Combinatorics and Physics

Chapter 6b
PASEP
and
Combinatorics of orthogonal polynomials

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The FV bijection
permutations -- Laguerre histories

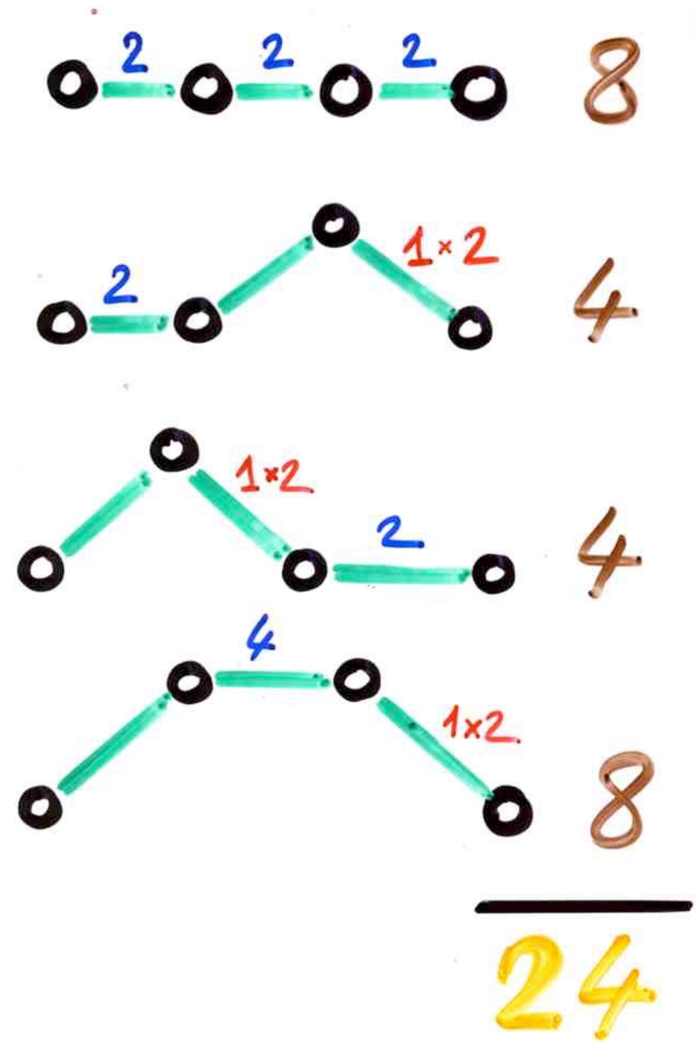
Combinatorial theory
of orthogonal polynomials

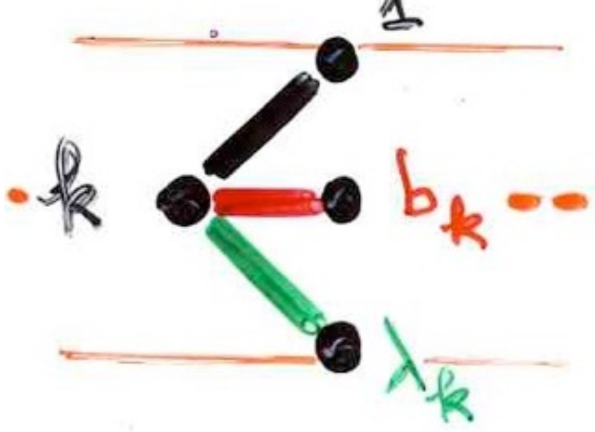
Laguerre $L_n^{(1)}(x)$

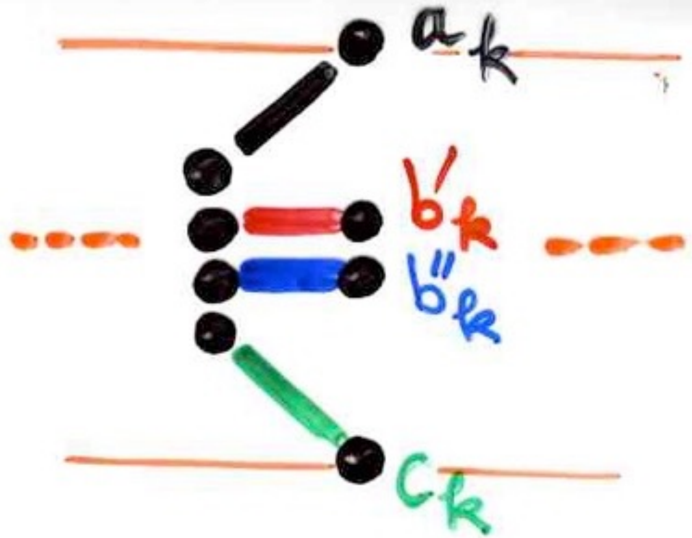
moment $\mu_n = (n+1)!$

$b_k = 2k+2$

$\lambda_k = k(k+1)$







$$b_k = b'_k + b''_k$$

$$a_{k-1} c_k = \lambda_k$$

Laguerre $L_n^{(1)}(x)$

$$\mu_n = (n+1)!$$

$$a_k = k+1$$
$$b'_k = k+1$$
$$b''_k = k+1$$
$$c_k = k+1$$

Laguerre history: definition

Bijection Permutations $n+1$

$\updownarrow$

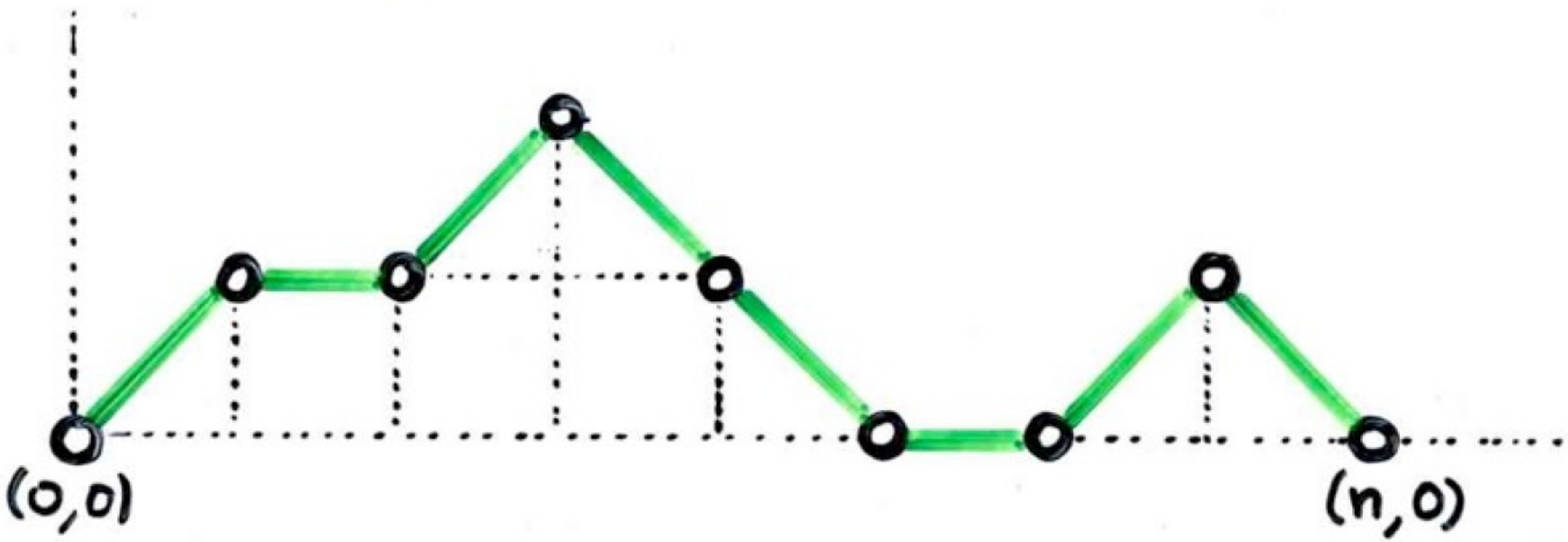
Histoires de Laguerre $(\gamma_c\ ;\ f)$ n

Bijection **Permutations** $n+1$

Histoires de Laguerre $(\gamma_c\ ; f)$ n

Chemin de Motzkin

n

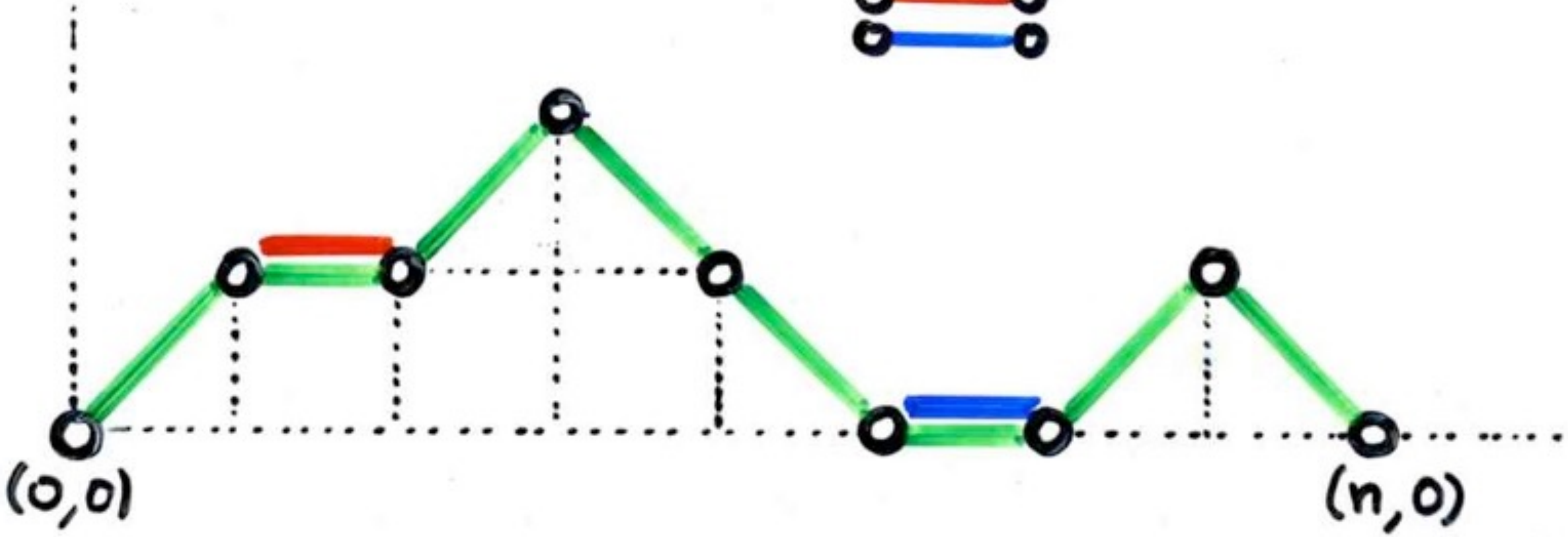


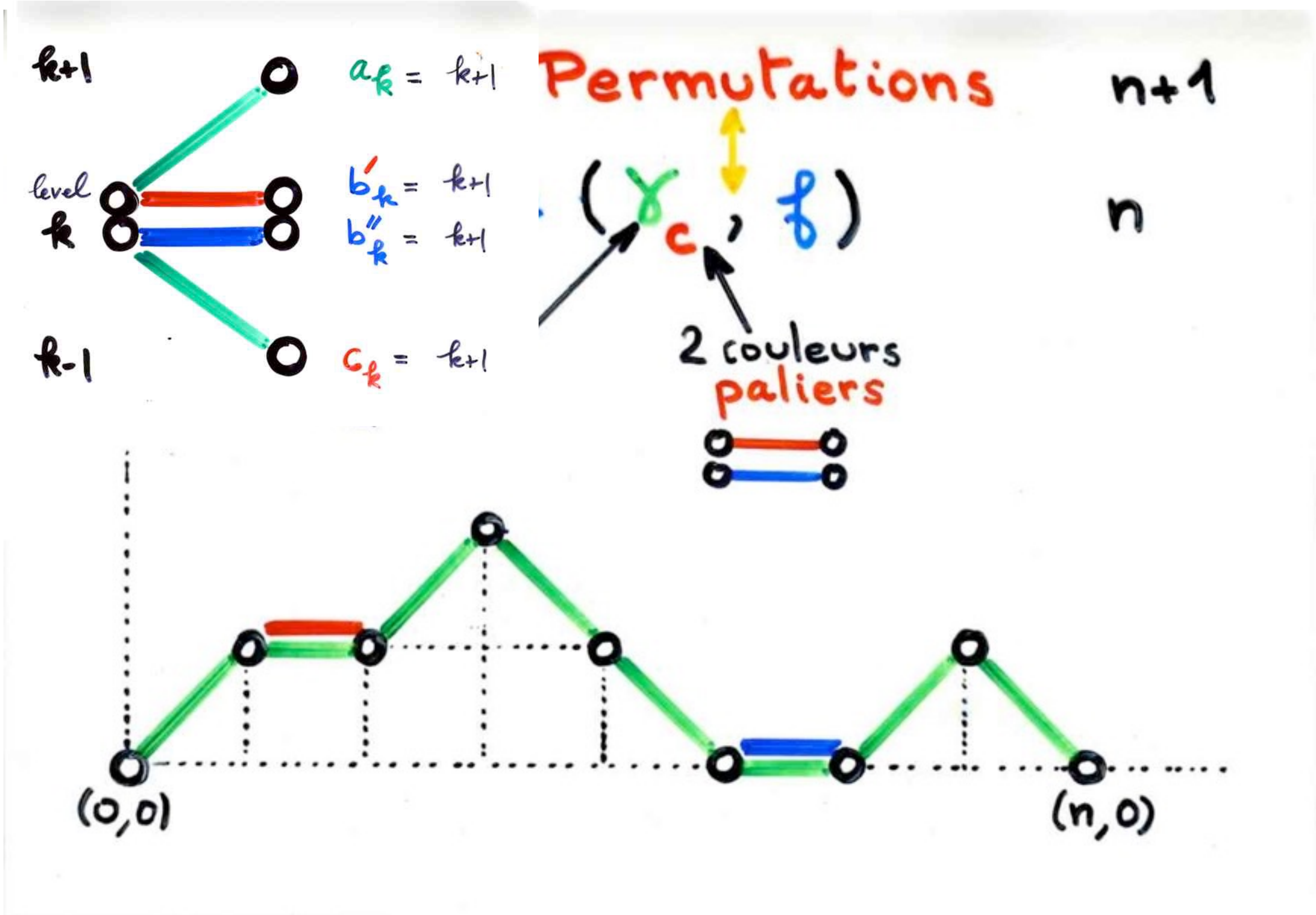
Bijection **Permutations** n+1

Histoires de Laguerre $(\gamma_c \, ; \, f)$ n

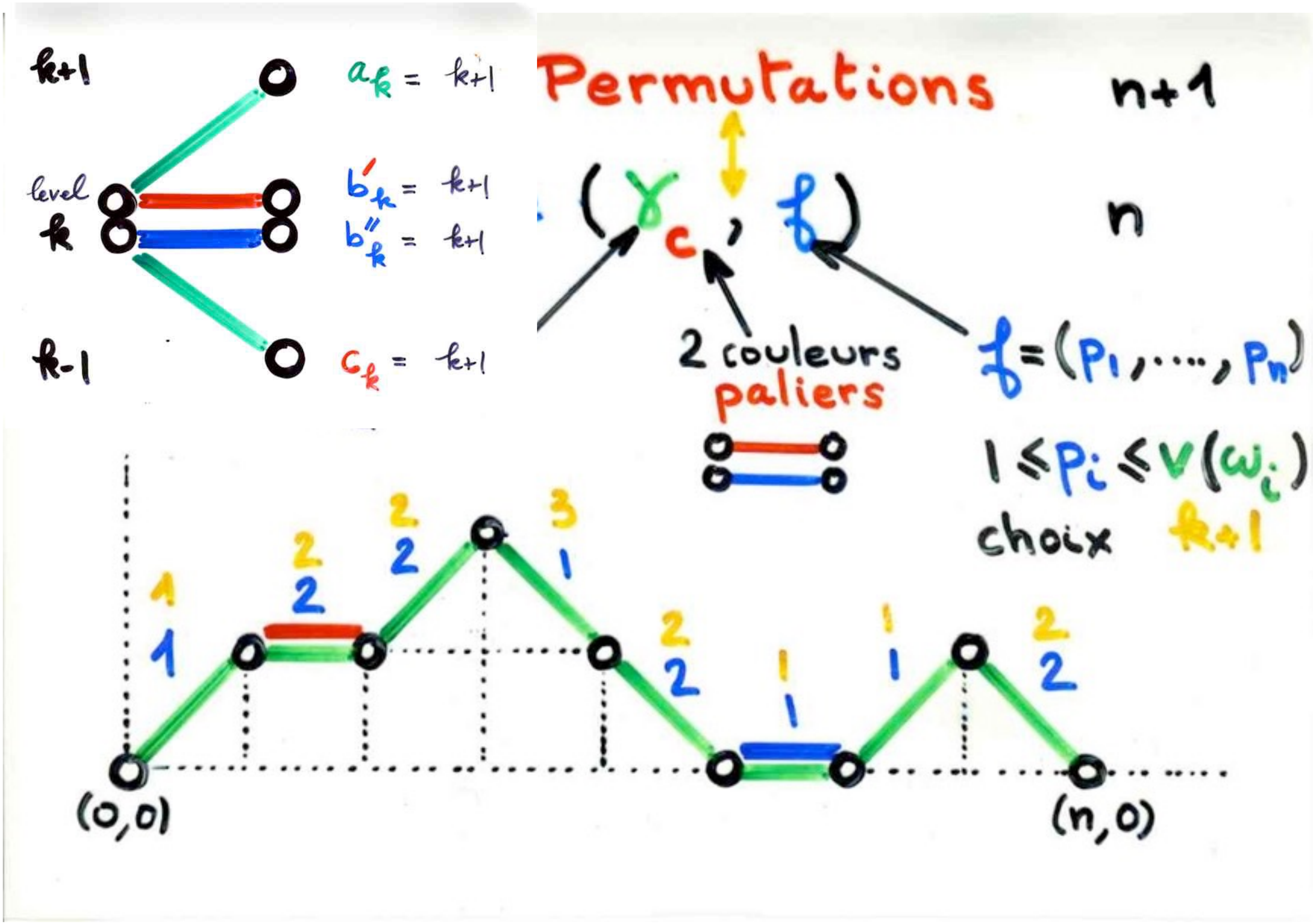
Chemin de Motzkin n-1

2 couleurs paliers





k+1
level
k
k-1
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
(γ_c , f)
n
2 couleurs
paliers
(0,0)
(n,0)


