Combinatorics and Physics

Chapter 5a
Combinatorics for the PASEP
(alternative tableaux)

IIT-Madras

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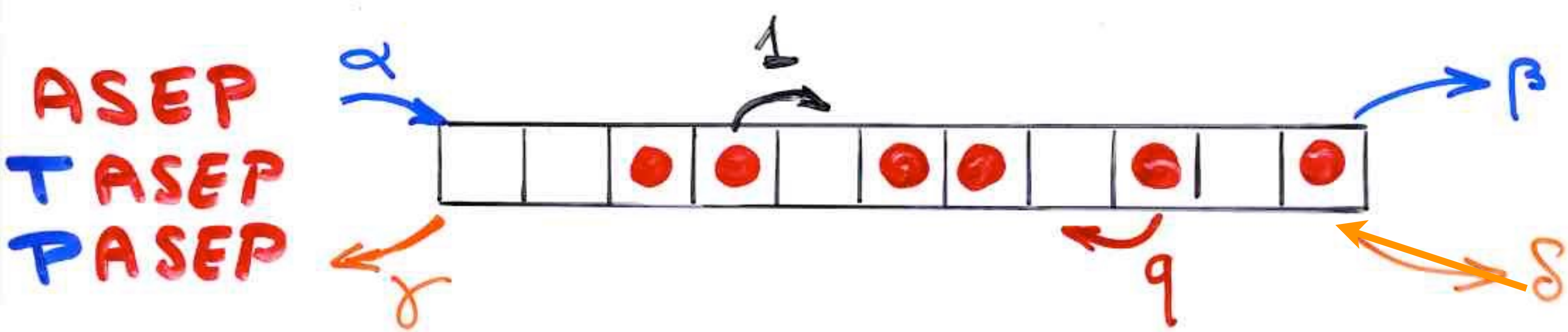
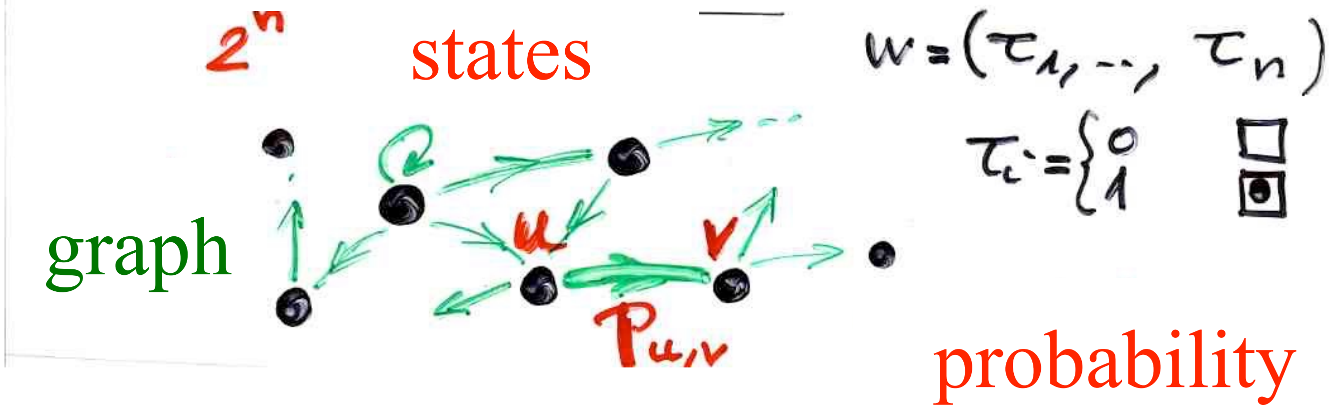
Xavier Viennot

CNRS, LaBRI, Bordeaux

The PASEP


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

Markov chains



S : states

$$M = (p_{u,v})_{u,v \in S}$$

probabilities matrix
(stochastic)

$$\pi = (p_u, \dots)$$

vector (time t)

$$\pi \cdot M$$

vector (time t+1)

$$p_v^{(t+1)} = \sum_u p_u^{(t)} \, p_{u,v}$$

t+1 time t

stationary probabilities

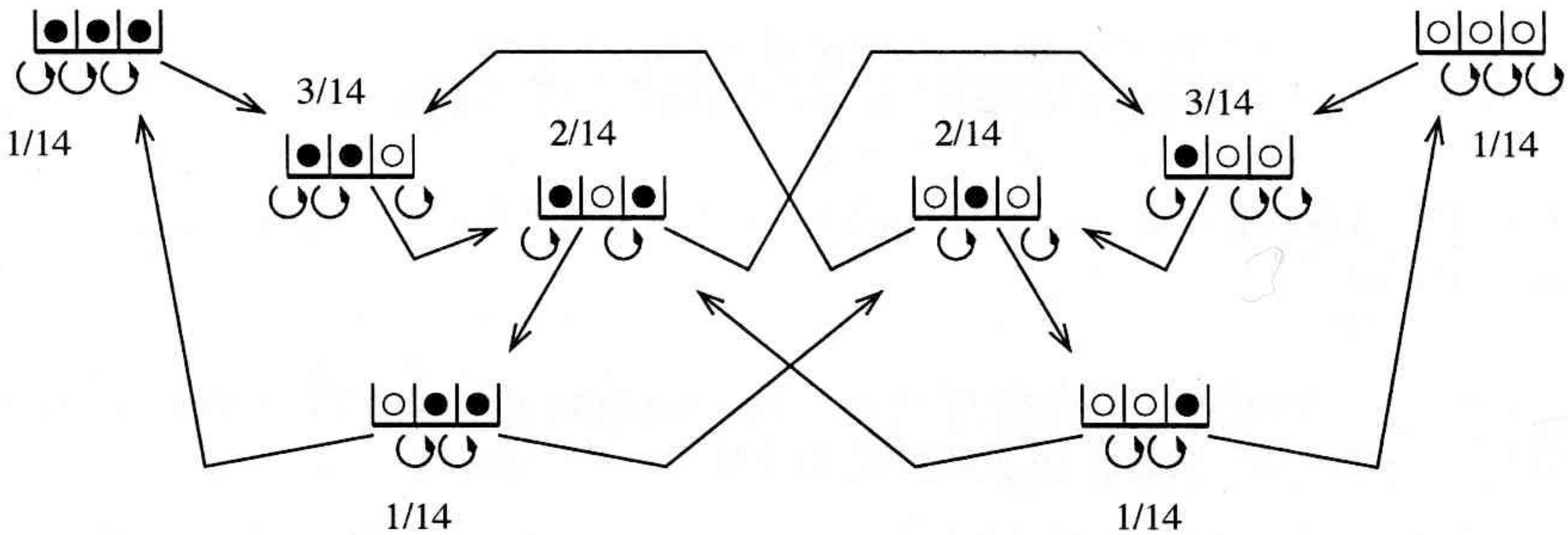
$\pi \cdot M = \pi$

unicity

eigenvector $\qquad M^T \qquad$ eigenvalue 1

time $\longrightarrow \infty$

$$p_v = \sum_u p_u \, p_{u,v}$$



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non-equilibrium statistical mechanics

... relaxation → stationary state

states $\tau = (\tau_1, \tau_2, \dots, \tau_n)$

$$\tau_i = \begin{cases} 1 & \text{site } i \text{ occupied} \\ 0 & \text{site } i \text{ empty} \end{cases}$$

unique stationary state $\frac{d}{dt} P_n(\tau_1, \dots, \tau_n) = 0$

Derrida, Evans, Hakim, Pasquier (1993)

boundary induced phase transitions

molecular diffusion

linear array of enzymes

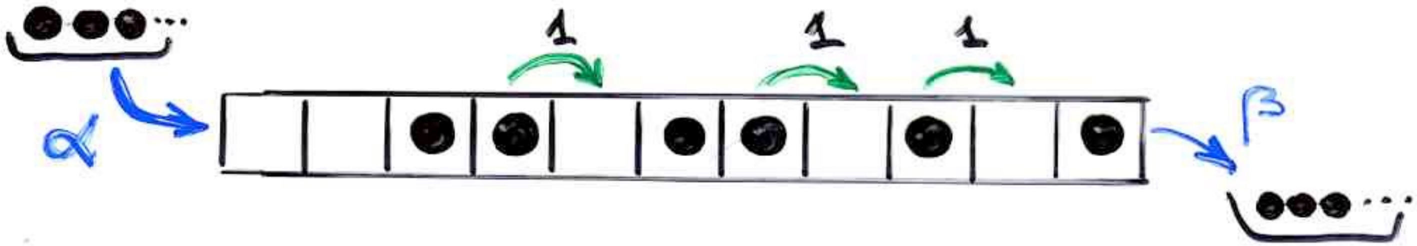
biopolymers

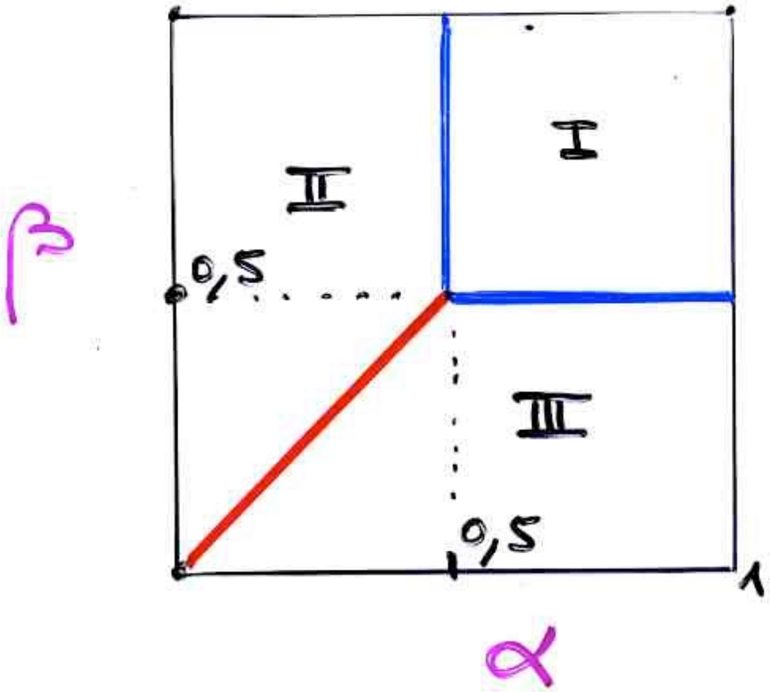
traffic flow

------ formation of shocks

TASEP

"totally asymmetric exclusion process"





$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1 - \beta$

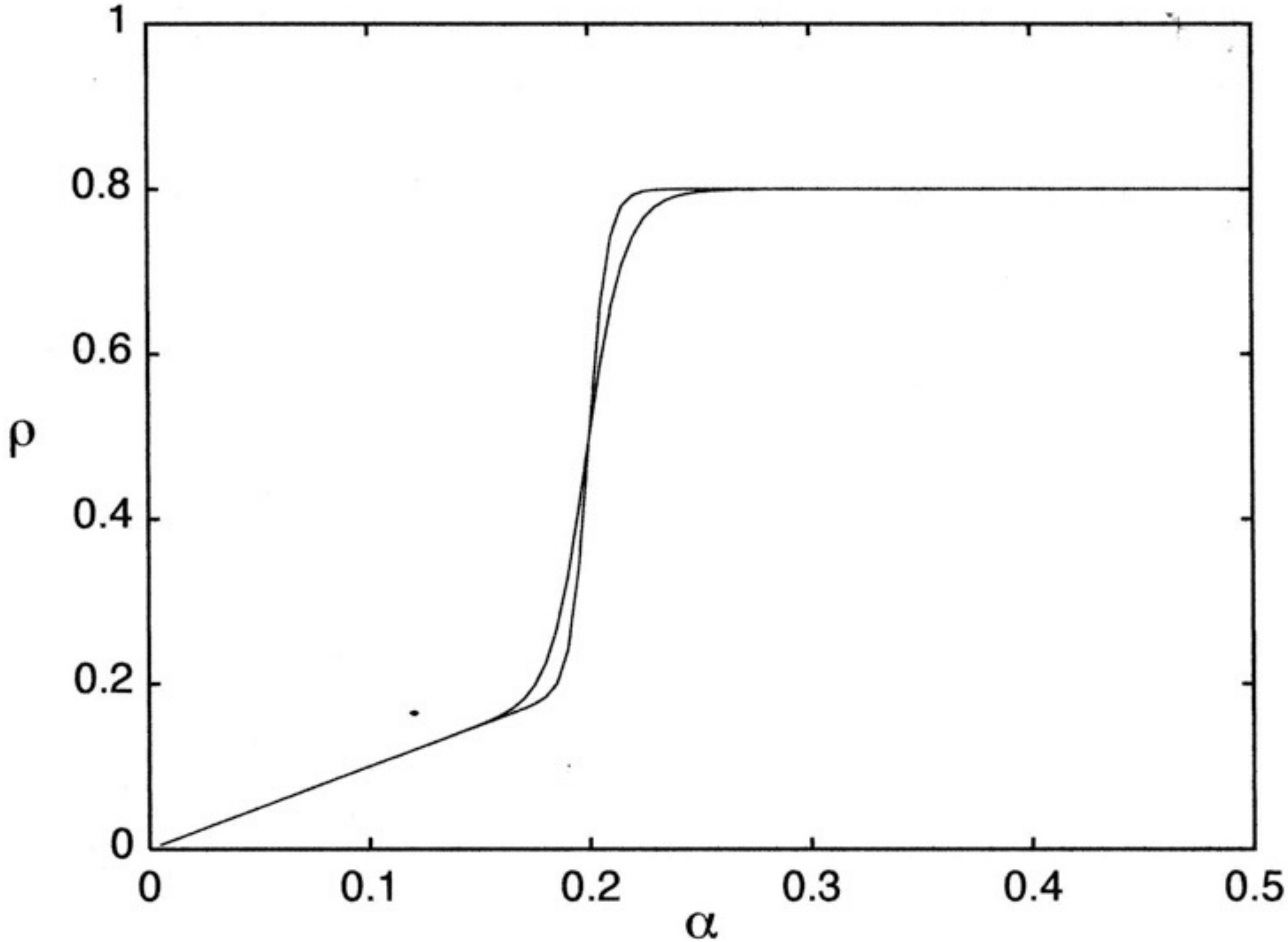
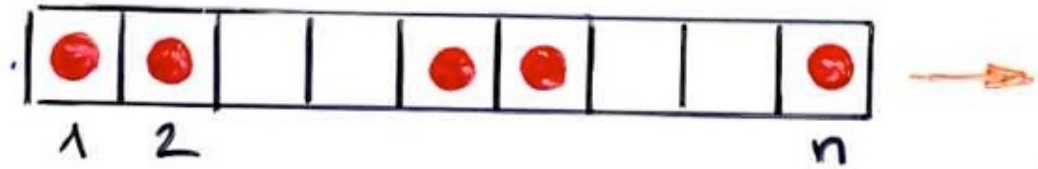
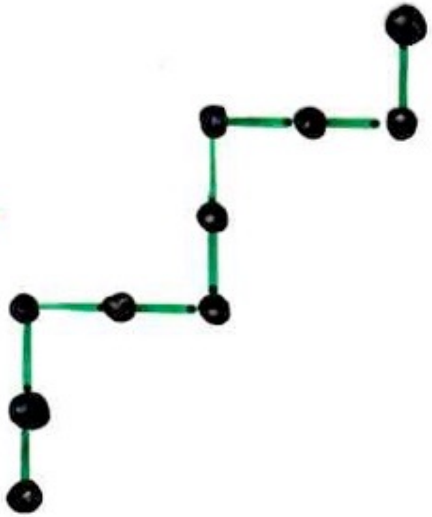


Figure 2: The average occupation $\rho = \langle \tau_{(N+1)/2} \rangle$ of the central site versus α for $N = 61$ and $N = 121$ when $\beta = .2$.

state $s = (\tau_1, \dots, \tau_n)$

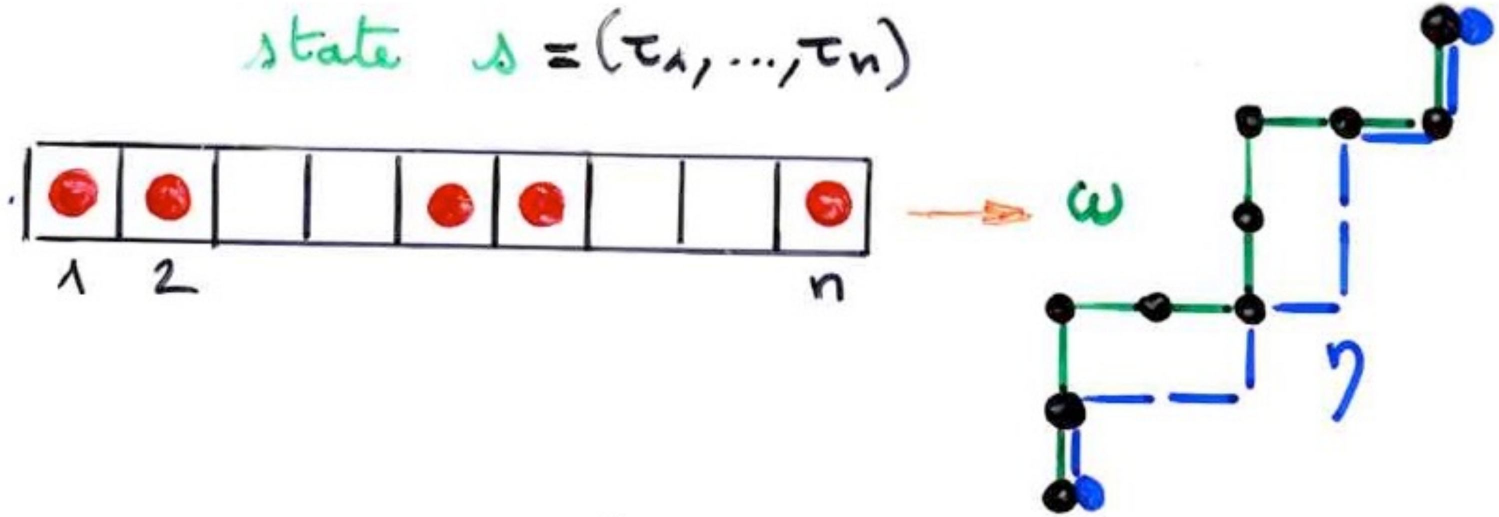




$\mathcal{P}_n(s) =$

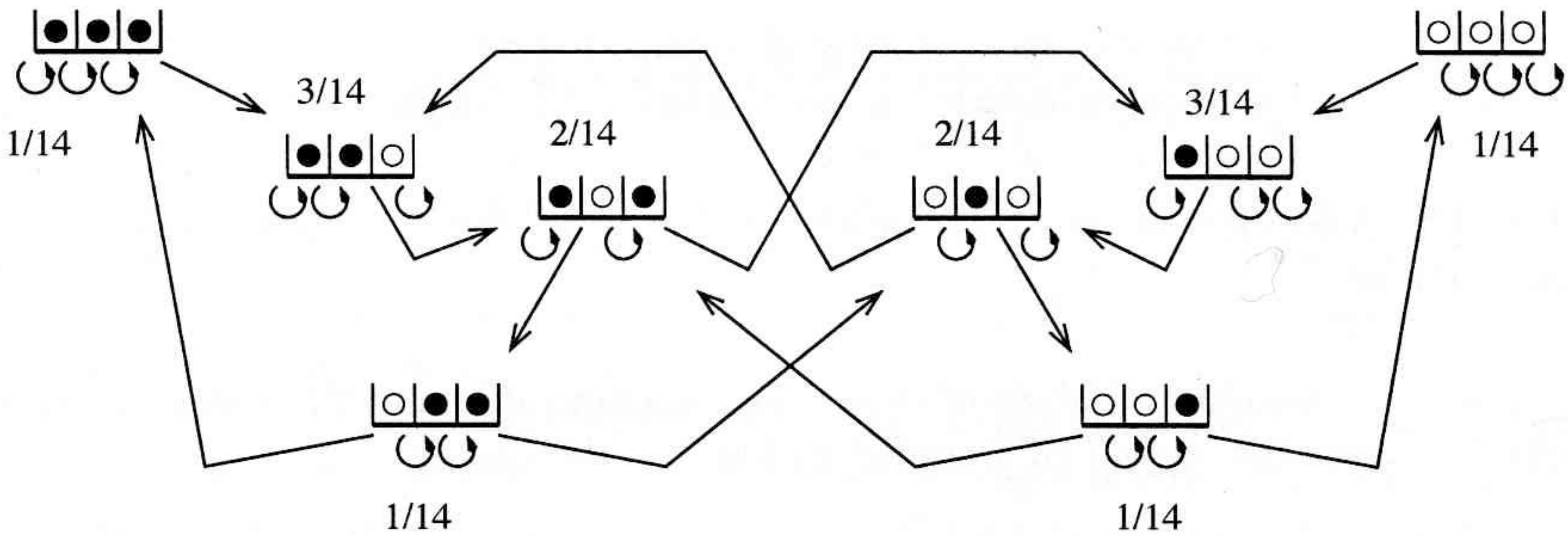
Shapiro, Zeilberger, 1982

state $s = (\tau_1, \dots, \tau_n)$



$$\mathcal{P}_n(s) = \frac{1}{C_{n+1}} \left(\text{number of paths } \eta \text{ below the path } \omega \text{ associated to } s \right)$$

