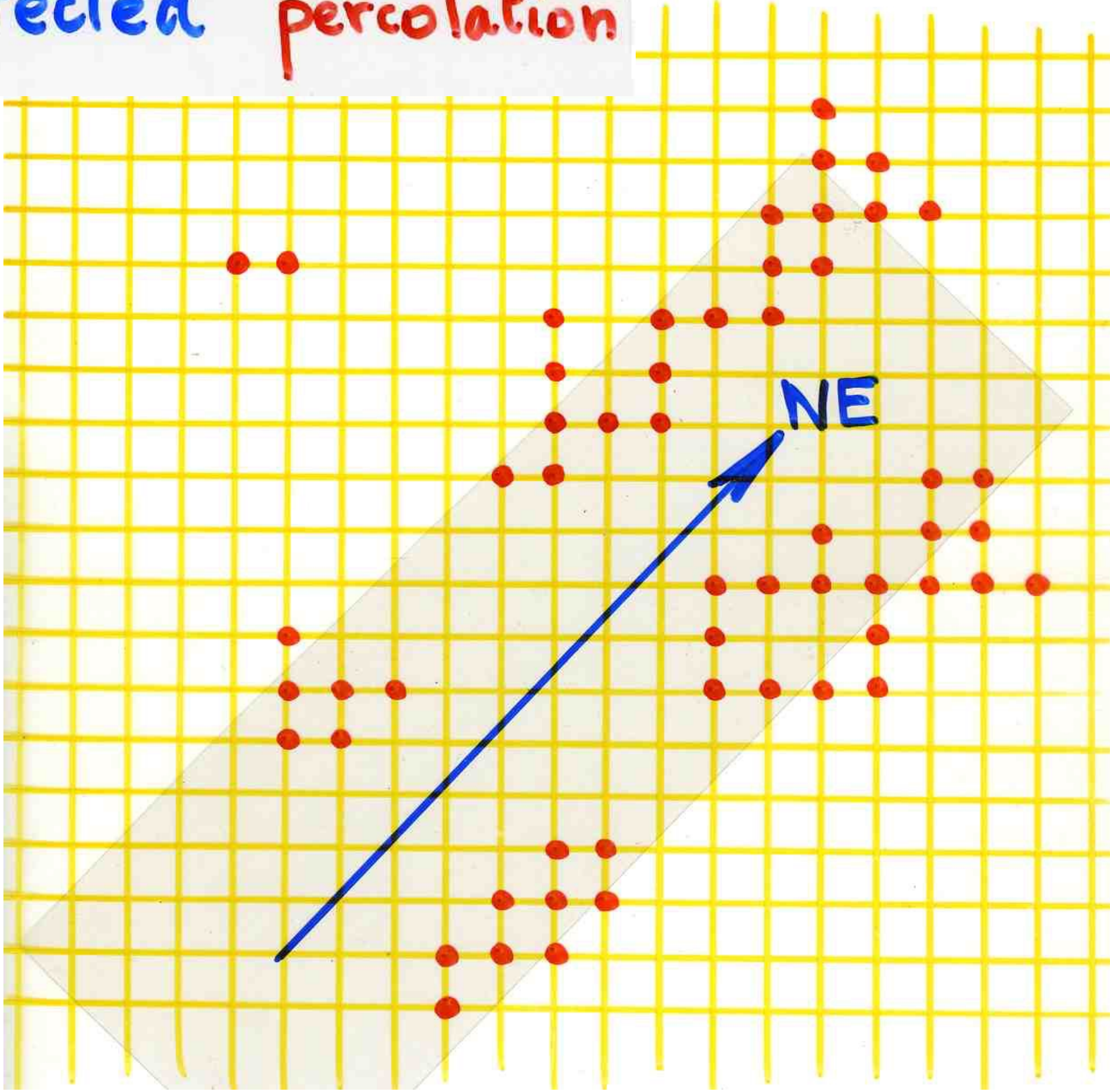
(Baxter 1981)

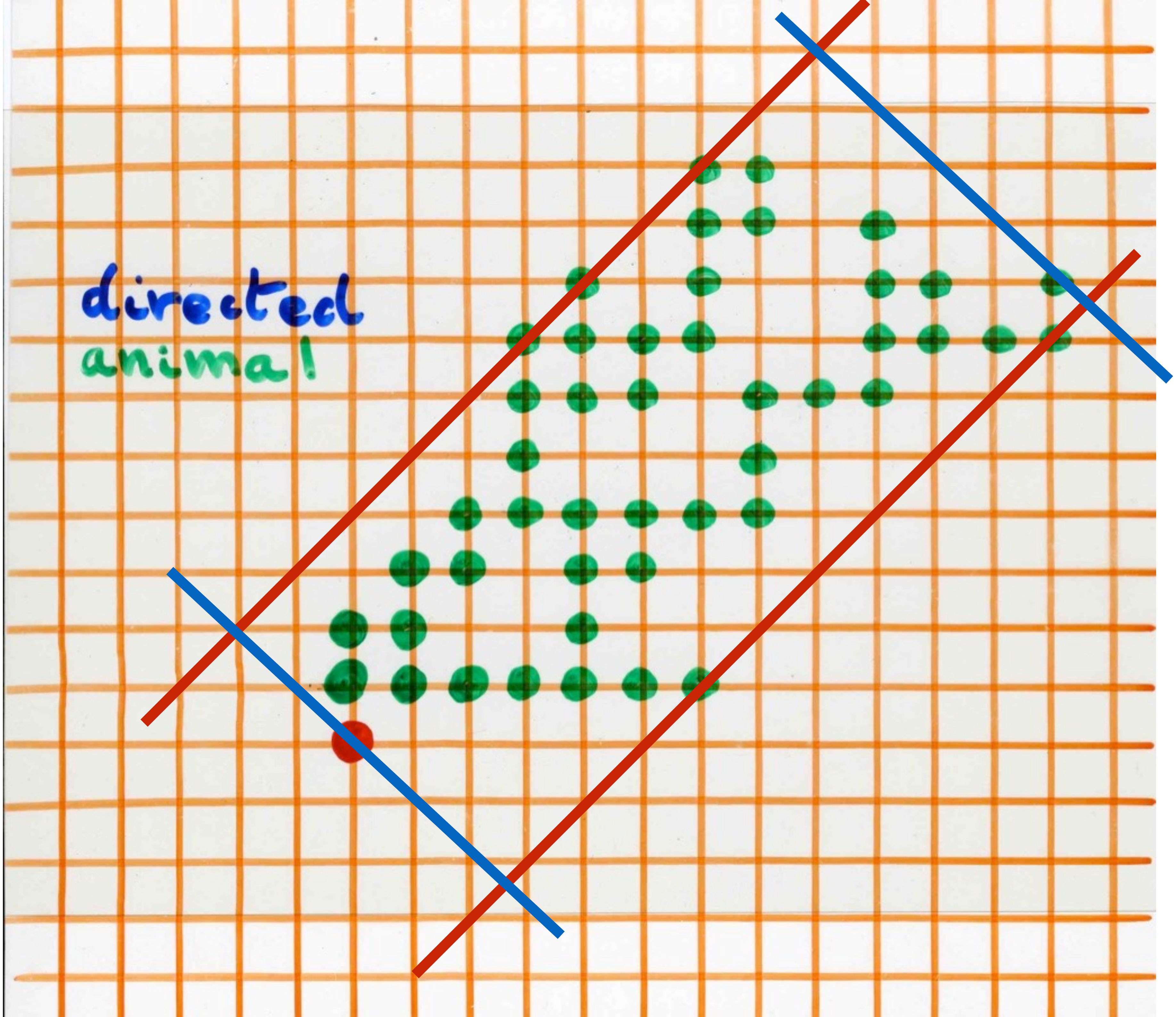
(Baxter, Tsang 1980)

(Baxter, Pearce 1982)

(Verhagen 1976)

directed percolation




directed
animal

$$a_n \sim \mu^n \, n^{-\theta}$$

number of directed animals n points

$$\ell_n \sim n^{\nu_\perp}$$

average width

$$L_n \sim n^{\nu_\parallel}$$

average length

$$a_n \sim \mu^n \, n^{-\theta}$$

number of directed animals n points

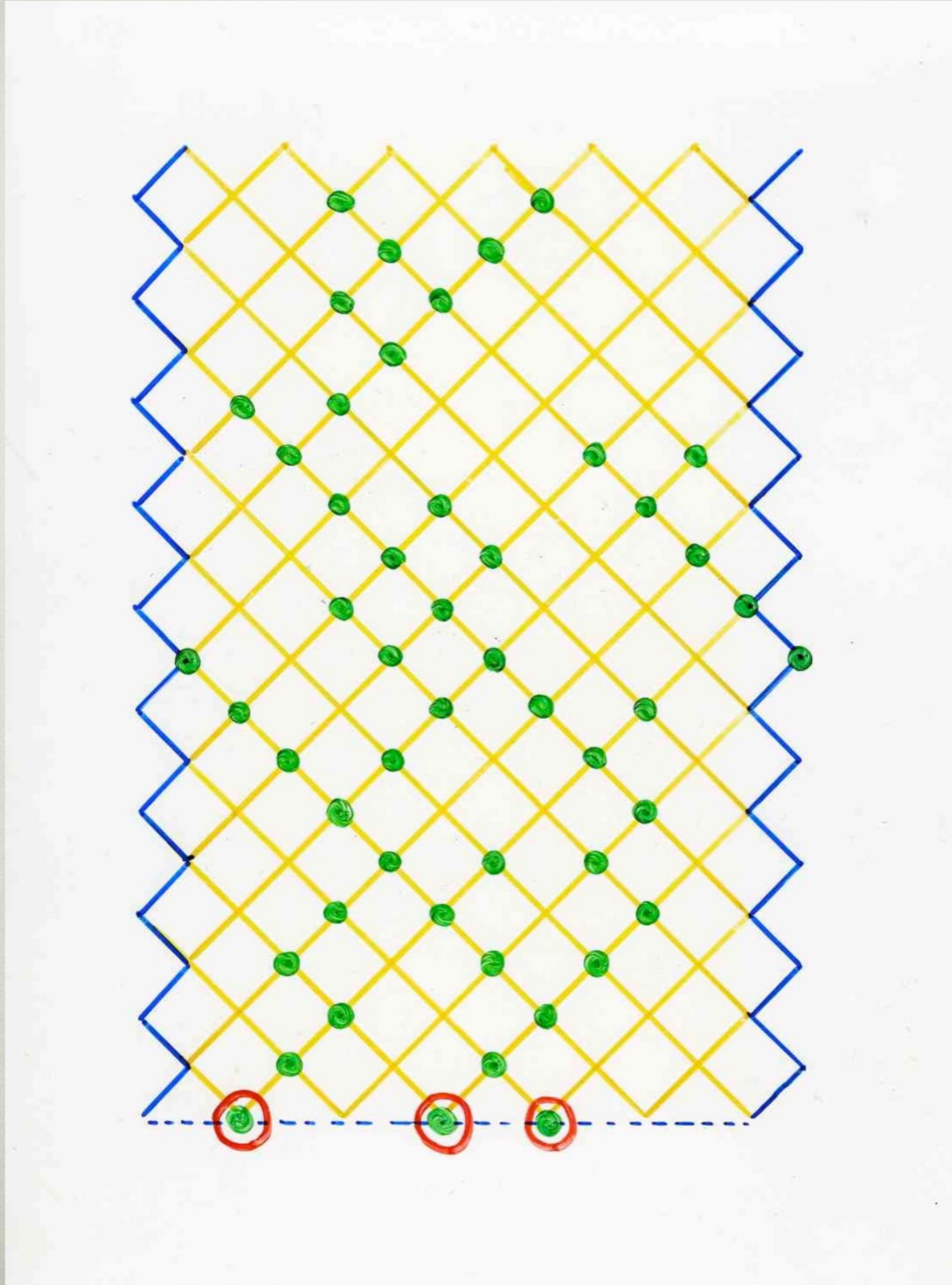
$$\ell_n \sim n^{\nu_\perp}$$

average width

$$L_n \sim n^{\nu_\parallel}$$

average length

Critical exponents



directed
animals
on a
circular
strip

Nadal, Derrida, Vannimenus 1982

Hakim, Nadal 1982

$$a_n^k \sim c\, \mu_k^n$$

$$\mu_k = 1 + 2\cos \pi/2k$$

$$a_n \sim c\, \mu^n\, n^{-\theta}$$

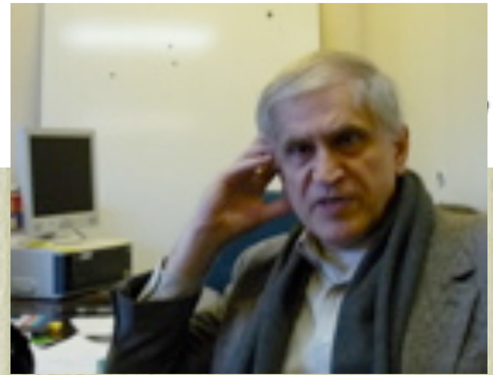
$$\mu = 3$$

Phenomenological Renormalization

$$\tilde{b}_n^{\leq k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin \alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin (i+\frac{1}{2}) \alpha_p}{\sin \frac{\alpha_p}{2}} \right)^{N_i} (1 + 2\cos \alpha_p)^{n-1}$$

animals
circular strip
width k

$$\alpha_p = \frac{2p+1}{2k} \pi$$



J. Vannimenus

J.P. Nadal

Boltzman medal 2010

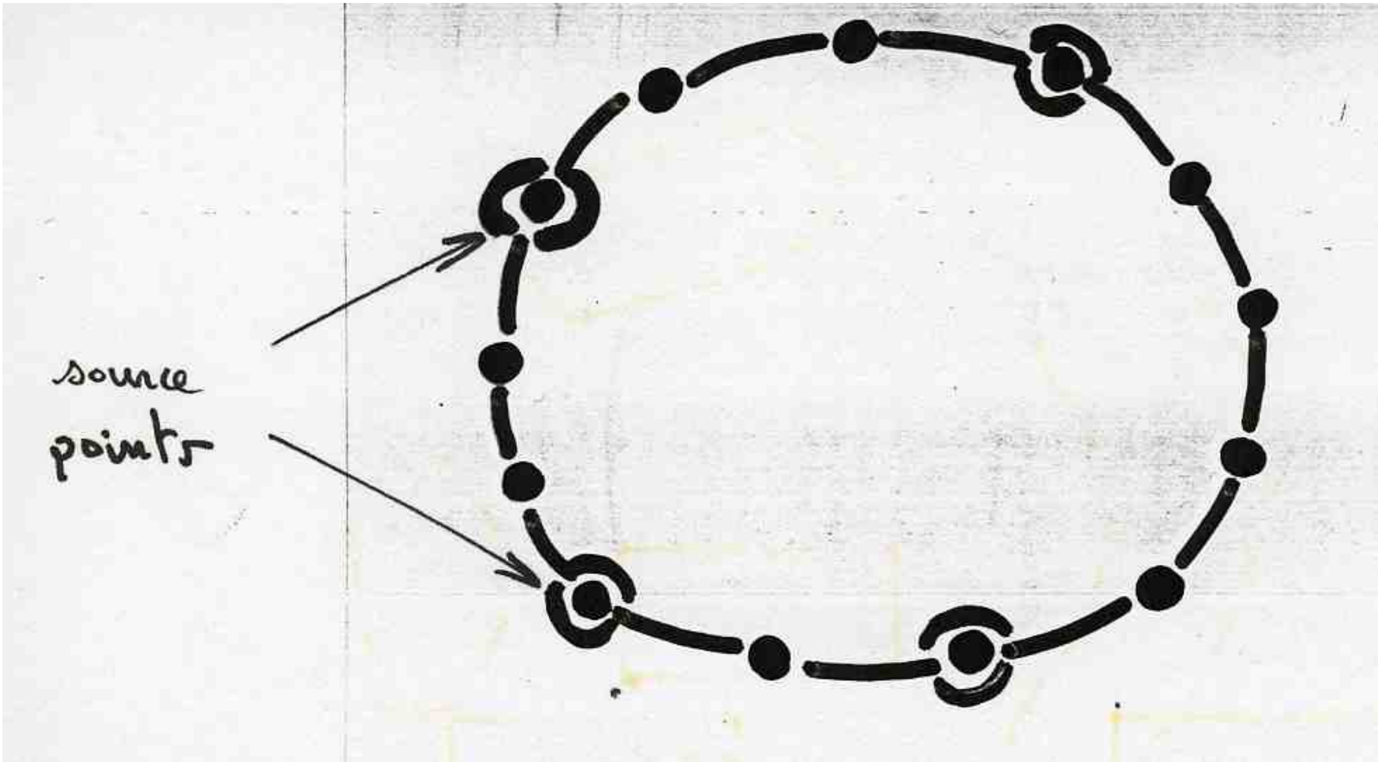
B. Derrida

$$\tilde{b}_n^{\le k} = \frac{1}{k}\sum_{p=0}^{k-1}(-1)^p \sin\alpha_p \prod_{i=1}^{k-1}\left(\frac{\sin\left(i+\frac{1}{2}\right)\alpha_p}{\sin\frac{\alpha_p}{2}}\right)^{N_i}\left(1+2\cos\alpha_p\right)^{n-}$$

animals
circular strip
width k

$$\alpha_p = \frac{2p+1}{2k}\pi$$

(Hakim, Nadal, 1982)


source
points

$$a_n \sim \mu^n \, n^{-\theta}$$

$\mu = 3 \quad \theta = 1/2$

$\nu_\perp = 1/2$

$\nu_{//} = \cancel{9/11}$?