k+1
level
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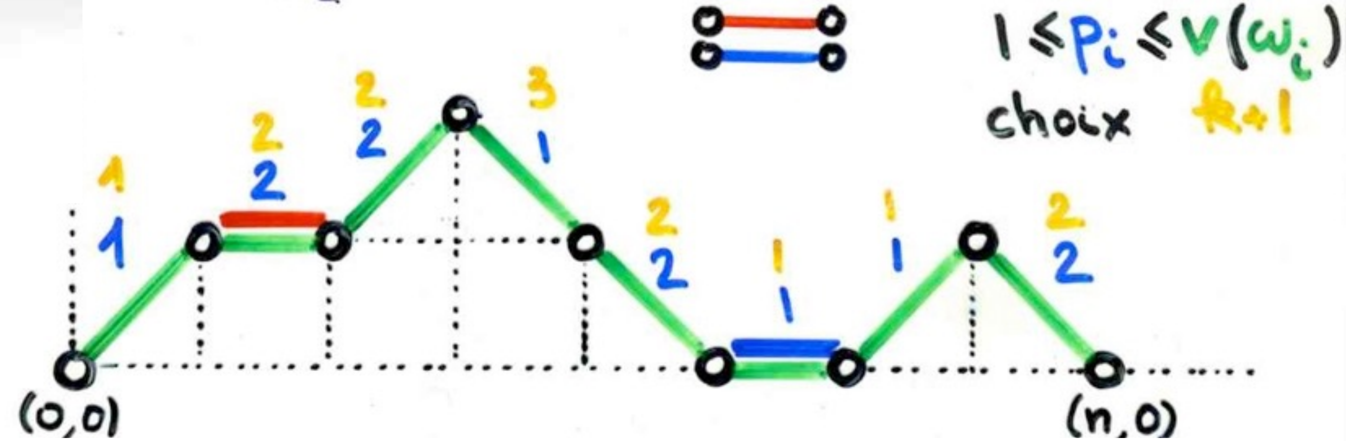
a_k = k+1
b'_k = k+1
b''_k = k+1
c_k = k+1
Permutations
n+1
(γ_c , f)
n
2 couleurs
paliers
f=(p1,...,pn)
1 ≤ pi ≤ v(ωi)
choix k+1
(0,0)
(n,0)

Bijection

Histoires de Laguerre $(\omega ; (p_1, \dots, p_n)) \longleftrightarrow$ permutations $(n+1)!$

description of the bijection permutations -- Laguerre histories with words

$h = (\omega_c ; (p_1, \ldots, p_n))$



x	ω_c	pos	v
1	↗	1	1
2	— (rouge)	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	— (bleu)	1	1
7	↗	1	1
n= 8	↘	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

Bijection

permutations -- Laguerre histories

reciprocal bijection

Françon-XGV., 1978

Bijection réciproque

$$\mathcal{L}_n \xrightarrow{\Pi \circ \Theta} \mathfrak{S}_{n+1}$$

- convention

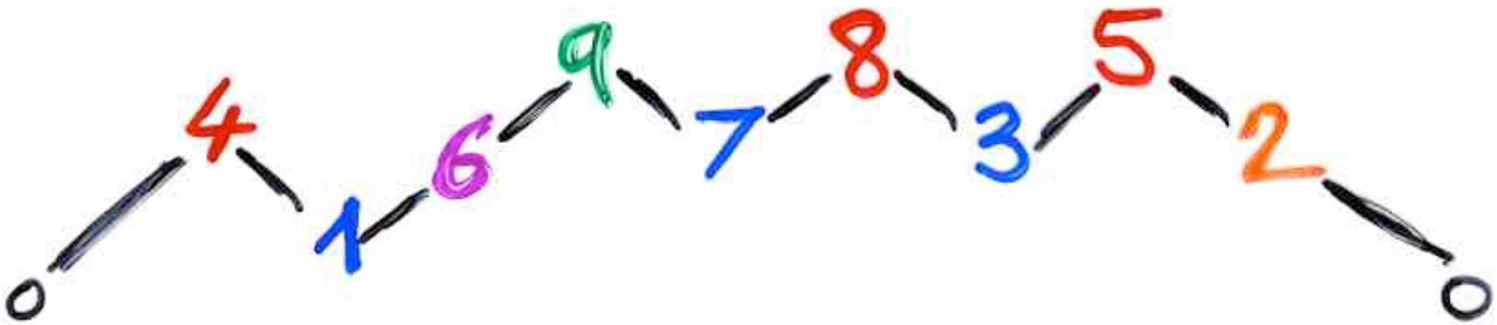
$\sigma \in \mathfrak{S}_n$, $\quad \sigma(0) = \sigma(n+1) = 0$

Def-

$x \in [1, n]$

(x valeur
i indice

pic	$\sigma(i-1) < x = \sigma(i) > \sigma(i+1)$
creux	$\sigma(i-1) > x = \sigma(i) < \sigma(i+1)$
double montée	$\sigma(i-1) < x = \sigma(i) < \sigma(i+1)$
double descente	$\sigma(i-1) > x = \sigma(i) > \sigma(i+1)$



$\sigma = 4\ 1\ 6\ 9\ 7\ 8\ 3\ 5\ 2$

Bijection réciproque

$$\mathcal{L}_n \xrightarrow{\pi \circ \theta} \mathfrak{S}_{n+1}$$

$$\sigma \in \mathfrak{S}_{n+1} \longrightarrow \left(\omega_c \,;\, (p_1, \dots, p_n)\right)$$

- ω_c

  $i^{\text{ème}}$ pas

i creux

i pic

i double montée

i double descente

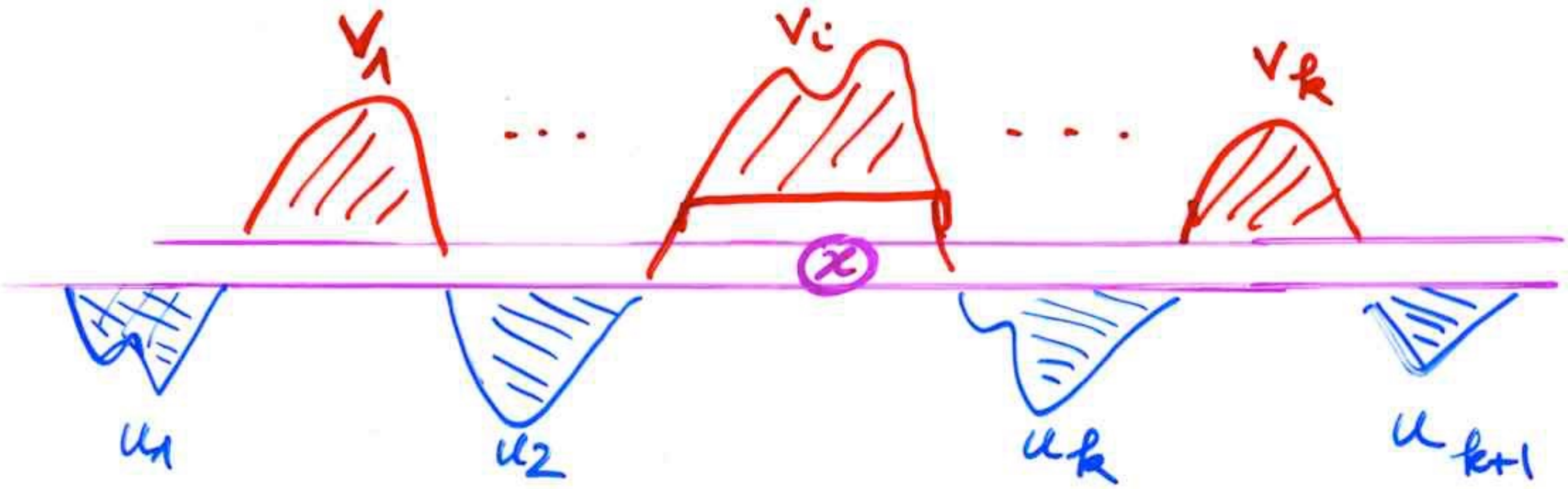
Def - $\sigma \in \mathfrak{S}_n$, $x \in [1, n]$

x- decomposition

- $\sigma = u_1 v_1 \ldots u_k v_k u_{k+1}$
- lettres $(u_i) < x$
- lettres $(v_j) \geq x$
- mots $v_1, u_2, \ldots, u_k, v_k$ non vides

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$

$u_1 = \varnothing$, $v_1 = 4$, $u_2 = 1$, $v_2 = 6\,9\,7\,8\,3\,5$, $u_3 = 2$



v_1
v_i
v_k
x
u_1
u_2
u_k
u_{k+1}

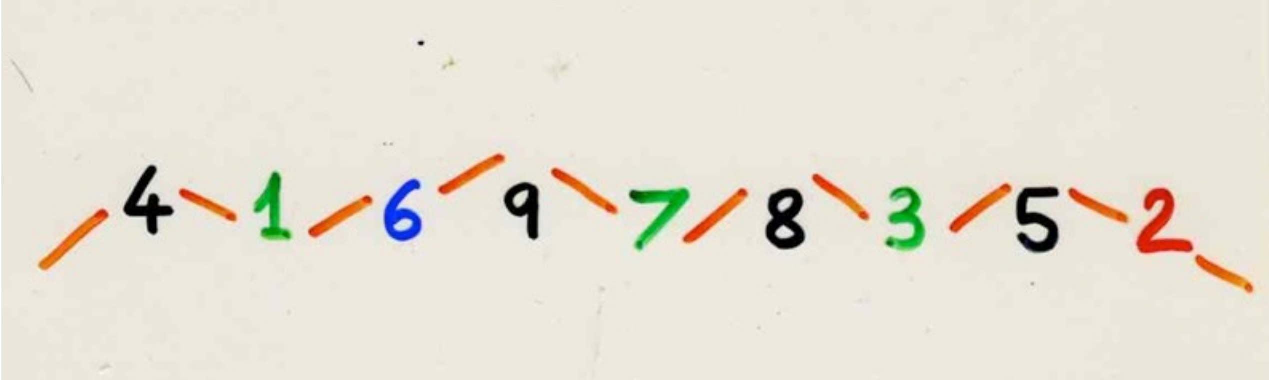
Bijection réciproque $\mathcal{L}_n \xrightarrow{\Pi \circ \Theta} \mathfrak{S}_{n+1}$

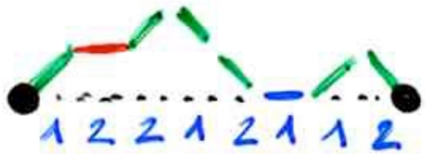
$$\sigma \in \mathfrak{S}_{n+1} \longrightarrow (\omega_c ; (p_1, \dots, p_n))$$

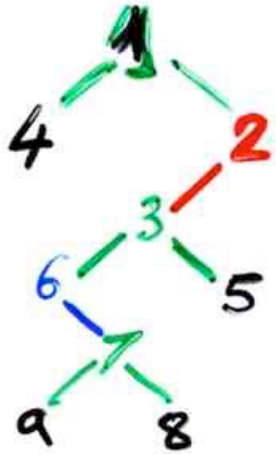
- ω_c — $i^{\text{ème}}$ pas

i creux
i pic
i double montée
i double descente

- $p_i = j$ ssi i lettre de v_j dans la i-décomposition de σ

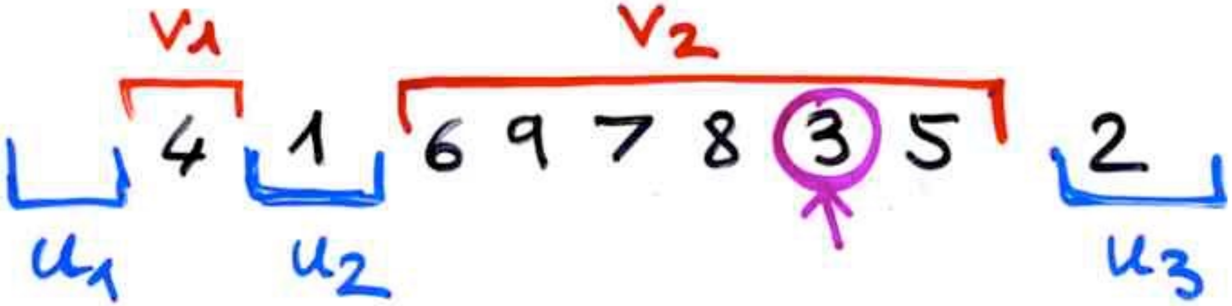






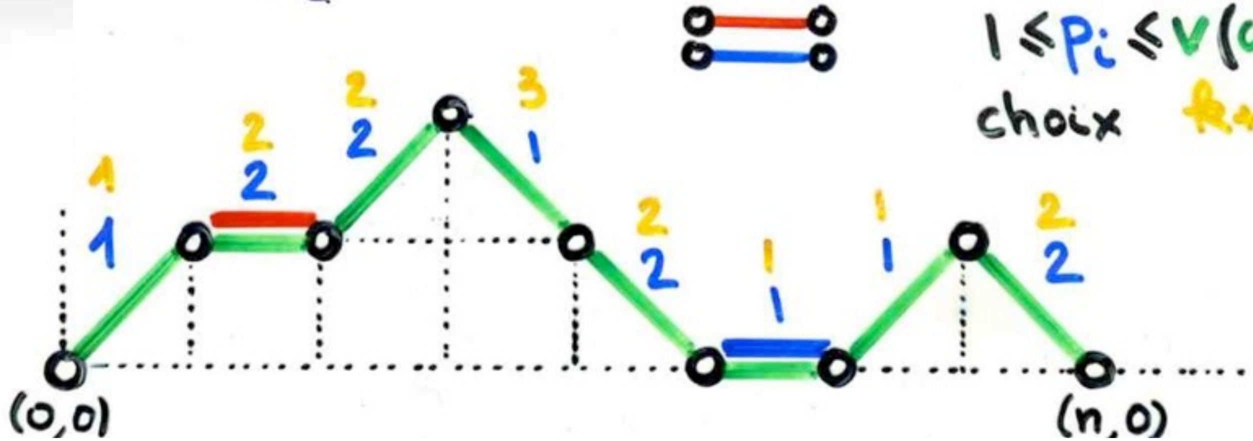
4 1 6 9 7 8 3 5 2

ex. $\sigma = 4\,1\,6\,9\,7\,8\,3\,5\,2$, $x = 3$



q-analogues of Laguerre histories

$h = (\omega_c ; (p_1, \dots, p_n))$



$1 \leq p_i \leq v(\omega_i)$

choix $k+1$

x	ω_c	pos	v
1	↗	1	1
2	—	2	2
3	↗	2	2
4	↘	1	3
5	↘	2	2
6	—	1	1
7	↗	1	1
$n =$ 8	↘	2	2
9	•		