Shapiro, Zeilberger, 1982



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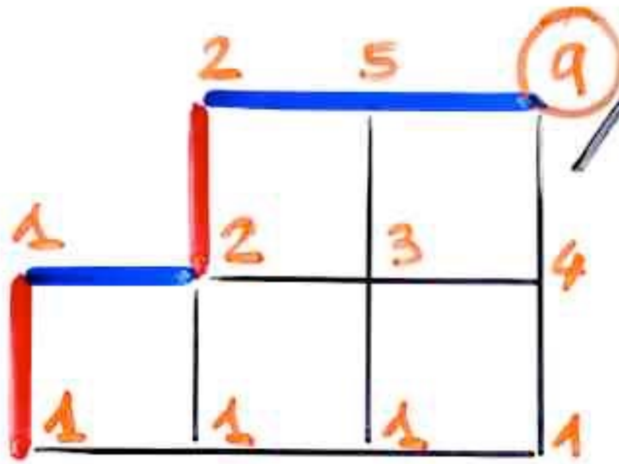
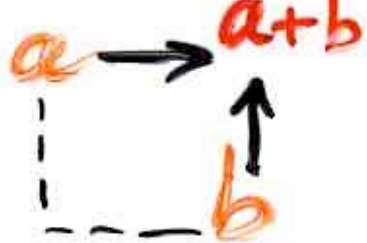
$s = (1,0,1,0,0)$ $\lambda = (1,2,2)$



$9 / C_6 = P(1,0,1,0,0)$



Combinatorics of the PASEP

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV, (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006) (2008) (2009) XGV, (2008)
Corteel, Stanton, Stanley, Williams (2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^+$$

$$\hat{a}\,\hat{a}^+ - q\,\hat{a}^+\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier 1993

$$\mathcal{P}_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = q\,ED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then

$f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

TASEP

q=0

$$\begin{cases} DE = \square + D + E \\ (\beta D - \square)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \square) = \langle W| \end{cases}$$

Then

$$f_n(\tau_1, \dots, \tau_n) = \langle W| \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) |V\rangle$$

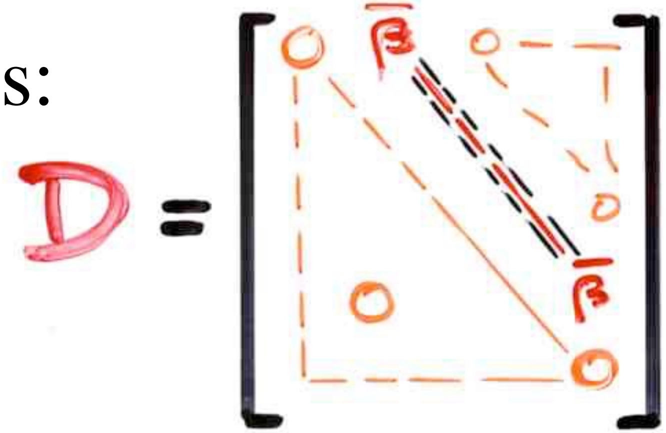
examples:

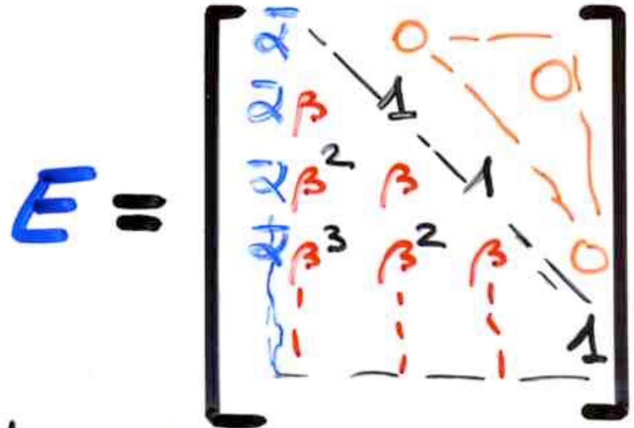
TASEP

$$\begin{cases} DE = D + E \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{cases}$$

examples:

TASEP

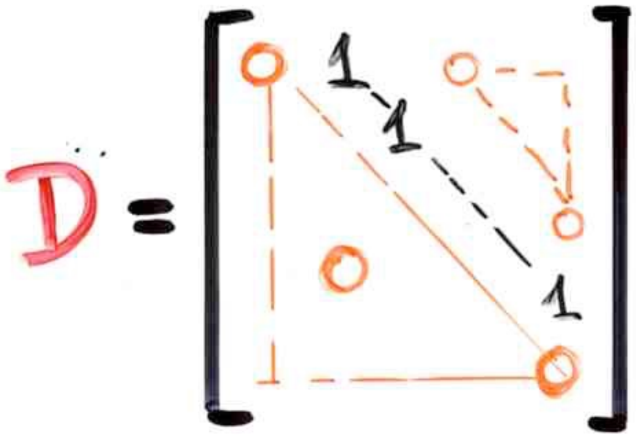


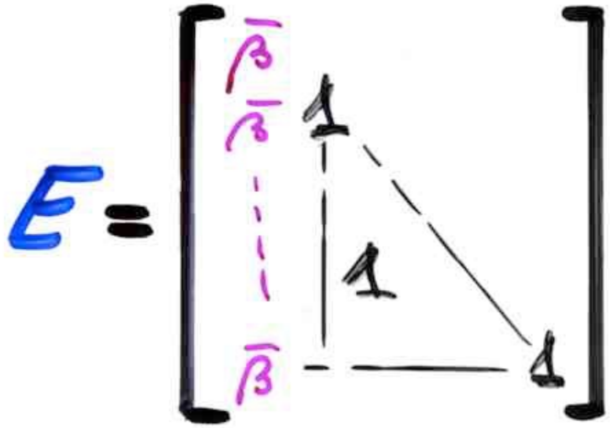


(infinite matrices)

$\bar{\beta} = \frac{1}{\beta}$, $\bar{\alpha} = \frac{1}{\alpha}$

$\langle W| = (1, 0, \ldots 0, \ldots)$ $\qquad$ $|V\rangle = (1, 1, \ldots 1, \ldots)^T$





(infinite matrices)

$\bar{\beta} = \frac{1}{\beta}$, $\bar{\alpha} = \frac{1}{\alpha}$

$\langle W| = (1, 0, \ldots)$ $\qquad$ $|V\rangle = (1, \bar{\alpha}, \bar{\alpha}^2, \ldots)^T$

examples:

TASEP

$$D = \begin{bmatrix} \bar{\beta} & \kappa & 0 & \cdots \\ 0 & 1 & 1 & \ddots \\ \vdots & \ddots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} \bar{\alpha} & 0 & \cdots & 0 \\ \kappa & 1 & \ddots & \vdots \\ 0 & 1 & 1 & 0 \\ 0 & \cdots & 1 & 1 \end{bmatrix}$$

(infinite matrices)

$$\langle W| = (1, 0, \ldots, 0) \qquad |V\rangle = (1, 0, \ldots, 0)$$

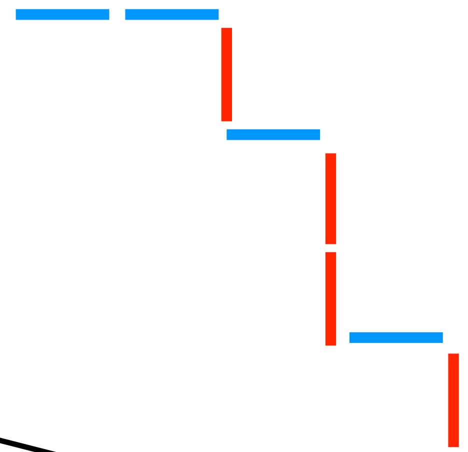
$$\bar{\alpha} = \frac{1}{\alpha} \qquad \bar{\beta} = \frac{1}{\beta} \qquad \kappa^2 = \bar{\alpha} + \bar{\beta} - \bar{\alpha}\bar{\beta}$$

The PASEP algebra

$$DE = qED + E + D$$

D D E D E E D E

D DE (D E) E D E



DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

$$DE = qED + E + D$$

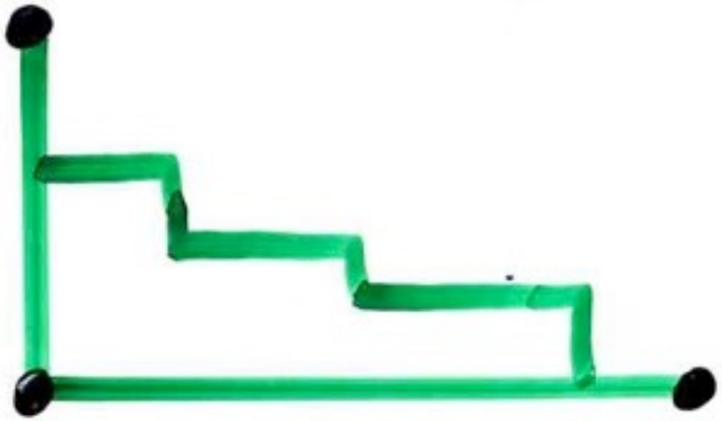
$$w(E,D) = \sum q^{k(T)} E^{i(T)} D^{j(T)}$$

unicity

alternative tableaux

alternative tableau

- Ferrers diagram F

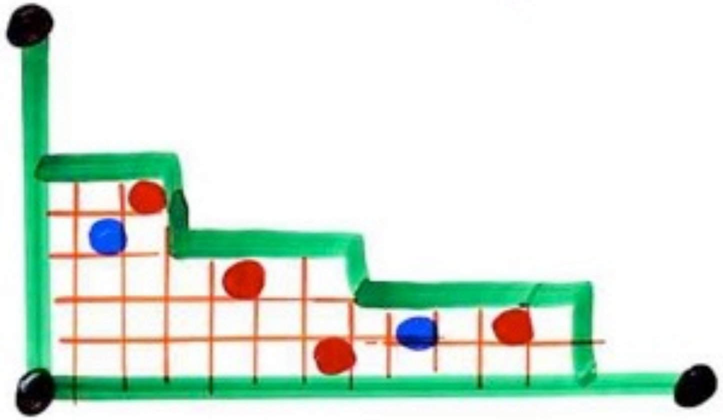


(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram F (possibly empty rows or column)

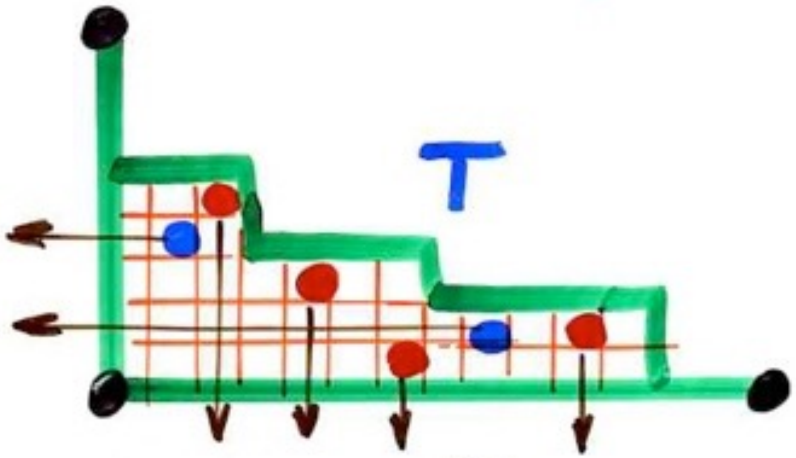


(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F (possibly empty rows or column)



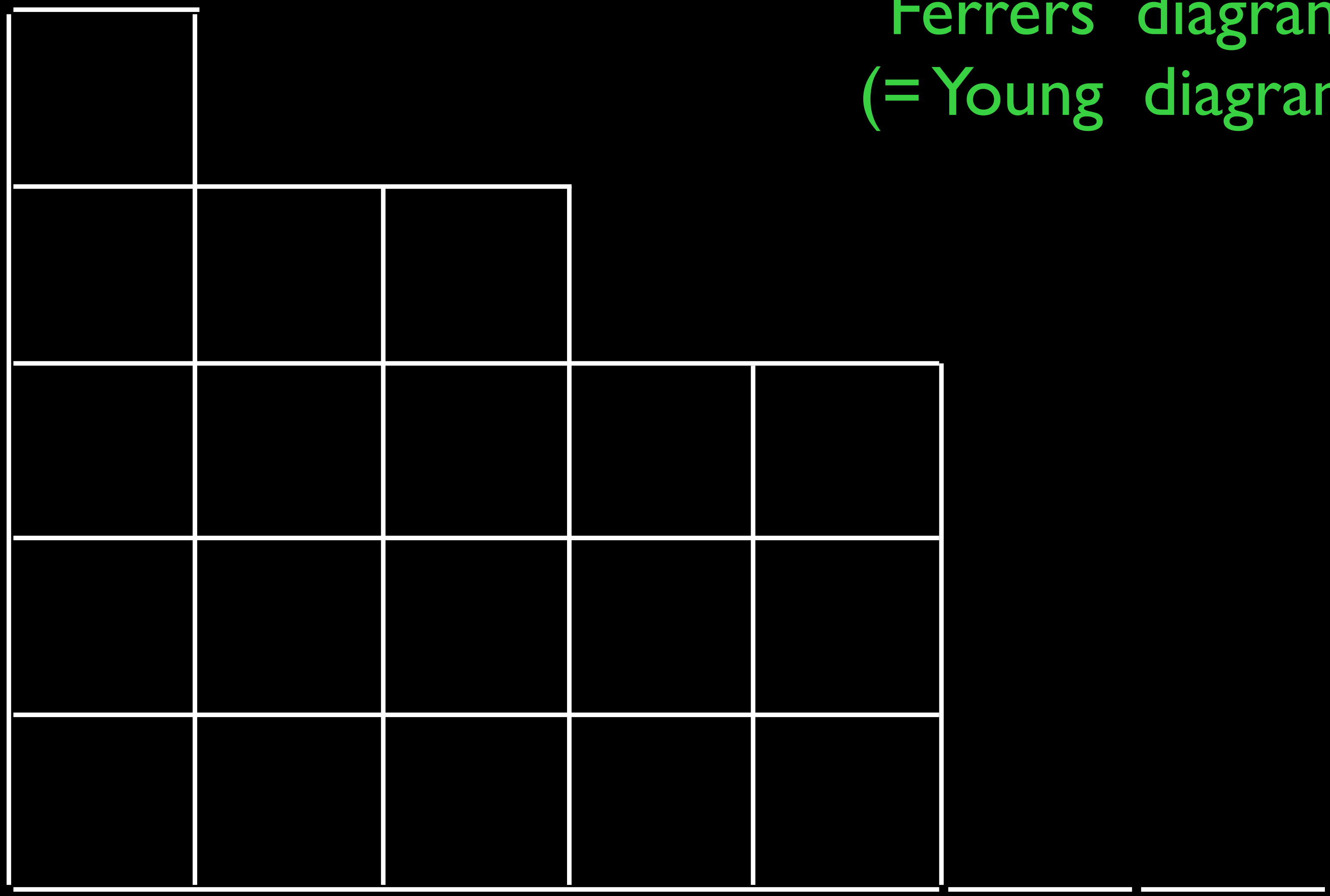
(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**
- { no coloured cell at the left of (blue cell)
  { no coloured cell below (red cell)