D.Dhar

equivalence with
hard gas model

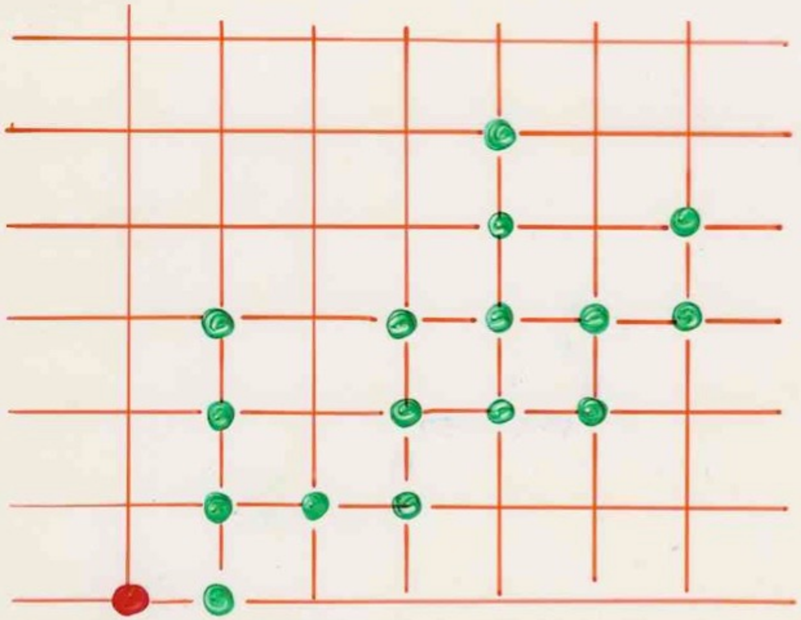
relation with
crystal growth model
stochastic lattice gas

V.Hakim, J.P. Nadal

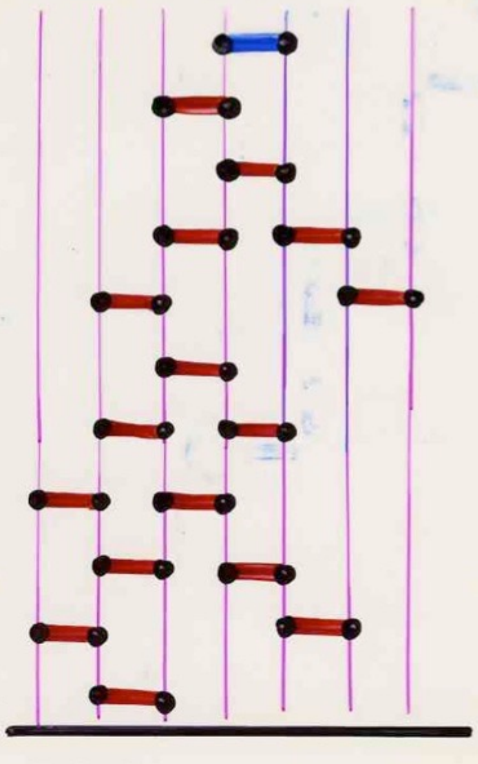
D.Gouyou-Beauchamps, X.V.
direct bijection with paths

generating functions
for
directed animals

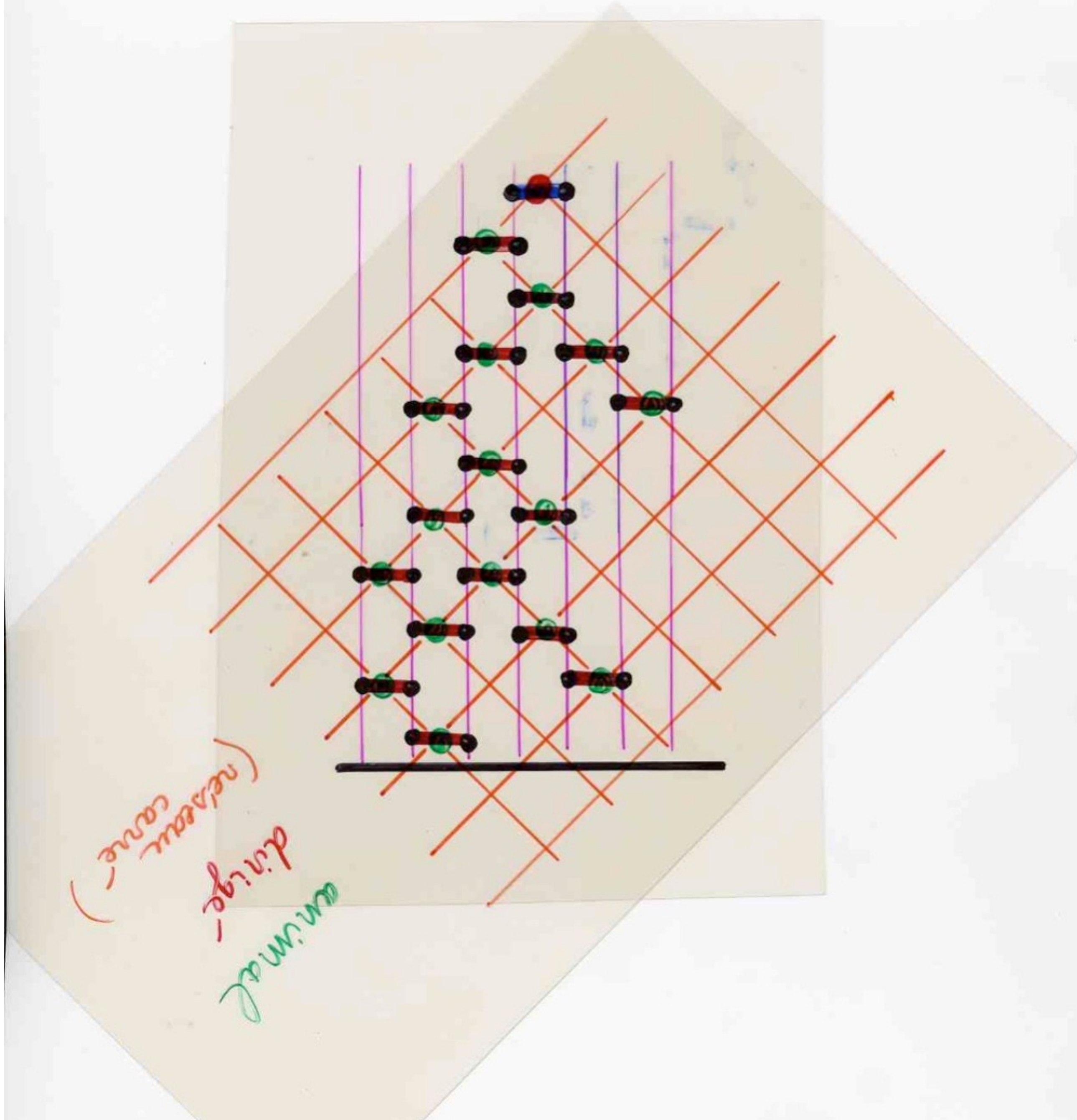
with pyramids of dimers

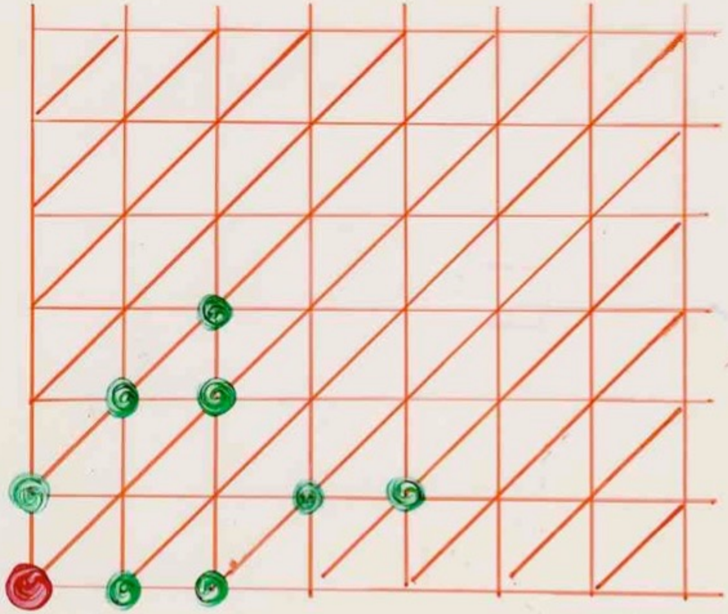


animal
dirigé
(réseau carré)



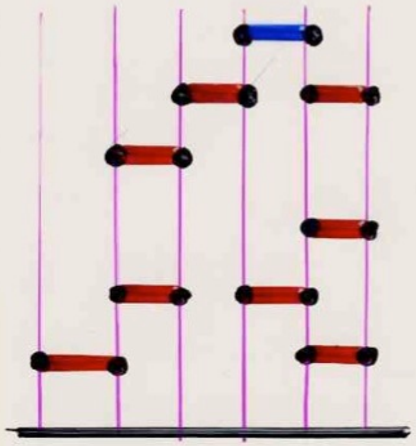
strict
heap

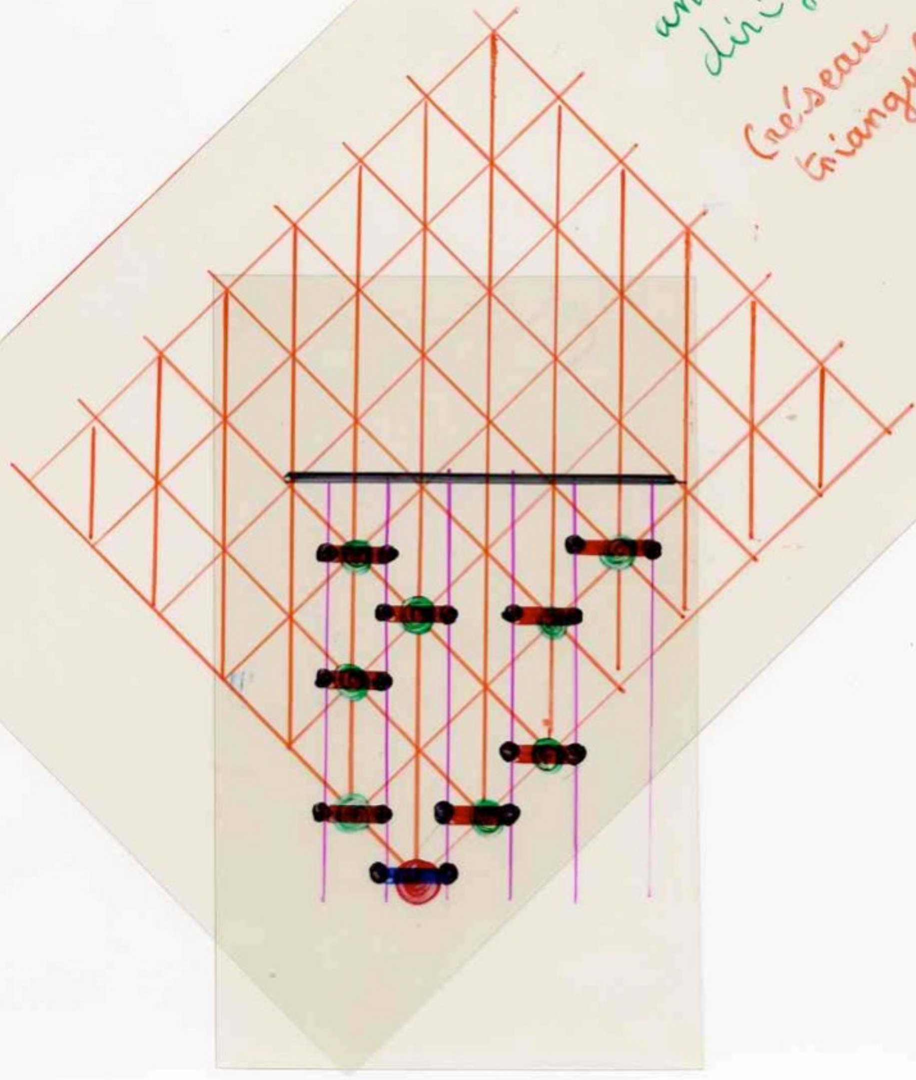

(réseau carré)
dirigé
animal



animal
dirigé

(réseau
triangulaire)



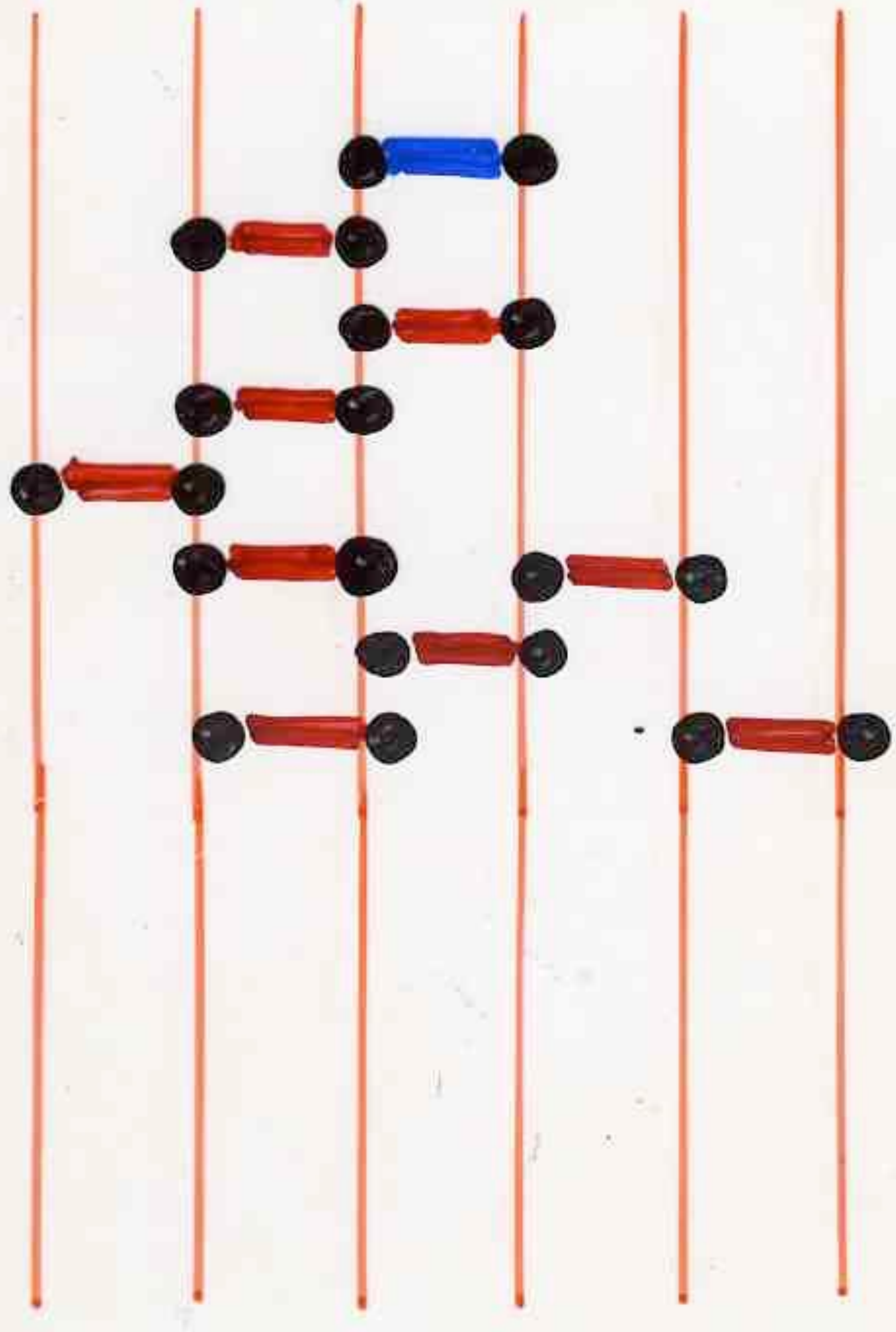


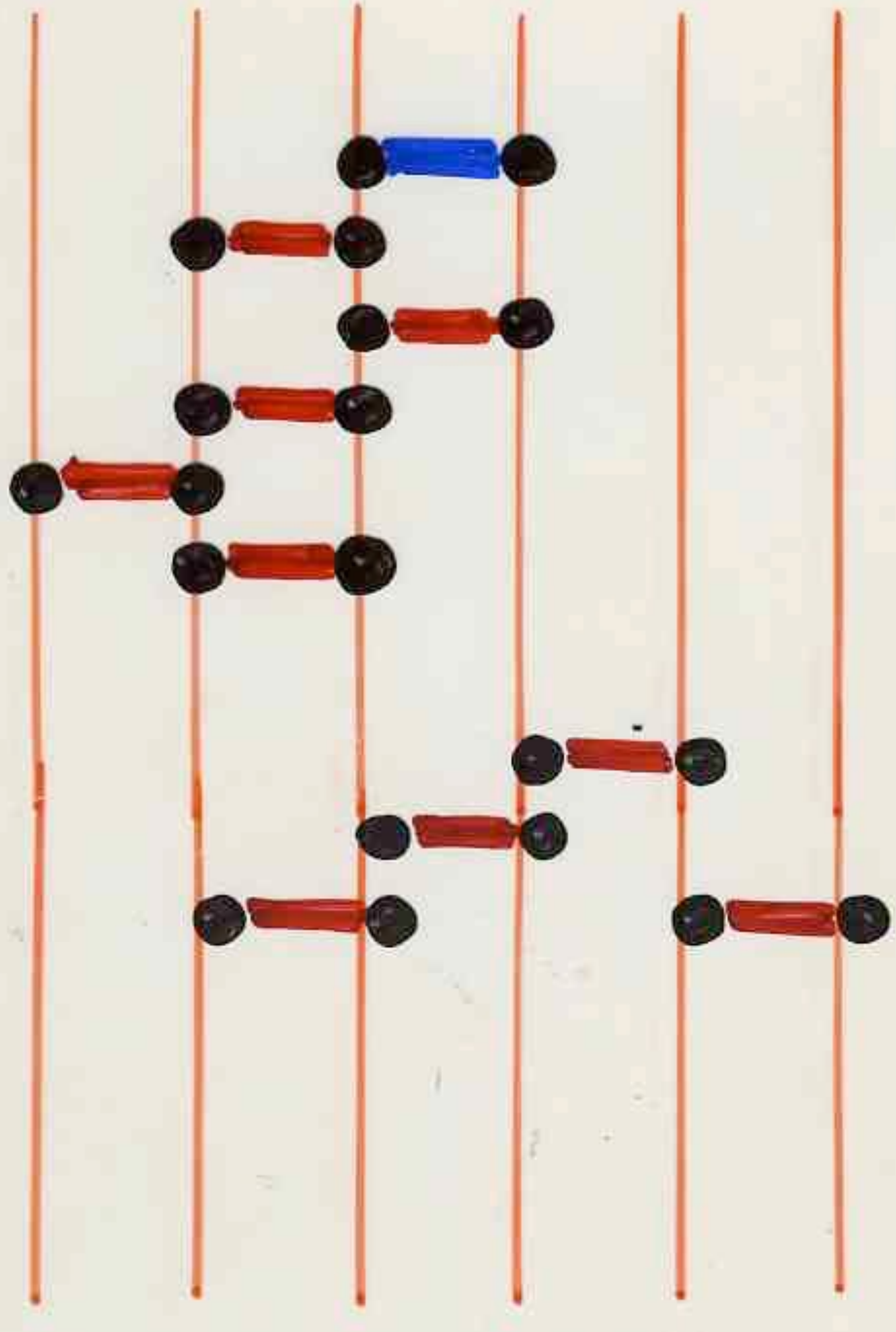
animal dirigé

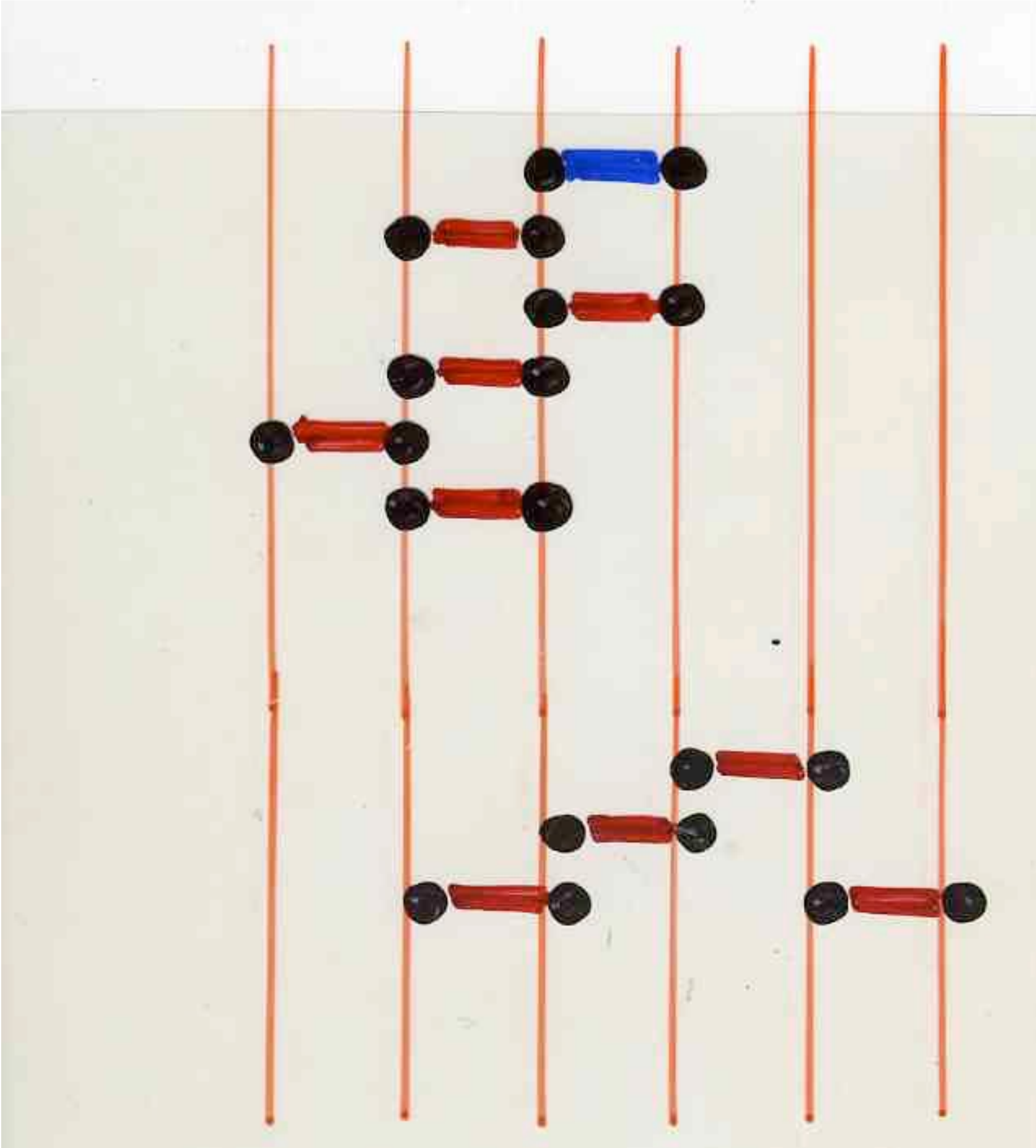
(réseau triangulaire)