␣

␣ 1 ␣

␣ 1 ␣ 2

␣ 1 ␣ 3 ␣ 2

4 1 ␣ 3 ␣ 2

4 1 ␣ 3 5 2

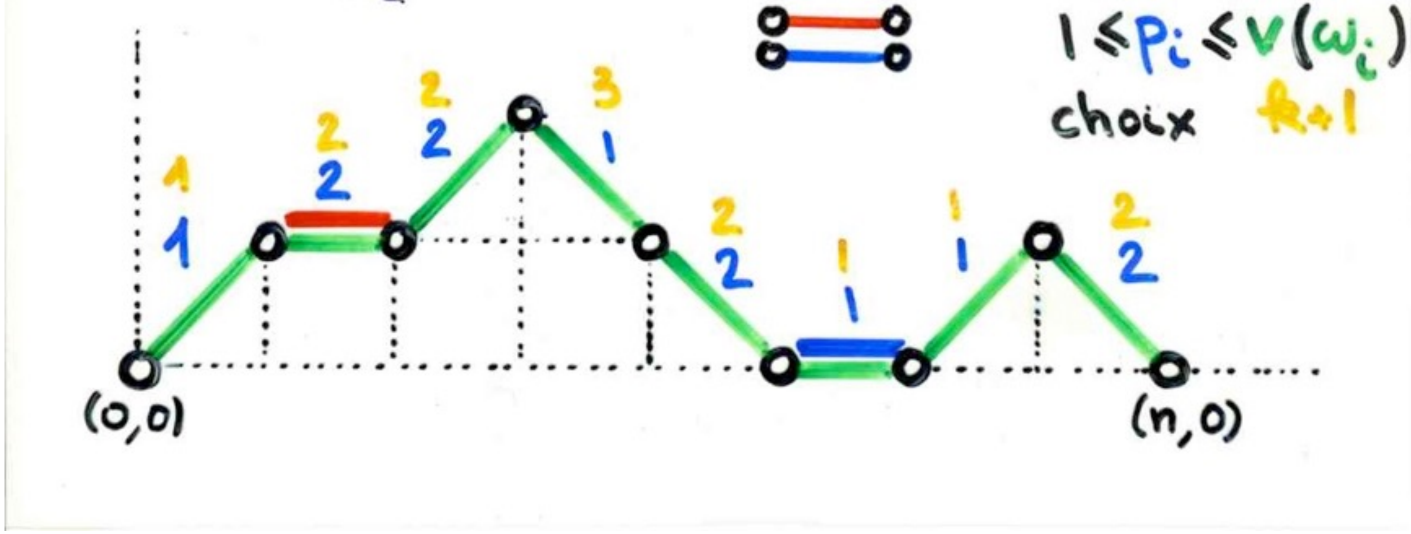
4 1 6 ␣ 3 5 2

4 1 6 ␣ 7 ␣ 3 5 2

4 1 6 ␣ 7 8 3 5 2

4 1 6 9 7 8 3 5 2 $= \sigma \in \mathfrak{S}_{n+1}$

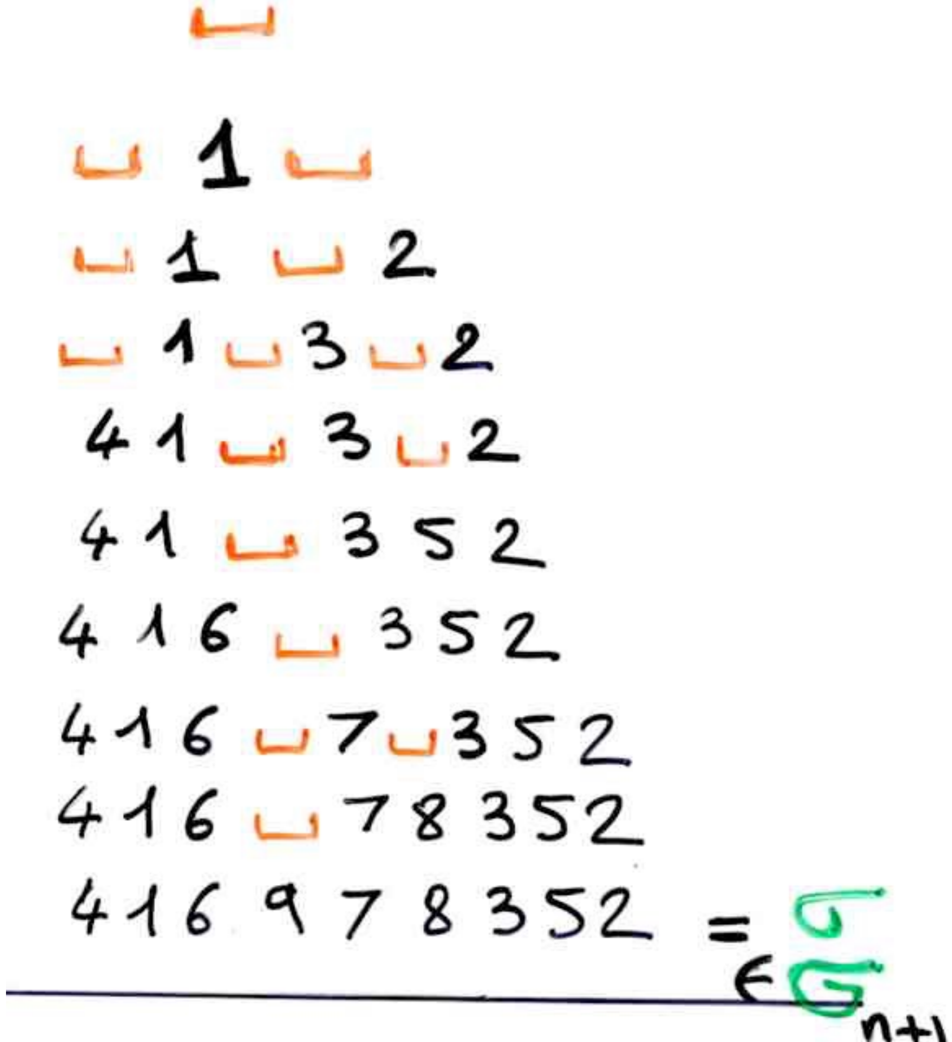
“q-analogue” of Laguerre histories



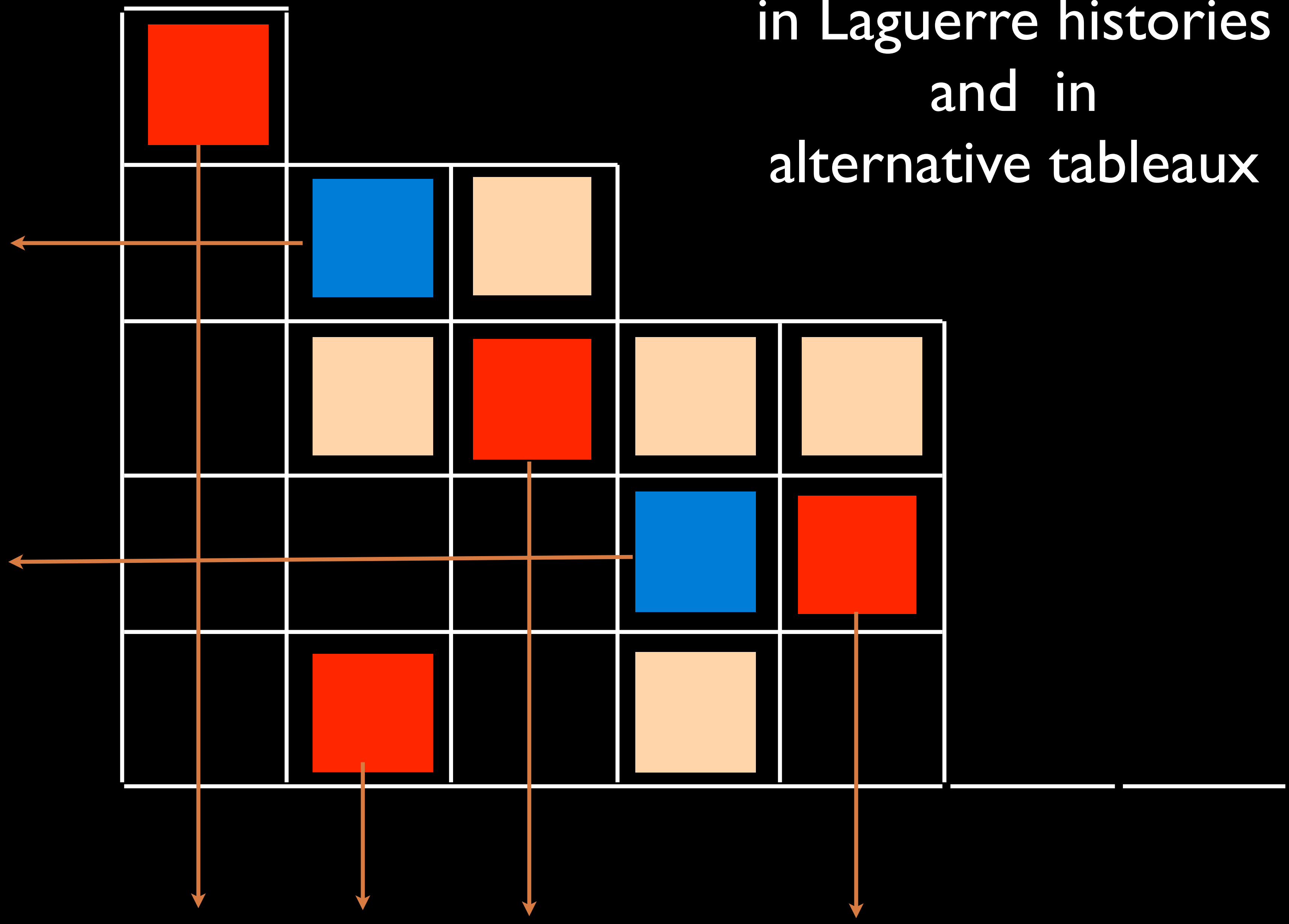
choices function

1	2	3	4	5	6	7	8
1	2	2	1	2	1	1	2
0	1	1	0	1	0	0	1

q-Laguerre : q^4



q parameter in the PASEP



in Laguerre histories
and in
alternative tableaux

Laguerre history

Lemme — Th. Ø

$h = (\omega_c\, ; (p_1, \dots, p_n)) \in \mathcal{L}_n$

$\downarrow$

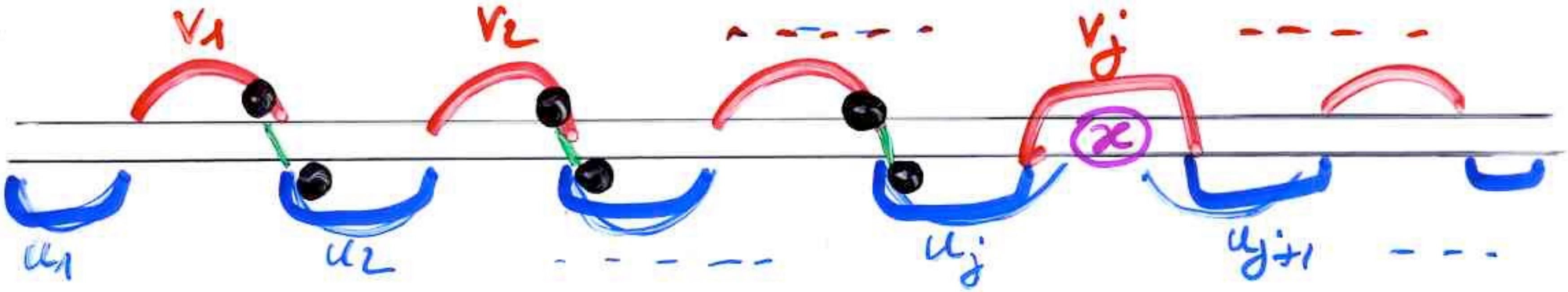
permutation $\sigma \in \mathfrak{S}_{n+1}$

$p_x = j$ est aussi :

$j = 1 +$ nb de triplets (a, b, x) ayant le "motif" (31-2) c.à.d :

$a = \sigma(i)$, $b = \sigma(i+1)$, $x = \sigma(\ell)$

$i < i+1 < \ell$ $\qquad b < x < a$

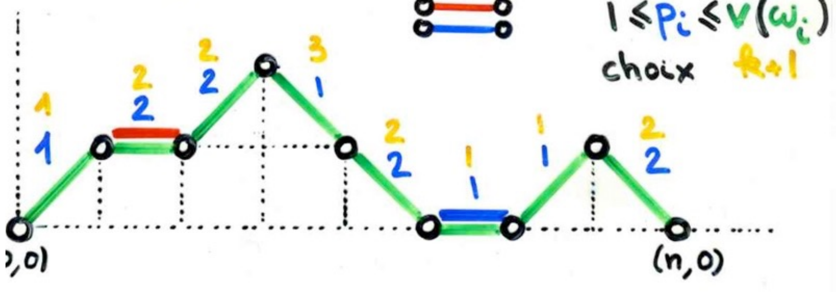


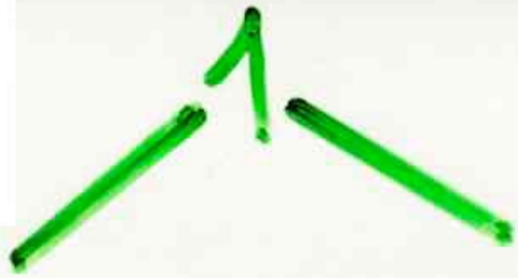
"q-analogue" of Laguerre histories

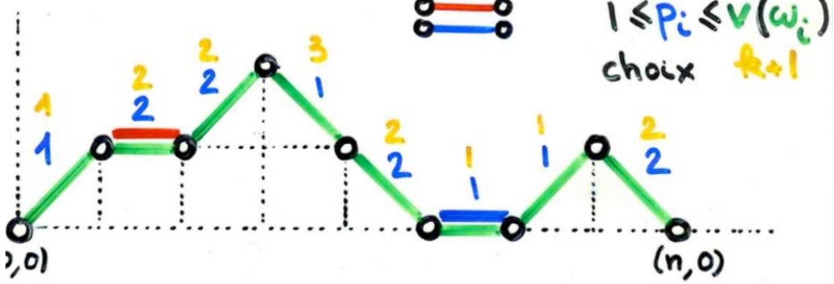
Bijection

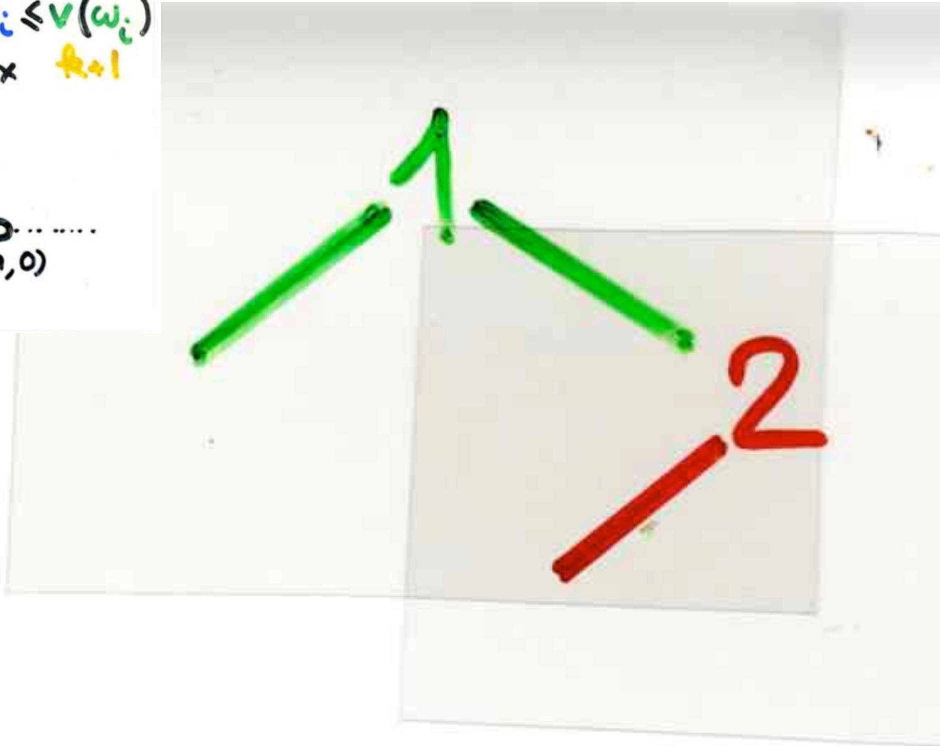
permutations -- Laguerre histories

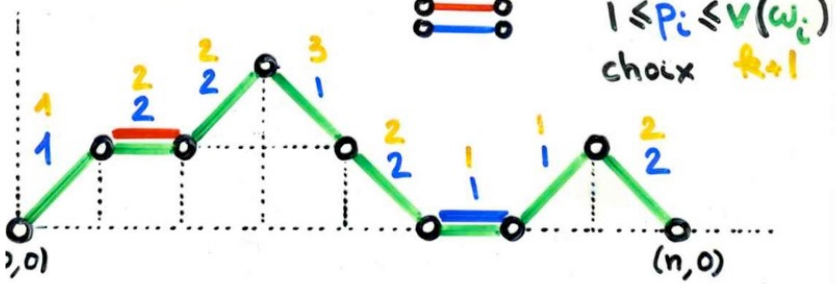
description with binary trees


1 ≤ p_i ≤ v(ω_i)
choix k+1
(n,0)