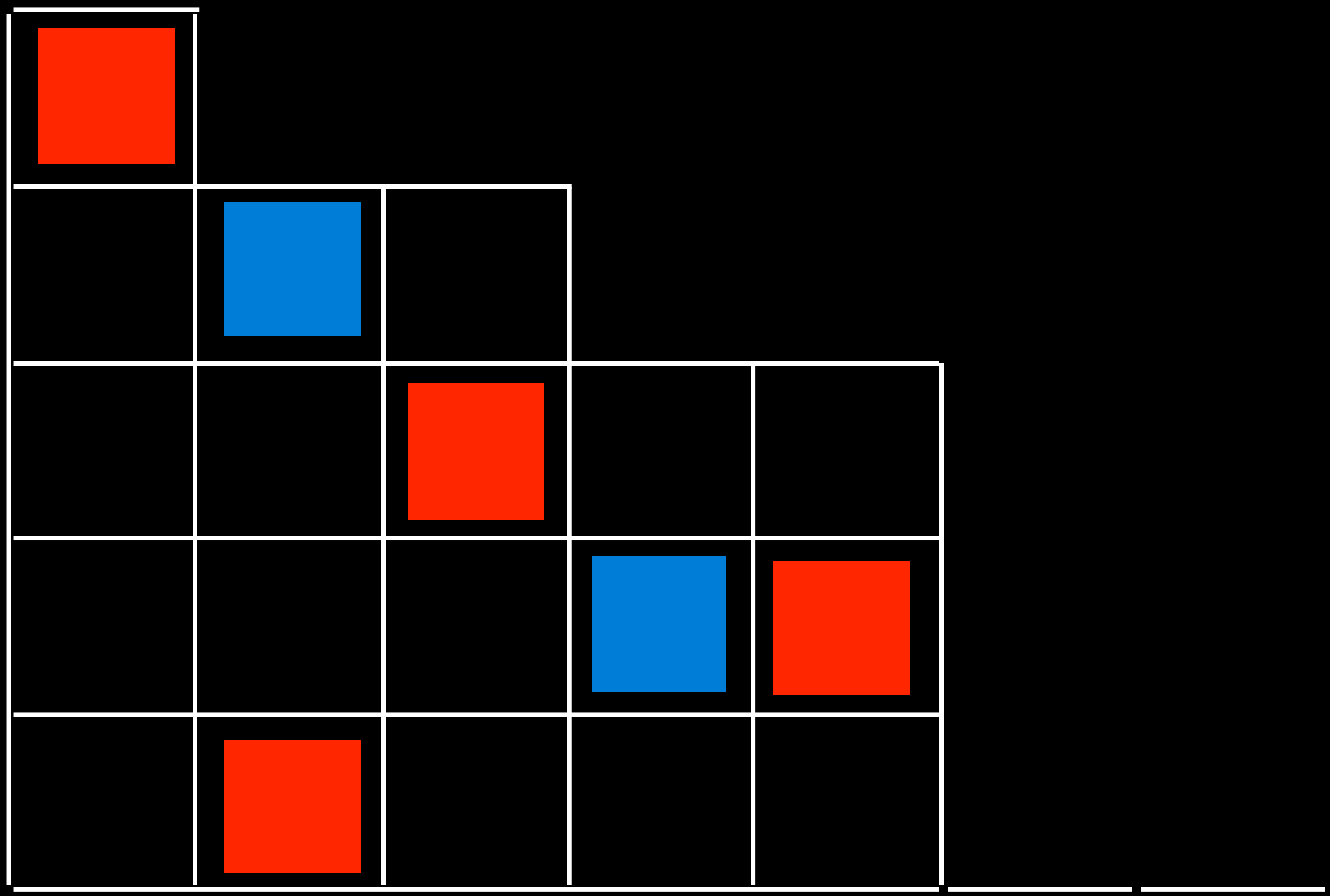
n size of T

alternative tableau

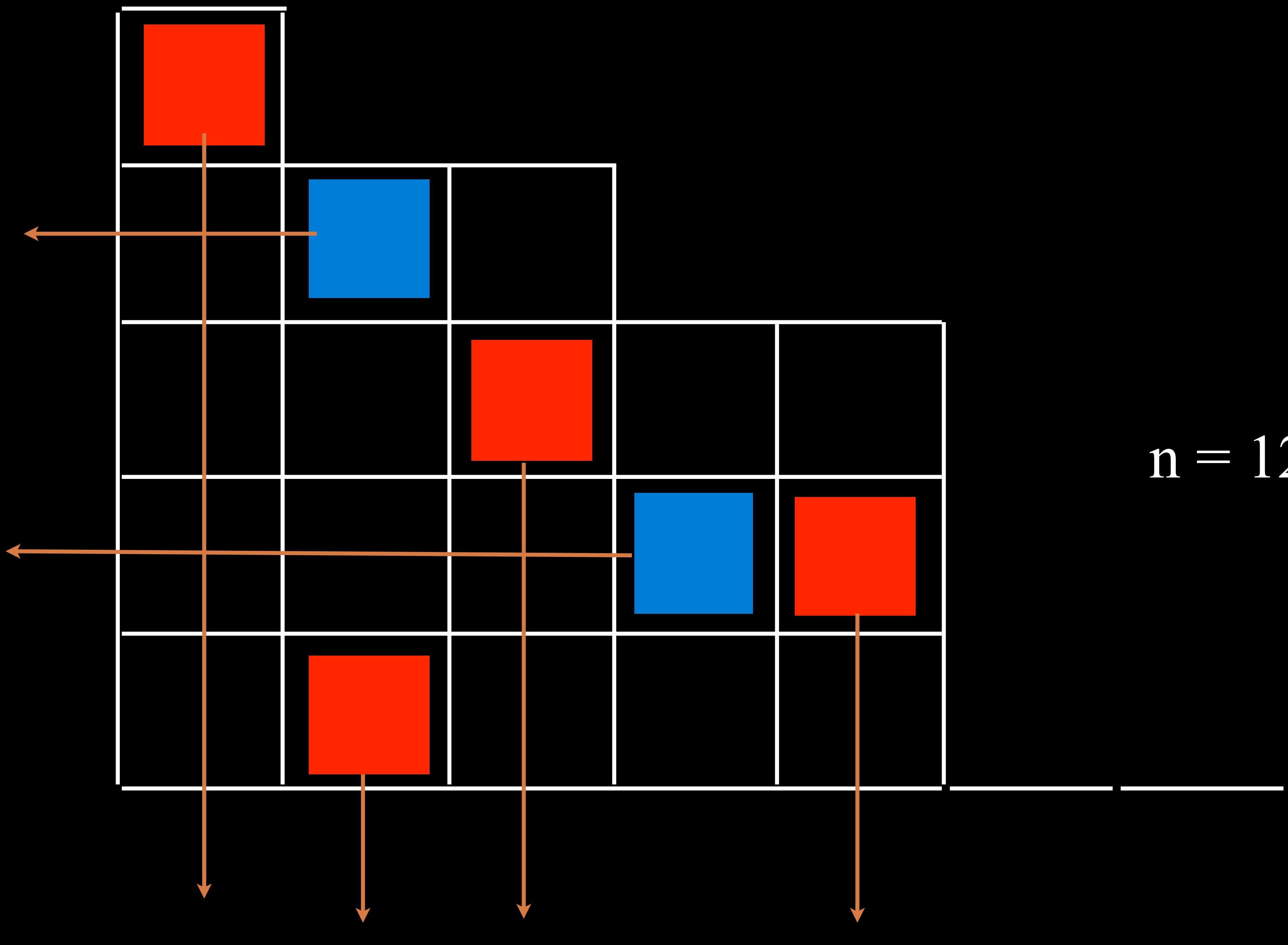
Ferrers diagram
(= Young diagram)