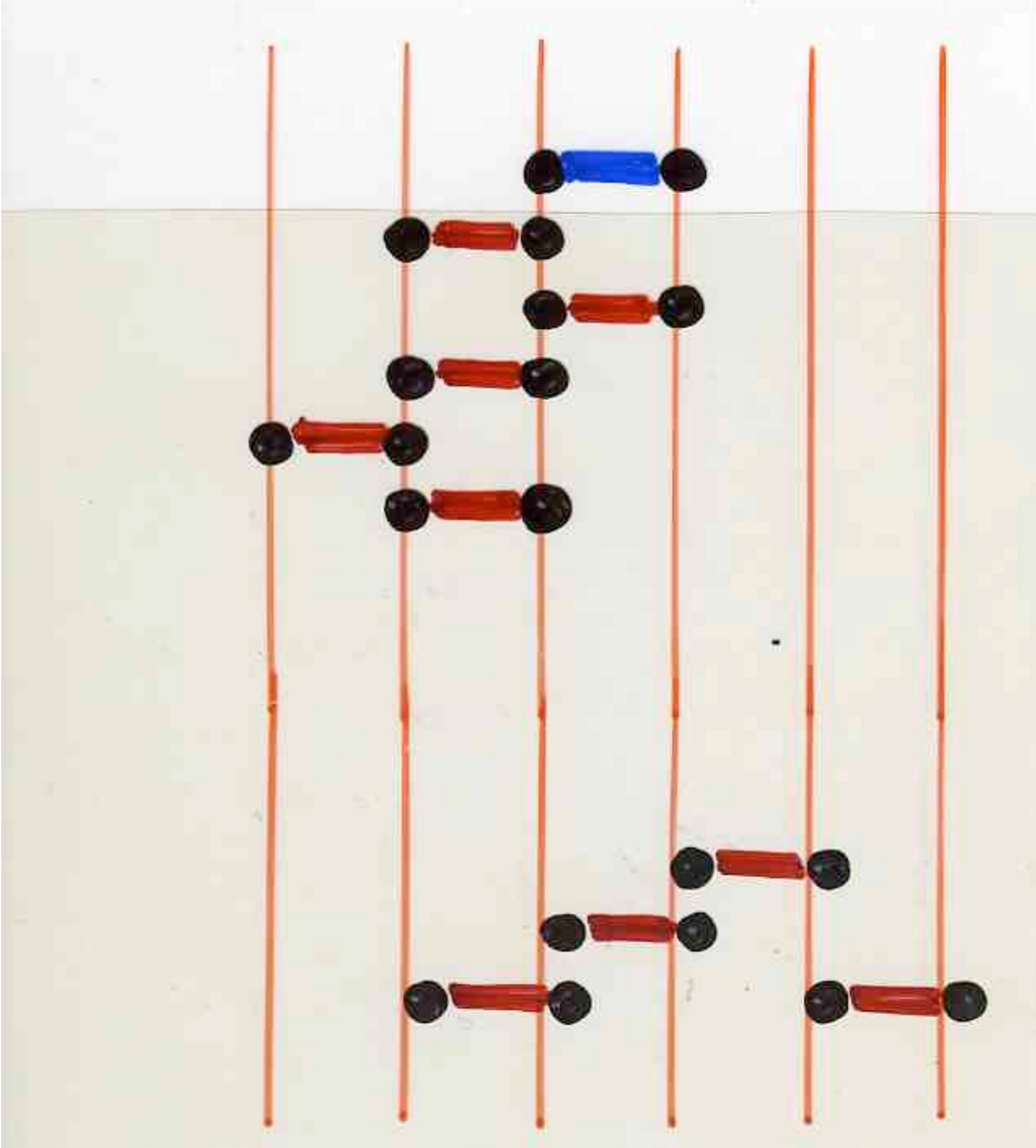
- strict pyramid $=$ $\begin{cases} \bullet \ \text{strict semi-pyramid} \\ \text{or} \\ \bullet \ (\text{strict pyramid}) \times (\text{strict semi-pyramid}) \end{cases}$

$$y = z + yz$$







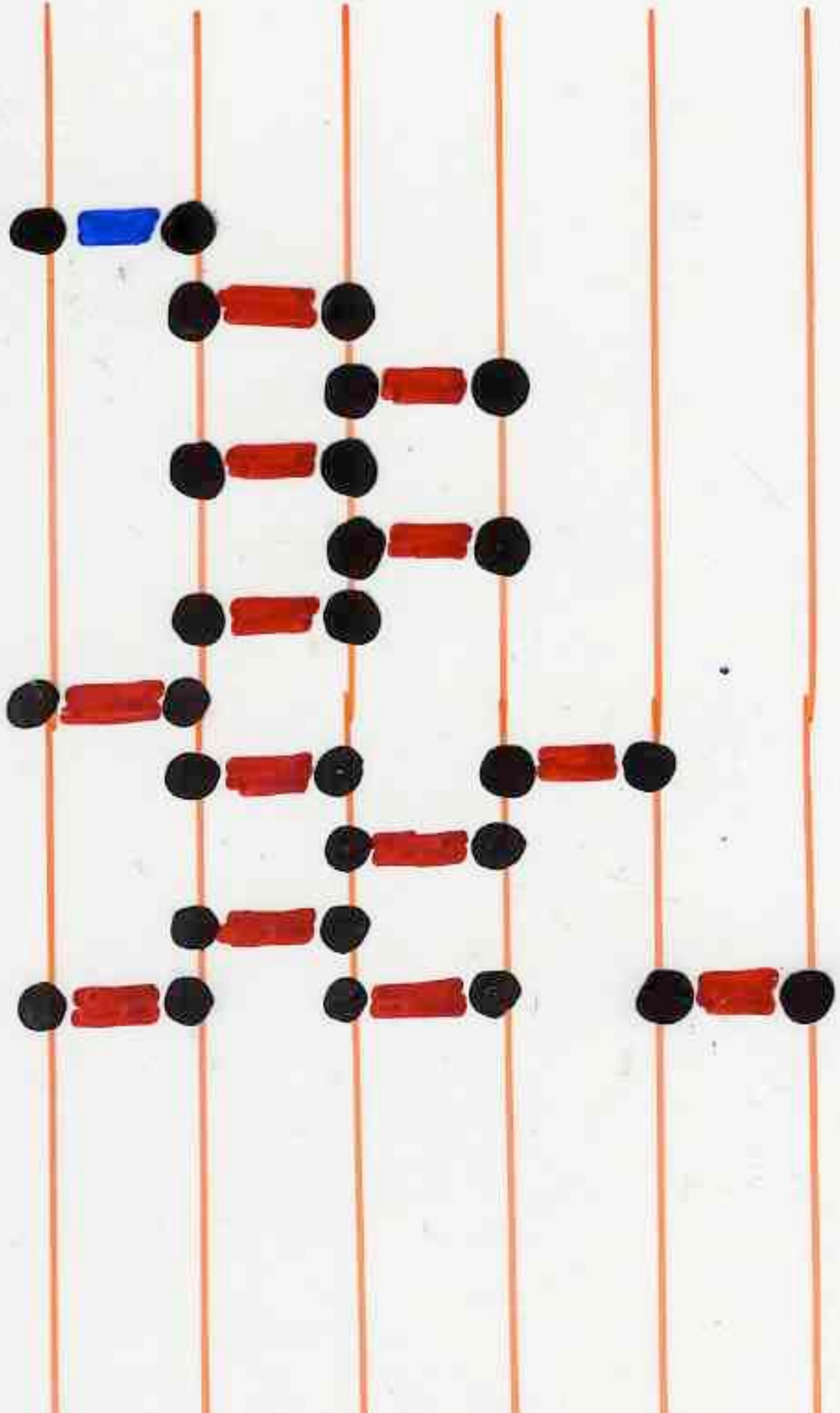


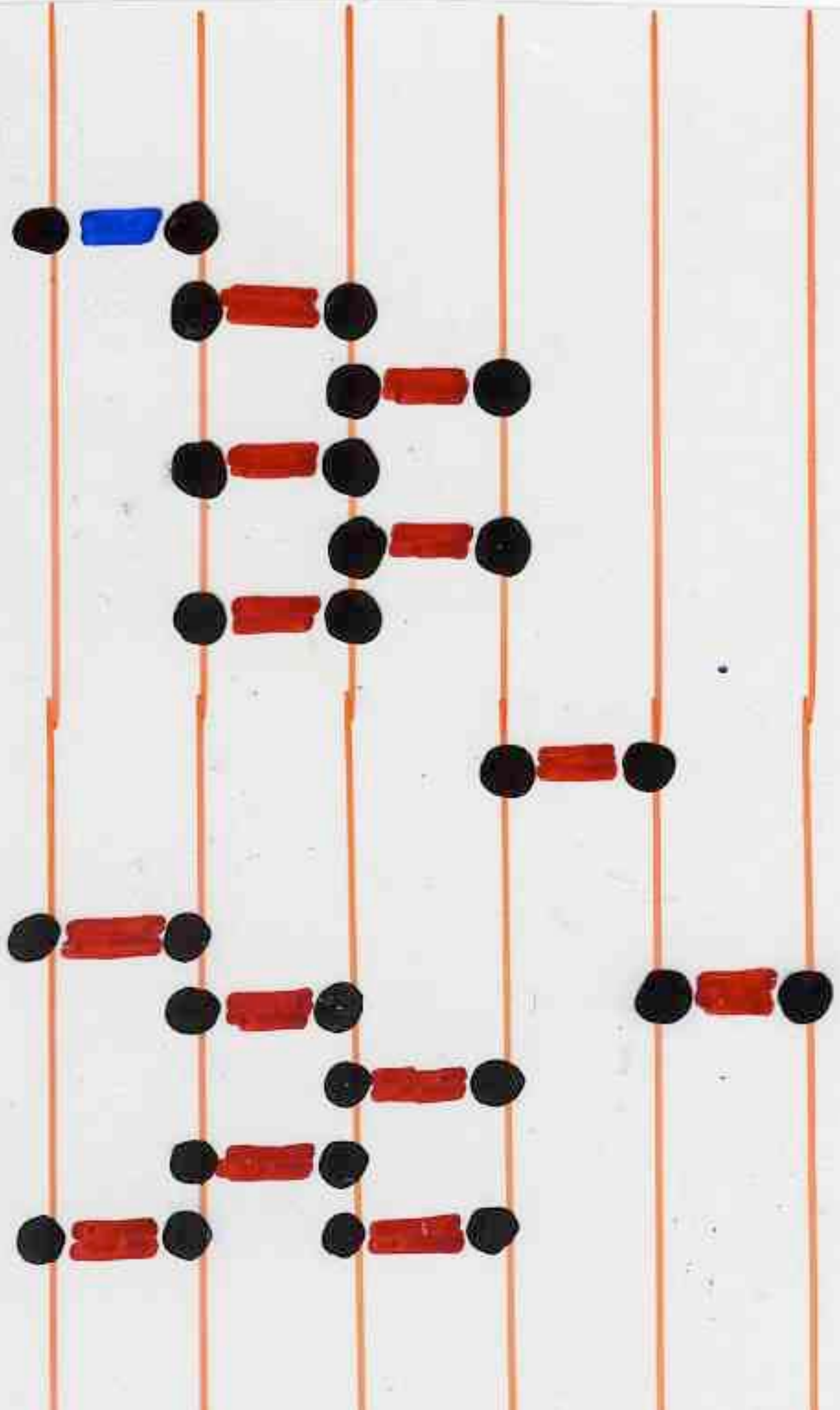
- strict pyramid = { • strict semi-pyramid
  or
  • (strict pyramid) × (strict semi-pyramid) }

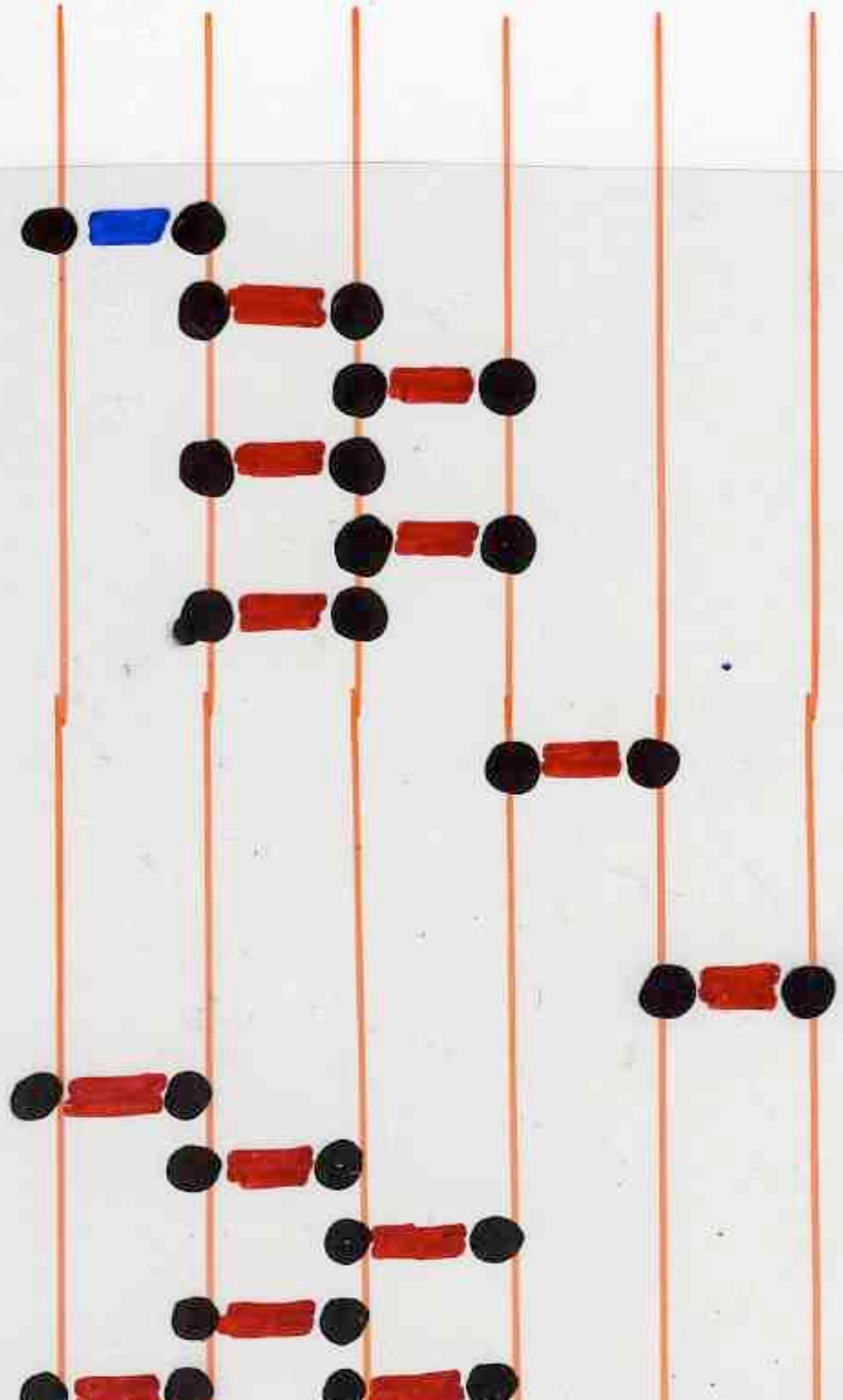
$$y = z + yz$$

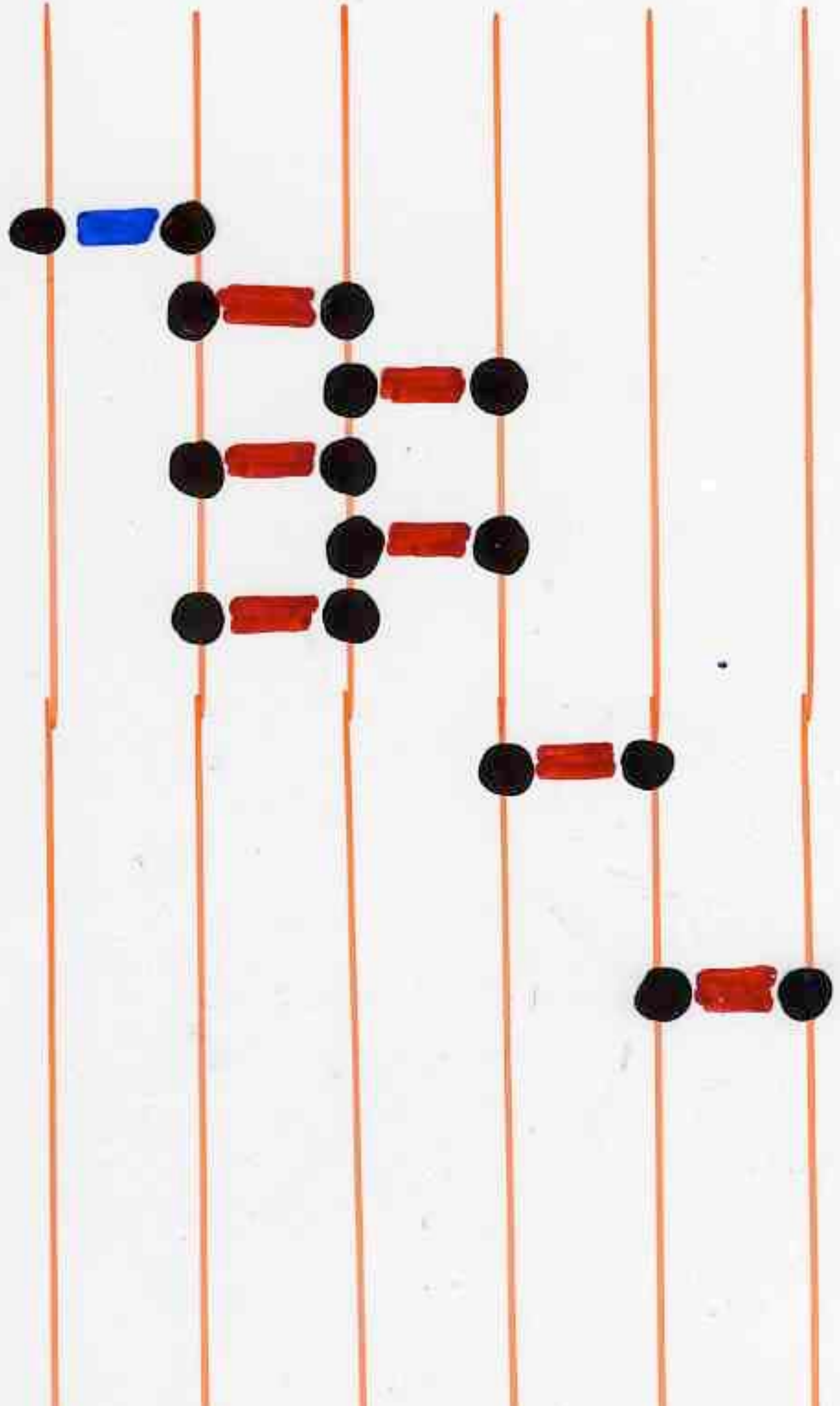
- strict semi-pyramid = { • 
  or
  • 
  or
  • (strict semi-pyramid)^2 × () }