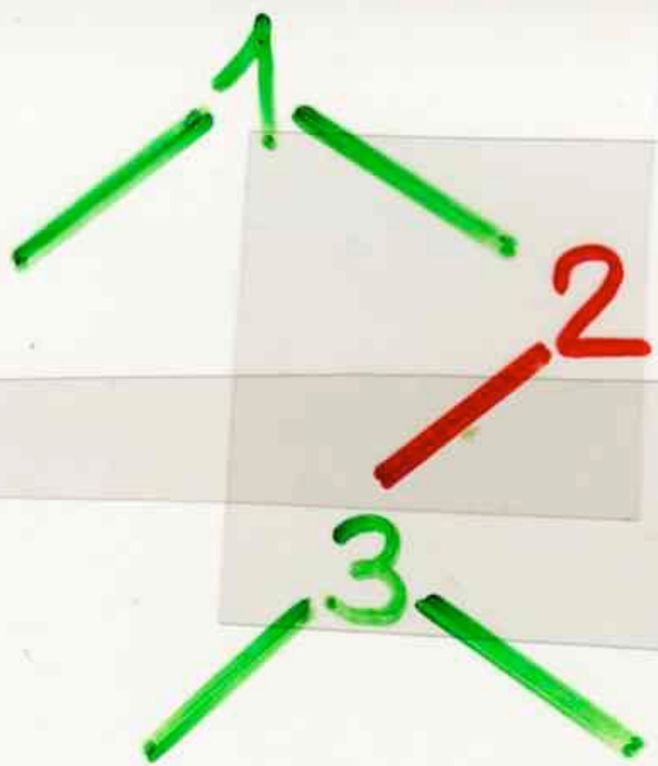



1 ≤ pi ≤ v(ωi)
choix
k+1
(n,0)

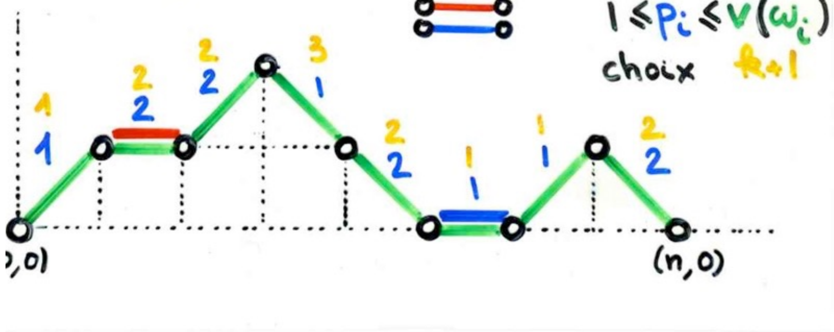

1
2

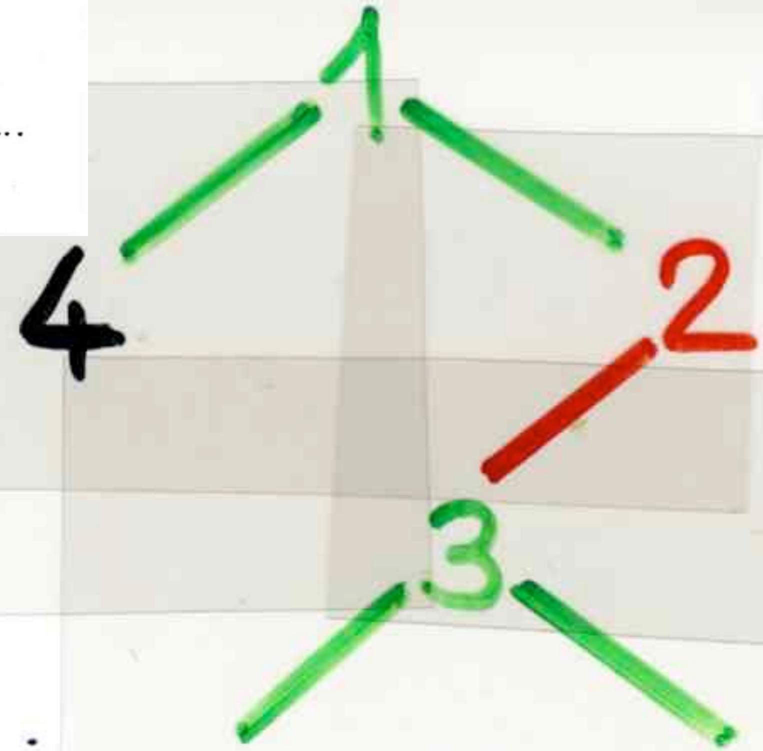


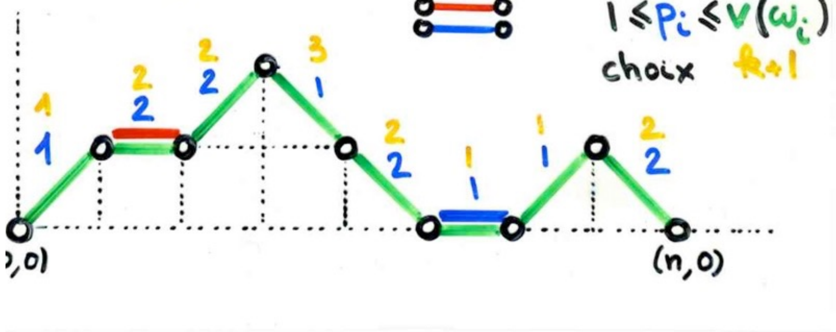
1 ≤ p_i ≤ v(ω_i)
choix
k+1
(n,0)

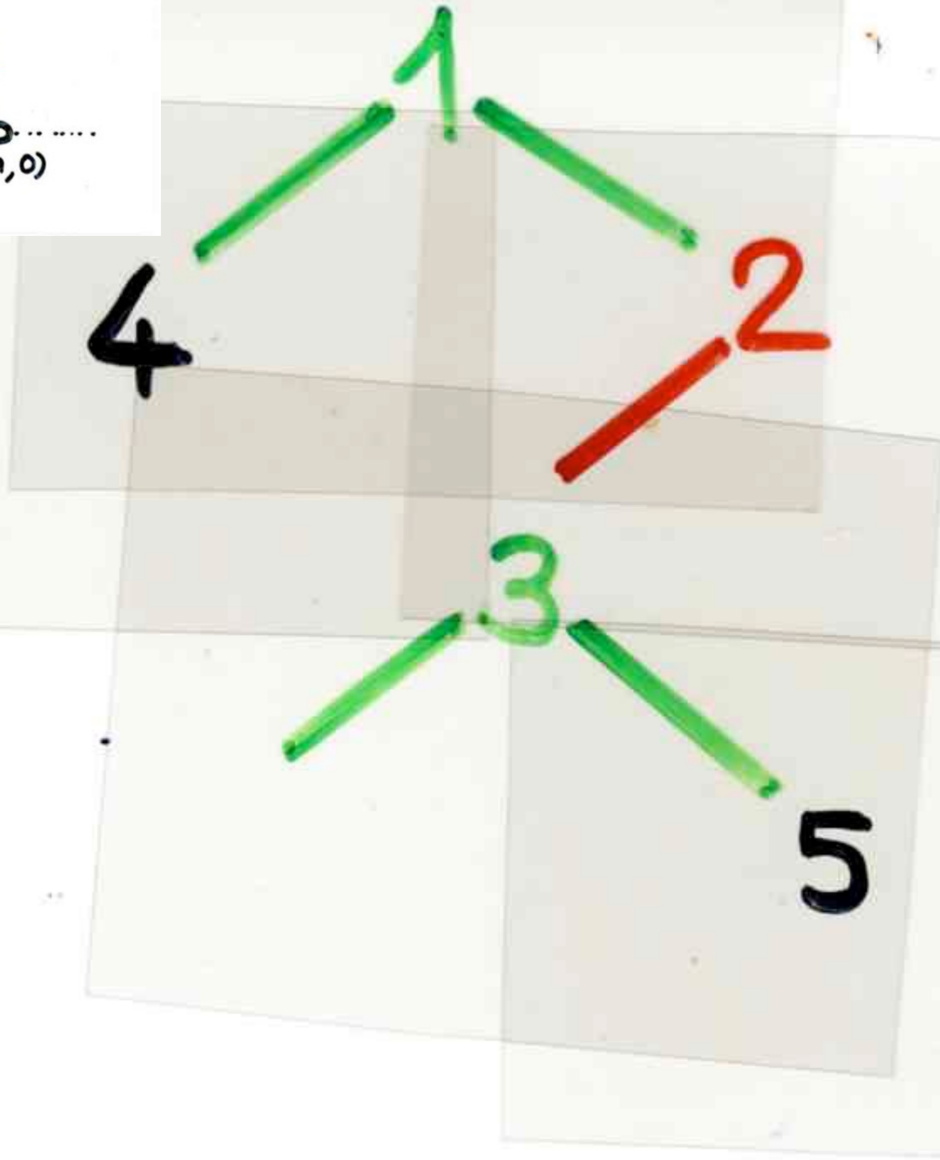


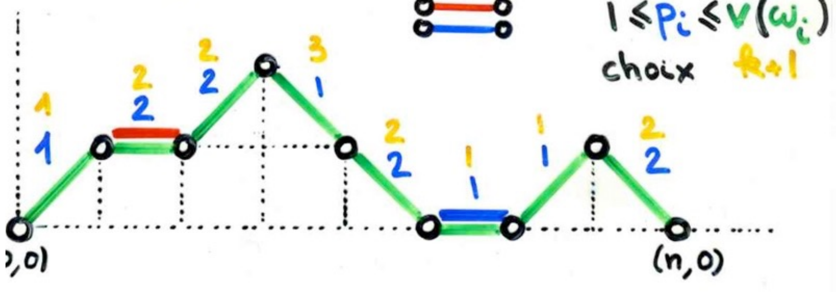
1
2
3


1 ≤ p_i ≤ v(ω_i)
choix
(n,0)

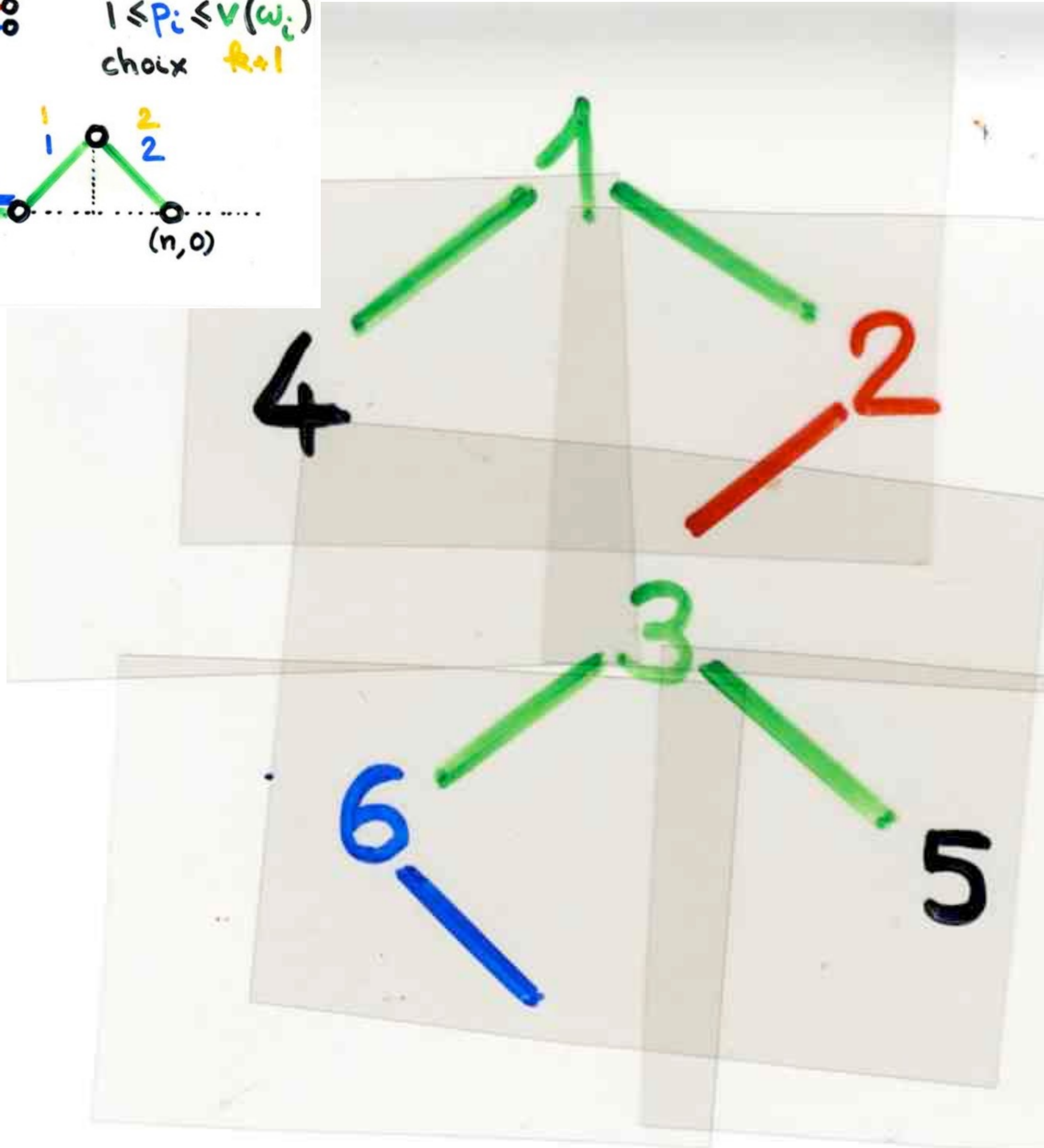

1
4
2
3


$1 \leqslant p_i \leqslant v(\omega_i)$
choix k+1
(n,0)


1
4
2
3
5



$1 \leq p_i \leq v(\omega_i)$
choix
k+1
(0,0)
(n,0)