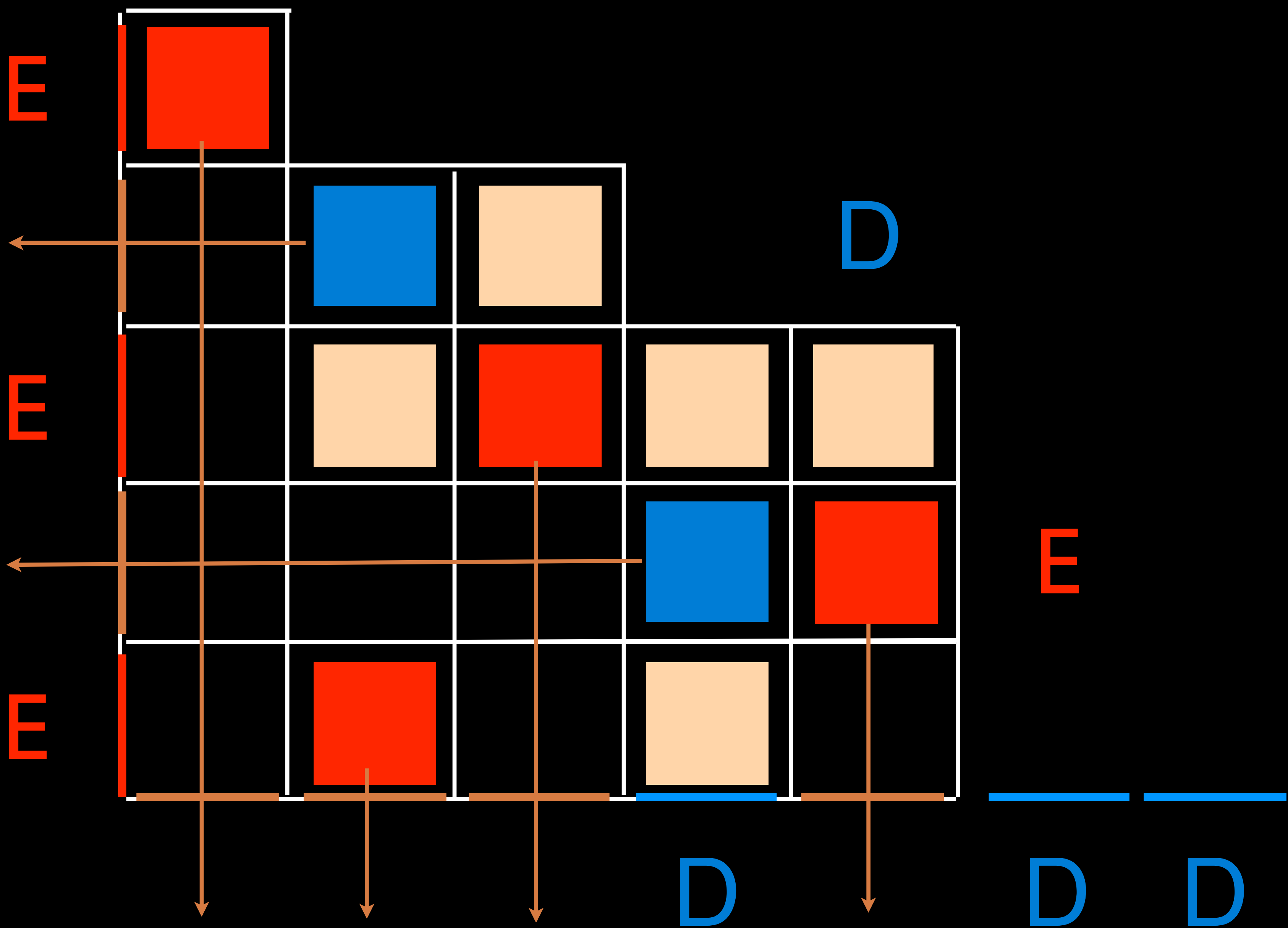
alternative tableau



alternative tableau



n = 12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D

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

unicity

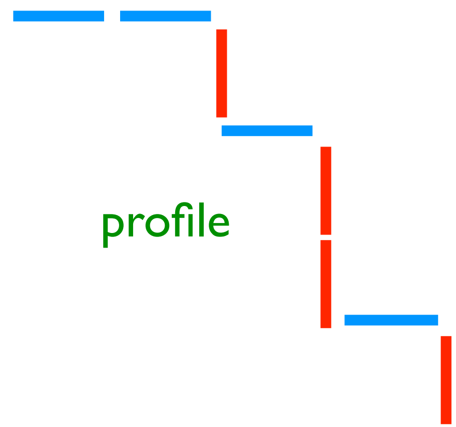
T alternative tableau with profile w

$k(T)$ = nb of $\square$

$i(T)$ = nb of rows without blue cell

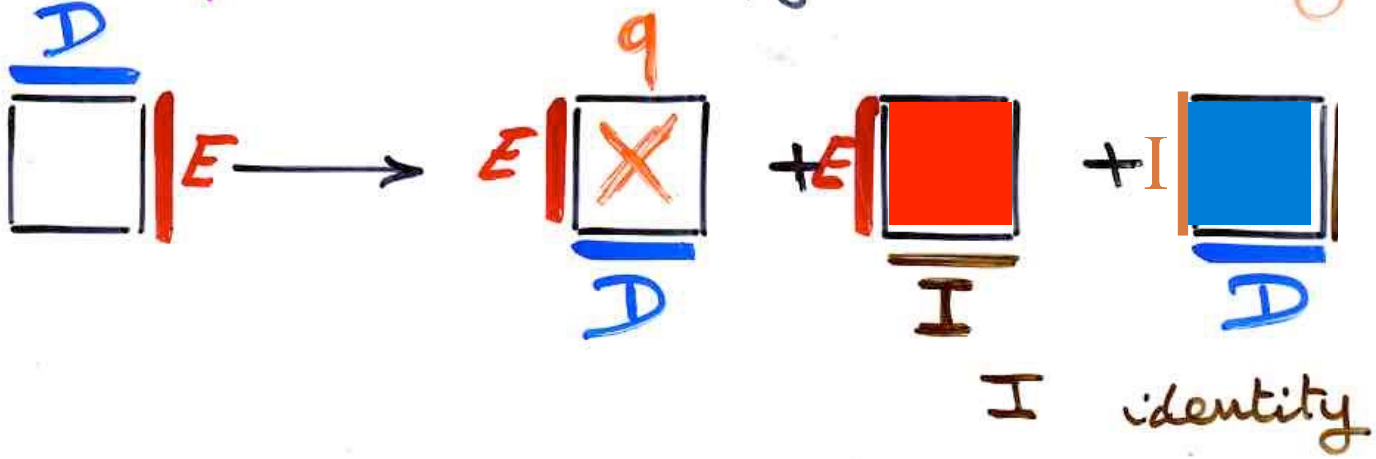
$j(T)$ = nb of columns without red cell

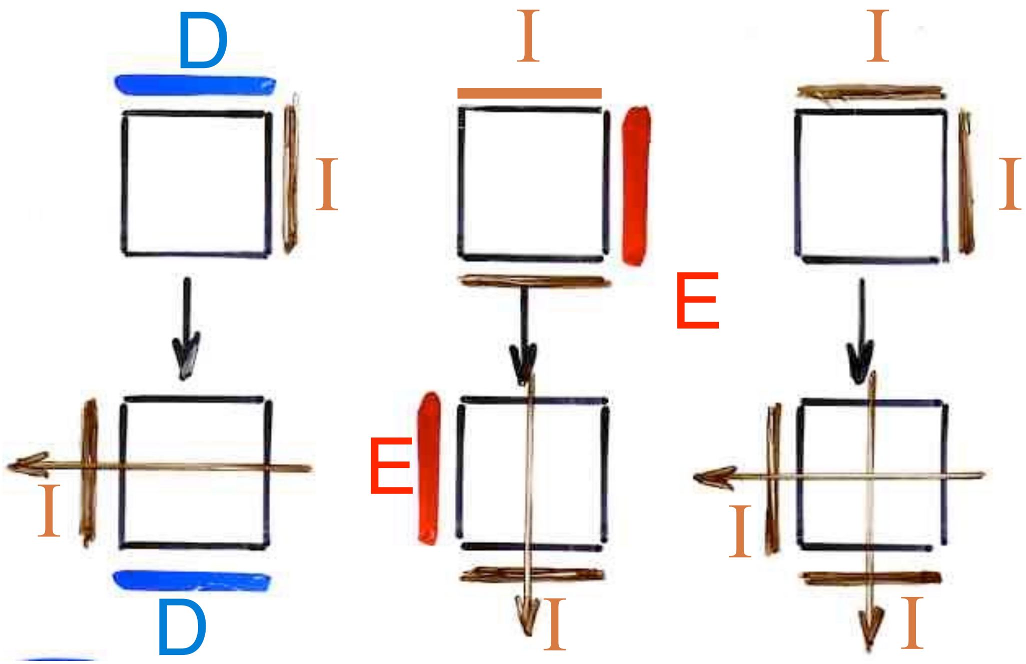
w = D D E D E E D E ⟶



idea of the proof

Proof: "planarization" of the rewriting rules



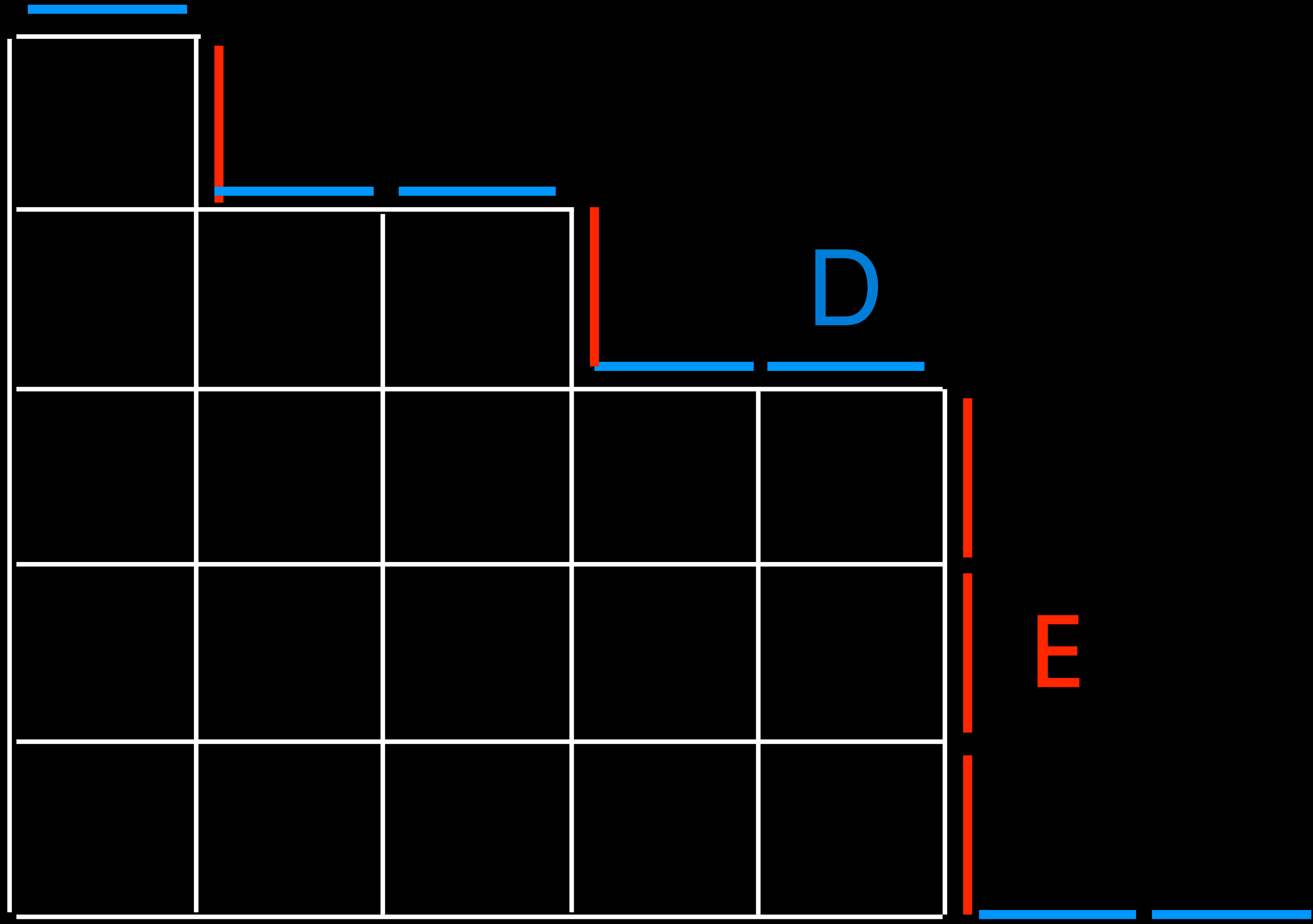


$$DE = qED + EI_h + I_vD$$