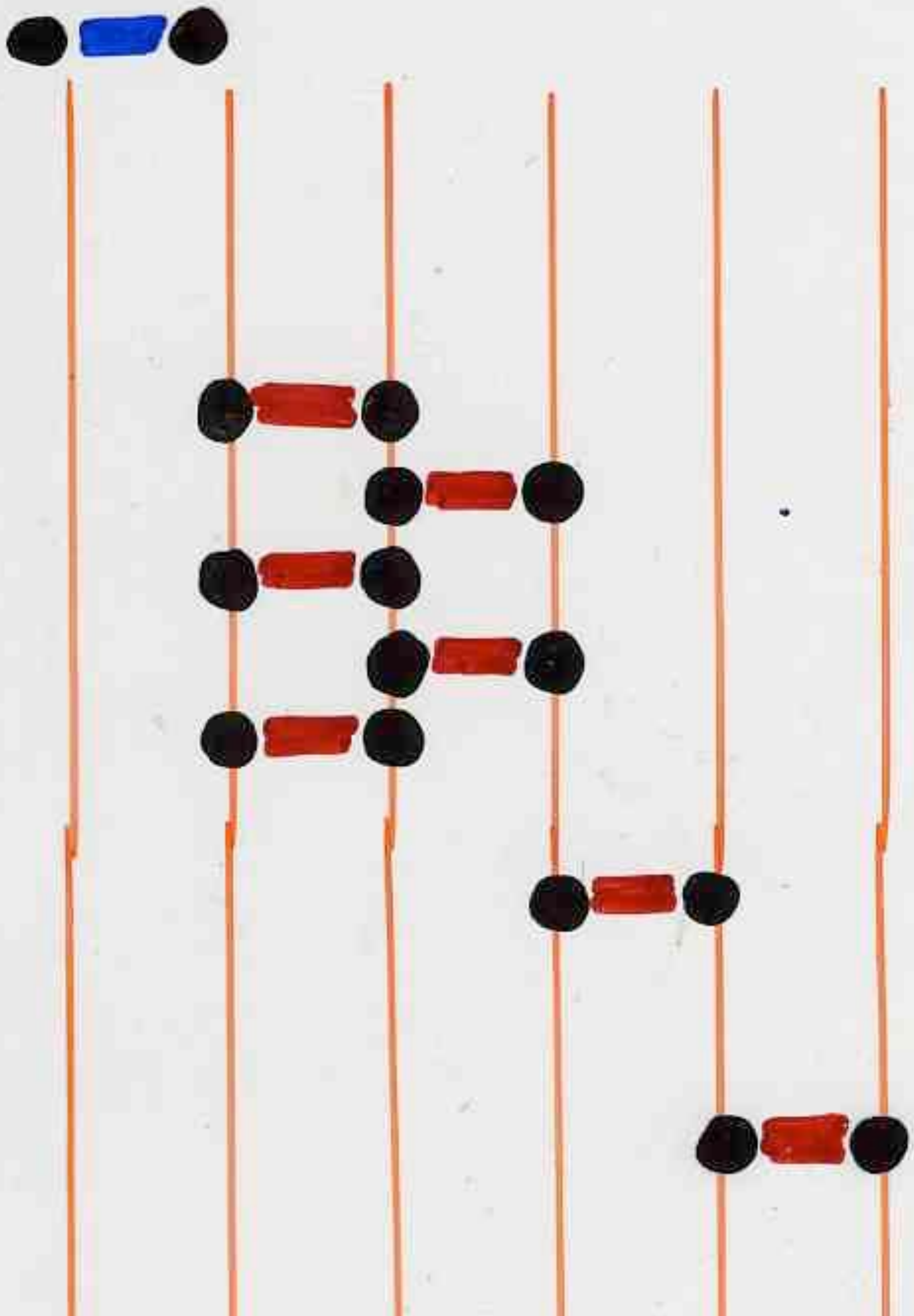
$$z = t + tz + tz^2$$











• strict semi-pyramid =

• 

or

• 

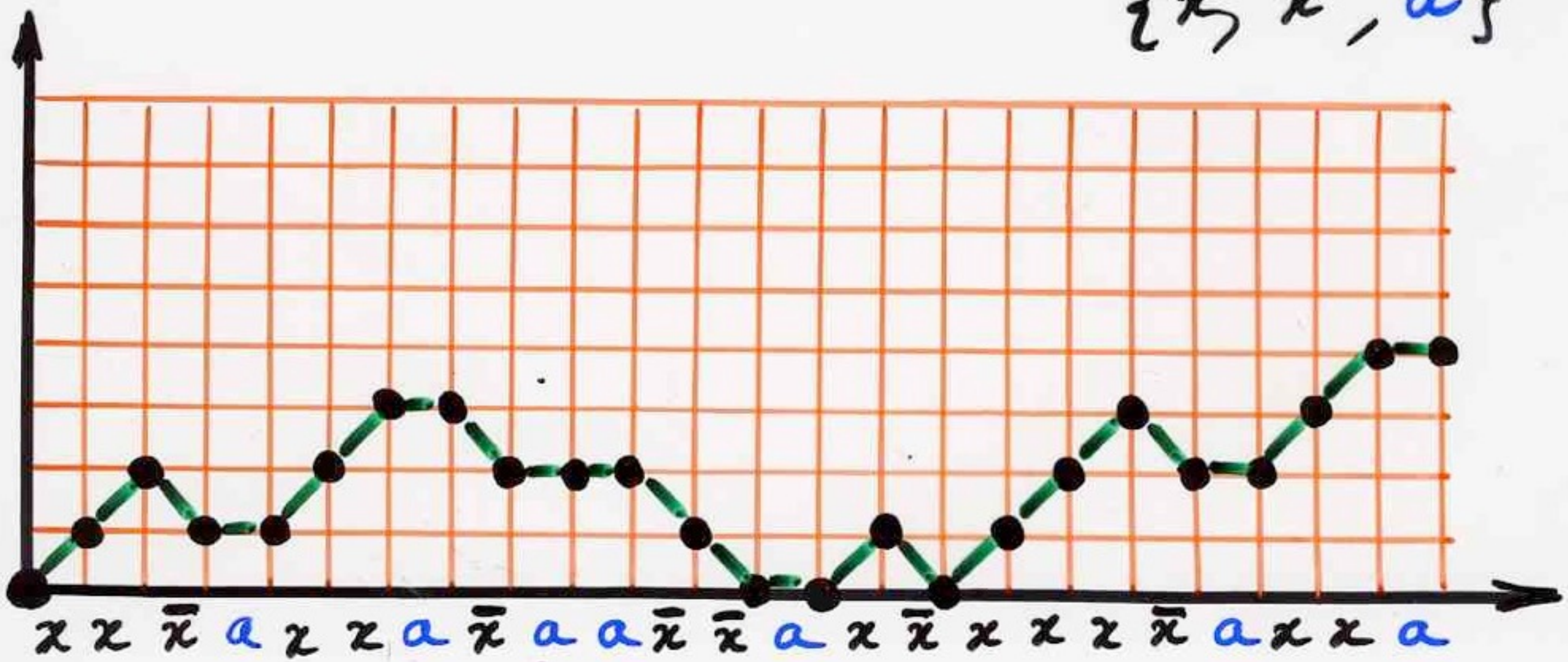
or

• (strict semi-pyramid)^2 × ()

$$z = t + tz + tz^2$$

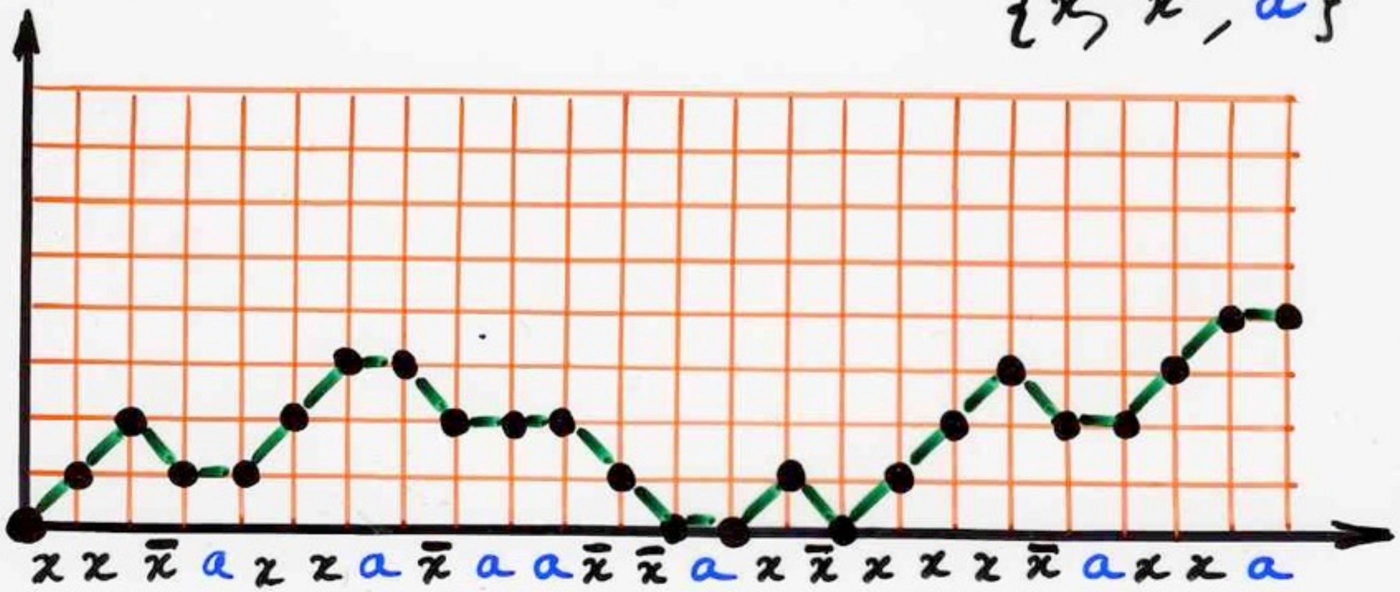
prefix (left factor) of a Motzkin path (word)

$\{x, \bar{x}, a\}$



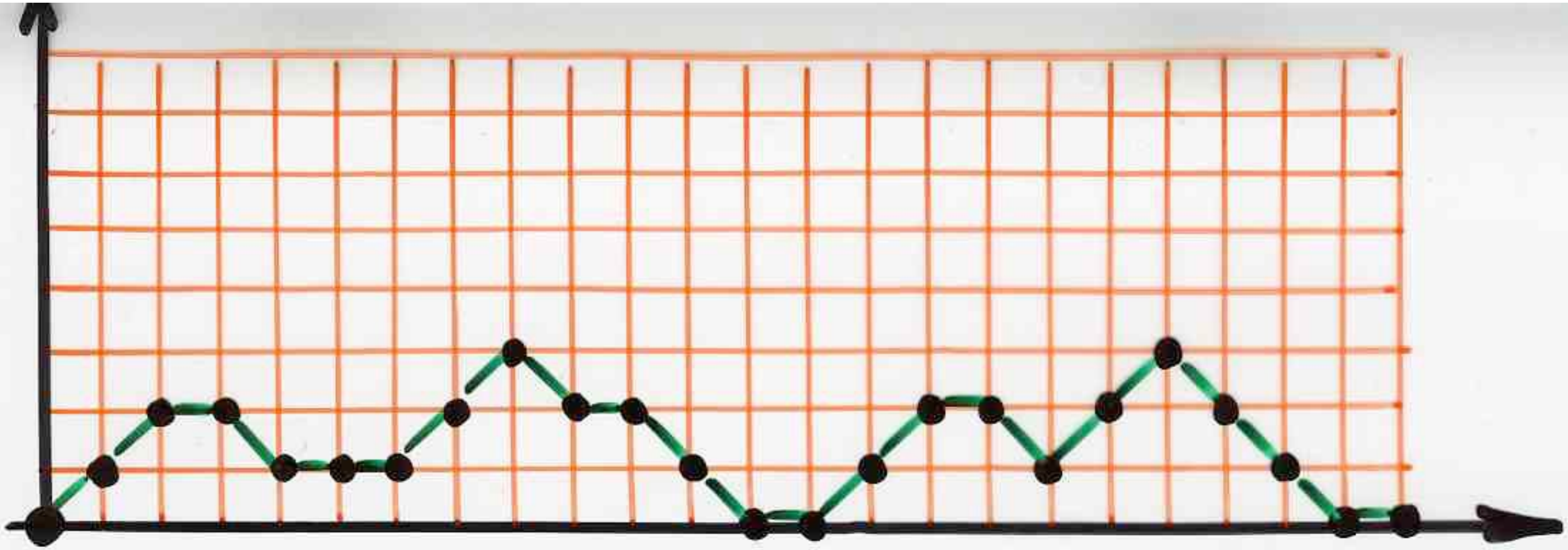
prefix (left factor) of a Motzkin path (word)

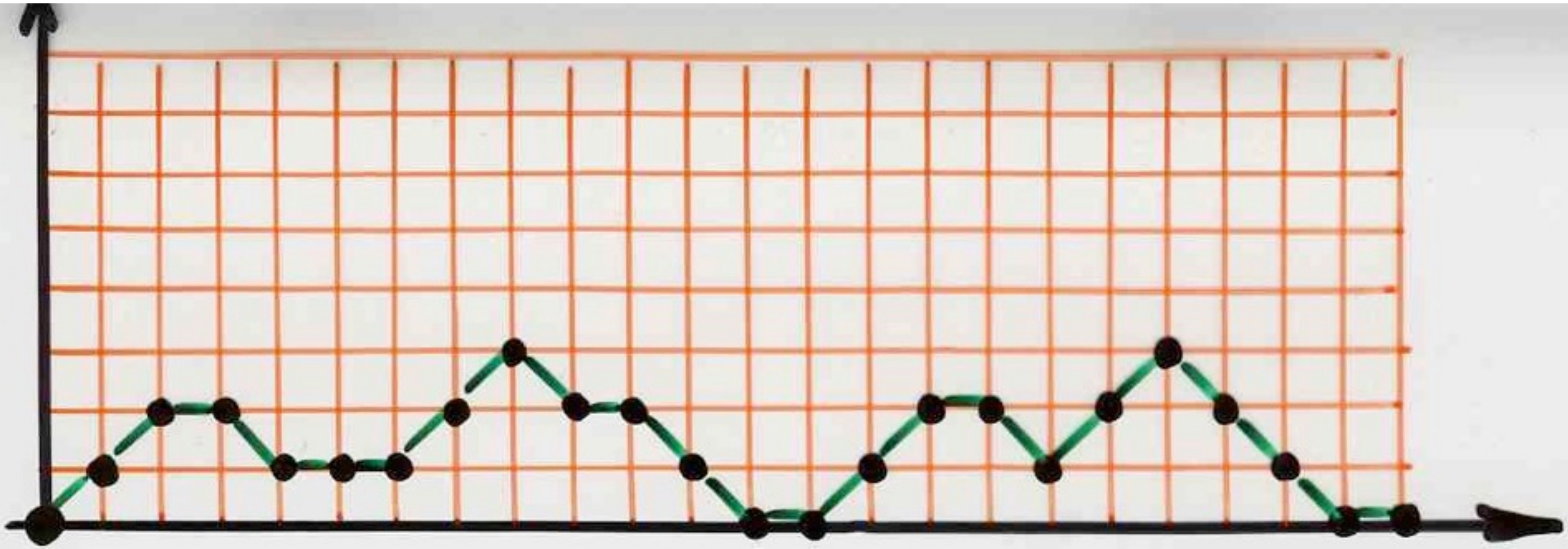
$\{x, \bar{x}, a\}$



- prefix Motzkin path $=$ { • Motzkin path
  • (Motzkin path) × ( ) × (prefix Motzkin path)

$$p = m + t\,p\,m$$





- Motzkin path $=$ $\begin{cases} \bullet \quad \emptyset \\ \bullet \quad (\bullet\!-\!\bullet) \times (\text{Motzkin path}) \\ \bullet \quad (\nearrow) \times (\text{Motzkin path}) \times (\searrow) \times (\text{Motzkin path}) \end{cases}$