1
4
2
3
6
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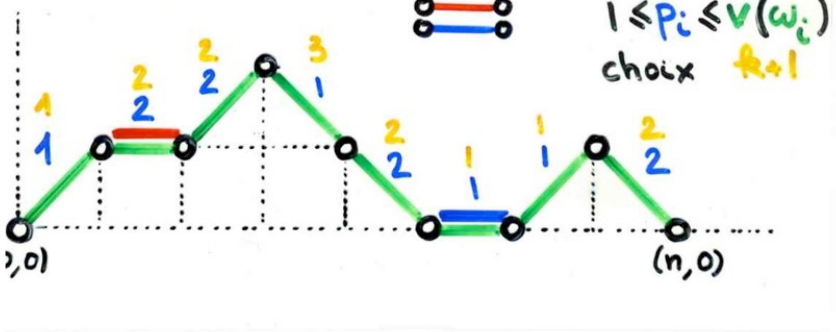
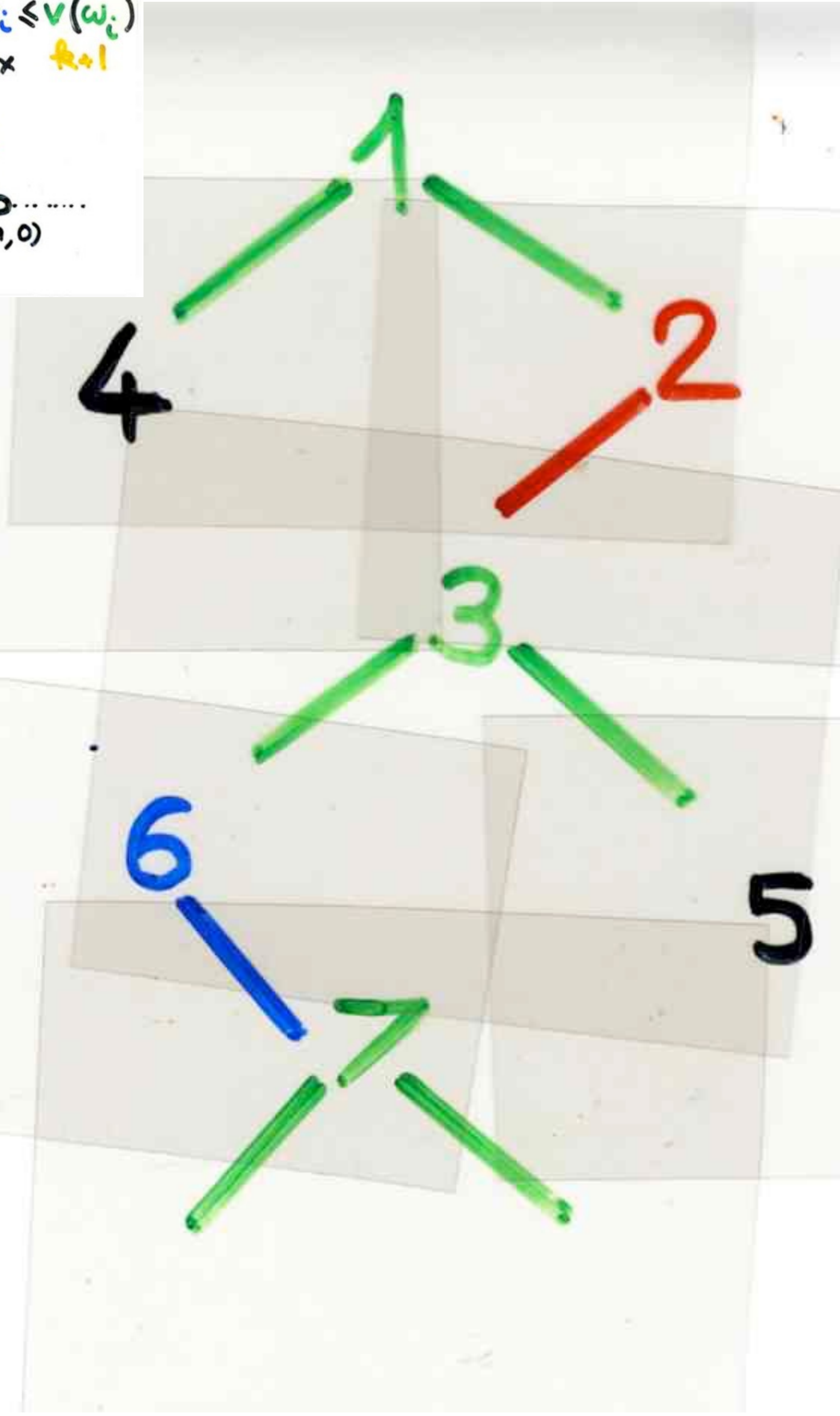


1 ≤ p_i ≤ $v(\omega_i)$
choix k+1
(n,0)



1
4
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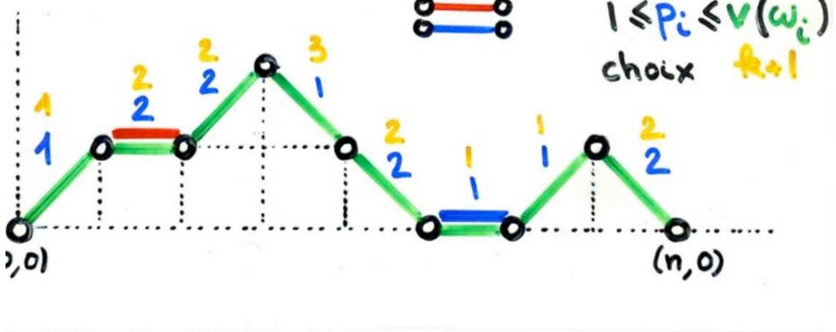
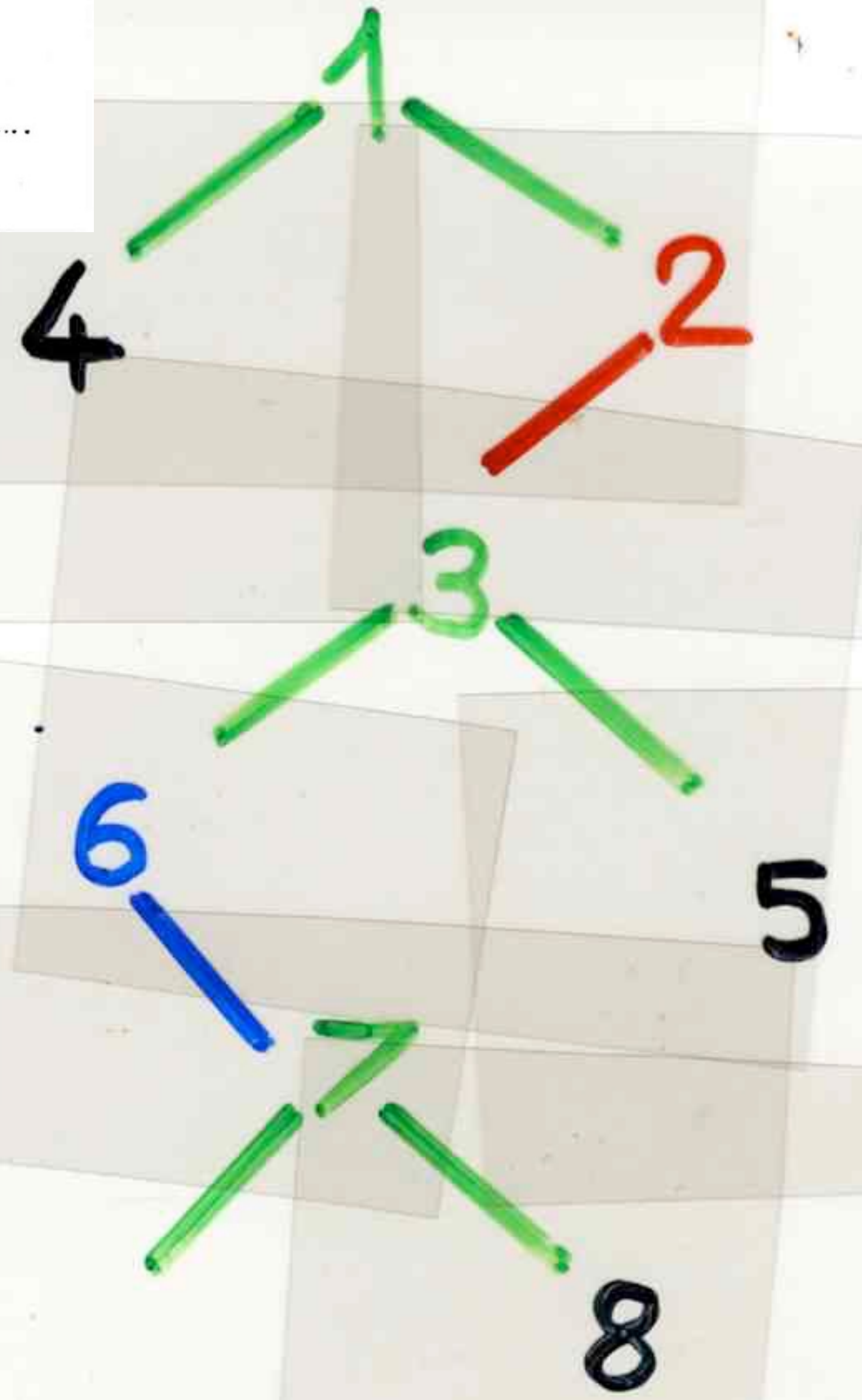

1 ≤ p_i ≤ v(ω_i)
choix k+1
(n,0)


1
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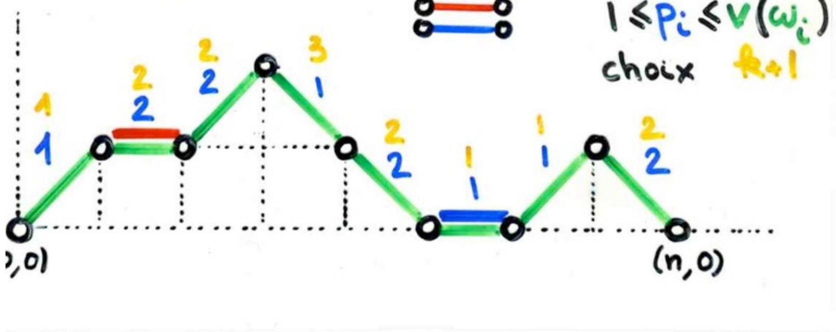
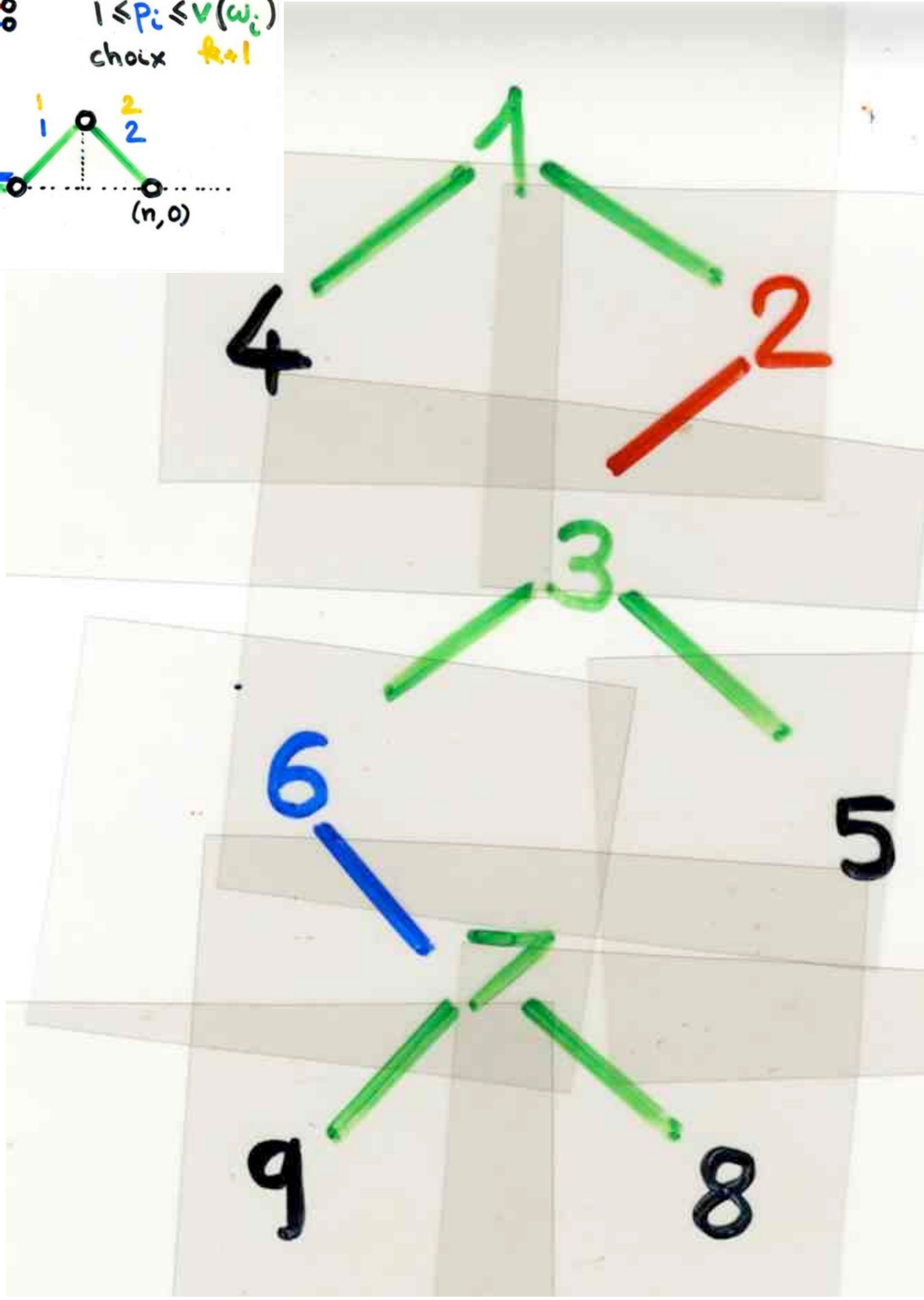


1 ≤ p_i ≤ v(ω_i)
choix k+1
(n,0)



1
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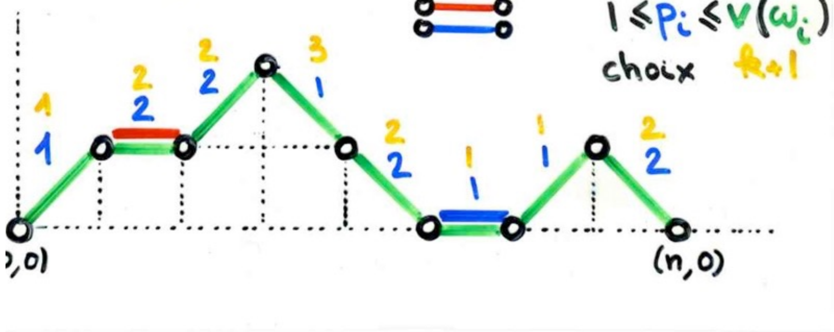
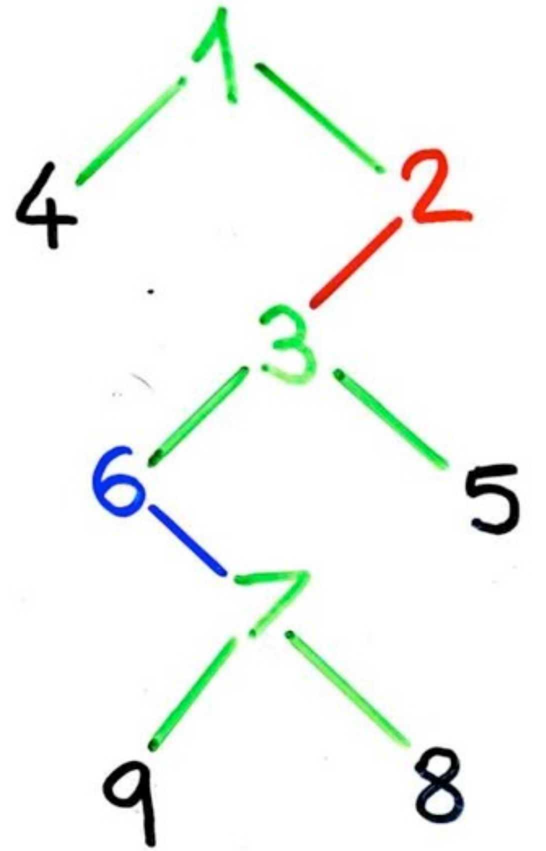




4 1 6 9 7 8 3 5 2

$$\mathcal{L}_n \xrightarrow{\ \theta\ } \mathcal{T}_{n+1} \xrightarrow{\ \pi\ } \mathfrak{S}_{n+1}$$

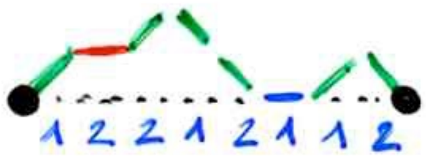
projection

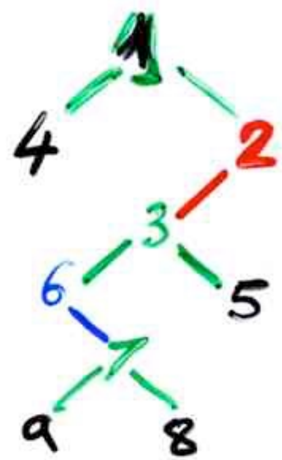
histoires de Laguerre — arbres binaires croissants — permutations

$$h = (\omega_c \,;\, (p_1, \dots, p_n))$$

↑ chemin Motzkin coloré

↑ fonction de possibilité





4 1 6 9 7 8 3 5 2

Hermite histories

Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

moments
Hermite
polynomials

$$\cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{3t}{\cdots}}}}$$

atque feries infinita ita fe habebit:

$$z = x - 1x^3 + 1.3x^5 - 1.3.5x^7 + 1.3.5.7x^9 - \text{etc.}$$

quae aequalis eft huic fractioni continuae:

$$z = \cfrac{x}{1 + \cfrac{1xx}{1 + \cfrac{2xx}{1 + \cfrac{3xx}{1 + \cfrac{4xx}{1 + \cfrac{5xx}{1 + \cfrac{6xx}{1 + \text{etc.}}}}}}}}$$

Si itaque ponatur $x = 1$, vt fiat:

DE FRACTIONIBVS CONTINVIS. DISSERTATIO.