$$DI_v = I_vD$$

$$I_hE = EI_h$$

$$I_hI_v = I_vI_h$$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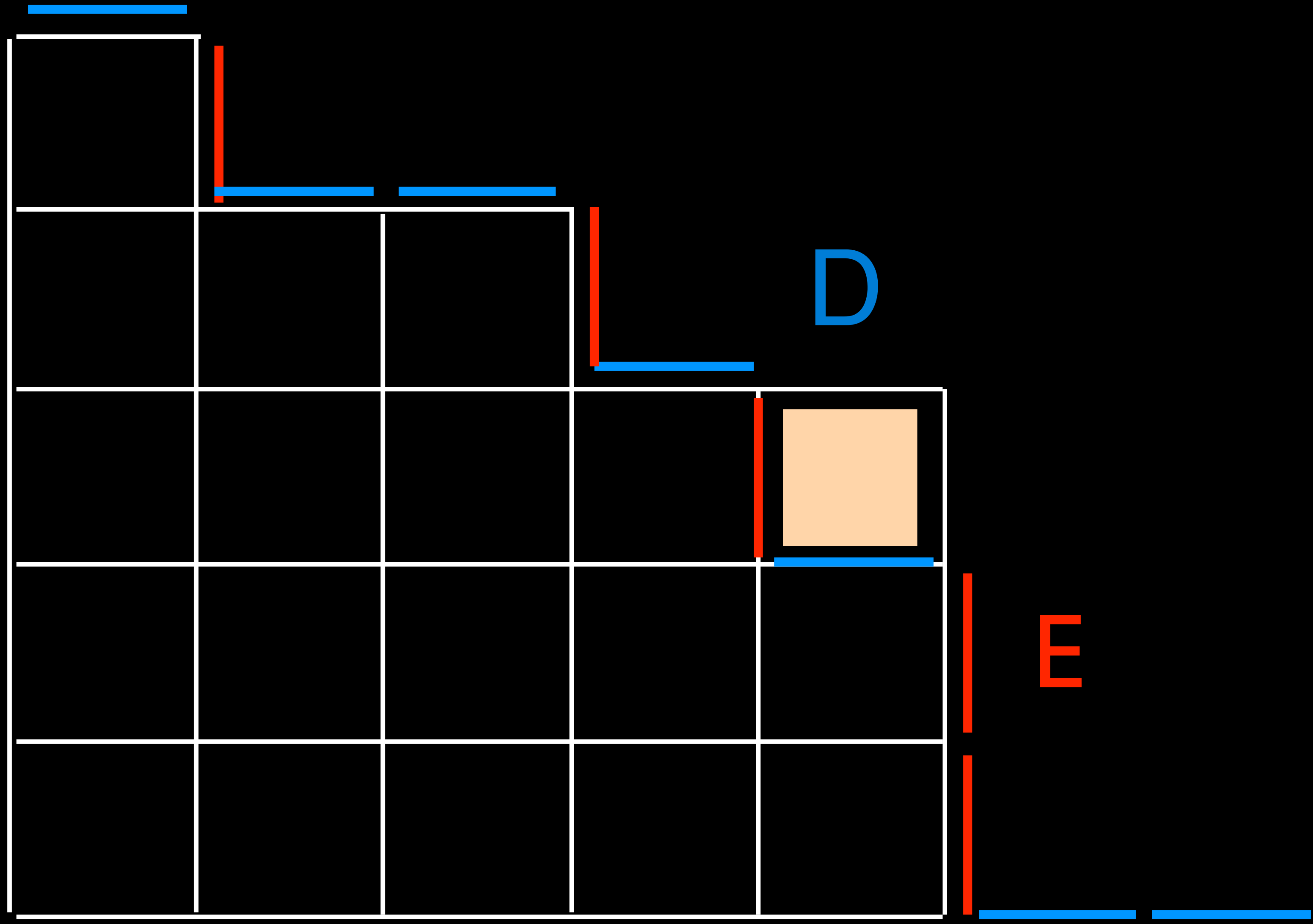
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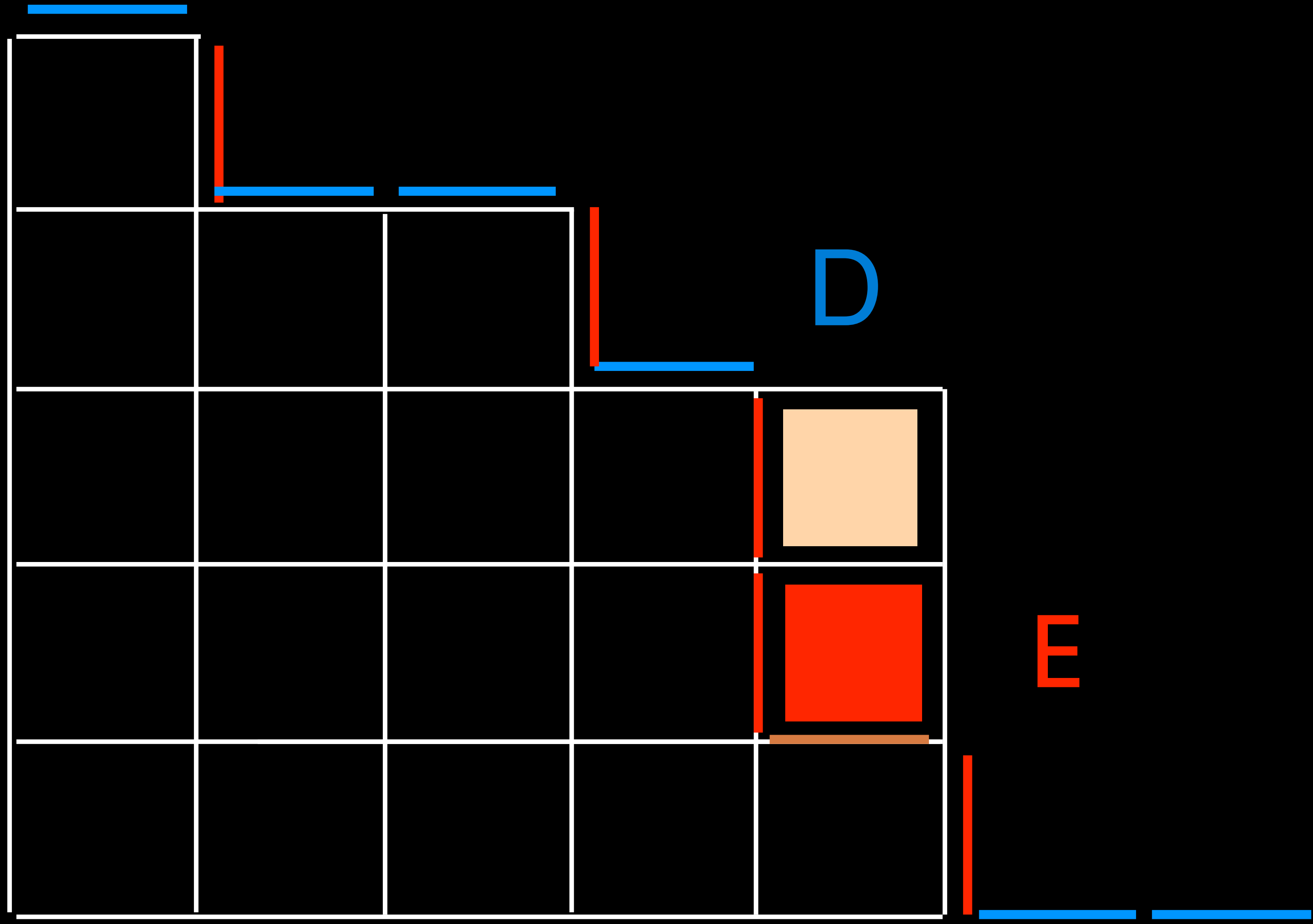
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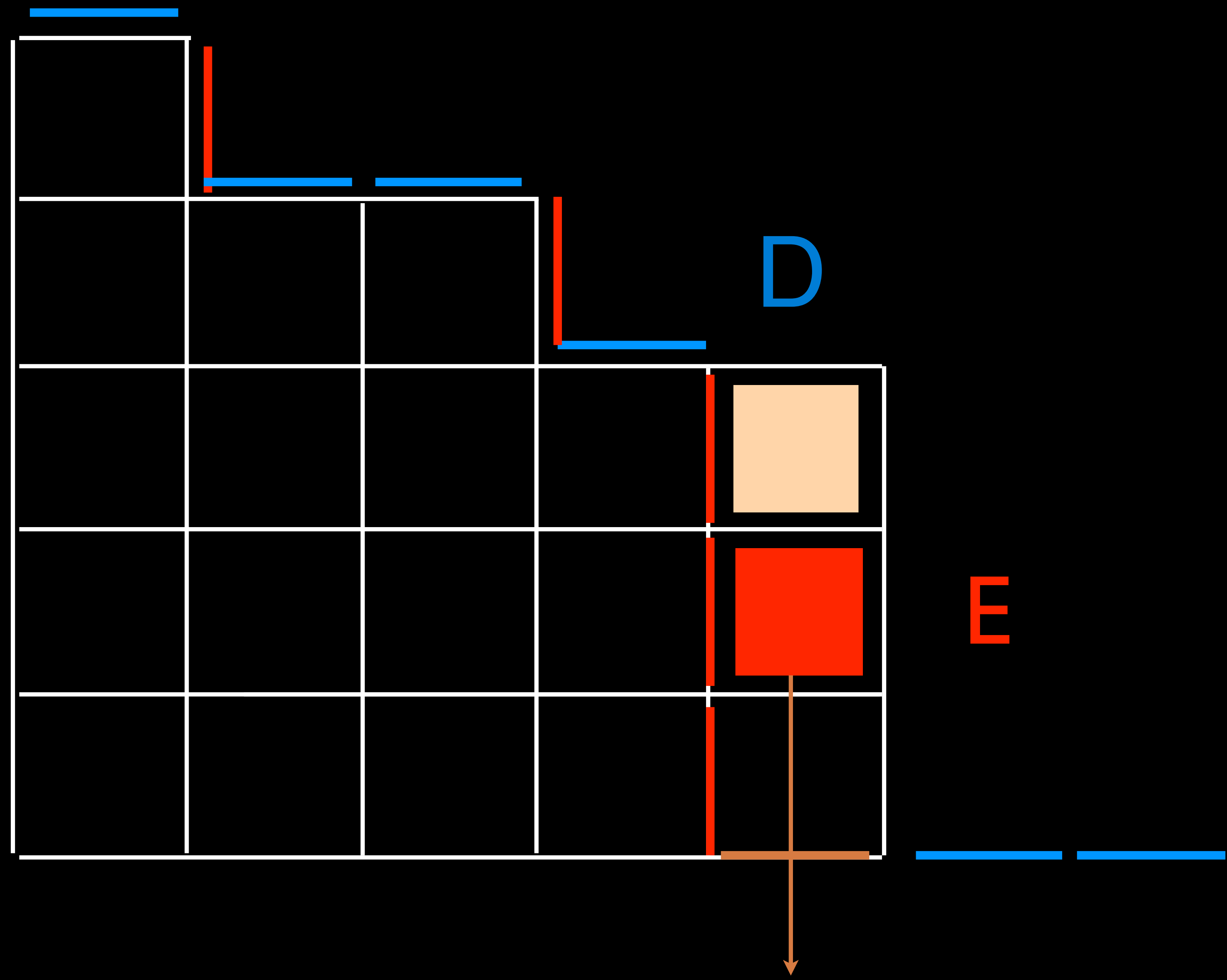
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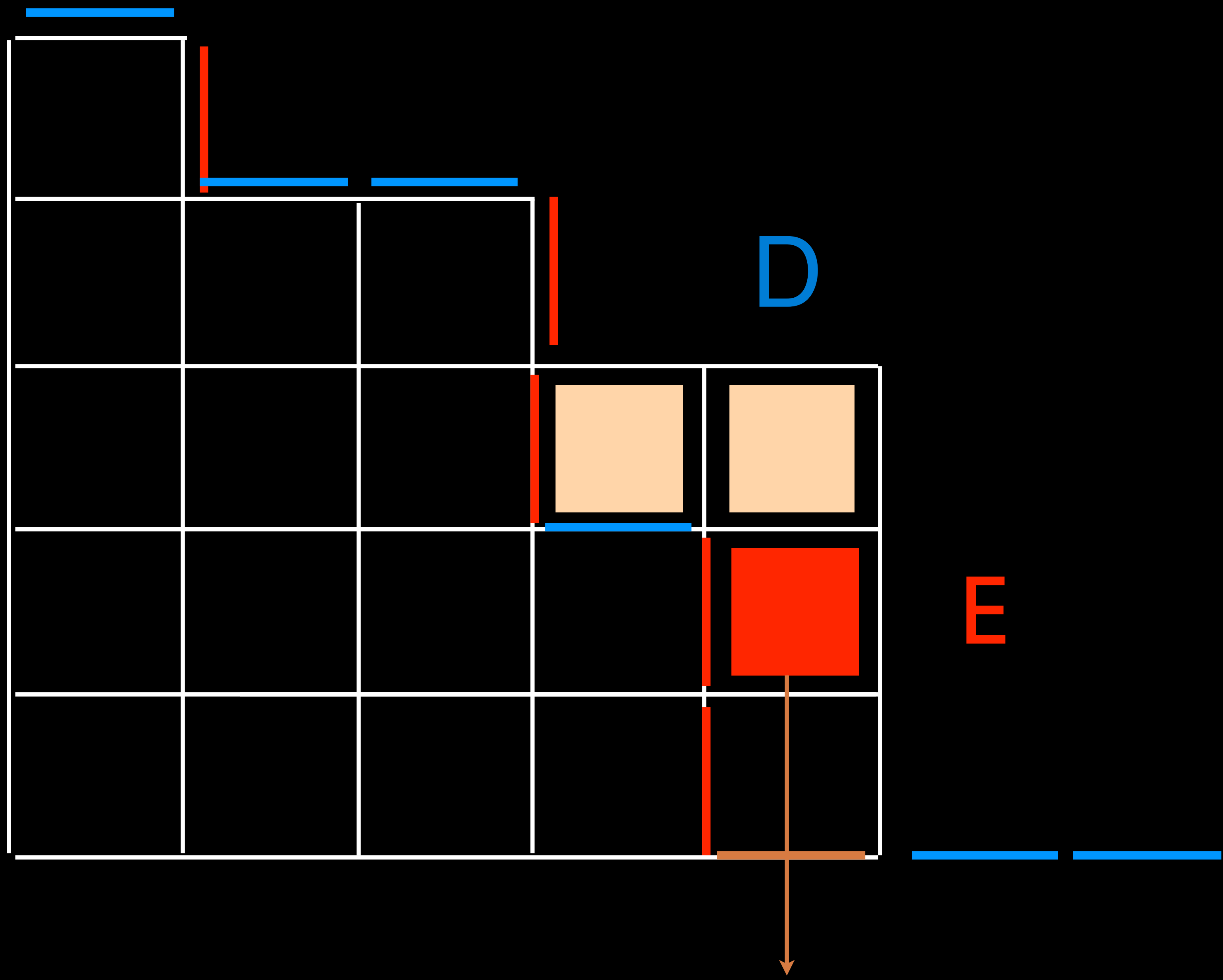
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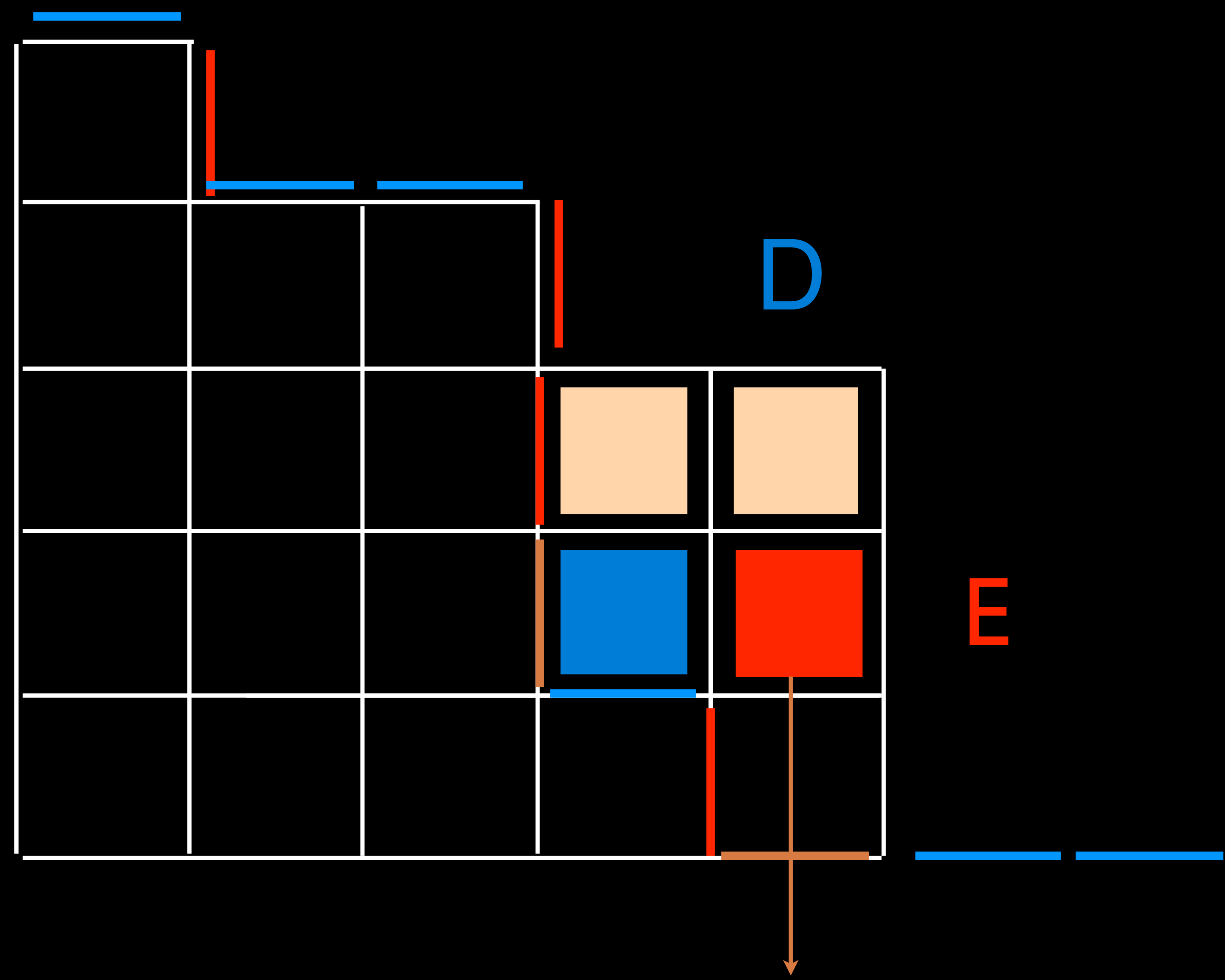
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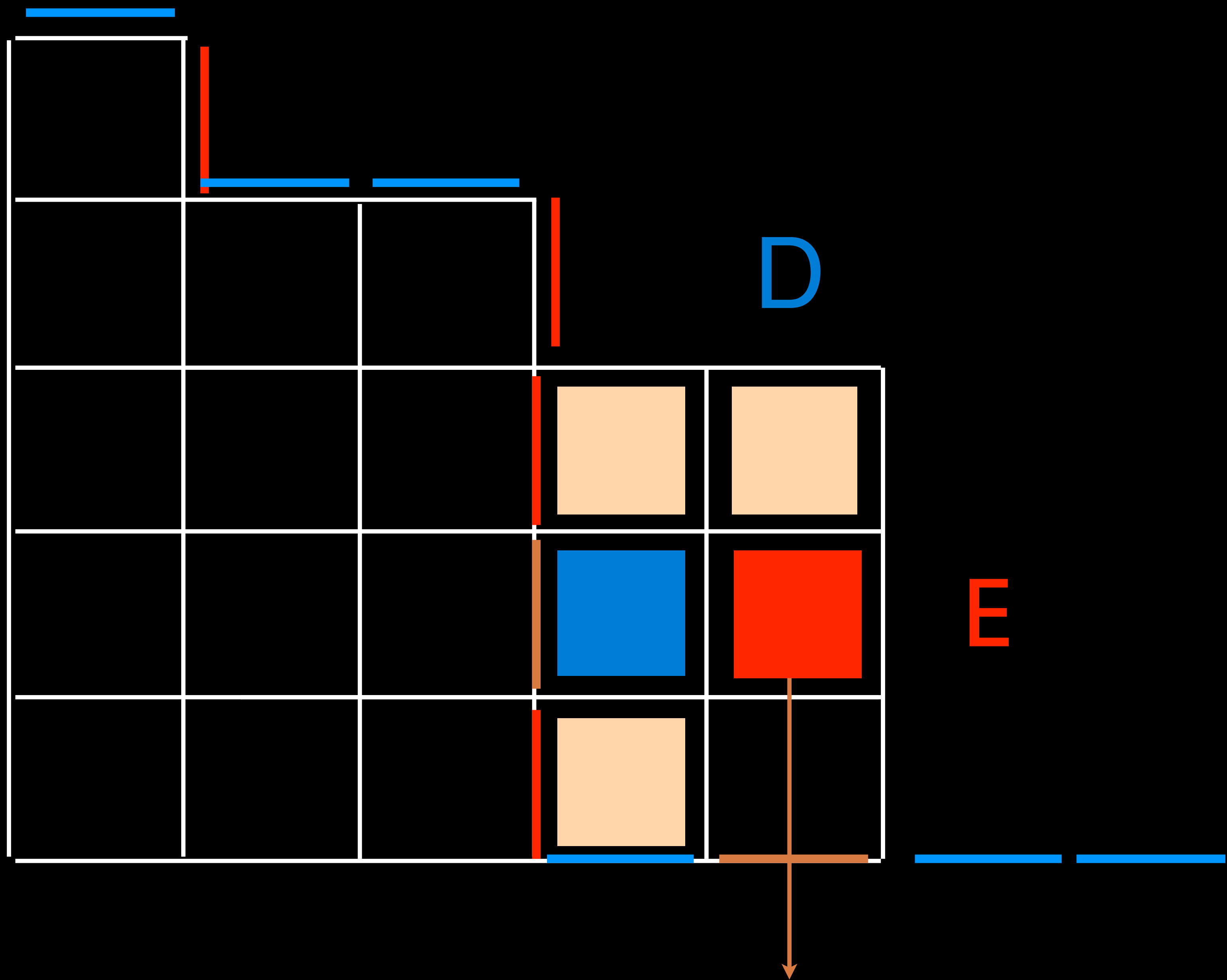