$$m = 1 + t\,m + t^2\,m^2$$

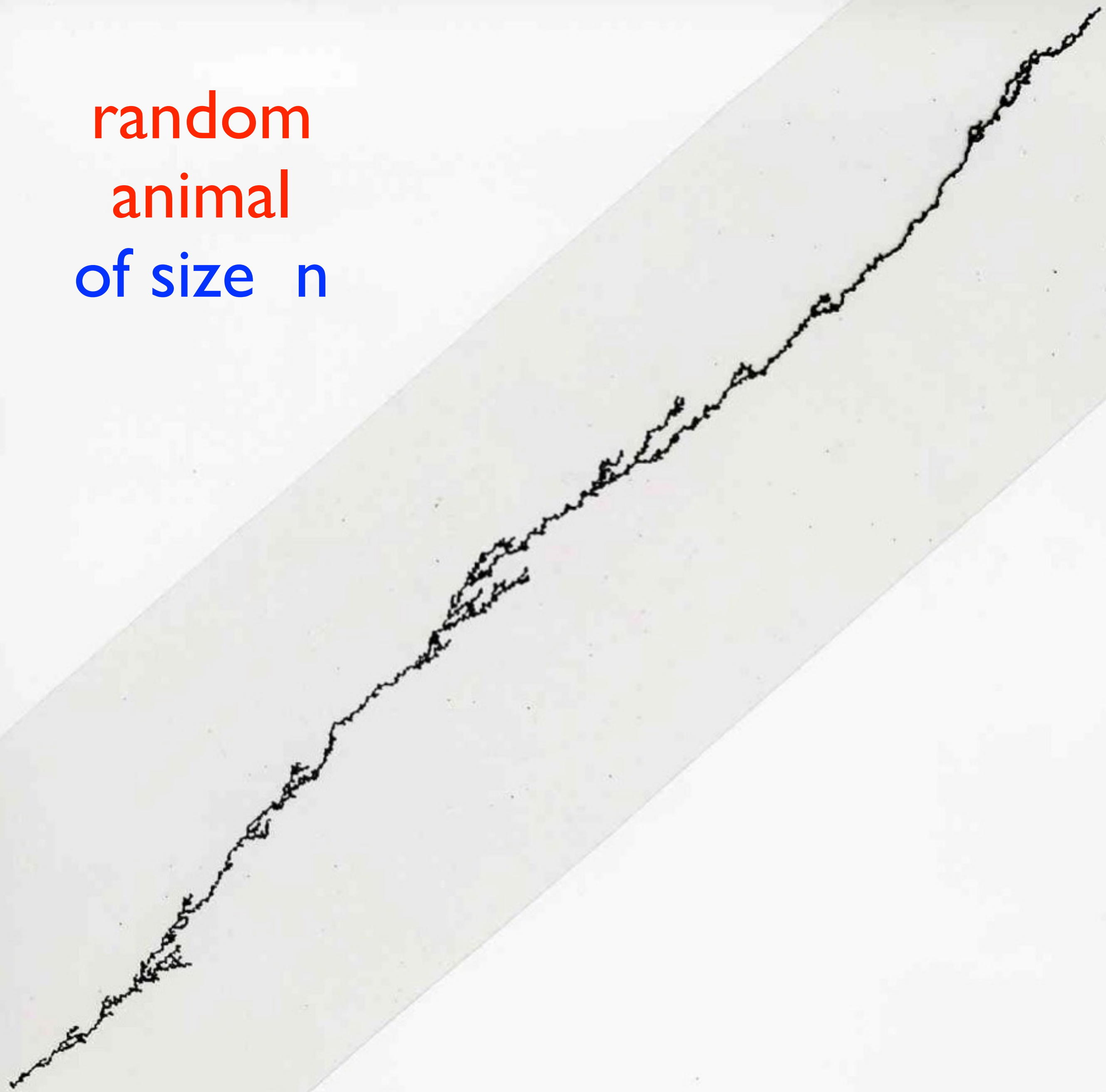
$y = t_p$

$z = t_m$

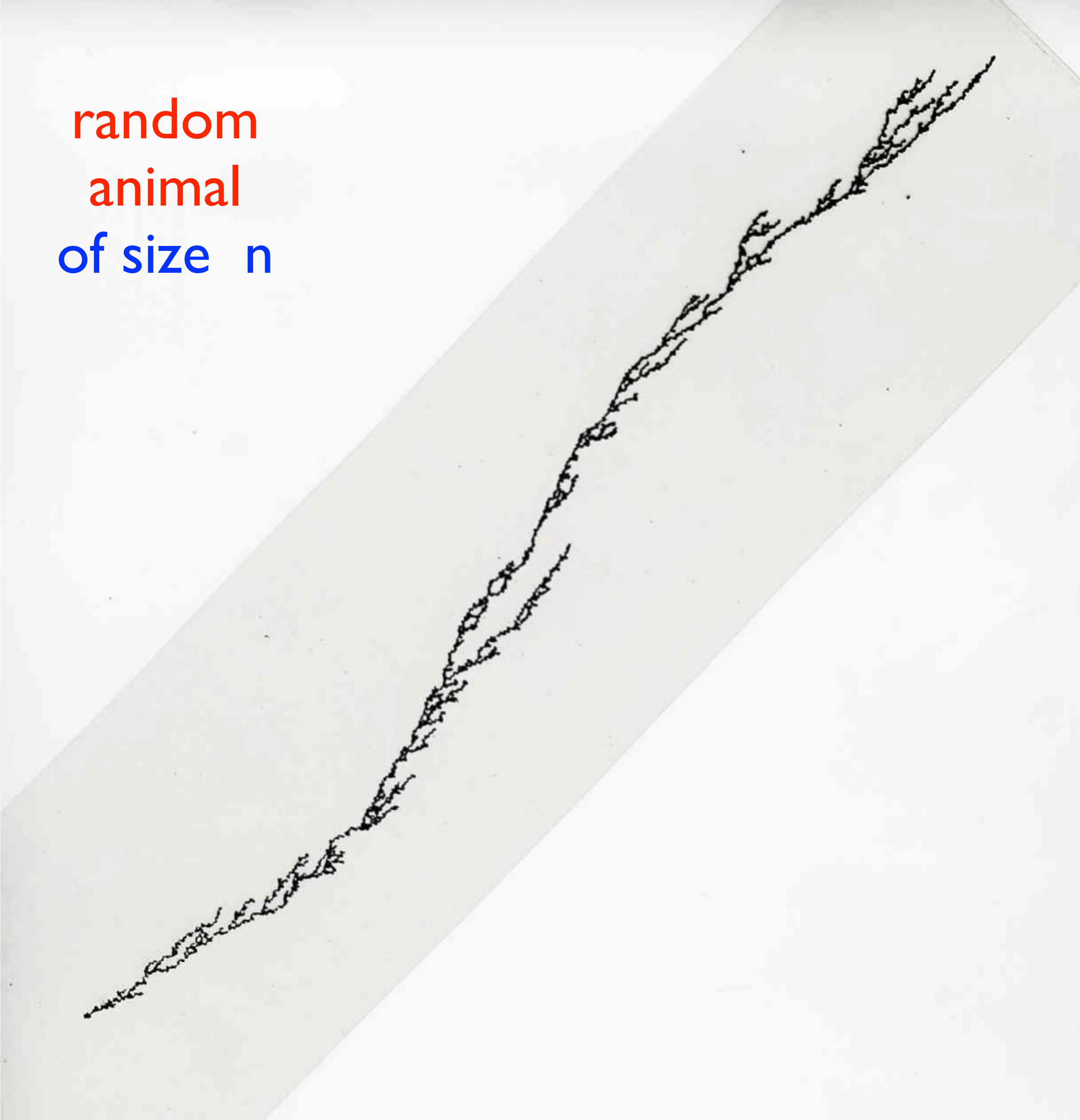
number of directed animals n points =

number of prefix of Motzkin paths length $(n-1)$

random
animal
of size n



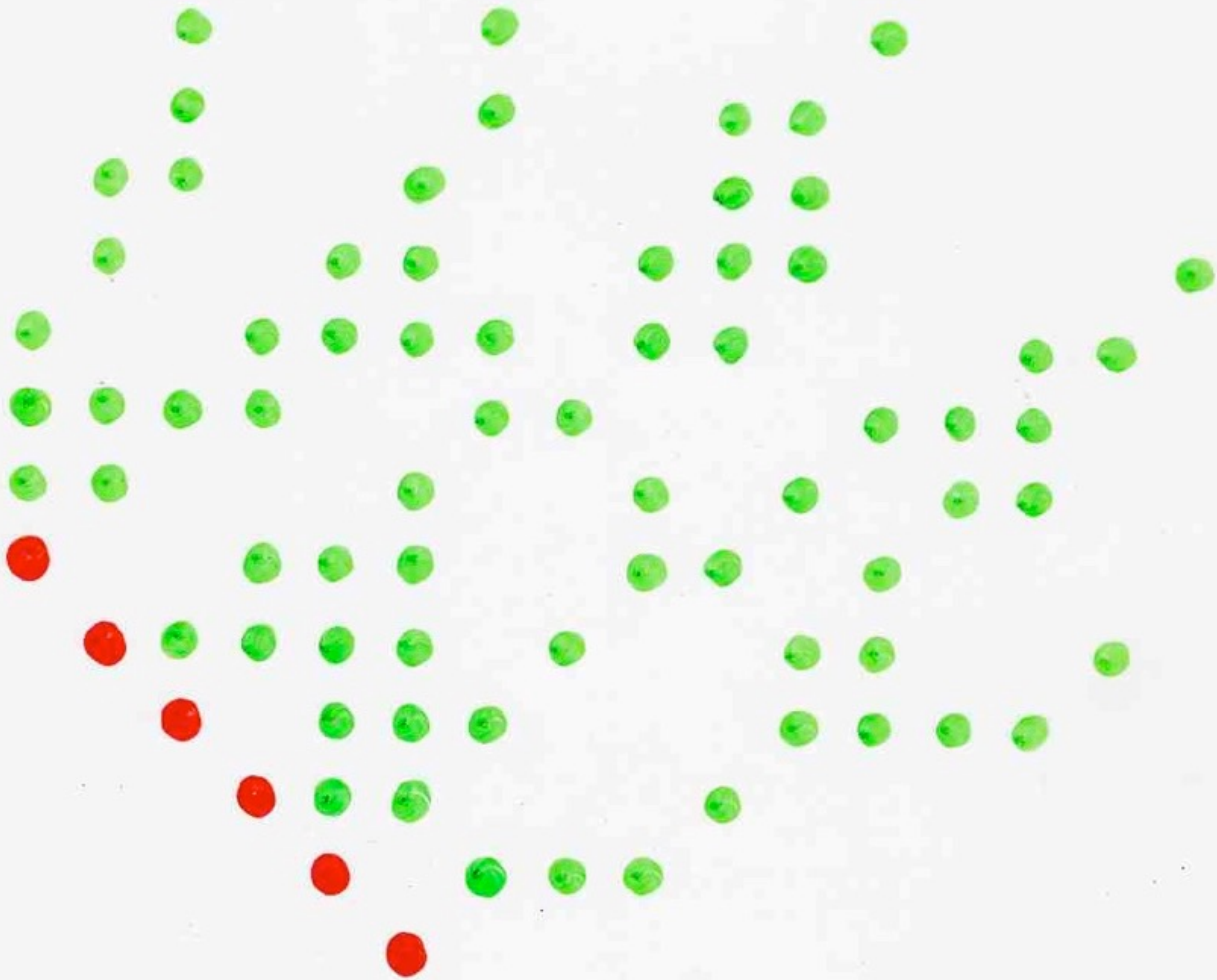
random
animal
of size n



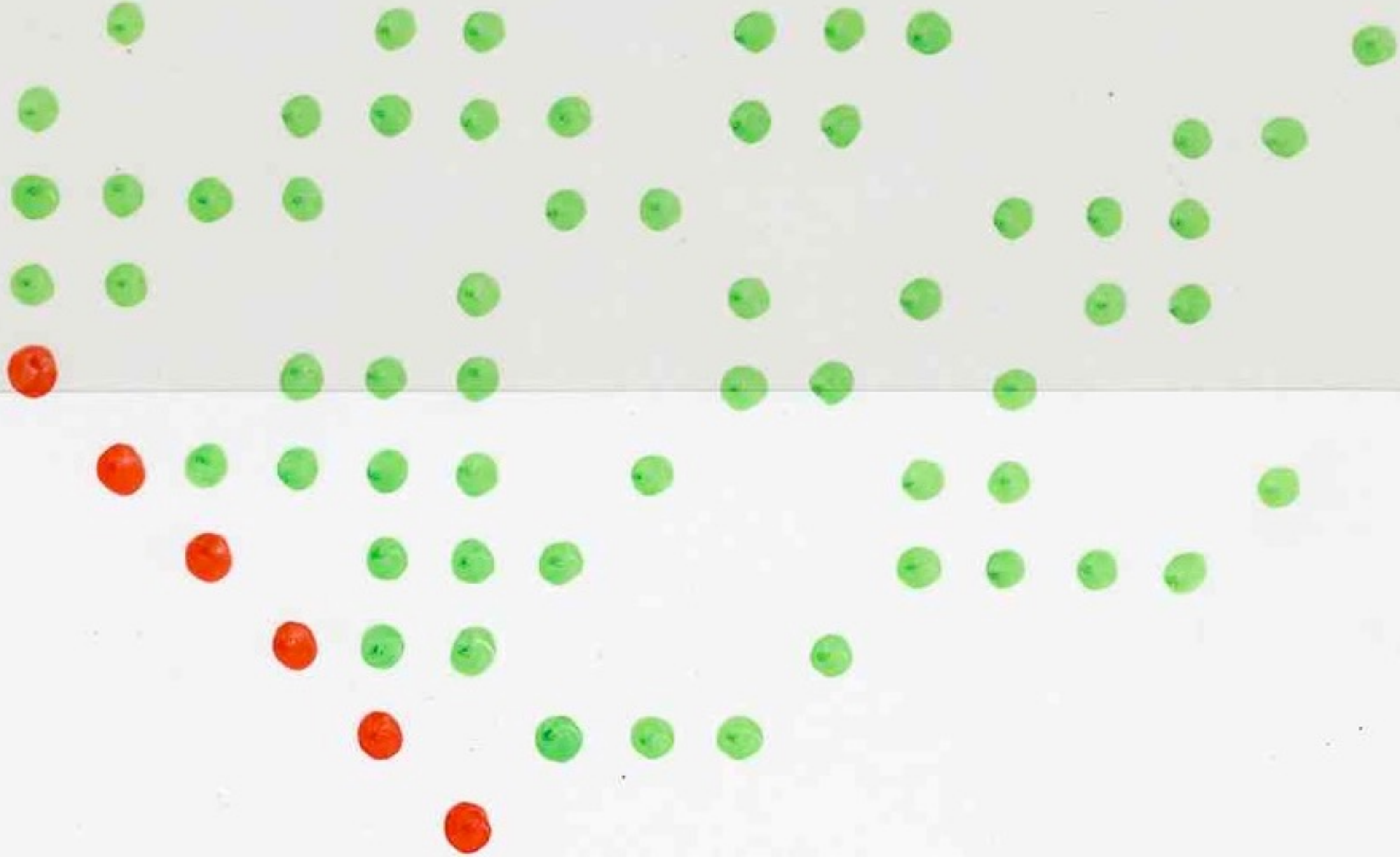
complements

exercise ?

compact source
directed animals



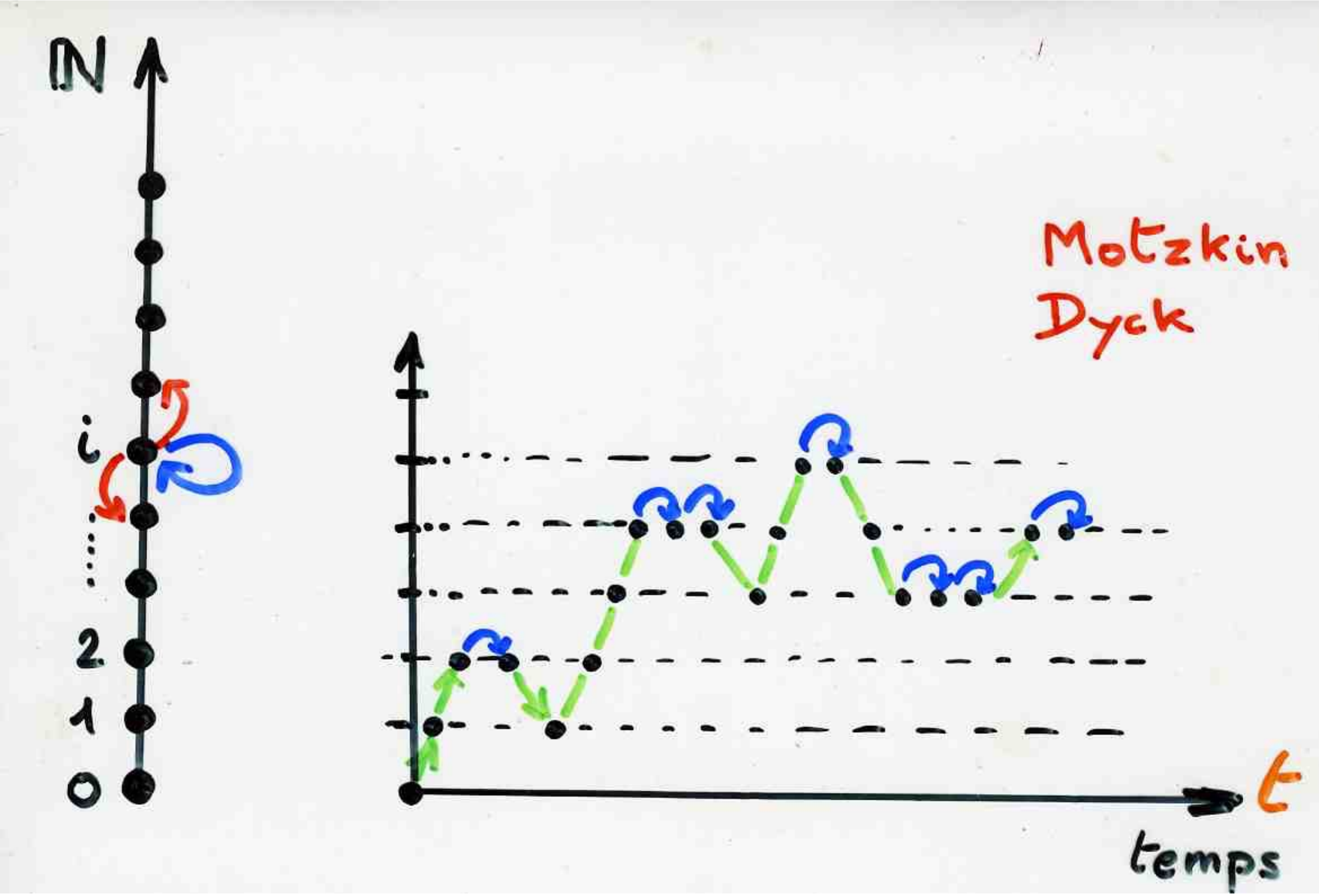
The number of directed animals size n+1, with compact source is



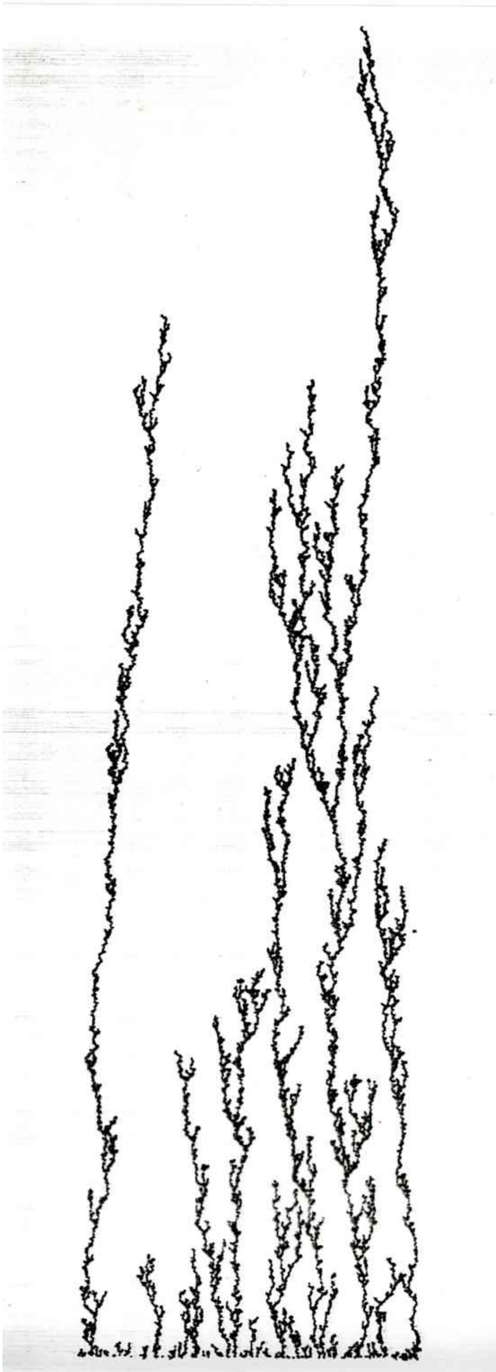
The number of directed animals size $n+1$, with compact source is