AVCTORE

Leonh. Euler.

§. 1.

Varii in Analyſin recepti ſunt modi quantitates, quae alias difficulter aſſignari queant, commode exprimendi. Quantitates ſcilicet irrationales et transcendentes, cuiusmodi ſunt logarithmi, arcus circulares, aliarumque curuarum quadraturae, per ſeries infinitas exhiberi ſolent, quae, cum terminis conſtent cognitis, valores illarum quantitatum ſatis diſtincte indicant. Series autem iſtae duplicis ſunt generis, ad quorum prius pertinent illae ſeries, quarum termini additione ſubtractioneue ſunt connexi; ad poſterius vero referri poſſunt eae, quarum termini multiplicatione coniunguntur. Sic vtroque modo area circuli, cuius diameter eſt $= 1$, exprimi ſolet; priore nimirum area circuli aequalis dicitur $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$ etc. in infinitum; poſteriore vero modo eadem area aequatur huic expreſſioni $\frac{2.4.4.6.6.8.8.10.10}{3.3.5.5.7.7.9.9.11}$ etc. in infinitum. Quarum ſerierum illae reliquis merito praeferuntur, quae maxime conuergant, et pauciſſimis ſumendis terminis valorem quantitatis quaeſitae proxime praebeant.

§. 2. His duobus ſerierum generibus non immerito ſuperaddendum videtur tertium, cuius termini continua diui-

moments

Hermite

polynomials

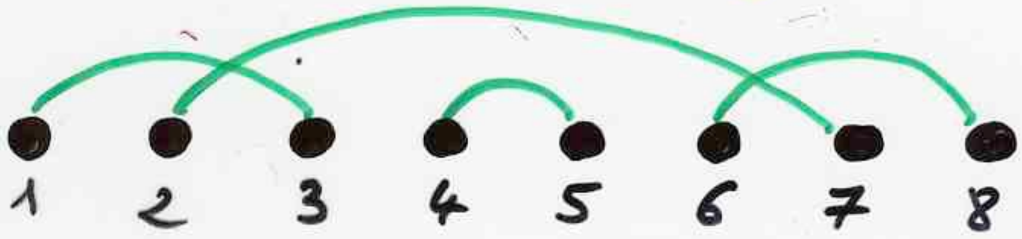
Hermite $\begin{cases} b_k = 0 \\ \lambda_k = k \end{cases}$

$\mu_{2n+1} = 0$

$\mu_{2n} = 1 \cdot 3 \cdot \ldots \cdot (2n-1)$

number of
involutions
no fixed point
on $\{1, 2, \ldots, 2n\}$

chord diagrams
perfect matching



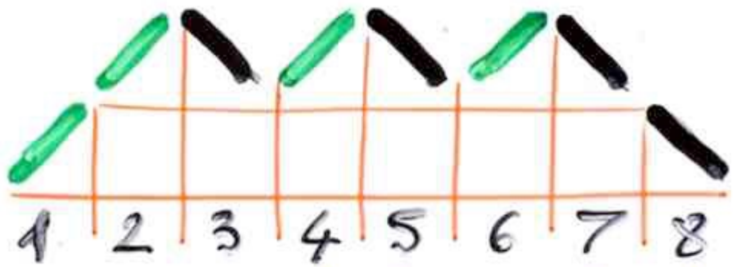
Hermite history

$$h = (\omega \; ; \; f)$$

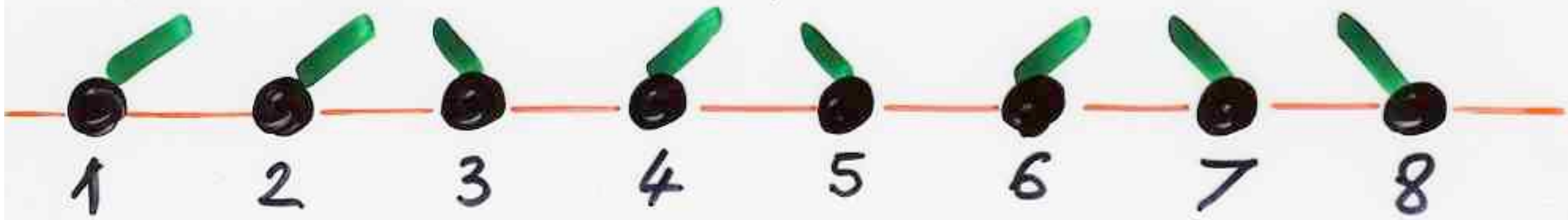
ω: Dyck path f: choice function

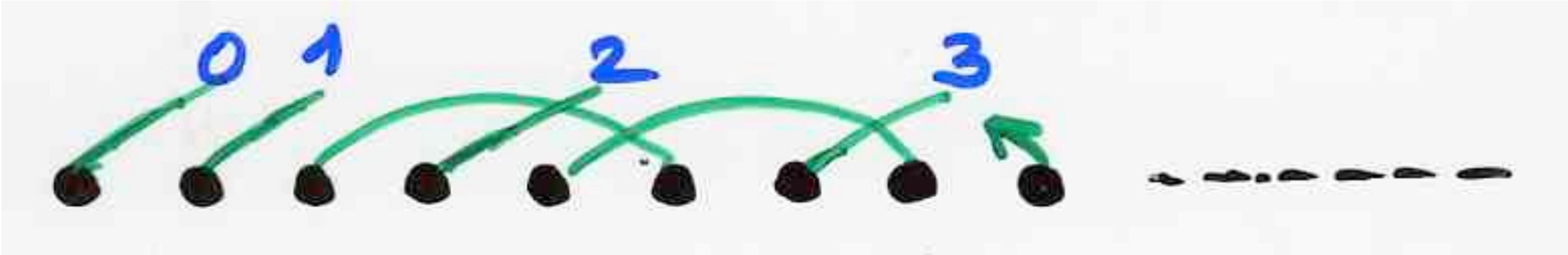
$\omega = \omega_1 \dots \omega_{2n}$ $f = (p_1, \dots, p_{2n})$

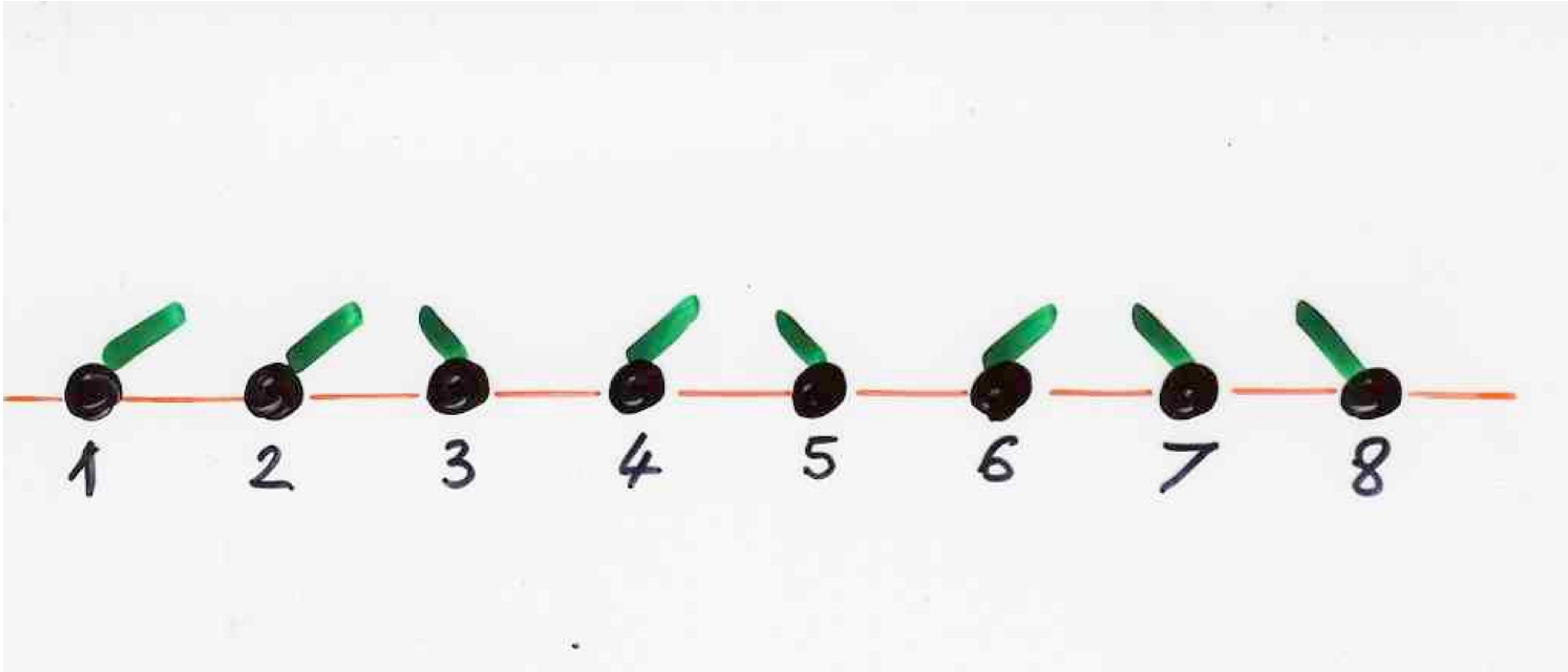
$p_i = 1$ $1 \le p_i \le v(\omega_i) = \lambda_{k_i}$