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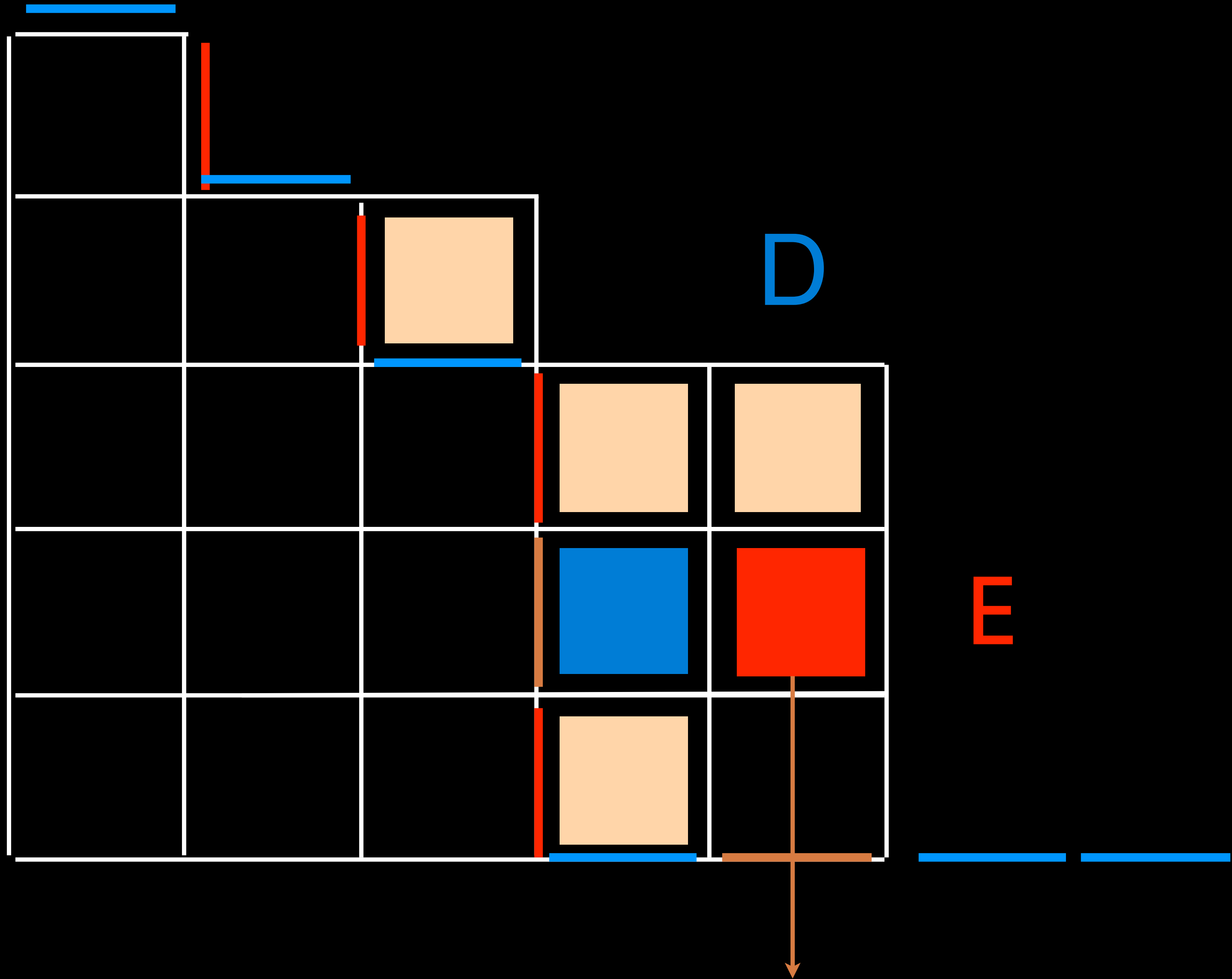
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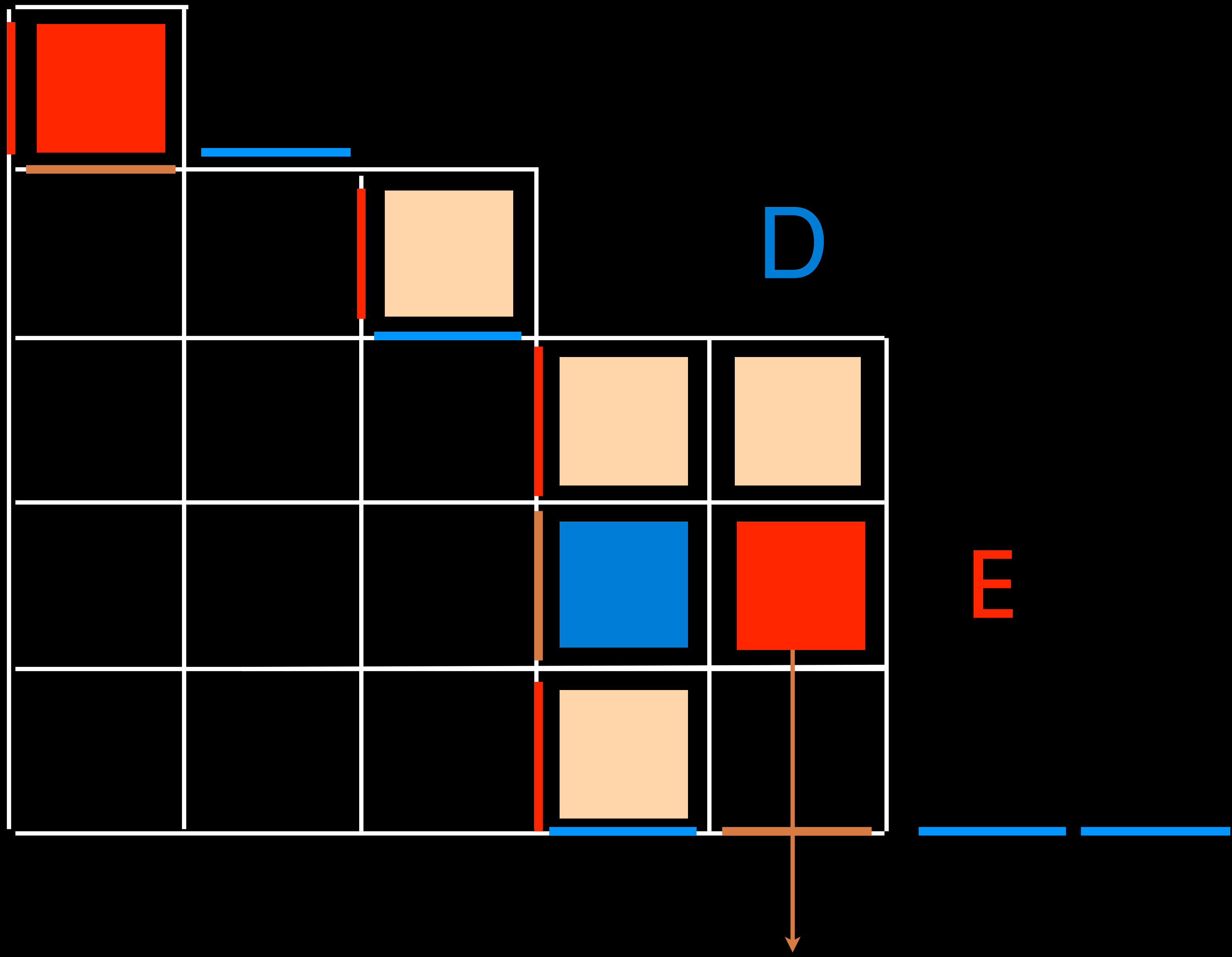
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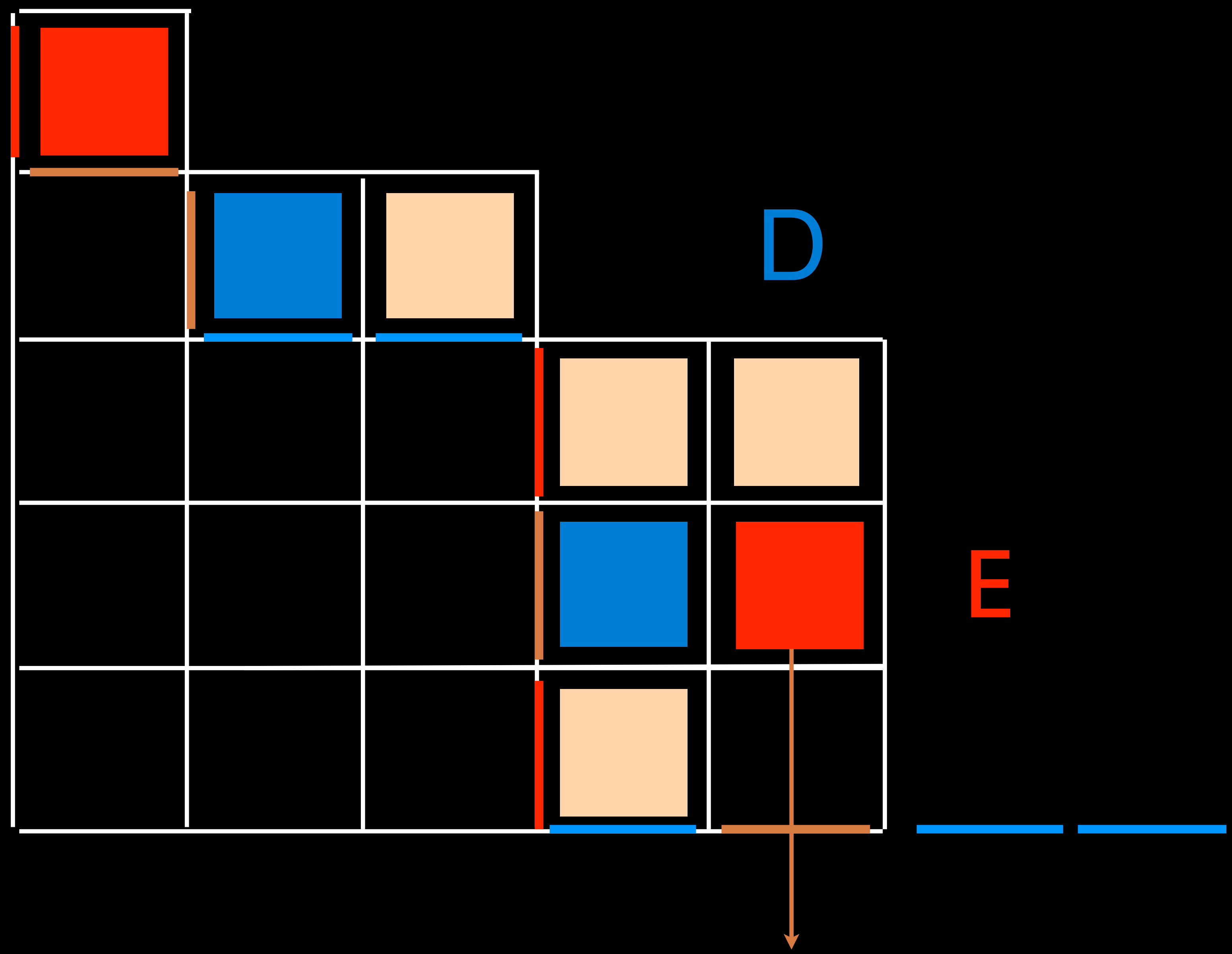
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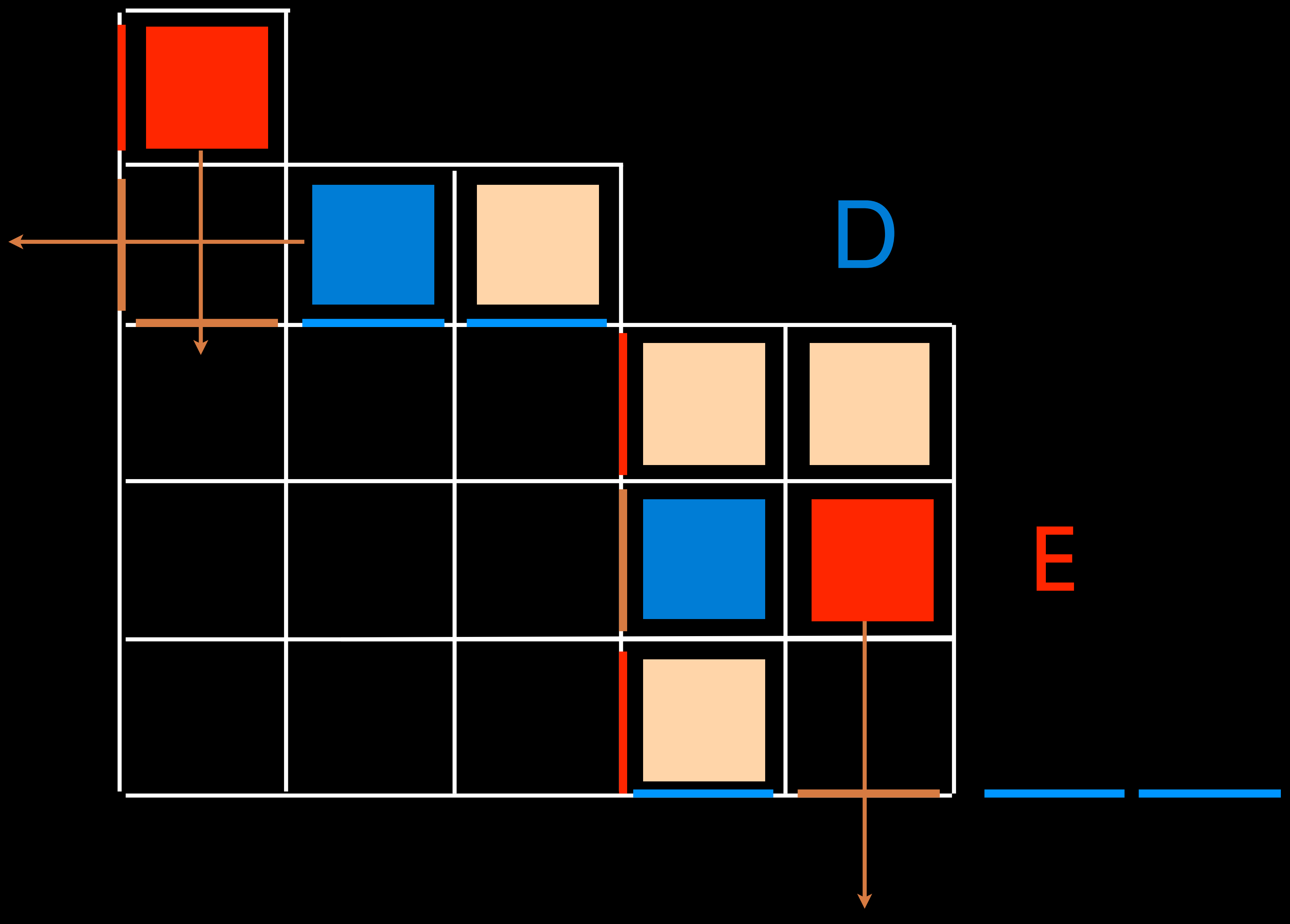

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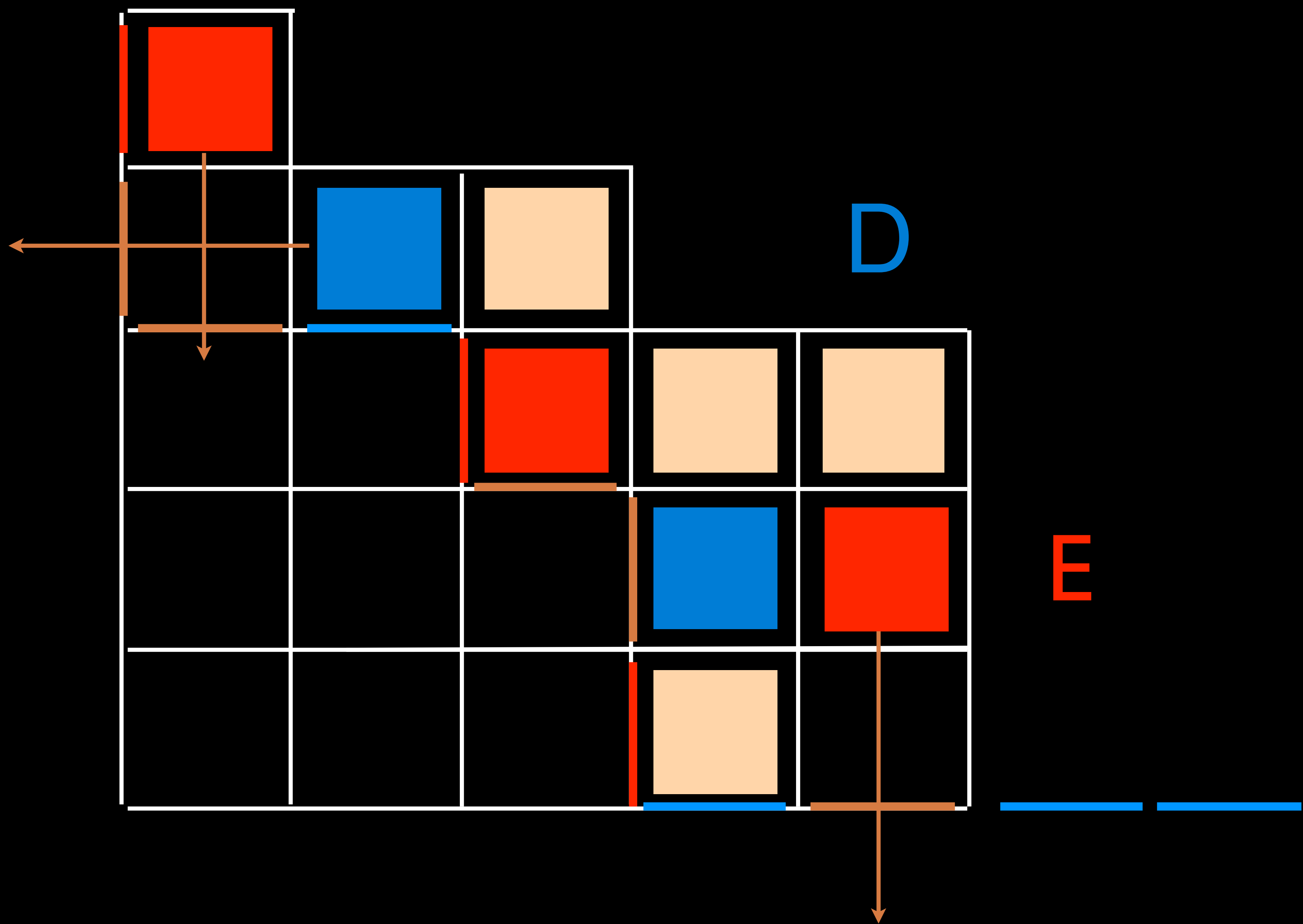
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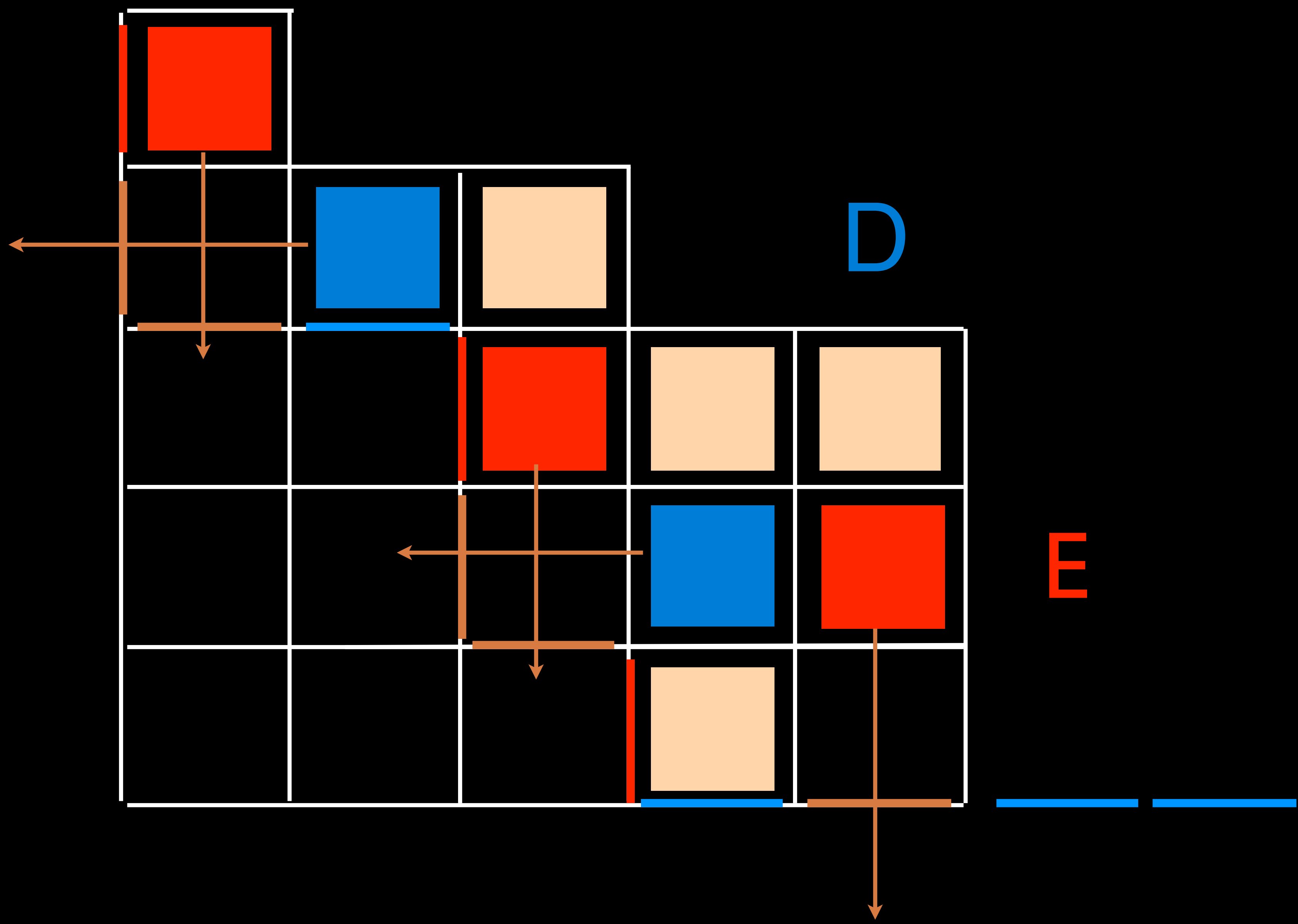