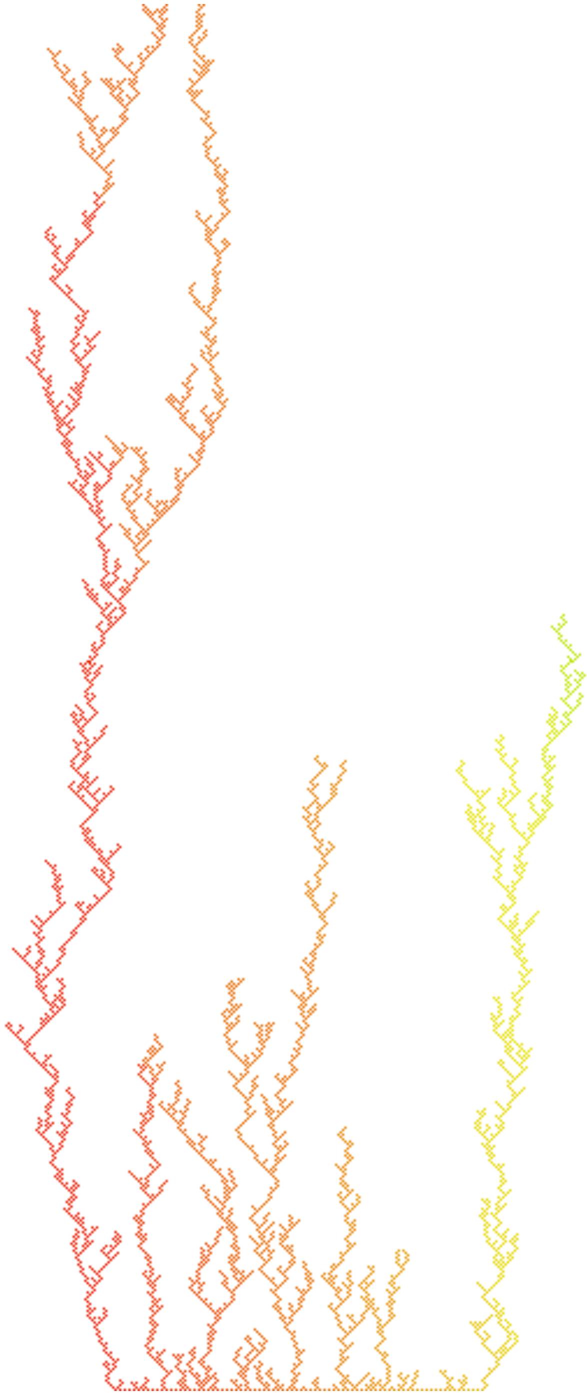
$$3^n$$

D.Gouyou-Beauchamps, X.V.
direct bijection with words in a,b,c (1988)



ℕ
i
2
1
0
Motzkin
Dyck
t
temps





exercise

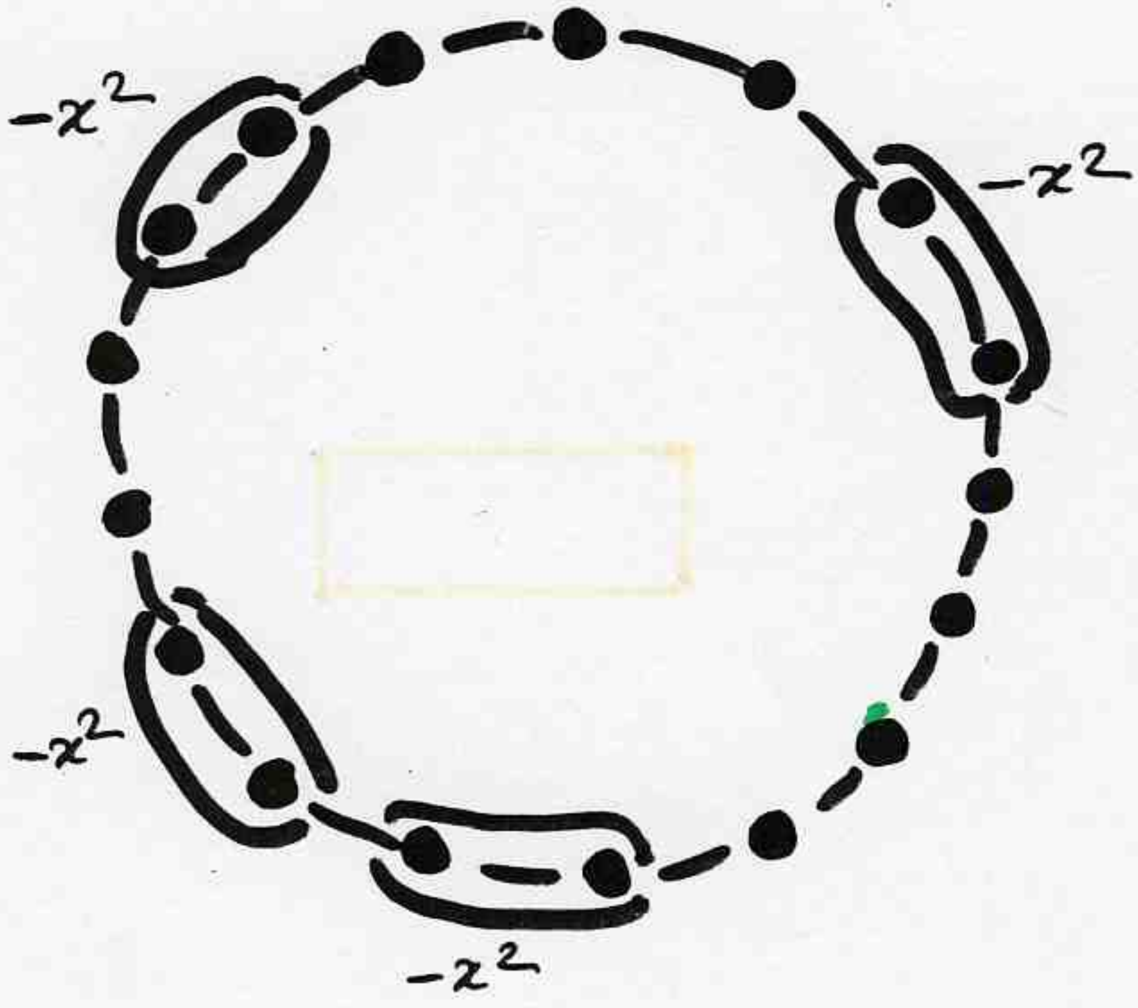
generating function for
directed animals
on a circular strip

$$\frac{N}{D} = \frac{\prod_{\ell} U_{\ell}^{*}\left(\frac{x}{2}\right)}{T_{k}^{*}\left(\frac{x}{2}\right)}$$

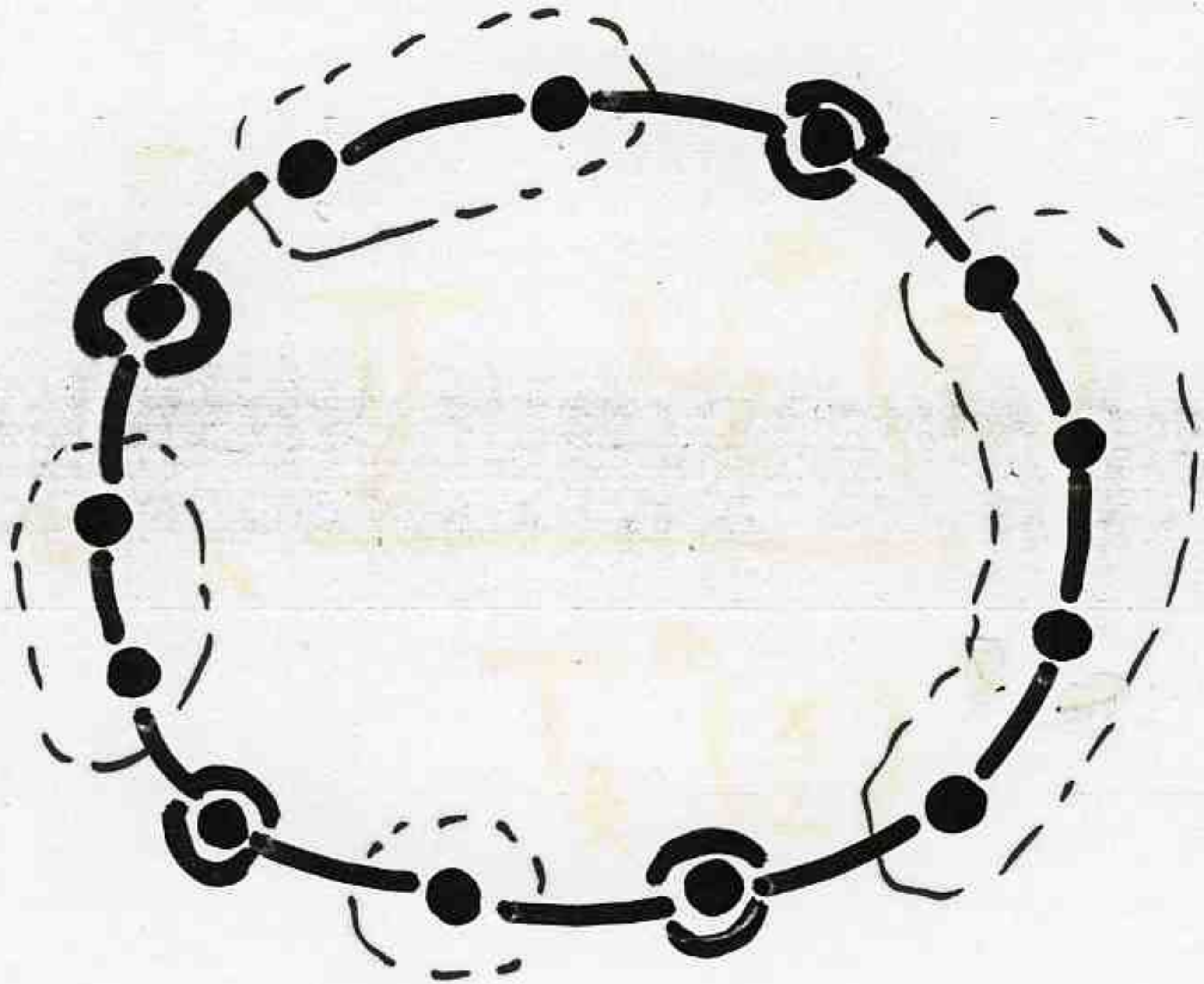
$$D = T_n^*\left(\frac{x}{2}\right)$$

$\cos n\theta = T_n(\cos\theta)$

Tchebycheff 1^{st} kind



$$T_n^*\left(\frac{x}{2}\right)$$

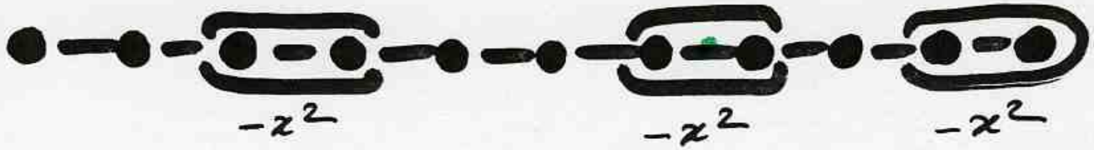


$$N = \prod_{\ell} U^{*}_{\ell}\left(\frac{x}{2}\right)$$

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta)$$

(circular bounded strip)
triangular lattice

Tchebycheff 2nd kind



$$U^{*}_{n}\left(\frac{x}{2}\right)$$

$$\tilde{b}_n^{\le k} = \frac{1}{k} \sum_{p=0}^{k-1} (-1)^p \sin\alpha_p \prod_{i=1}^{k-1} \left(\frac{\sin(i+\frac{1}{2})\,\alpha_p}{\sin\frac{\alpha_p}{2}} \right)^{N_i} (1+2\cos\alpha_p)^{n-}$$

animals
circular strip
width k

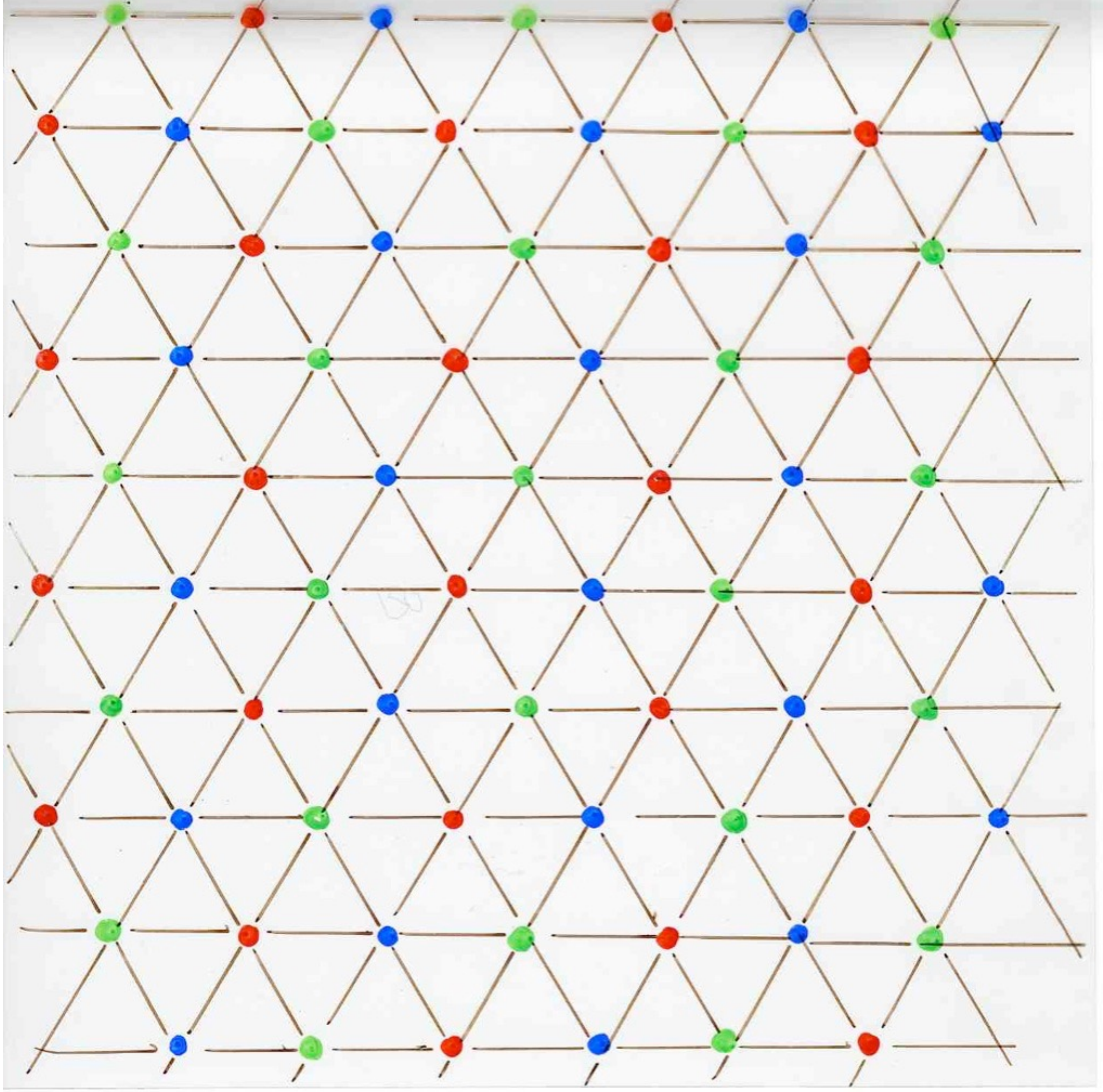
$$\alpha_p = \frac{2p+1}{2k}\pi$$

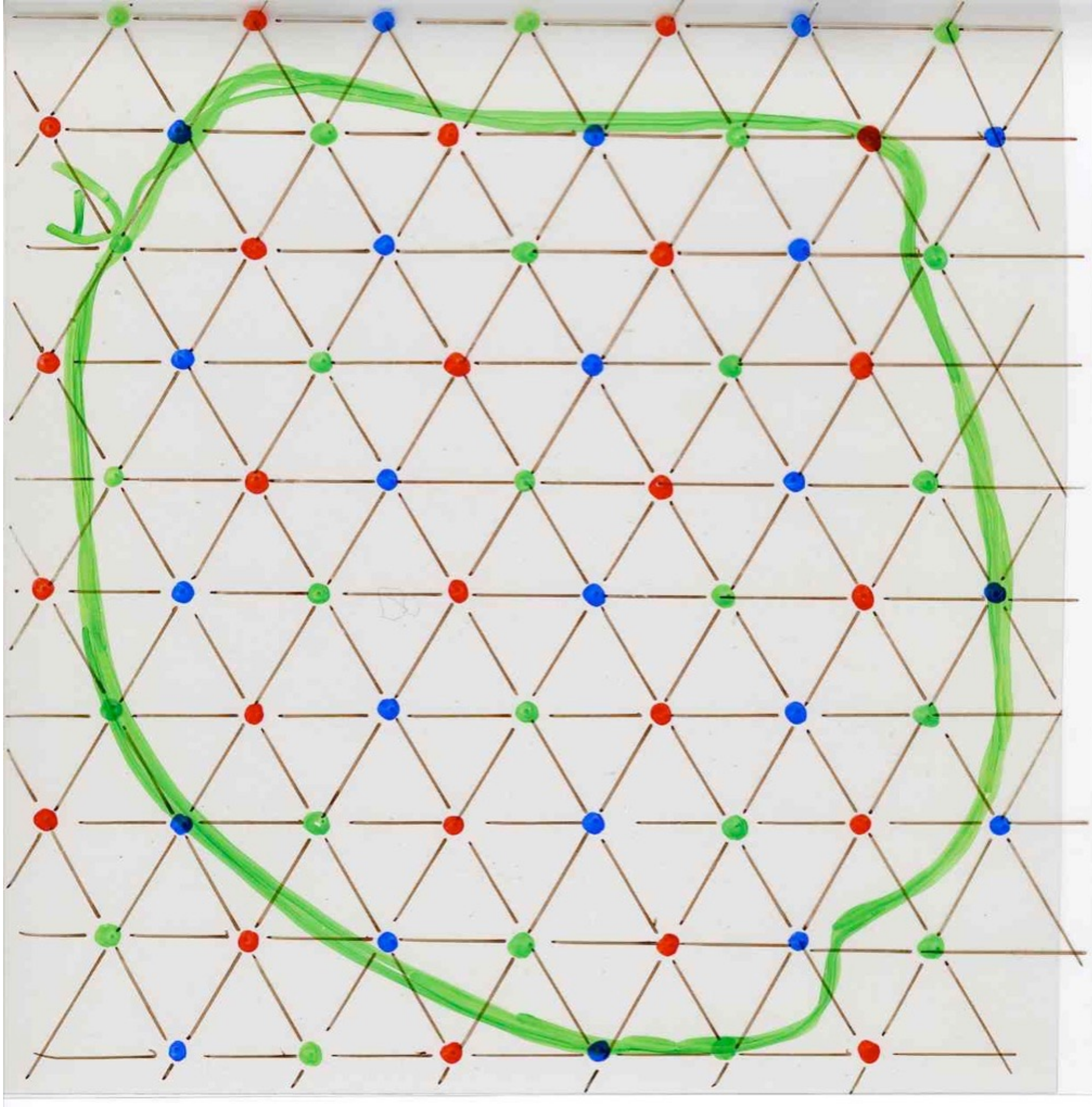
(Hakim, Nadal, 1982)