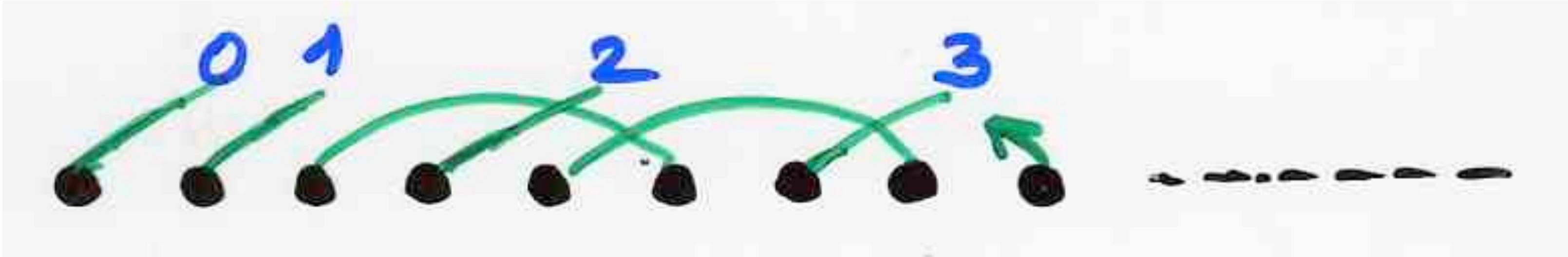


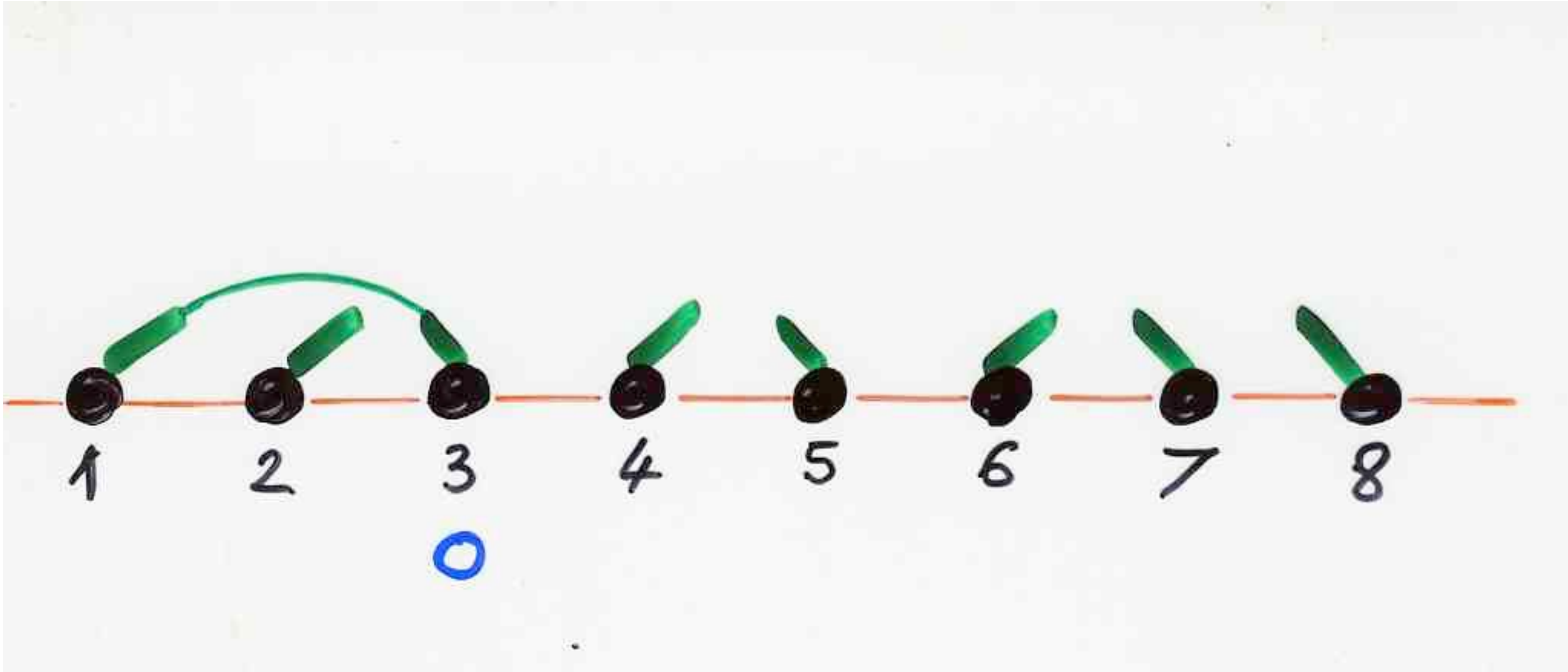
pos 1 1 1 1 2 1 1 1

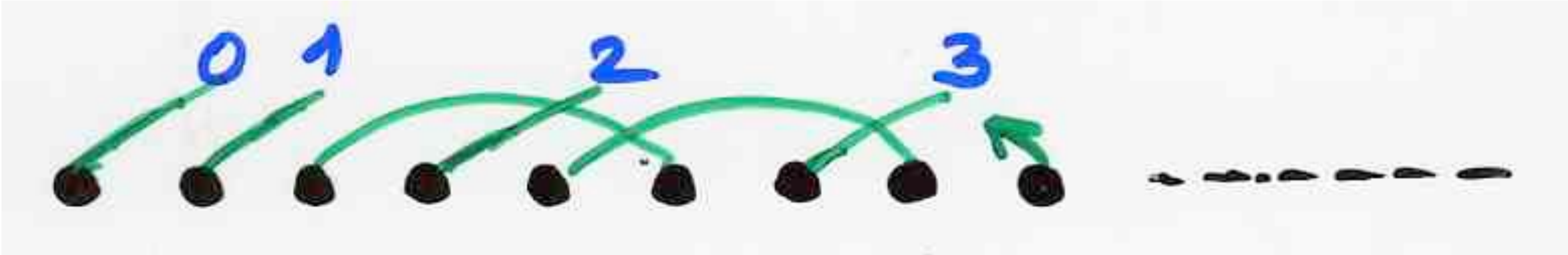

1
2
3
4
5
6
7
8

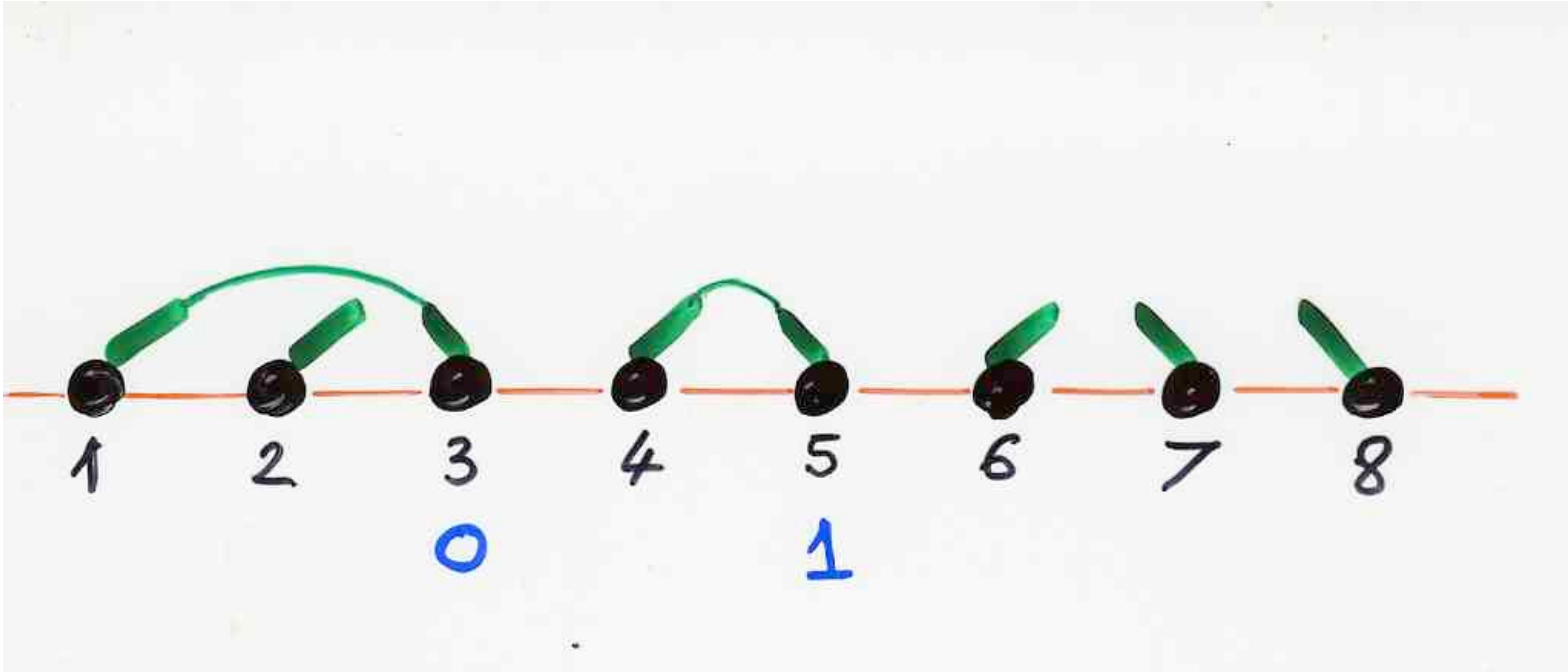

0
1
2
3

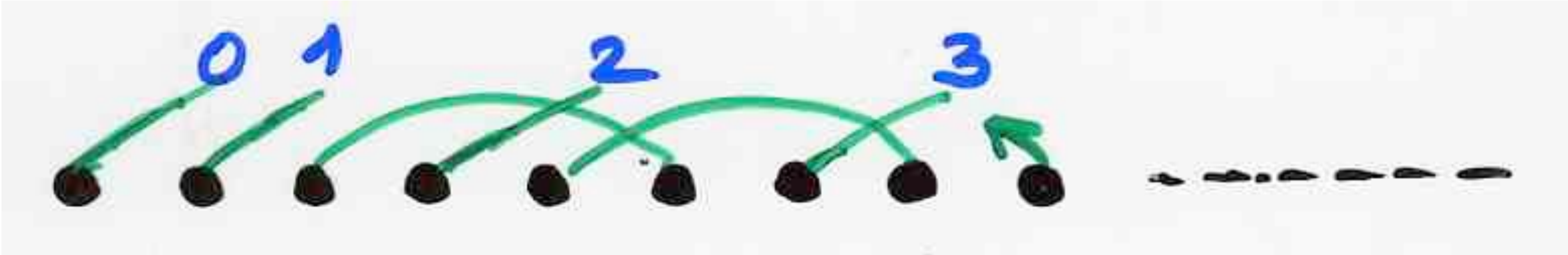

1
2
3
4
5
6
7
8

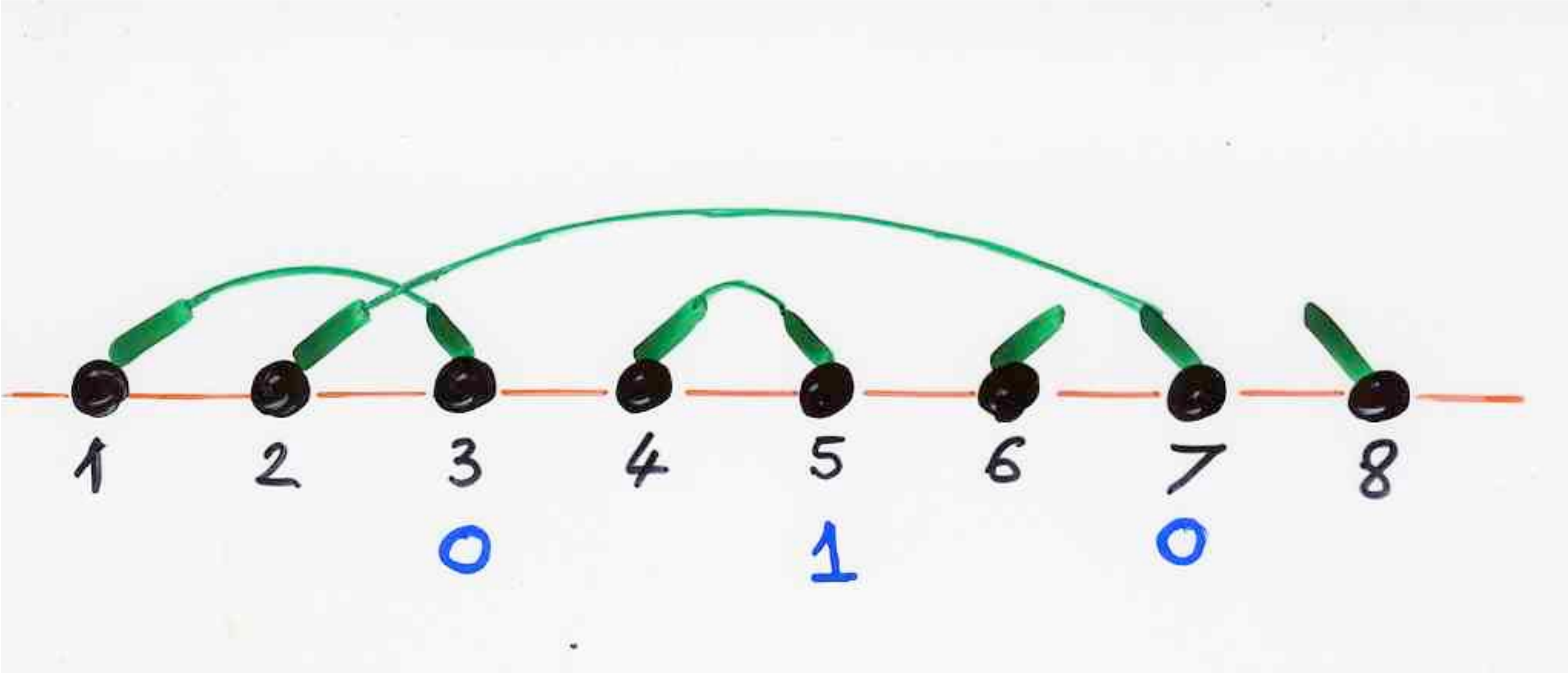

0
1
2
3

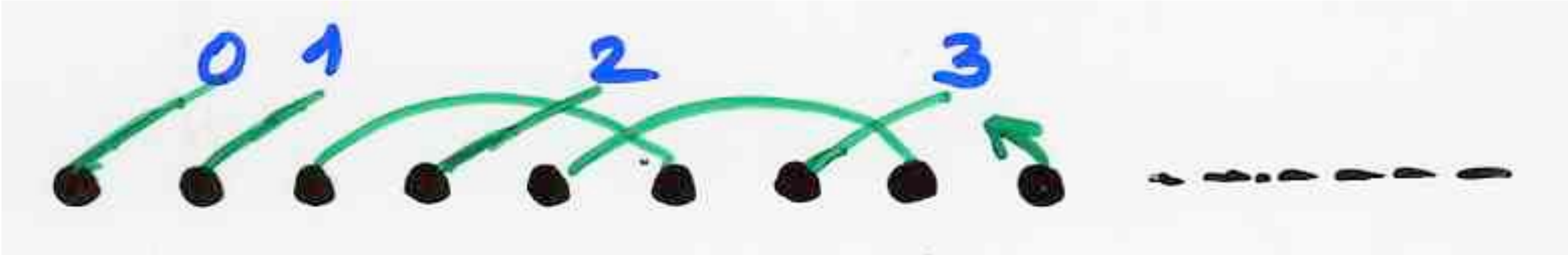

1
2
3
4
5
6
7
8
0


0
1
2
3

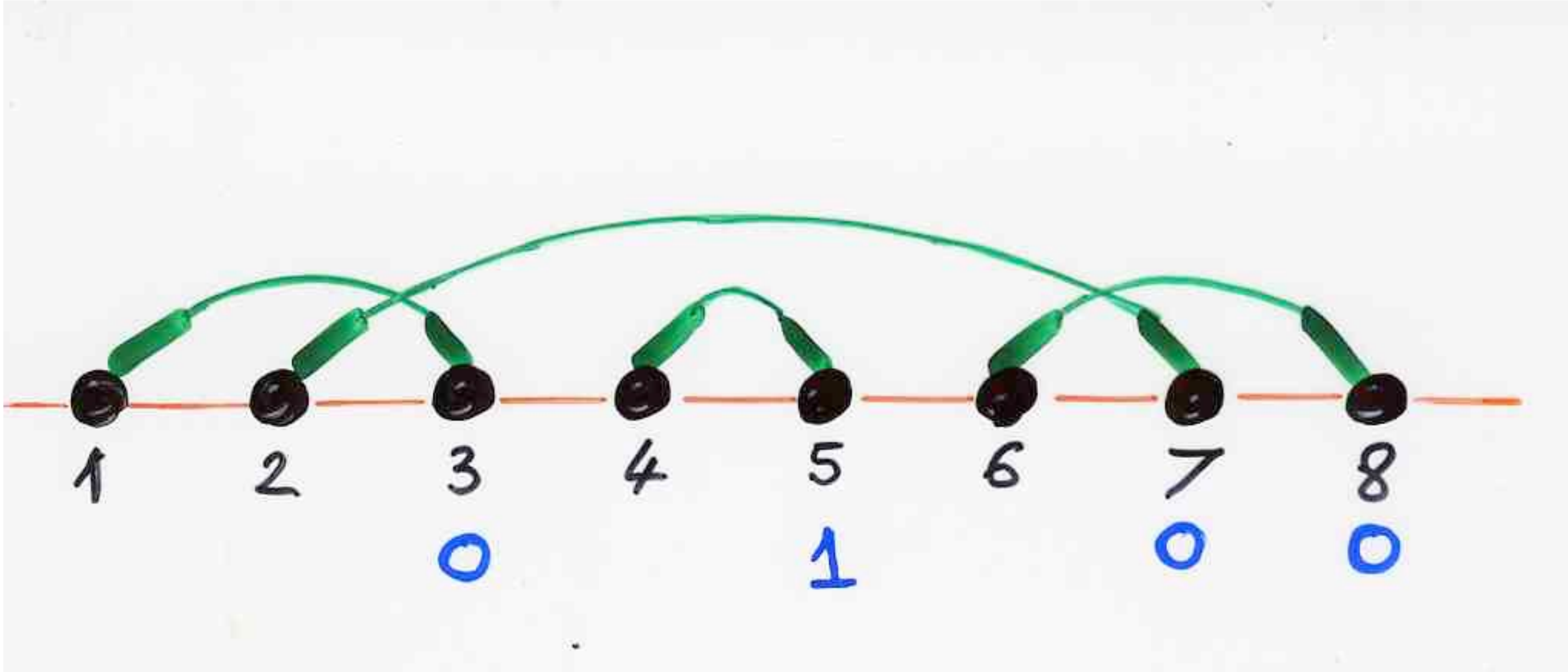

1
2
3
4
5
6
7
8
0
1


0
1
2
3


1
2
3
4
5
6
7
8
0
1
0