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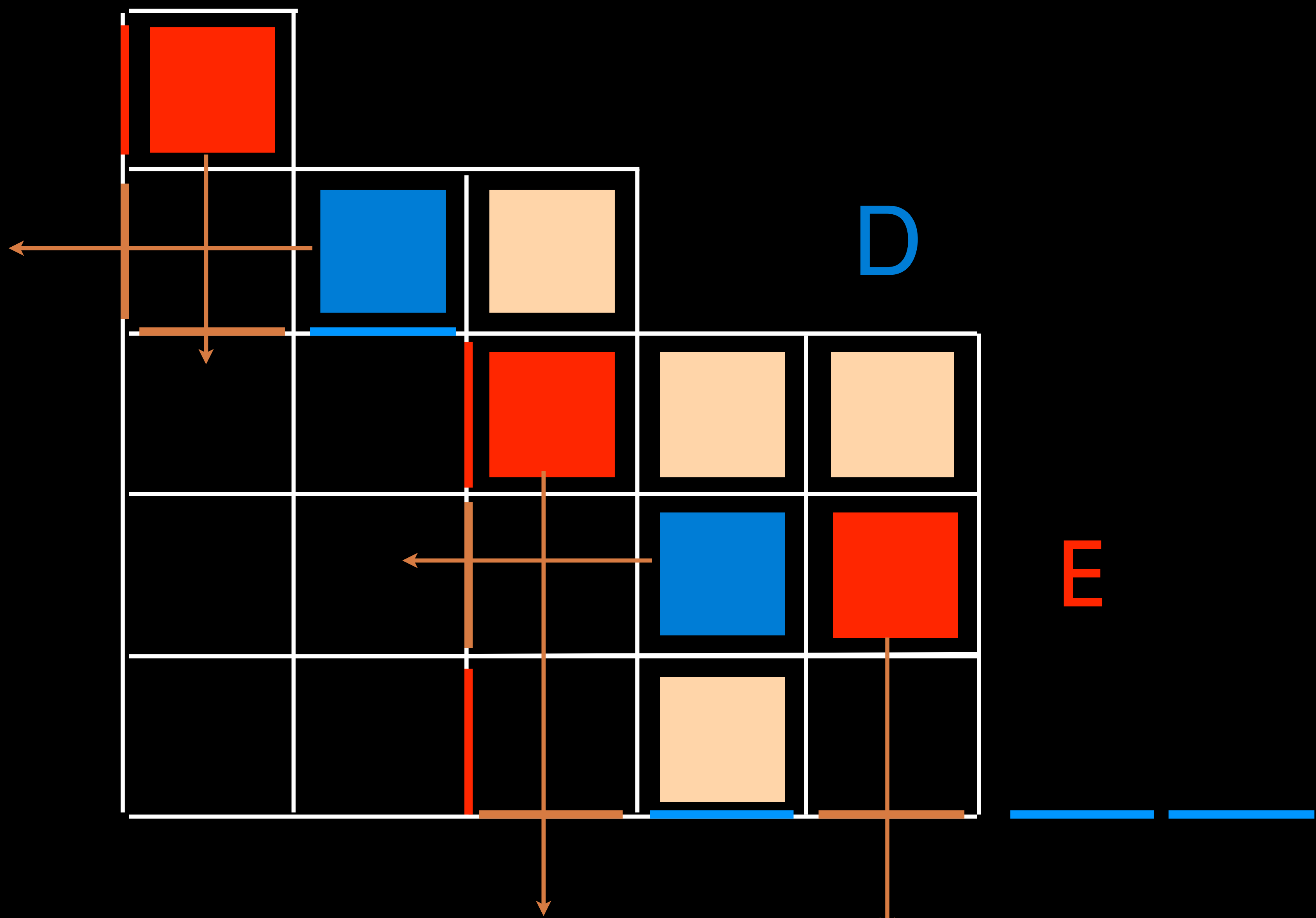
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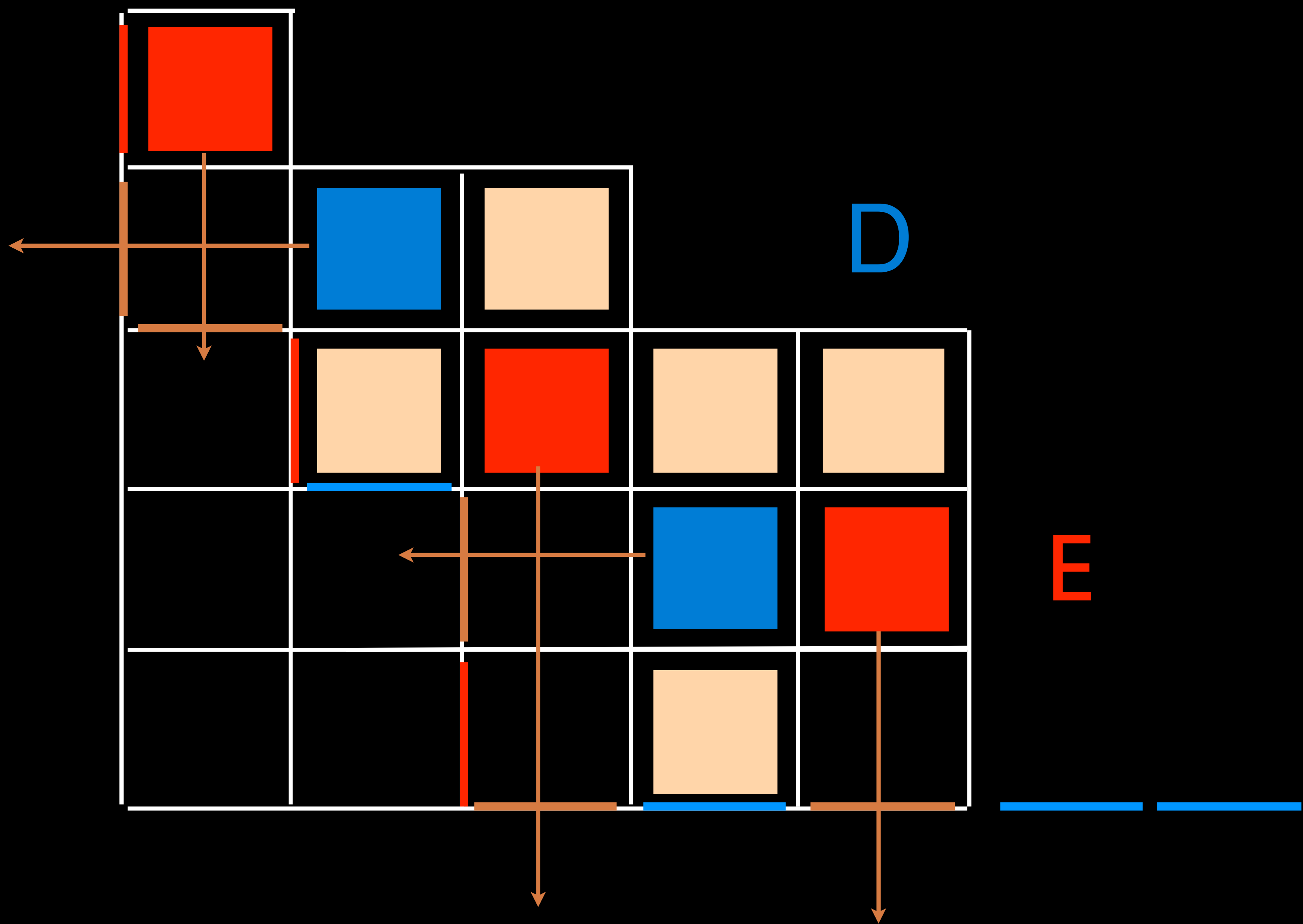
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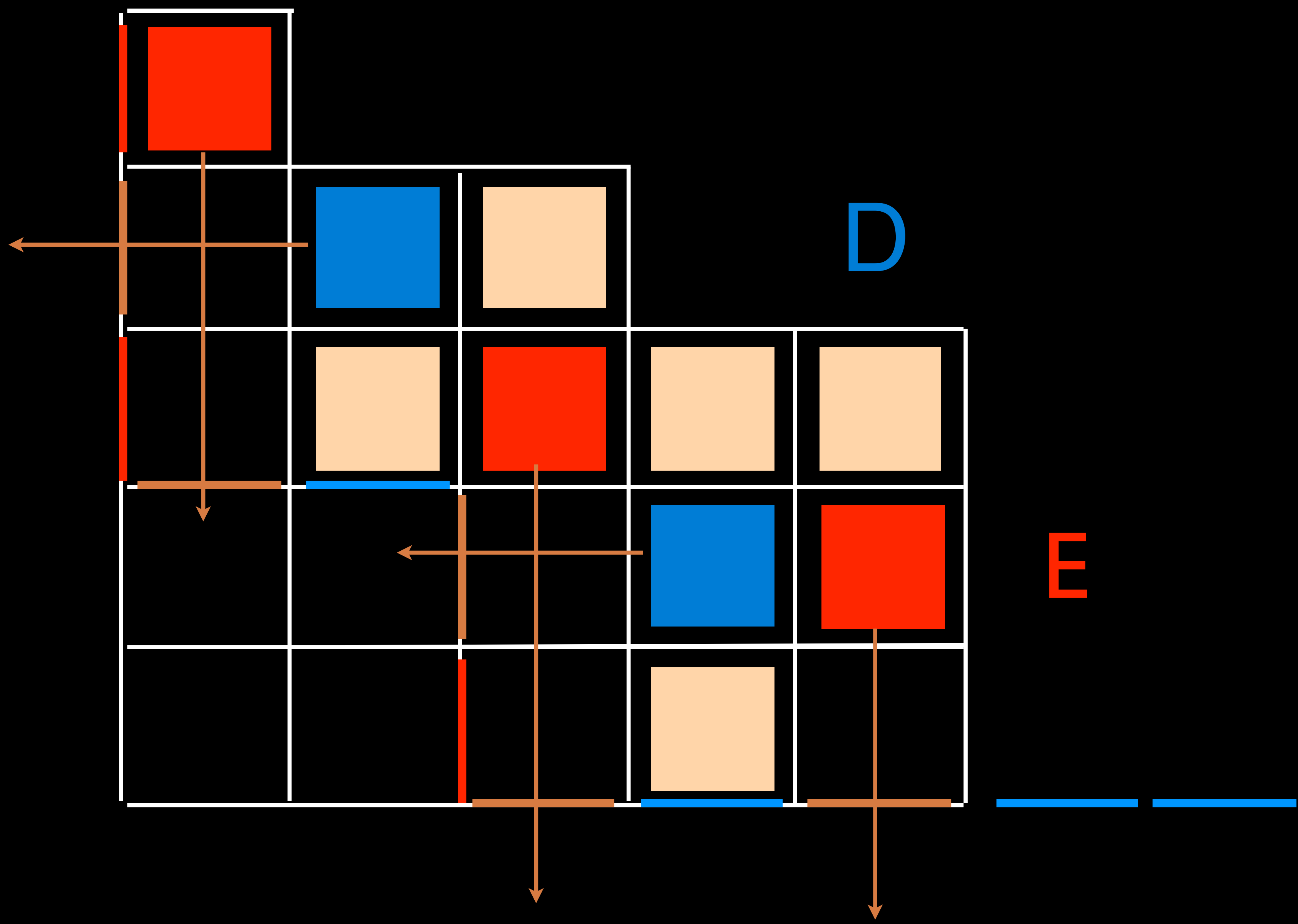
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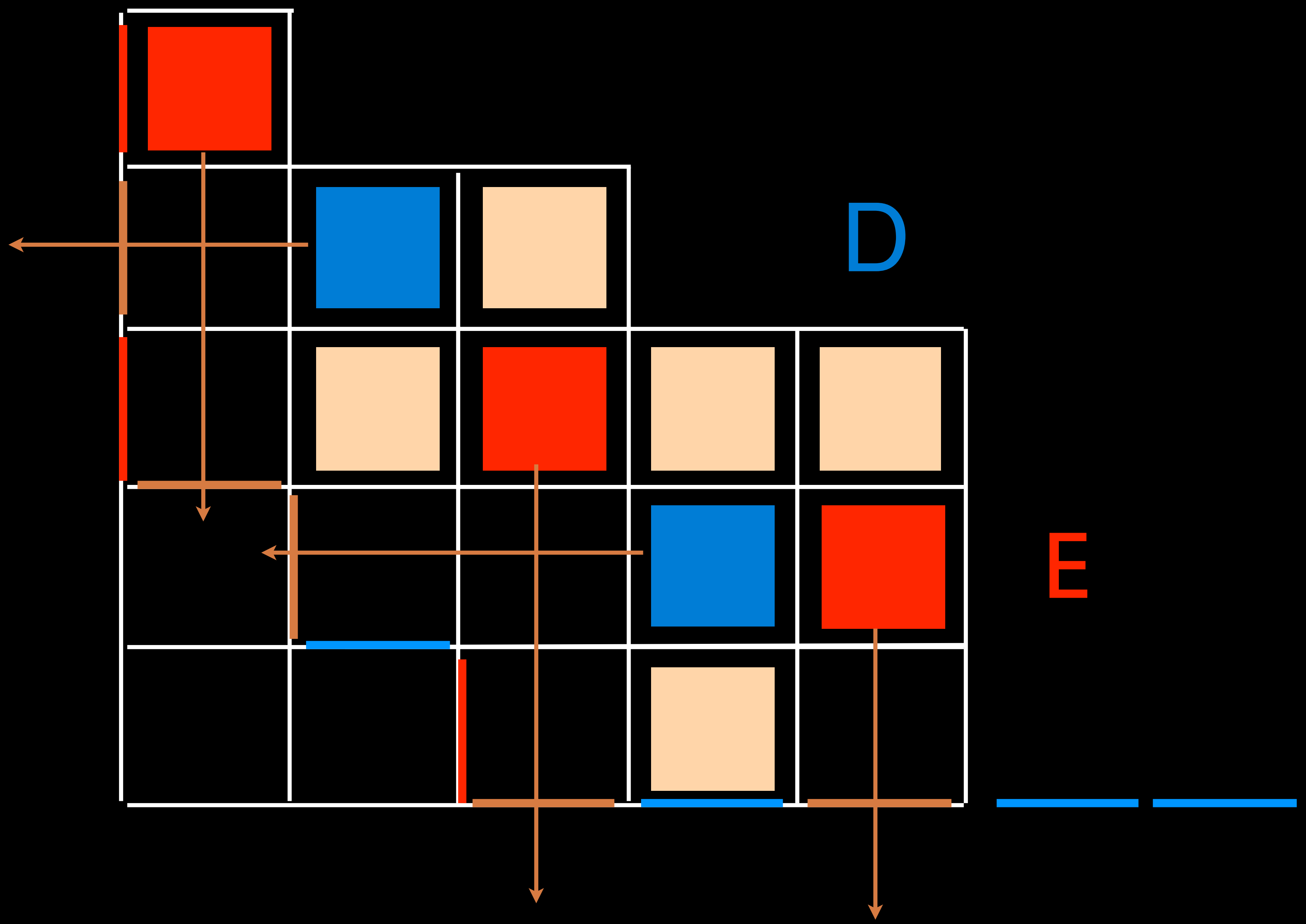
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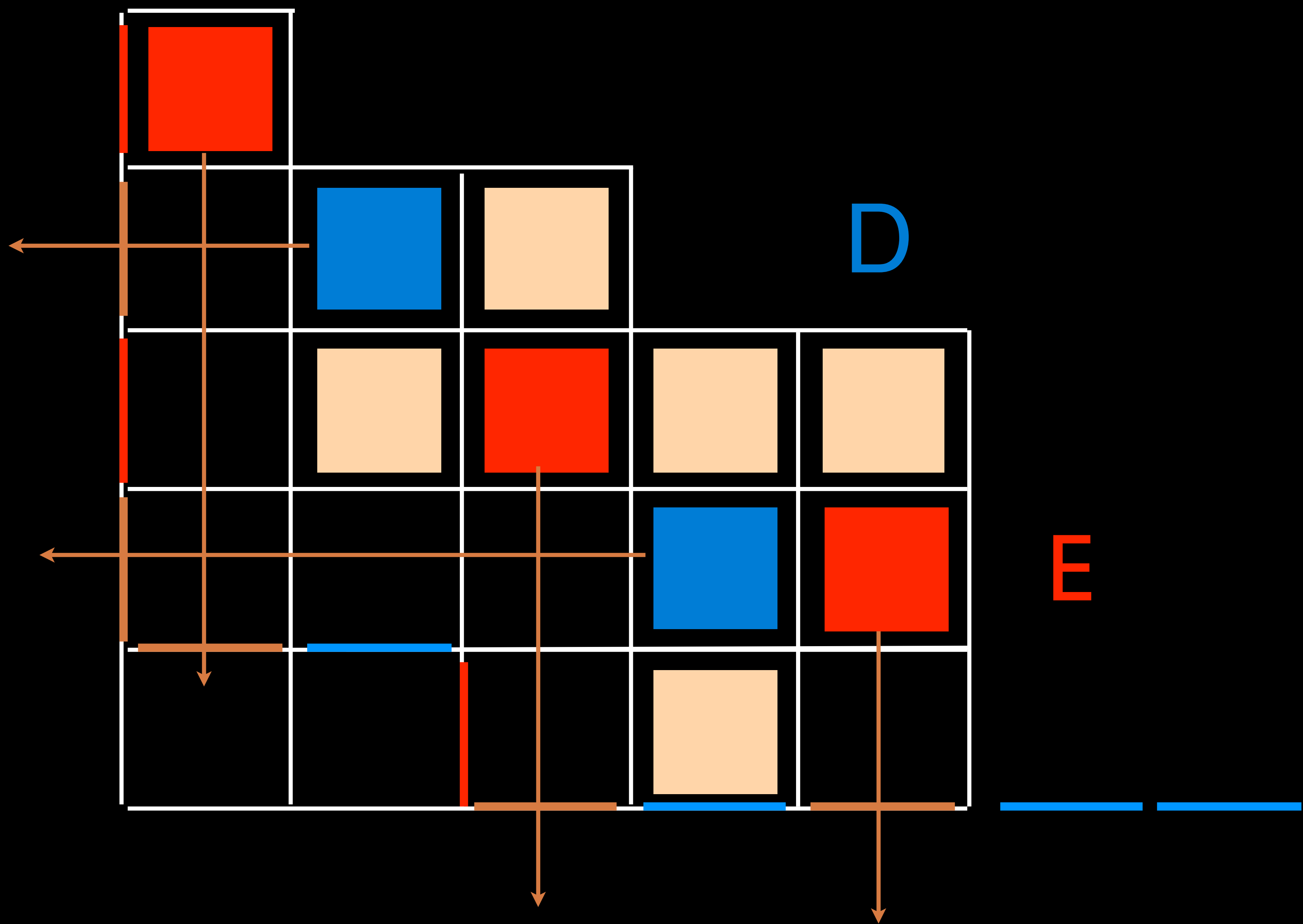
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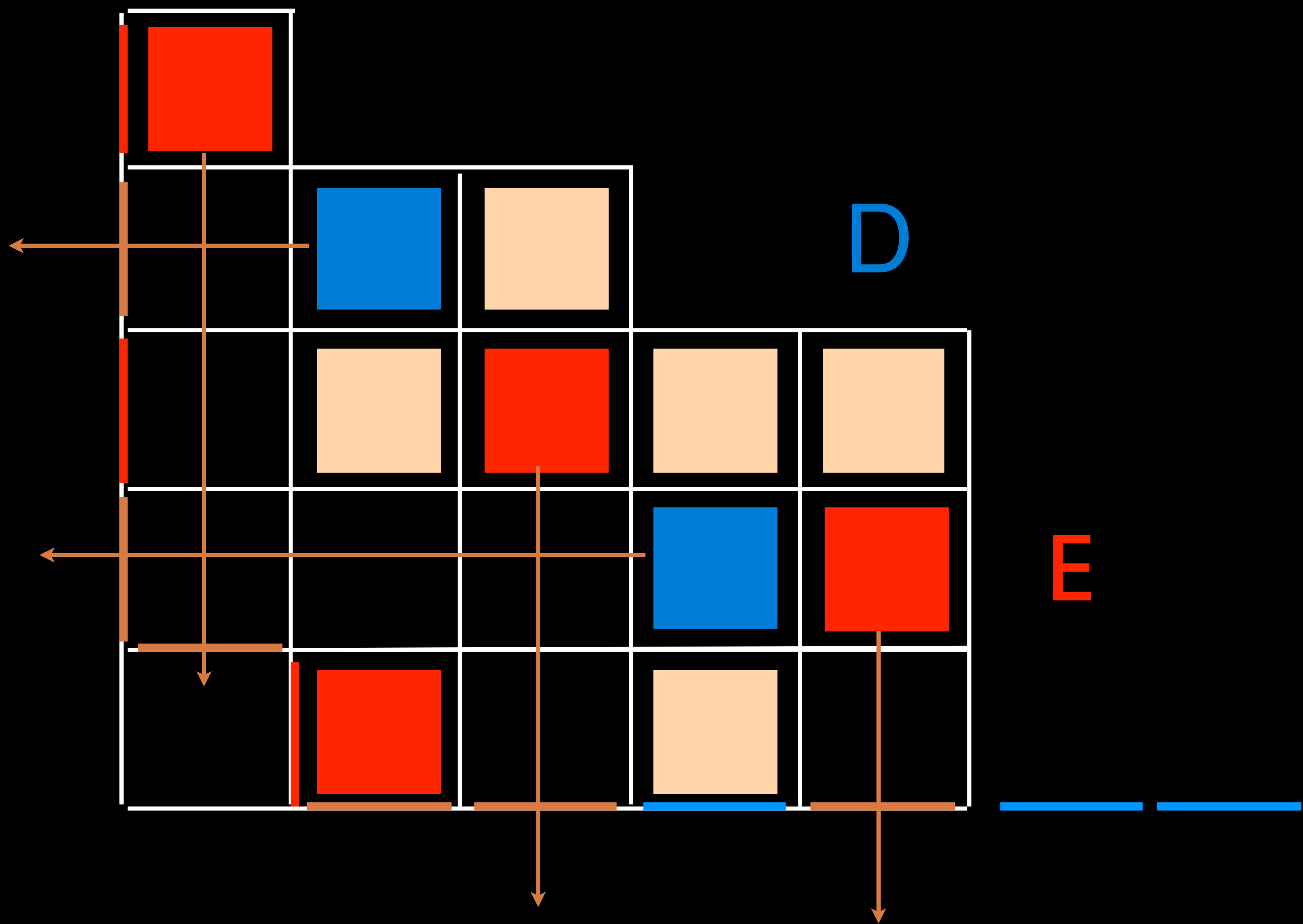
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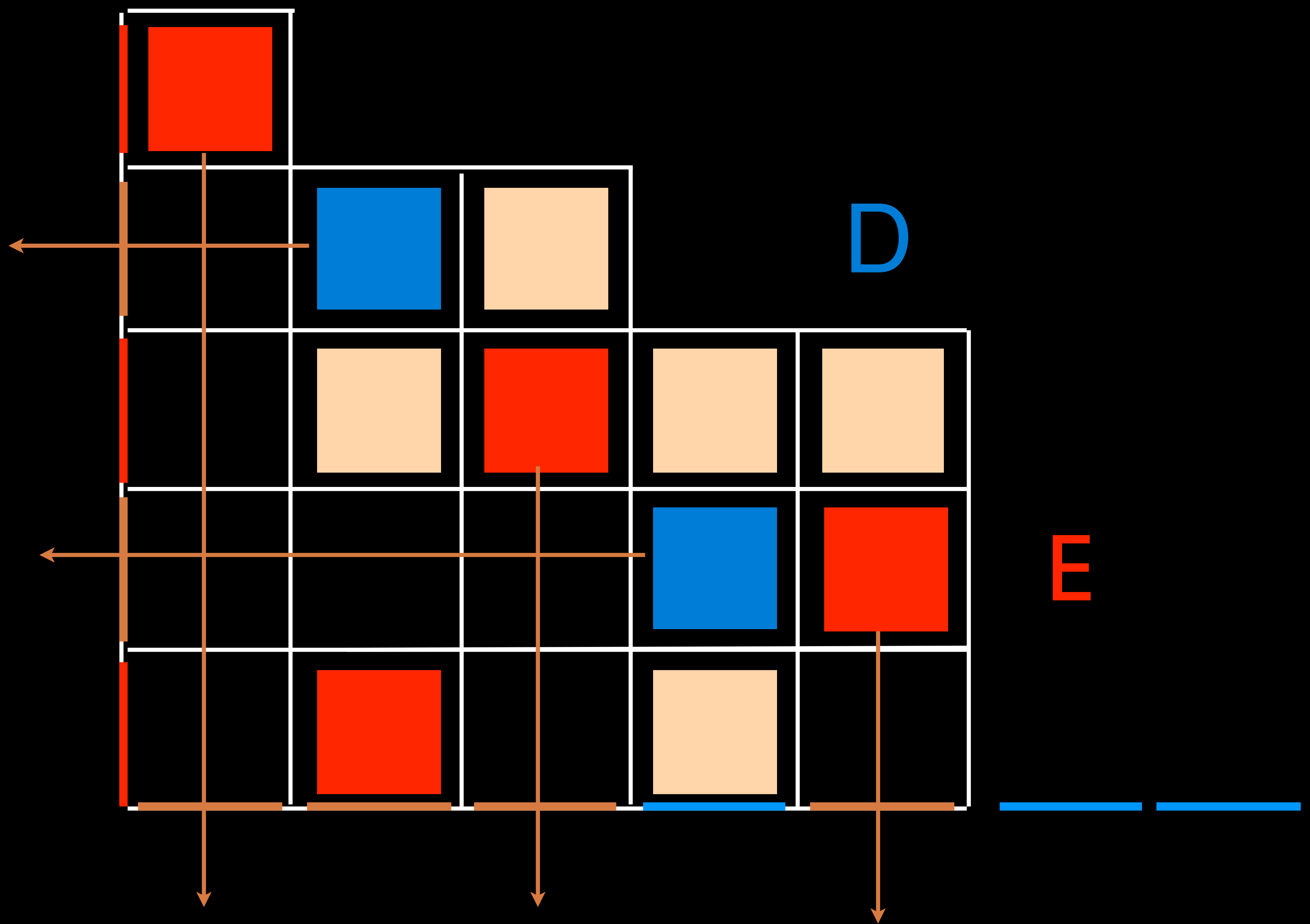
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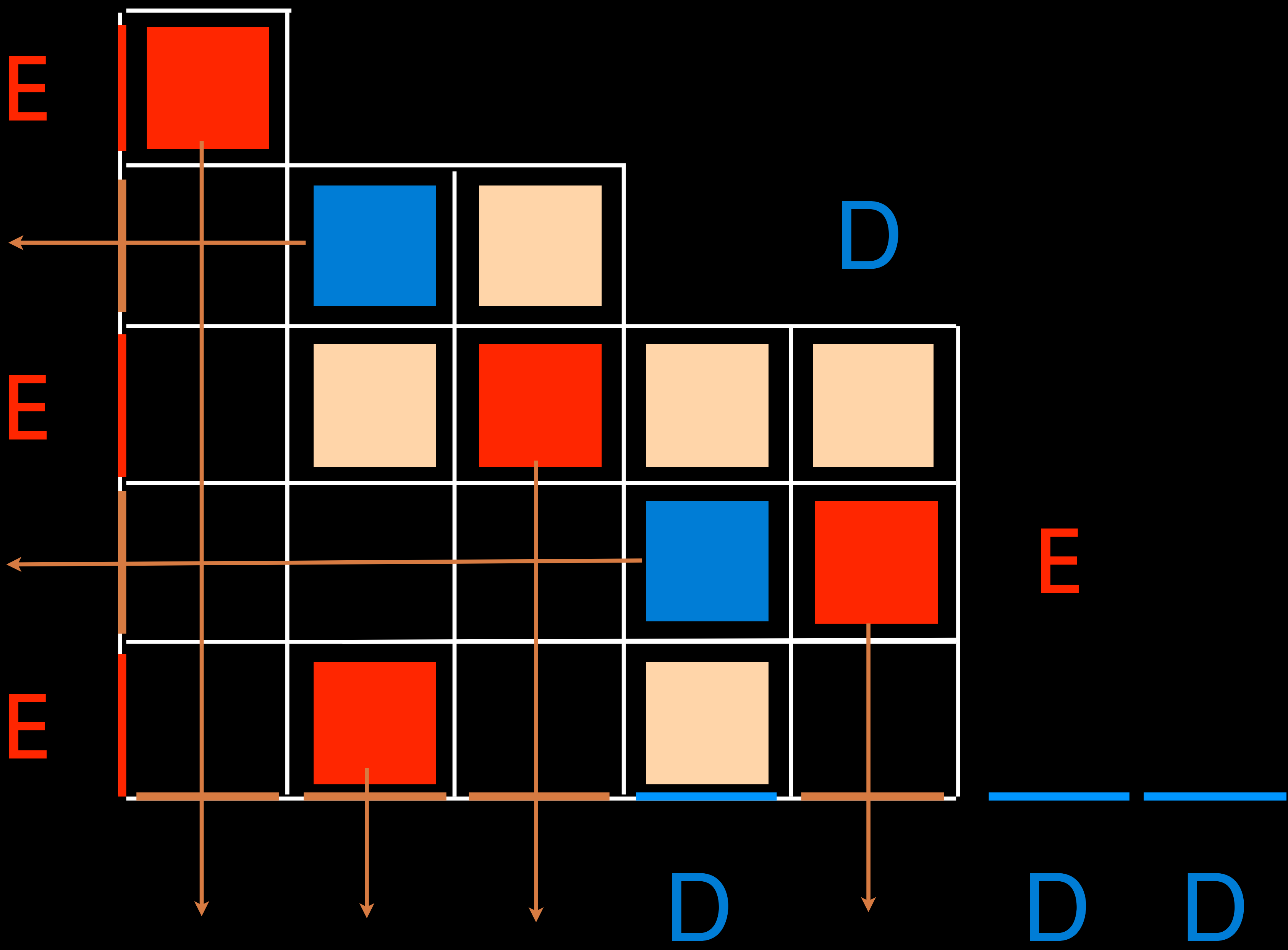
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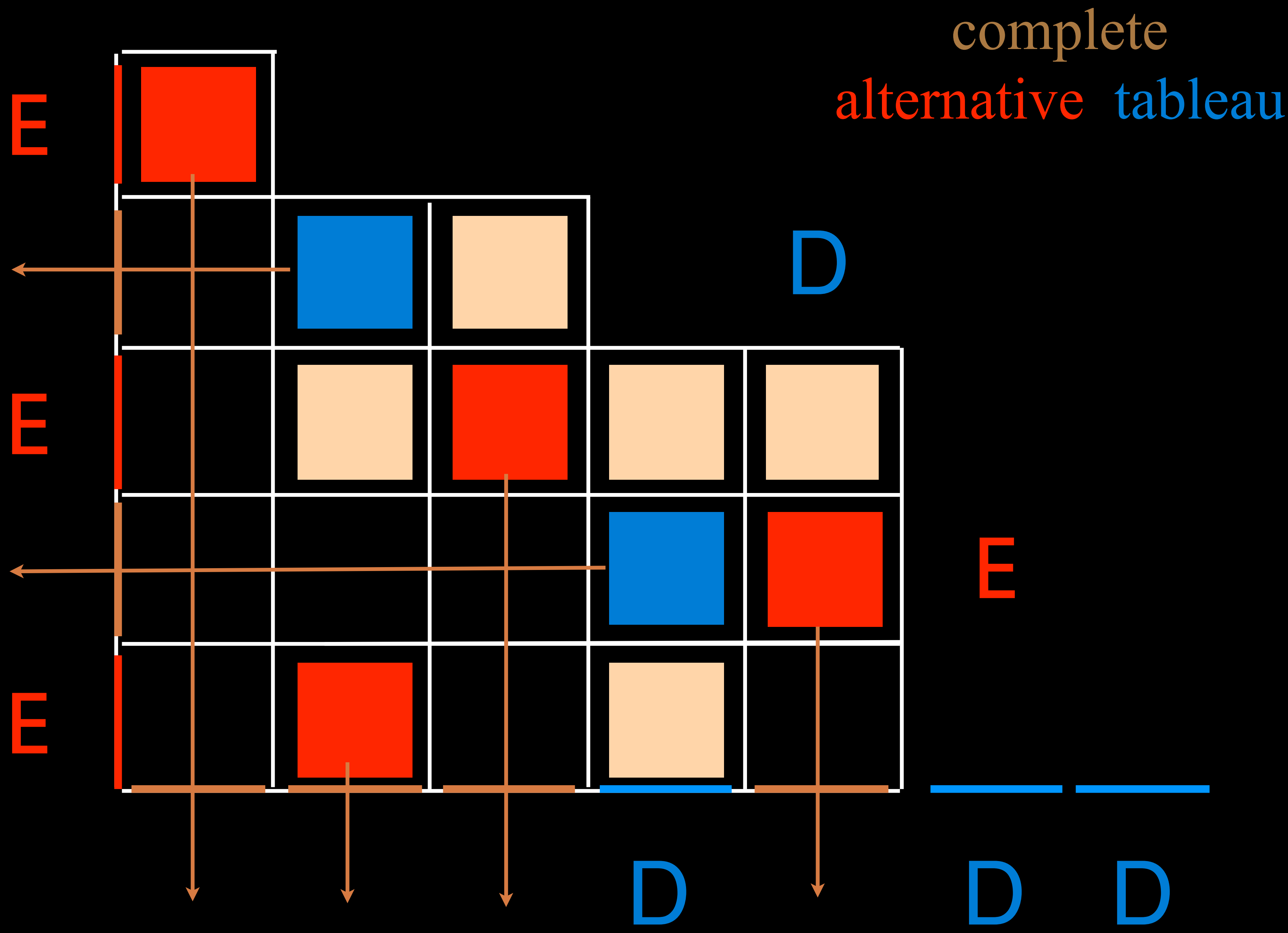