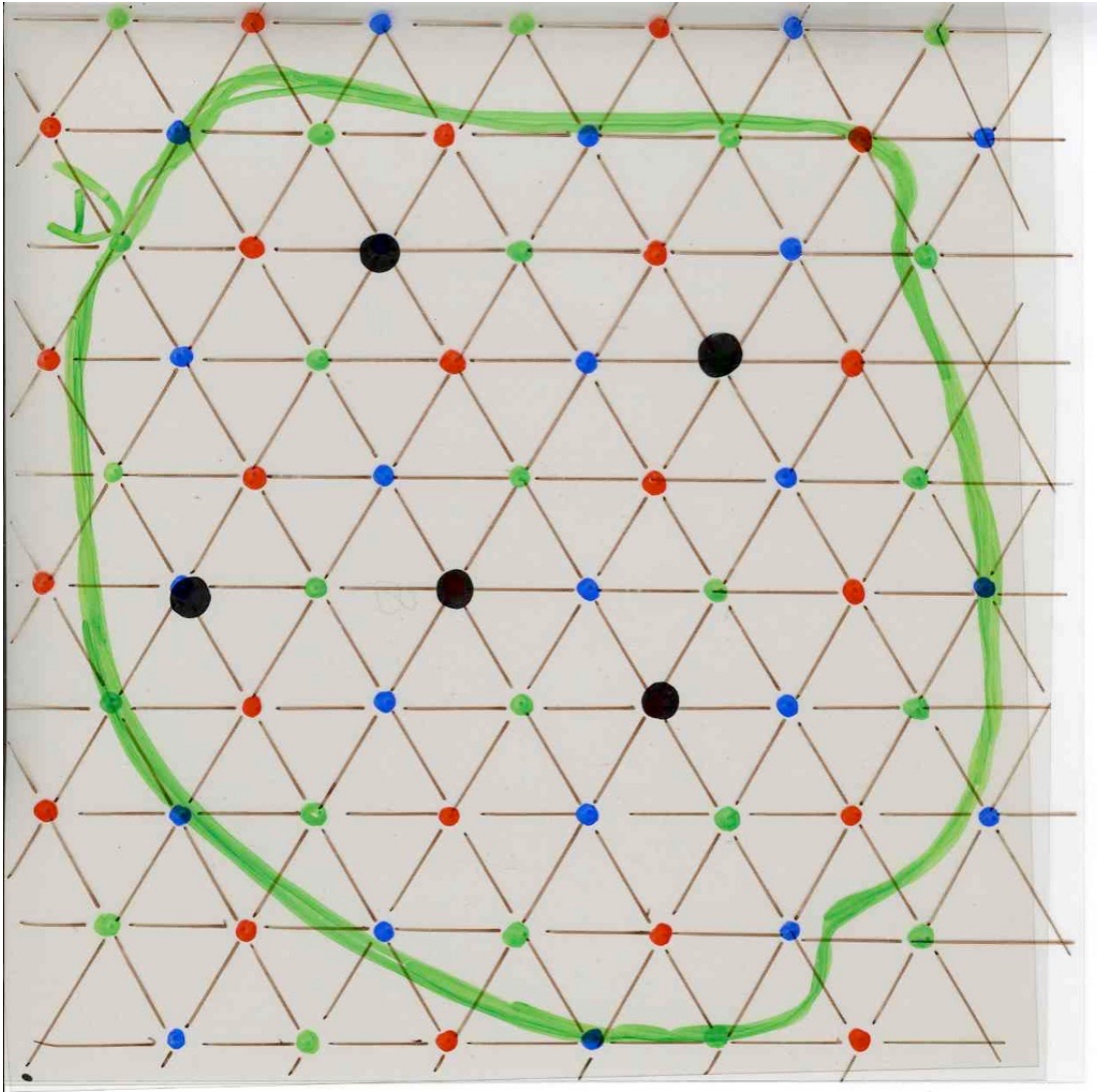
Hard
Hexagons

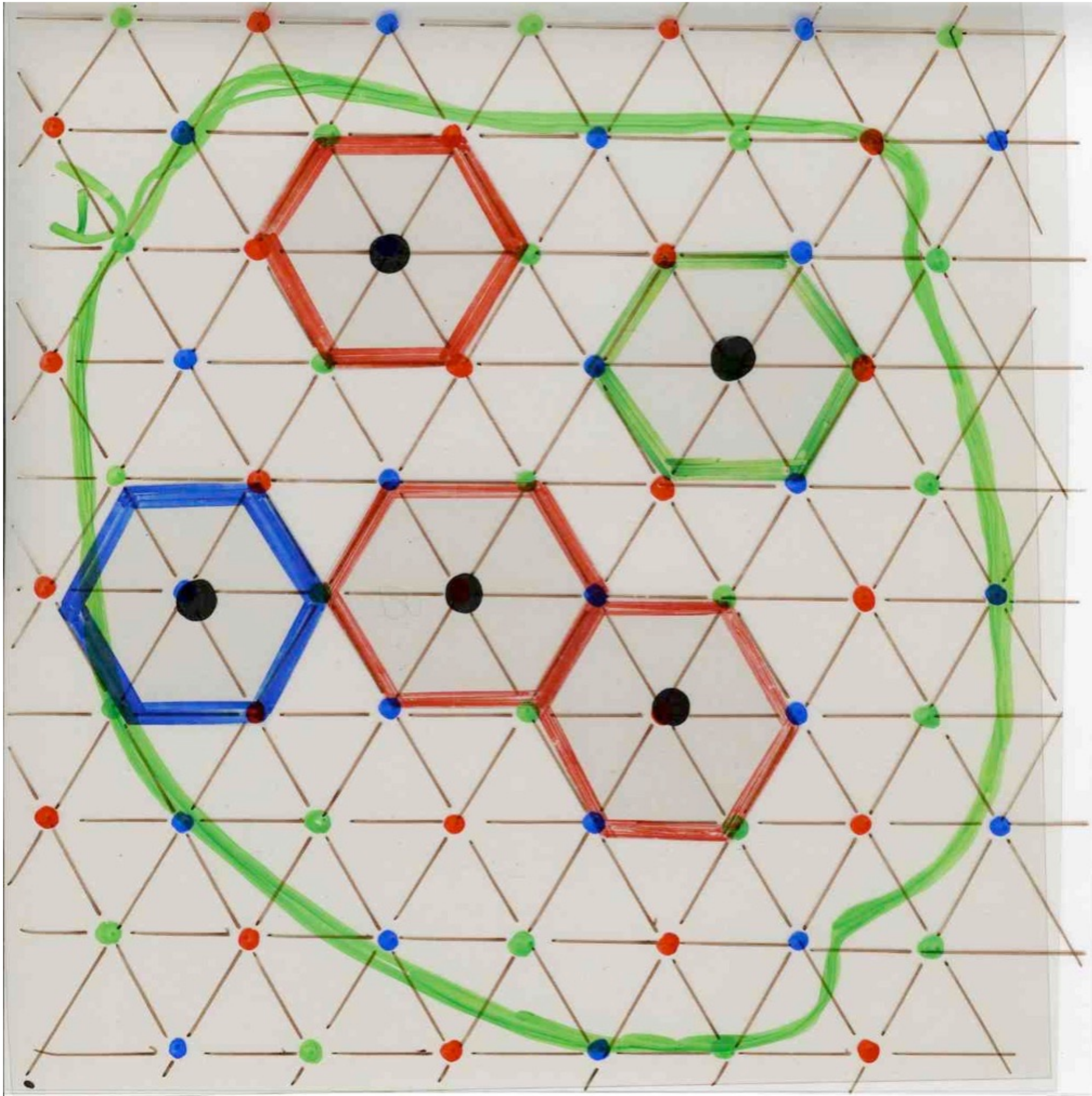
Baxter
1980
1982

gas model
with "hardcore interactions"









partition function

$$Z_D(t) = \sum_{n \geqslant 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{"}D \to \infty\text{"}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

$$R(q) = \prod_{n\geqslant 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$R(q) = \prod_{n\geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_I}$$

$$t = -q\left[R(q)\right]^5$$

$$R(q) = \prod_{n \geq 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$t = -q\left[R(q)\right]^5$$

$$Y(q) = \prod_{n \geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5n+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}$$

$$R(q) = \prod_{n \geqslant 0} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+3})(1-q^{5n+2})} = \frac{R_{II}}{R_{I}}$$

$$t = -q\left[R(q)\right]^{5}$$

$$Y(q) = \prod_{n \geqslant 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^{2}(1-q^{6n+4})(1-q^{5n+1})^{2}(1-q^{5n+4})^{2}(1-q^{5n})^{2}}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^{2}(1-q^{5n+2})^{3}(1-q^{5n+3})^{3}}$$

$$Z(t) = Y(q(t))$$

- critical temperature

$$T_c = \frac{11 + 5\sqrt{5}}{2}$$

- critical exponent $\frac{5}{6}$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{5}$$

$\frac{5}{6}$ critical exponent

Baxter

(1980)

$t_c = 11.09017..$

$\frac{1}{2}(11 + 5\sqrt{5})$

Gaunt 1967

critical temperature for hexagons:

$(\phi)^5$

golden ratio

$= \left(\frac{1+\sqrt{5}}{2}\right)^5$

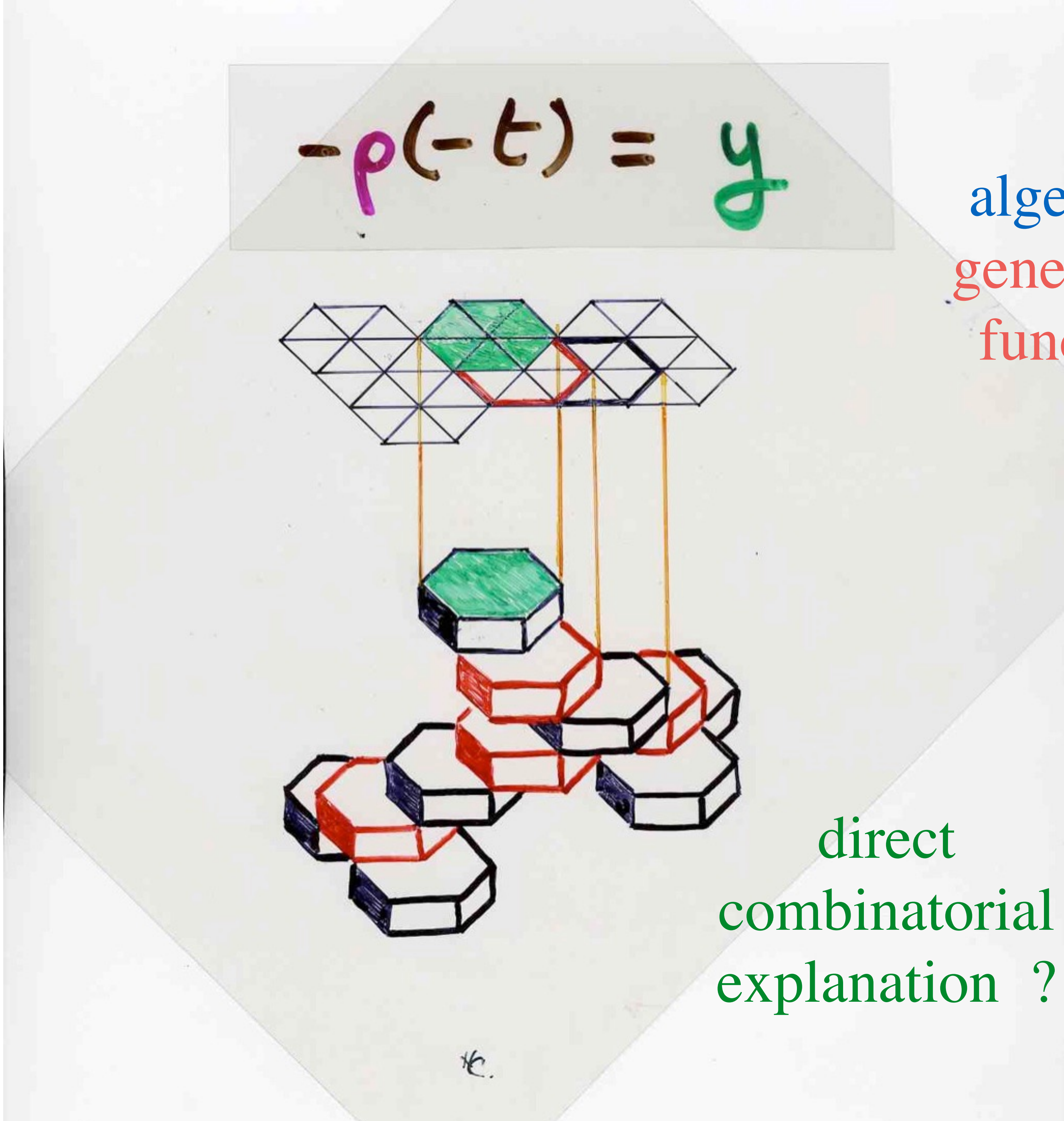
helium monolayer
absorbed onto
a graphite surface

(Riedel 1981)

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

density of a "hard-core" lattice gas model

t is the "activity" of the gas

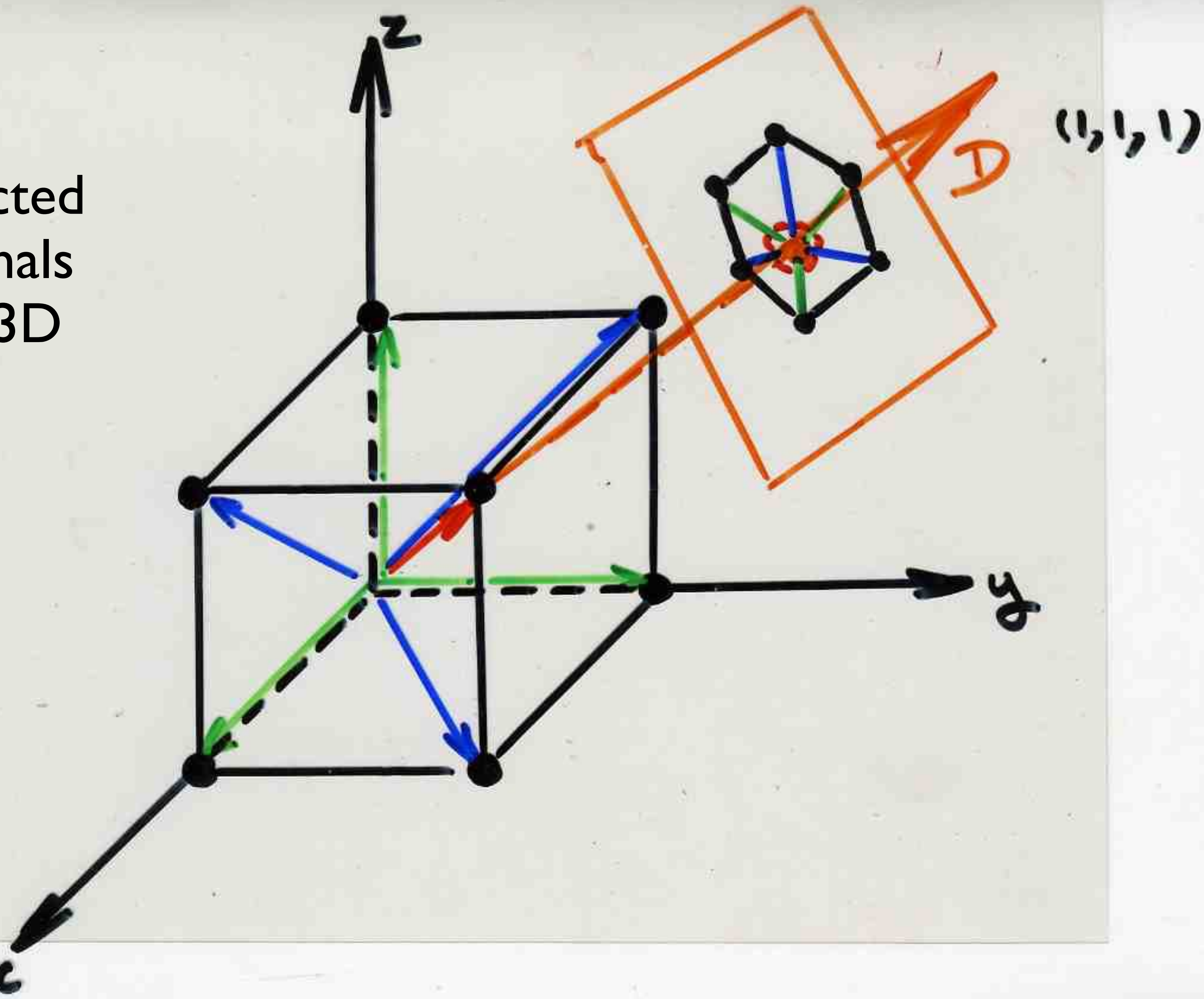


algebraic
generating
function

direct
combinatorial
explanation ?