complete
alternative tableau
E
D
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E
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D

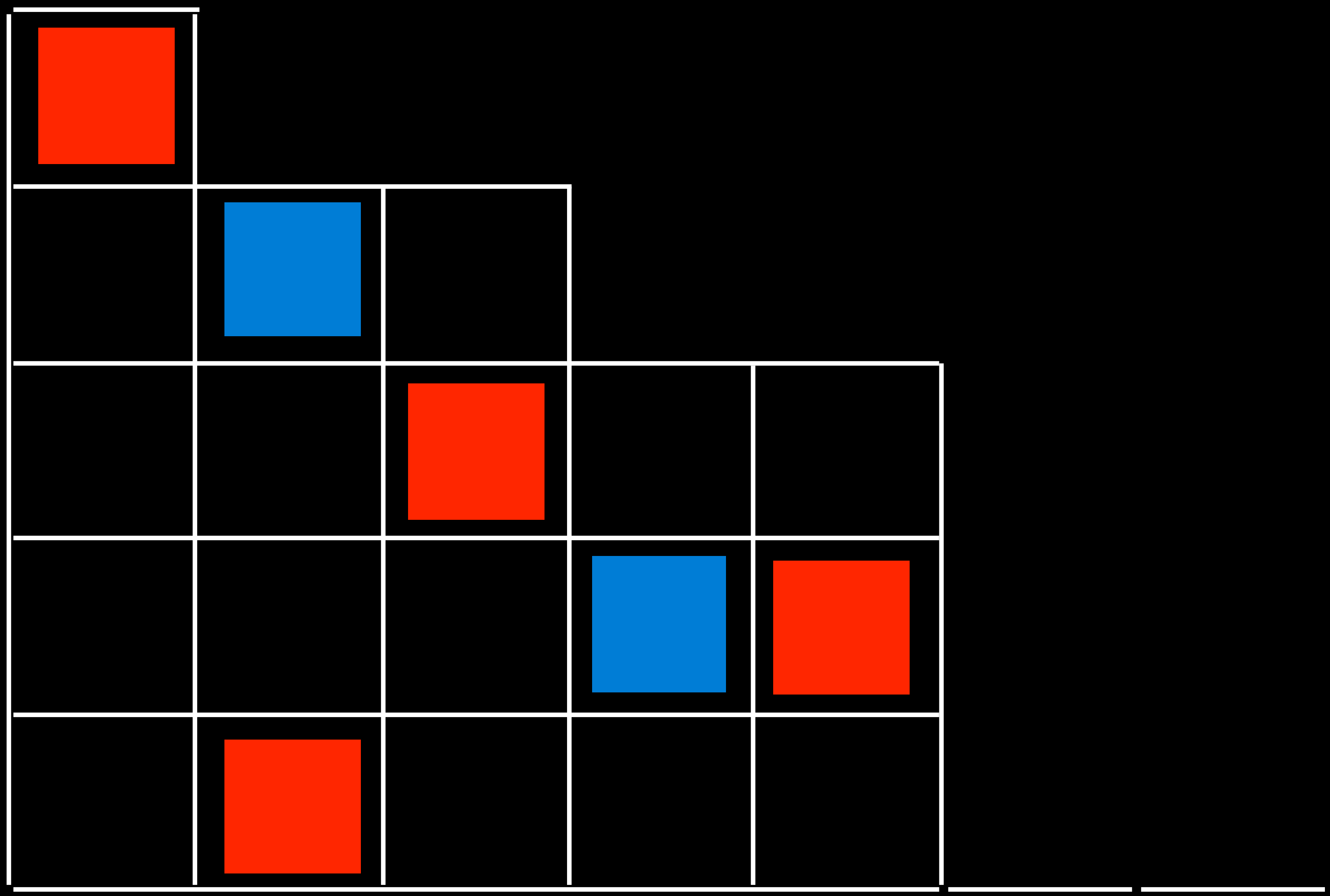
$$DE = qED + EI_h + I_vD$$

$$DI_v = I_vD$$

$$I_hE = EI_h$$

$$I_hI_v = I_vI_h$$

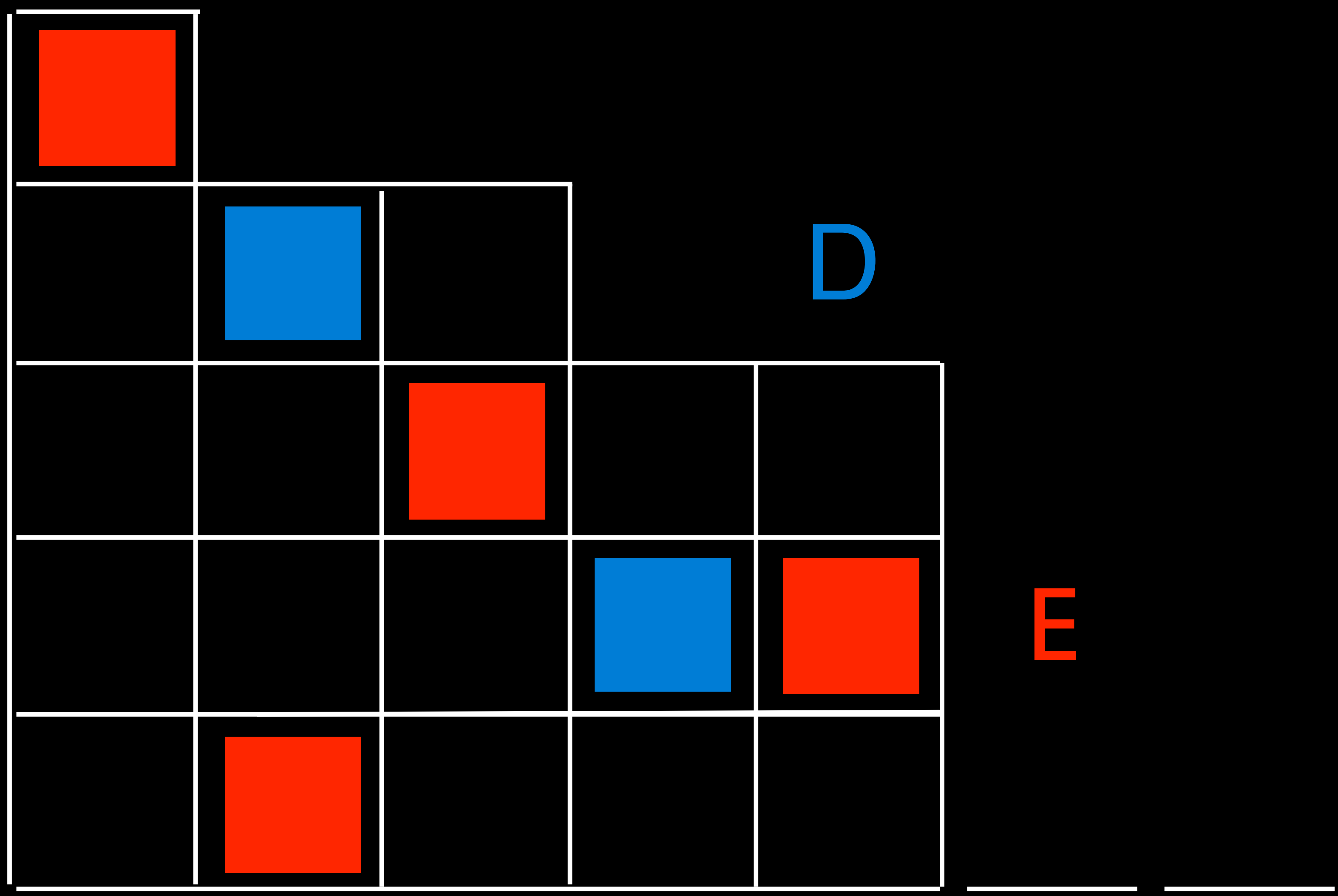
alternative tableau



complete

Q - tableau ⟷ alternative tableaux

Q PASEP algebra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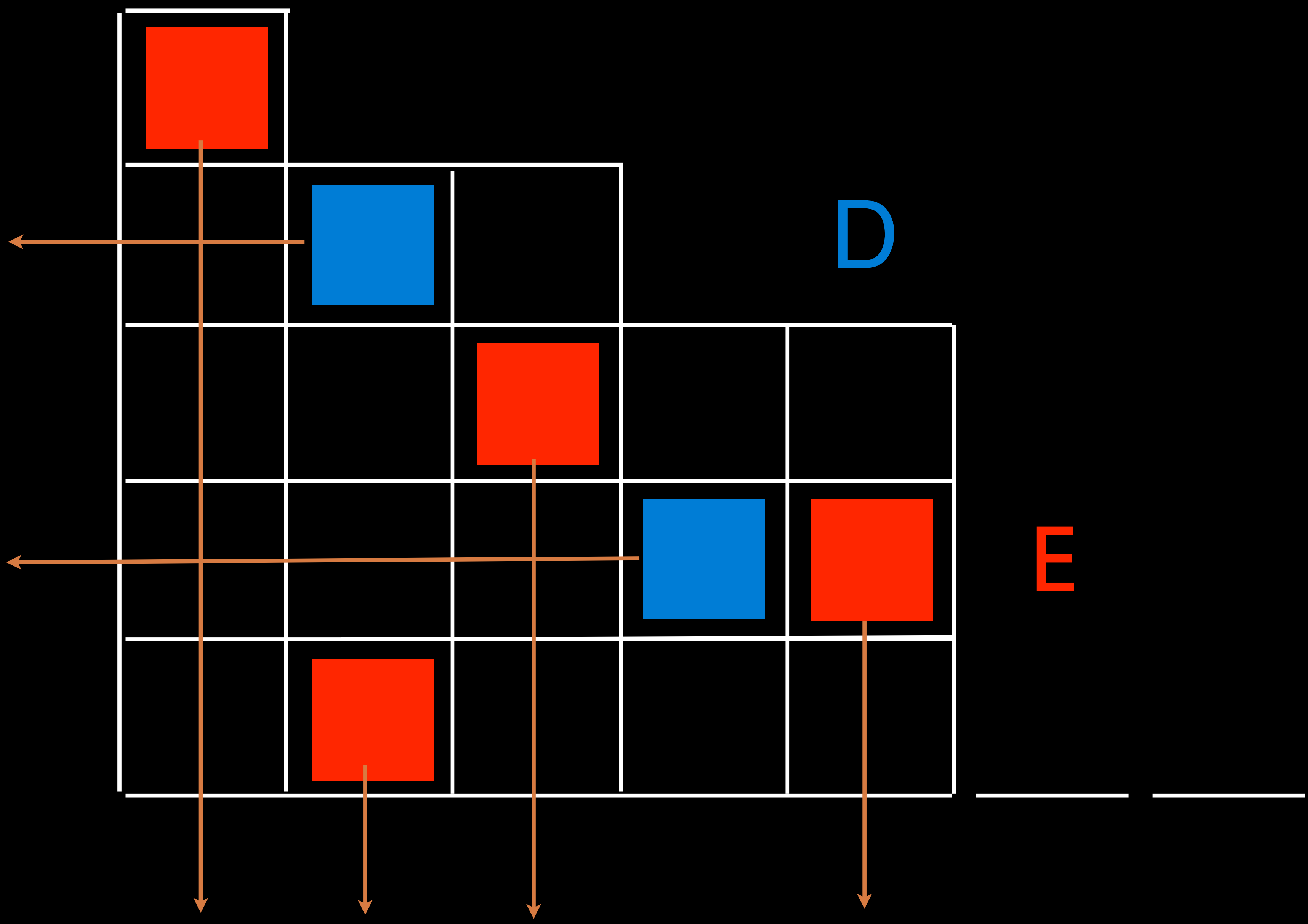
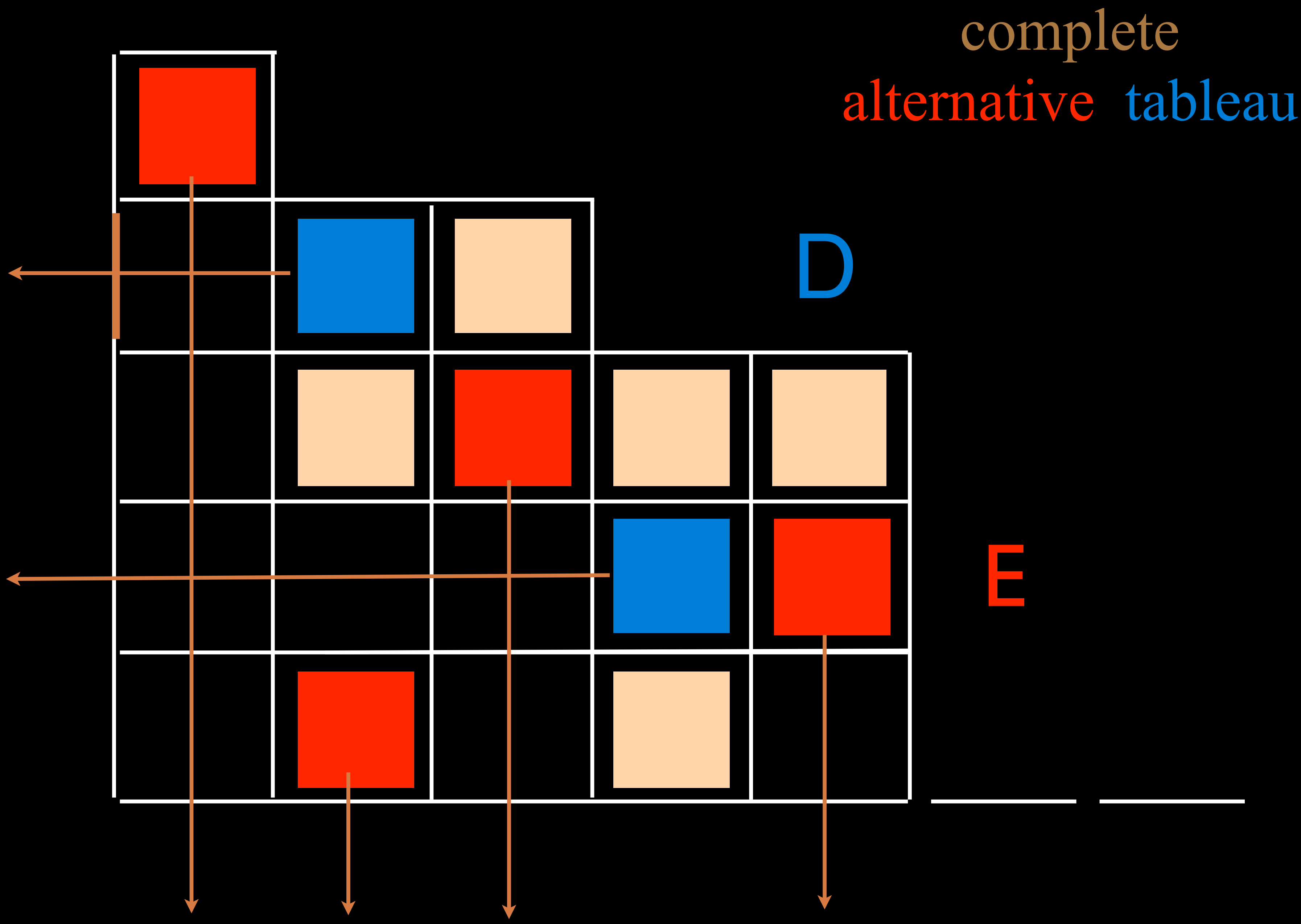
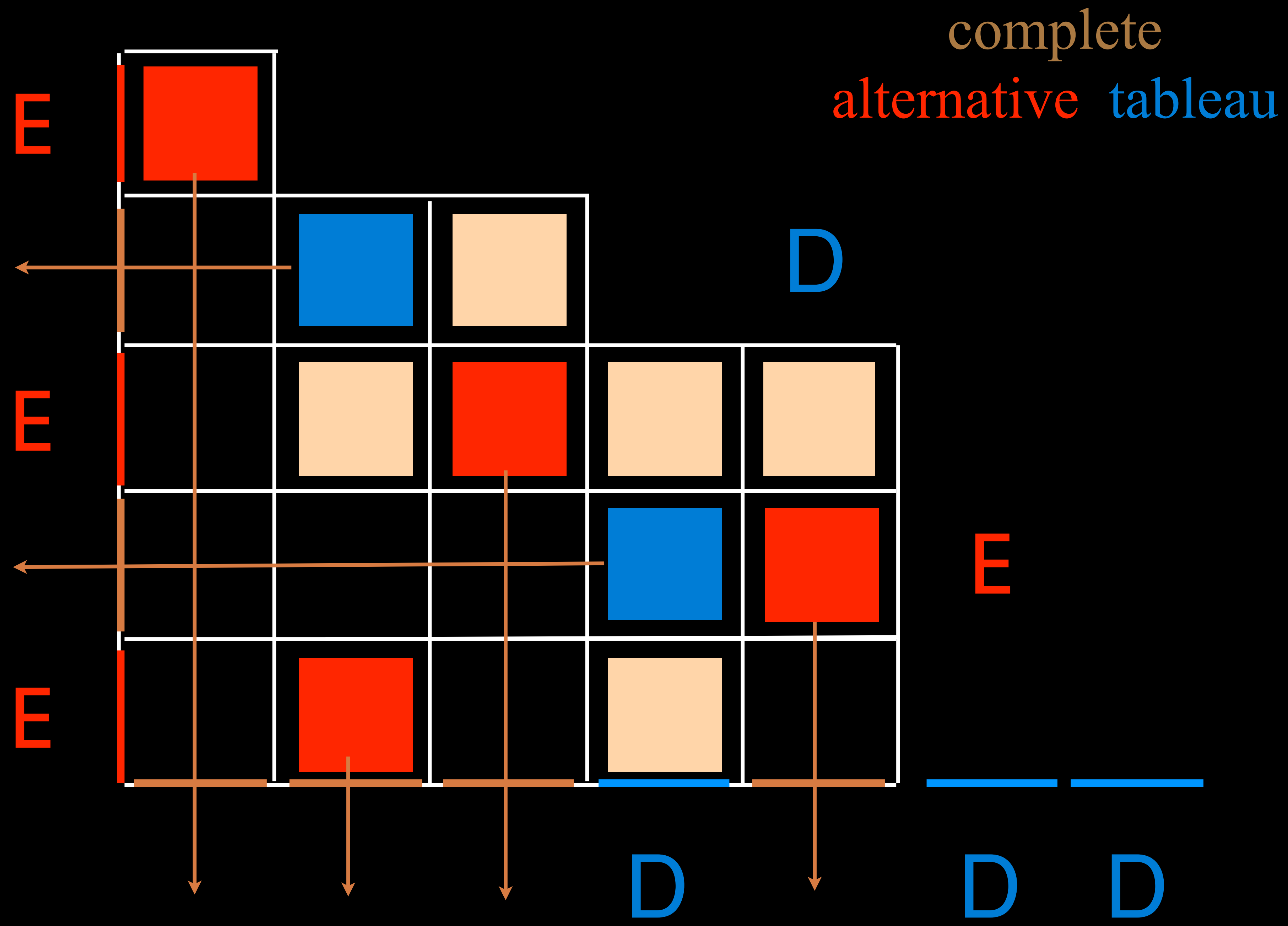

D
E


complete
alternative tableau
D
E



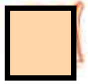
complete
alternative tableau
E
D
E
E
E
D
D
D

stationary probabilities for the PASEP

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

→ stationary probabilities

$$\begin{cases} DE = qED + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$$W E^i D^j V = \bar{\alpha}^i \bar{\beta}^j \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \ldots, \tau_n)$ is : (PASEP)

$$\text{proba}_\tau(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{k(\ )} \bar{\alpha}^{i(T)} \bar{\beta}^{j(T)}$$

T: alternative tableaux profile τ

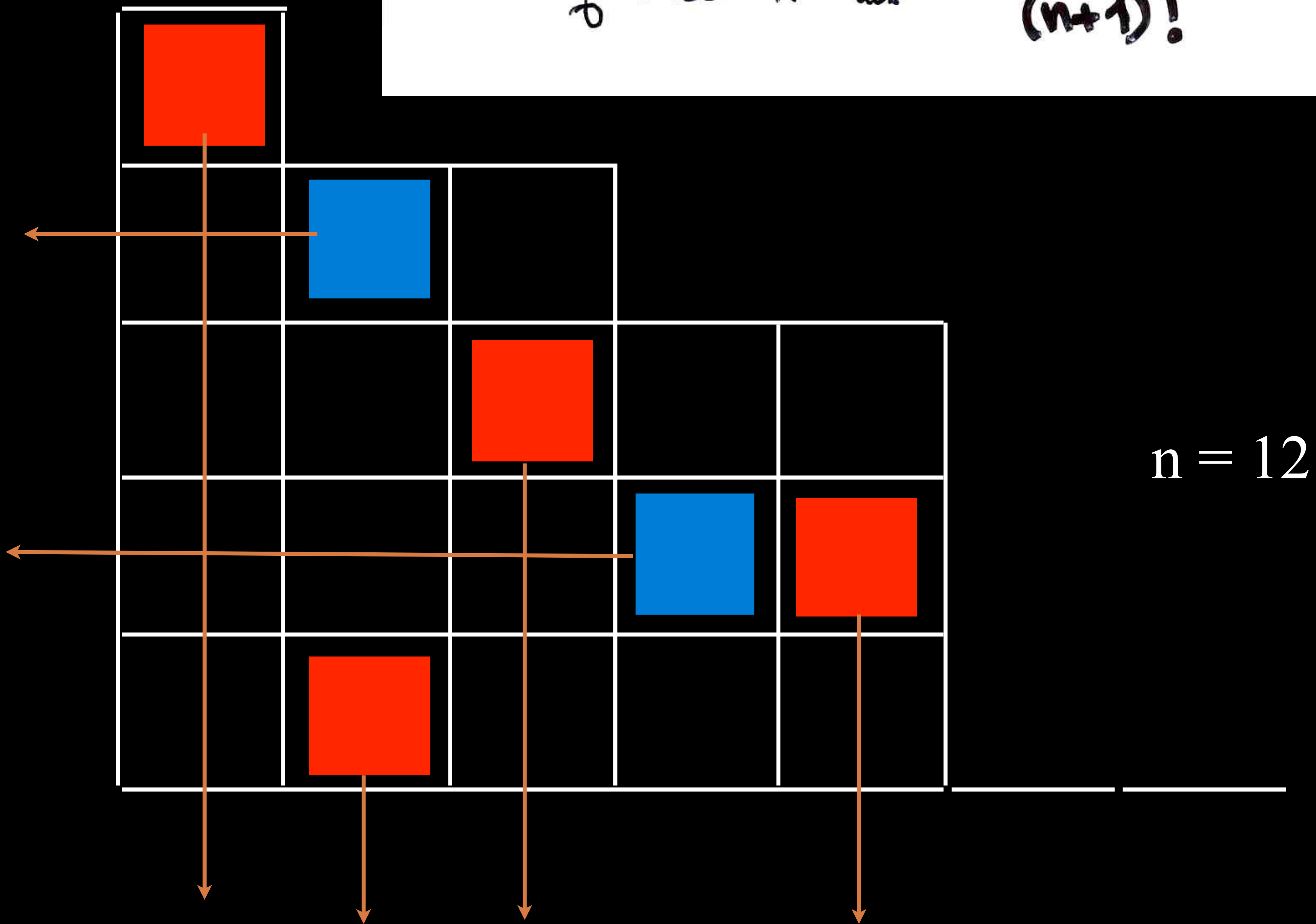
$$\begin{cases} k(T) = \text{nb of cells } □ \cdot (⊠) \\ i(T) = \text{nb of rows without } ● \text{ cell} \\ j(T) = \text{nb of columns without } ● \text{ cell} \end{cases}$$

permutation tableaux

S. Corteel, L. Williams (2007) (2008) (2009)

The number of alternating tableaux

Prop. The number of alternative tableaux of size n is (n+1)!



The total number of permutation tableaux (n fixed, $1 \leq k \leq n$) is

$$n!$$

bijection permutations $\longleftrightarrow$ permutation tableaux

- Postnikov, Steingrímsson, Williams (2005)
- Corteel (2006)
- Corteel, Nadeau (2007)

bijection $\updownarrow$ alternative tableaux size n / permutation tableaux size $(n+1)$

Two main bijections

alternative tableaux

exchange-fusion algorithm

tree-like tableaux

insertion algorithm