?

directed
animals
in 3D



thermodynamic limit

partition function

$$Z_D(t) = \sum_{n \geq 0} a_{n,D}\, t^n$$

$$Z(t) = \lim_{\text{"} D \to \infty \text{"}} \left(Z_D(t) \right)^{1/|D|}$$

thermodynamic limit

Thus denoting

$$\rho(t) = t \frac{d}{dt} \log Z(t)$$

density

partition function

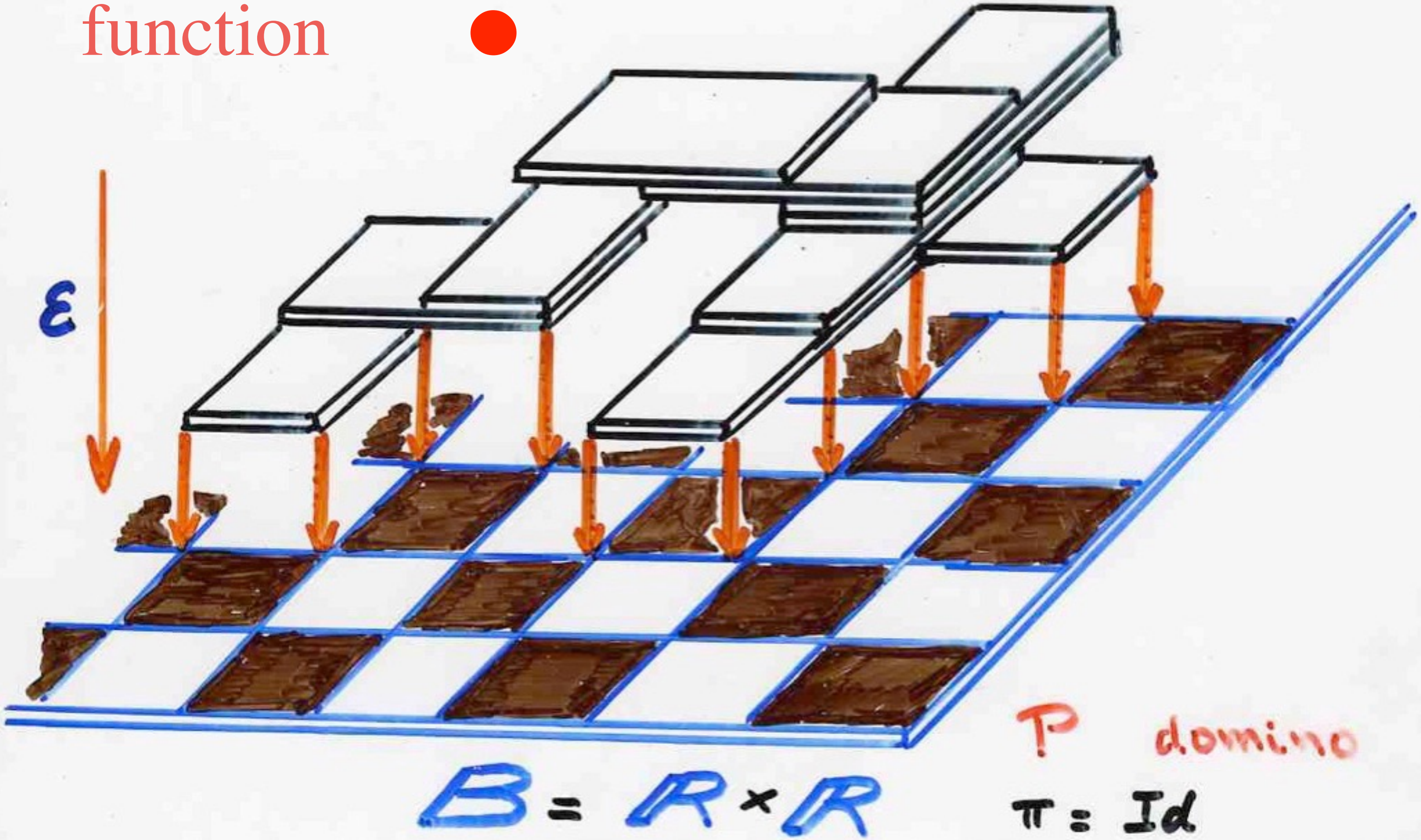
Thm. $-\rho(-t) = \sum a_n t^n$

nb of pyramid on Hex with n hexagons

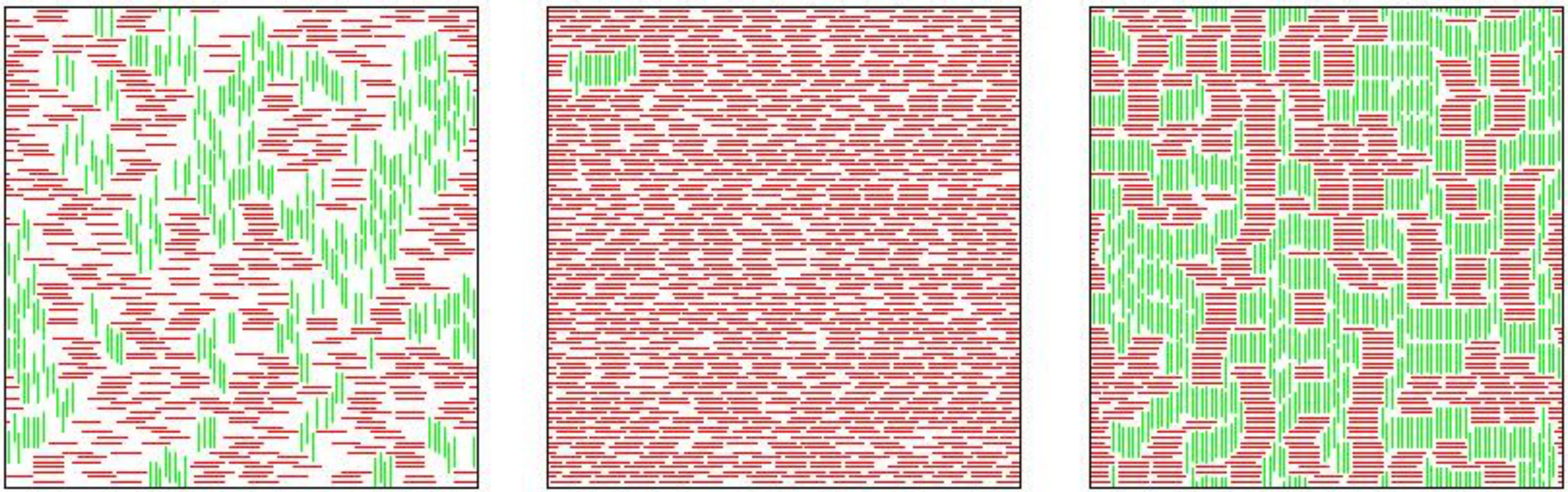
Hard core
lattice gas models

interpretation of the density

algebraic
generating
function ?

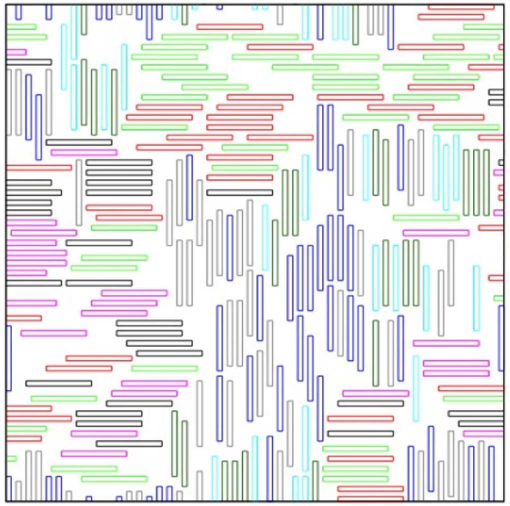


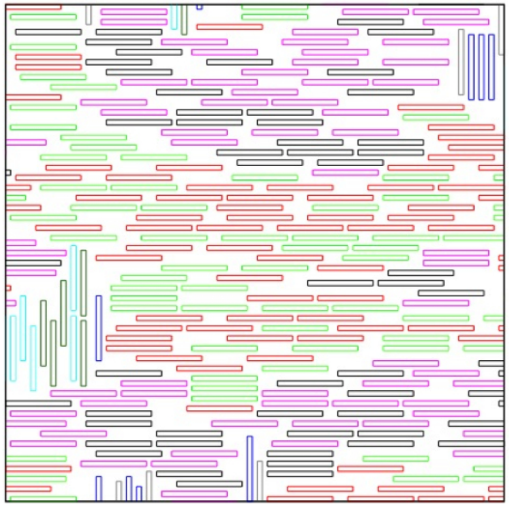
Rajesh

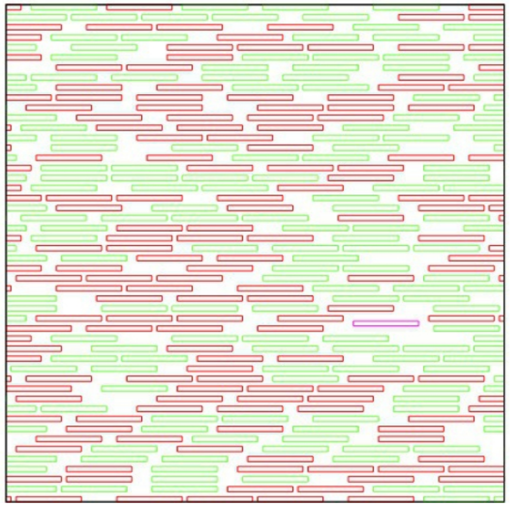


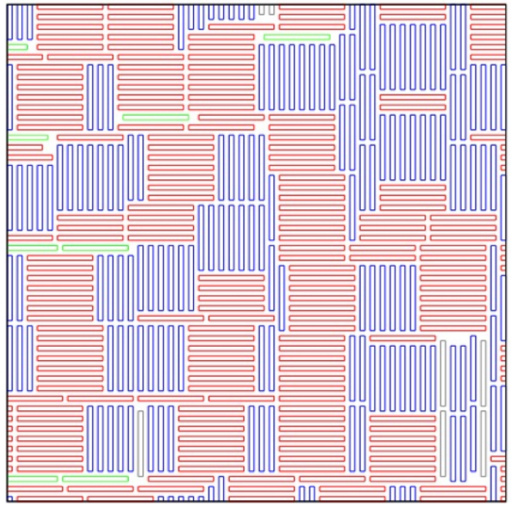
Low density | Intermediate density | High density

Rajesh









thermodynamic limit

complement:
the proof

$\mathcal{D} \subseteq \mathrm{Hex}$

finite domain

$s \in \mathcal{D}$ $\qquad \mathbf{P}_{\mathcal{D},s}(t)$ generating function for pyramid

- projection in $\mathcal{D}$
- maximal piece s

Compute: $\dfrac{1}{|\mathcal{D}|} \displaystyle\sum_{s \in \mathcal{D}} \mathbf{P}_{\mathcal{D},s}(t) = \dfrac{1}{|\mathcal{D}|} \mathbf{P}_{\mathcal{D}}(t)$

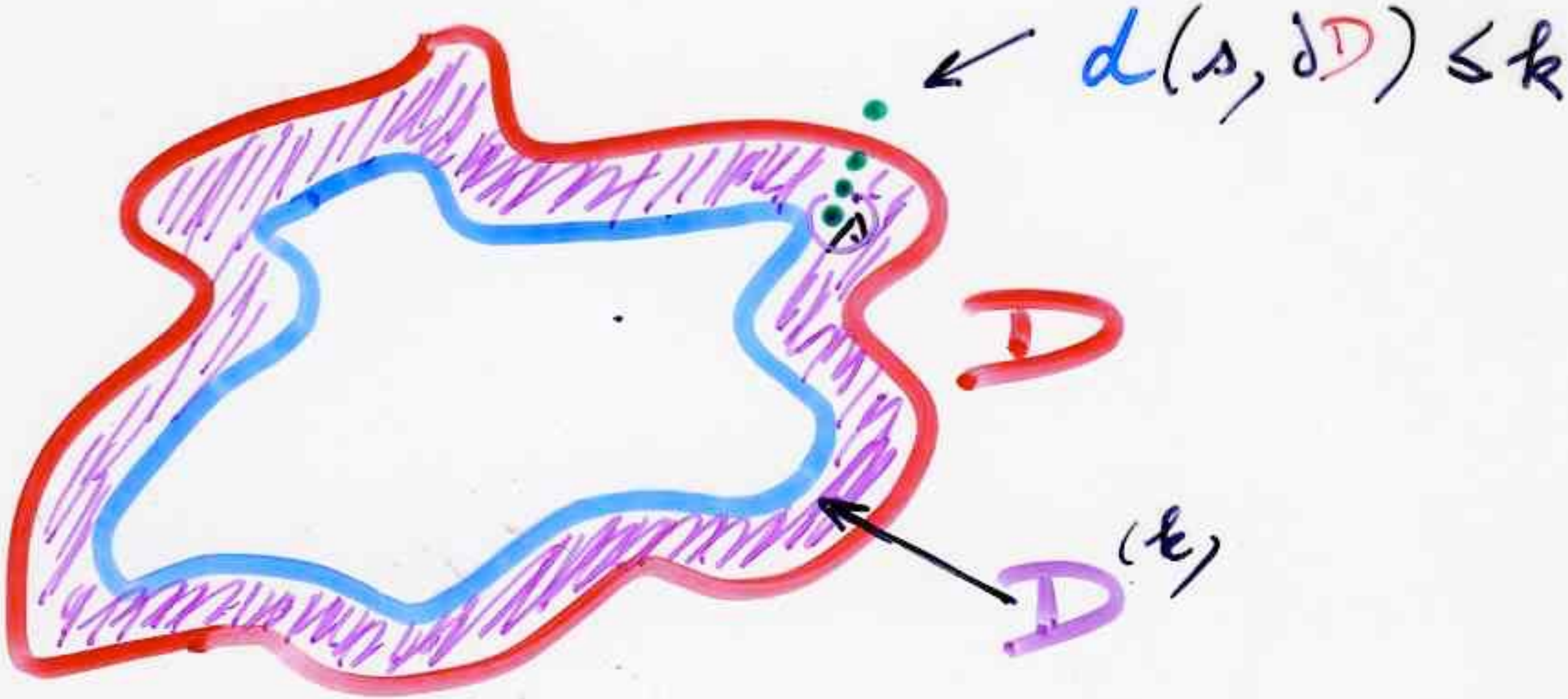
$\underbrace{\hphantom{xxxxxxxxxx}}$

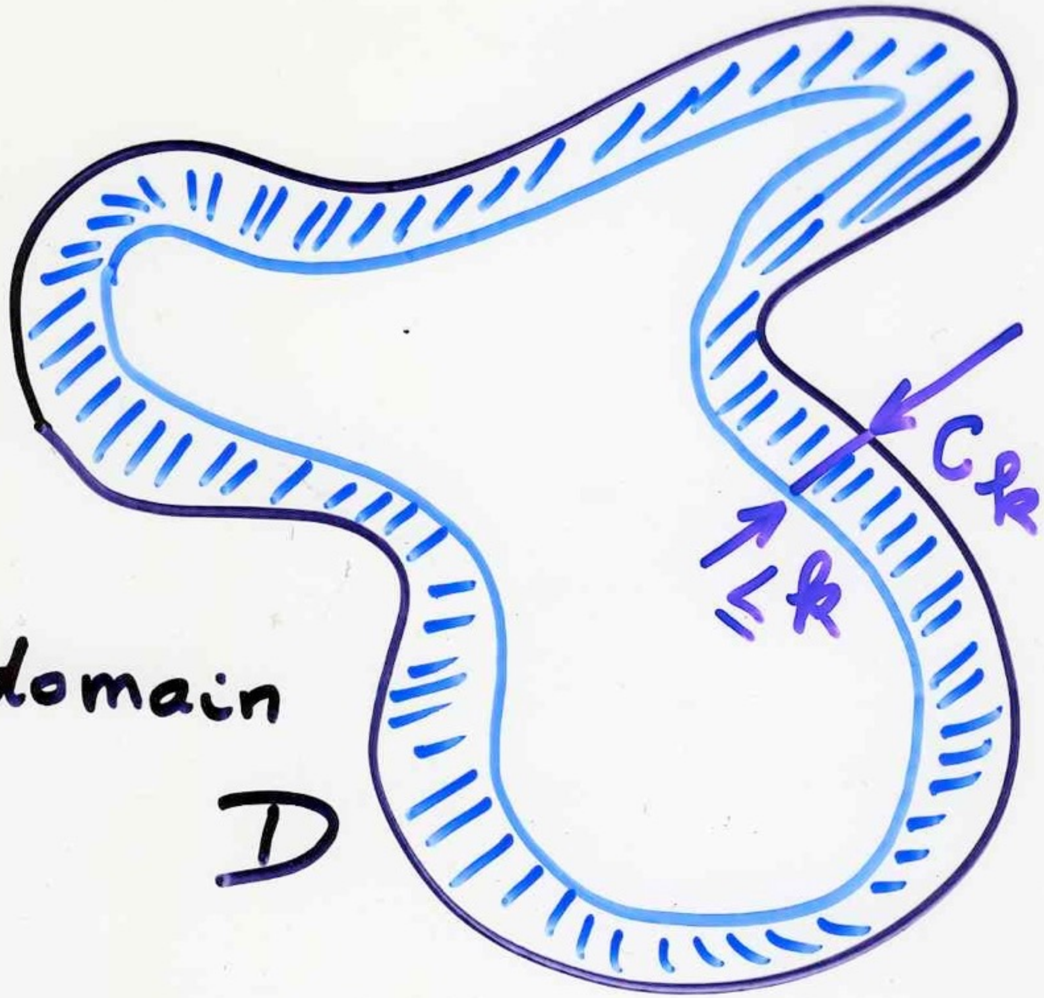
Hexagon Pyramids on a tube of base $\mathcal{D}$

$$\mathbf{P}_{\mathcal{D}}(t) = (-t) \frac{d}{dt} \log Z_{\mathcal{D}}^{-1}(-t)$$

Def. $d(s, \partial D)$

smallest length of paths (on *Hex*) to go from s to the outside of D





domain

D

$$\frac{|C_k|}{|D|} \to 0$$

Prop. Sequence of domains $D_1 \subseteq \dots \subseteq D_n \subseteq D_{n+1} \subseteq \dots$ such that : for every k,

$$\frac{\left|\{s \in D_n,\ d(s, \partial D_n) \leq k\}\right|}{|D_n|} \longrightarrow 0 .$$

(for $n \to \infty$)

Then : $\frac{1}{|D_n|} \mathbf{P}_{D_n}(t) \longrightarrow \mathbf{P}(t)$ generating function for Pyramid on Hex (up to translation)

$\longrightarrow$ means : $f_n(t) = \sum_{i \geq 0} a_{n,i}\, t^i$; $f(t) = \sum_{i \geq 0} a_i\, t^i$

then for every i, $a_{n,i} \to a_i$.

$\mathcal{D}$ domain $\quad k \geq 0$ fixed

$$\mathcal{D}^{(k)} = \{ s \in \mathcal{D} \, , \; d(s, \partial\mathcal{D}) \leq k \}$$

Let $\mathcal{P}^{(k)}_{\mathcal{D},s}(t)$; $\mathcal{P}^{(k)}_{\mathcal{D}}(t)$ polynomial of degree k truncation of the formal power series $\mathcal{P}_{\mathcal{D},s}(t)$ and $\mathcal{P}_{\mathcal{D}}(t)$.

<u>Fact</u> for every $s \in \mathbf{D} - \mathbf{D}^{(k)}$,

$$\mathbf{P}^{(k)}_{\mathbf{D},s}(t) = \underbrace{\mathbf{P}^{(k)}(t)}.$$

(no "border effect" up to degree k)

truncation of $\mathbf{P}(t)$

Pyramid on Hex

(enumerated up to translation)

$$\frac{1}{|D_n|} P_{D_n}^{(k)}(t) \longrightarrow P^{(k)}(t)$$

$$(n \to \infty)$$

Then

$$\frac{1}{|D|} P_D^{(k)}(t) = \frac{1}{|D|} \sum_{\Delta \in D} P_{D,\Delta}^{(k)}(t)$$

$$= \frac{|D - D^{(k)}|}{|D|} \cancel{\sum_{\Delta \in D - D^{(k)}}} P^{(k)}(t)$$

$$+ \frac{1}{|D|} \sum_{\Delta \in D^{(k)}} P_{D,\Delta}^{(k)}(t)$$

$$D = D_1, D_2, \dots, D_n, \dots$$

Then

$$\frac{1}{|D|}\,P^{(k)}_{D}(t) = \frac{1}{|D|}\sum_{\Delta\in D} P^{(k)}_{D,\Delta}(t)$$

$$= \underbrace{\frac{|D - D^{(k)}|}{|D|}}_{\to 1}\;\cancel{\sum_{\Delta\in D-D^{(k)}}}\;P^{(k)}(t)$$

$$\underbrace{+\frac{1}{|D|}\sum_{\Delta\in D^{(k)}} P^{(k)}_{D,\Delta}(t)}_{\to 0}$$

$$D = D_1, D_2, \ldots, D_n, \ldots$$

Thus denoting

$$\rho(t) = t \frac{d}{dt} Z(t)$$

density

partition function

Thm. $$-\rho(-t) = \sum a_n t^n$$

nb of pyramid on Hex with n hexagons

$$\rho(t) = t - 7t^2 + 58t^3 - 519t^4 + 4856t^5 - \ldots$$

$$Z(t) = 1 + t - 3t^2 + 16t^3 - 106t^4 + 789t^5 - 6318t^6 + \ldots$$