0
1
2
3



1
2
3
4
5
6
7
8
0
1
0
0

q-analog of Hermite histories

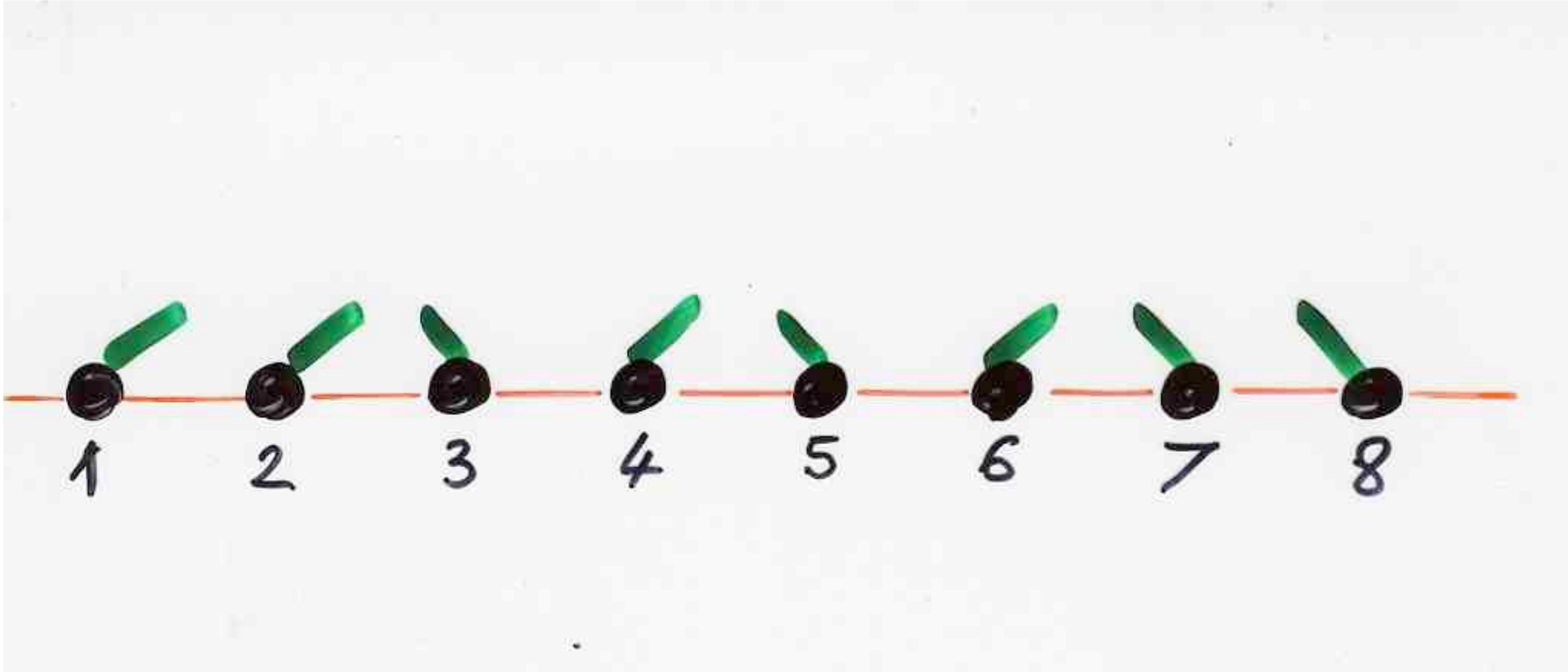
q-Hermite

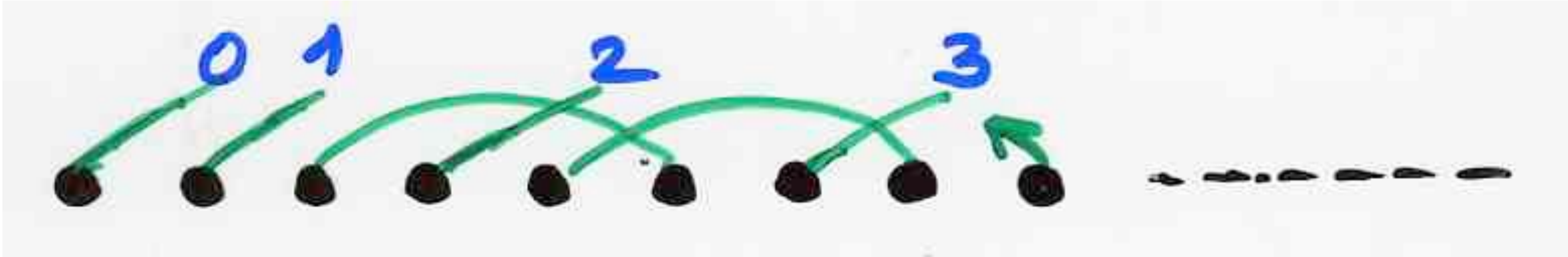
$H^{I}_{n}(x; q)$

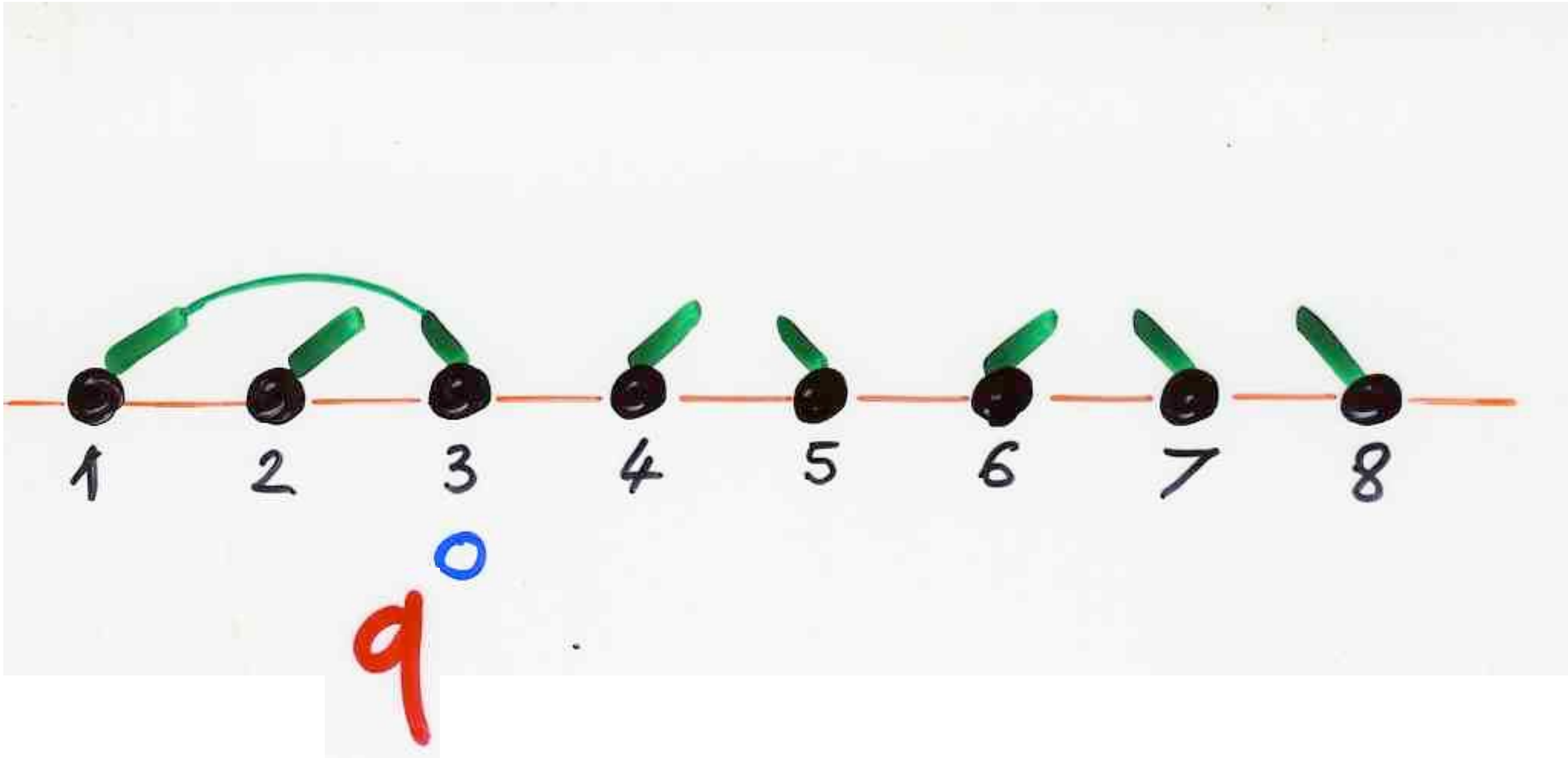
$b_k = 0$

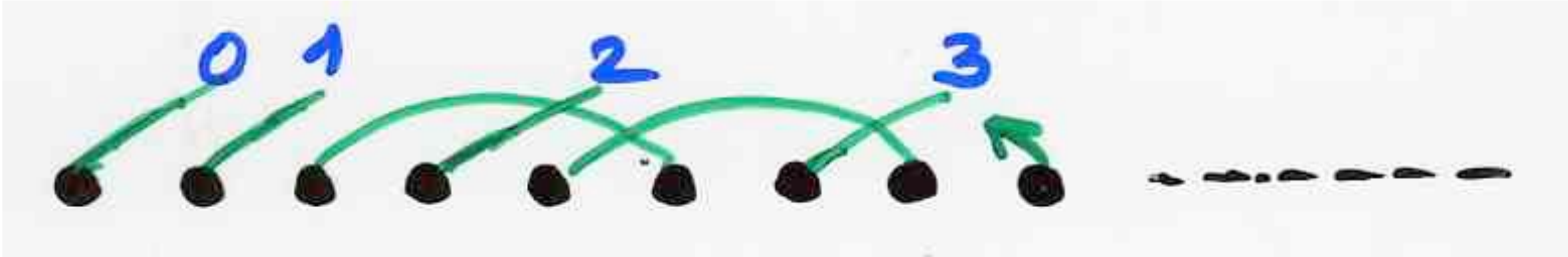
$\lambda_k = [k]_q$

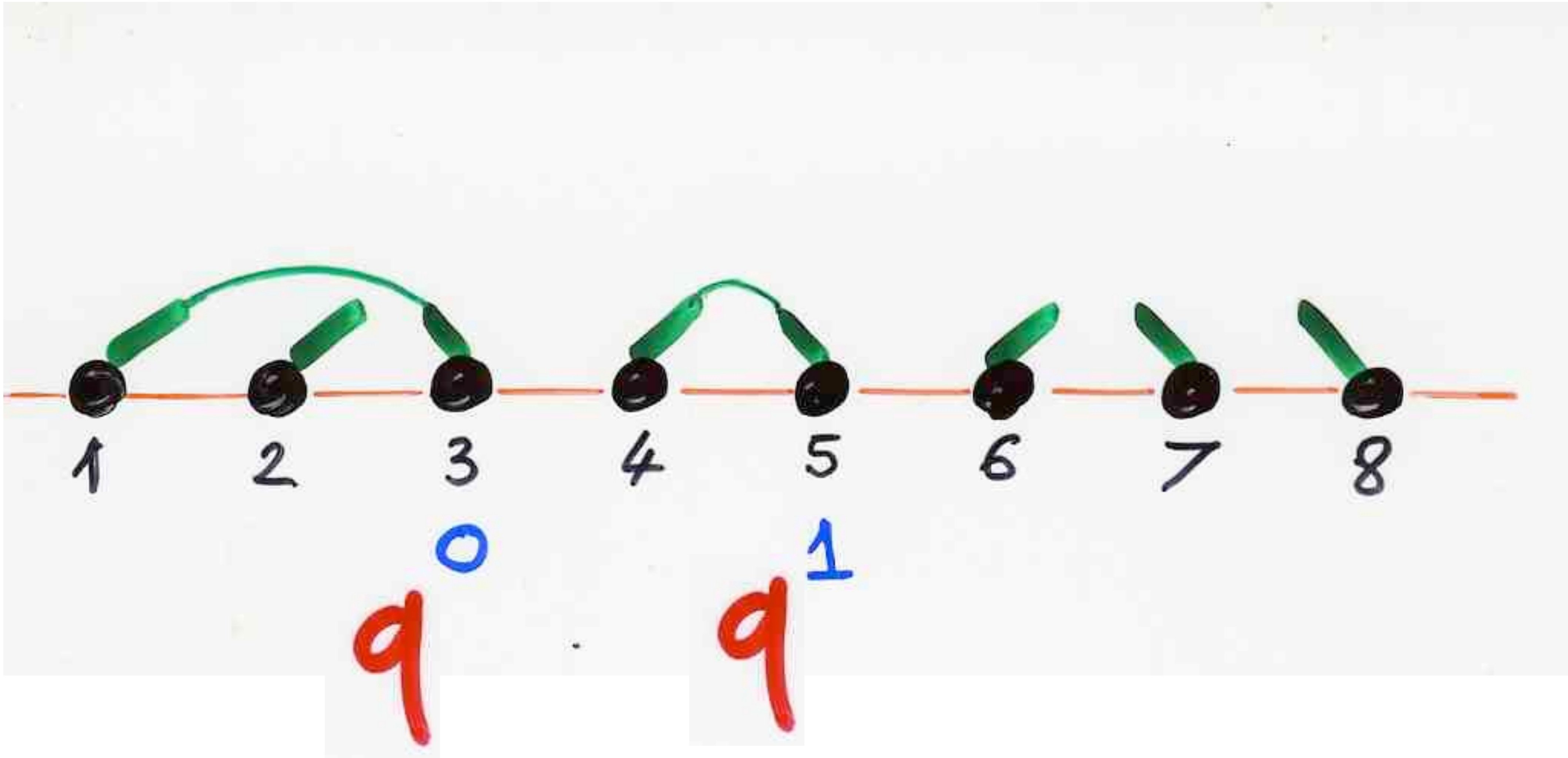
$= 1 + q + \ldots + q^{k-1}$

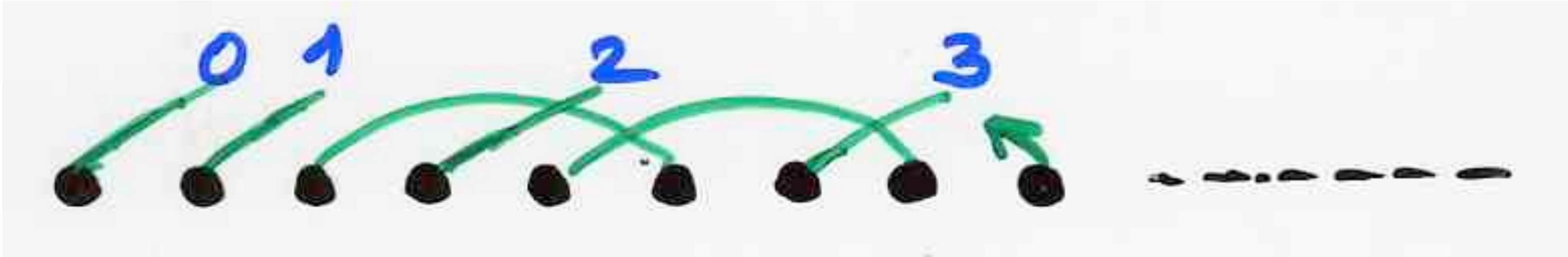

1
2
3
4
5
6
7
8

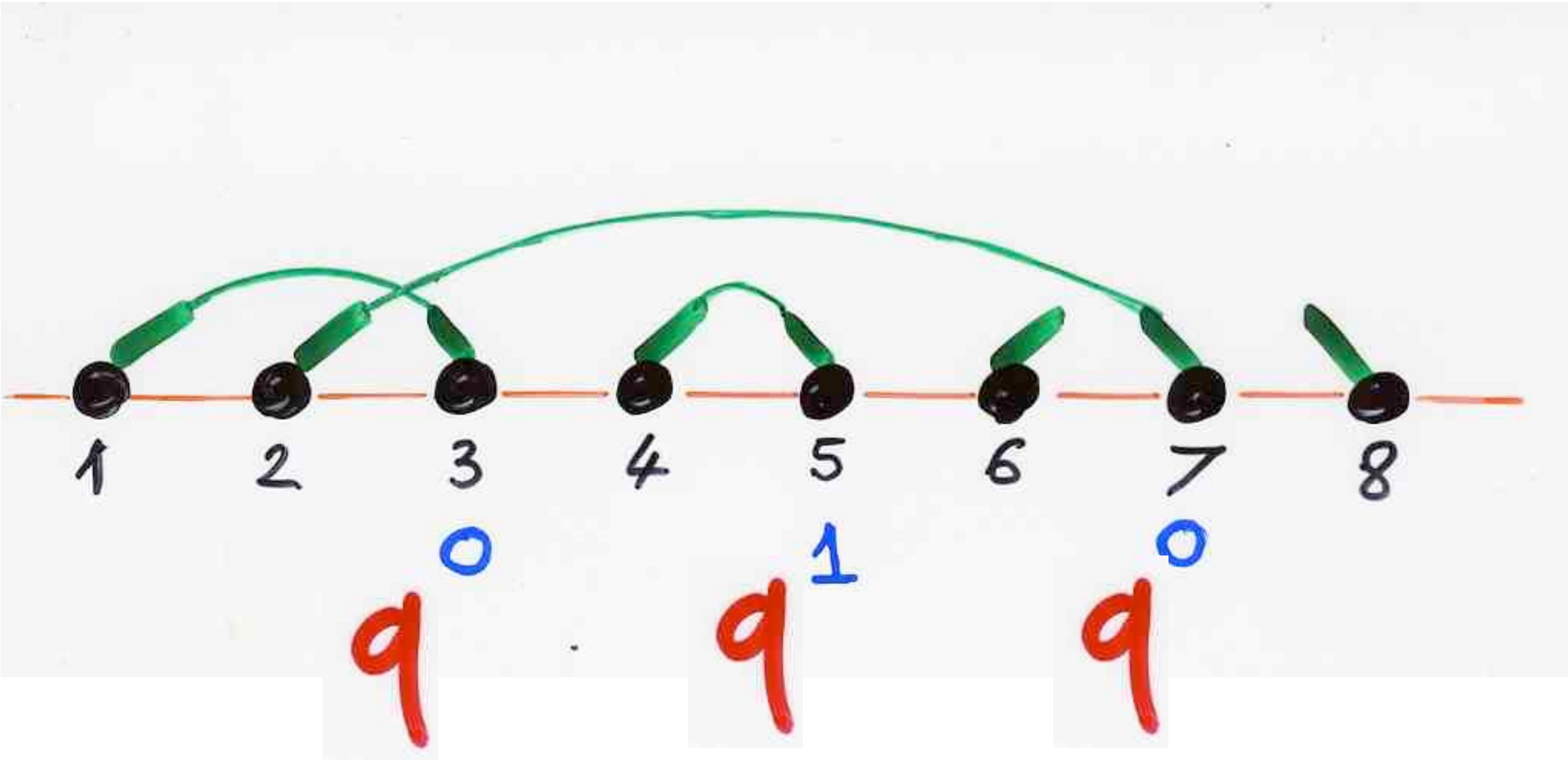

0 1 2 3


1 2 3 4 5 6 7 8
9
0

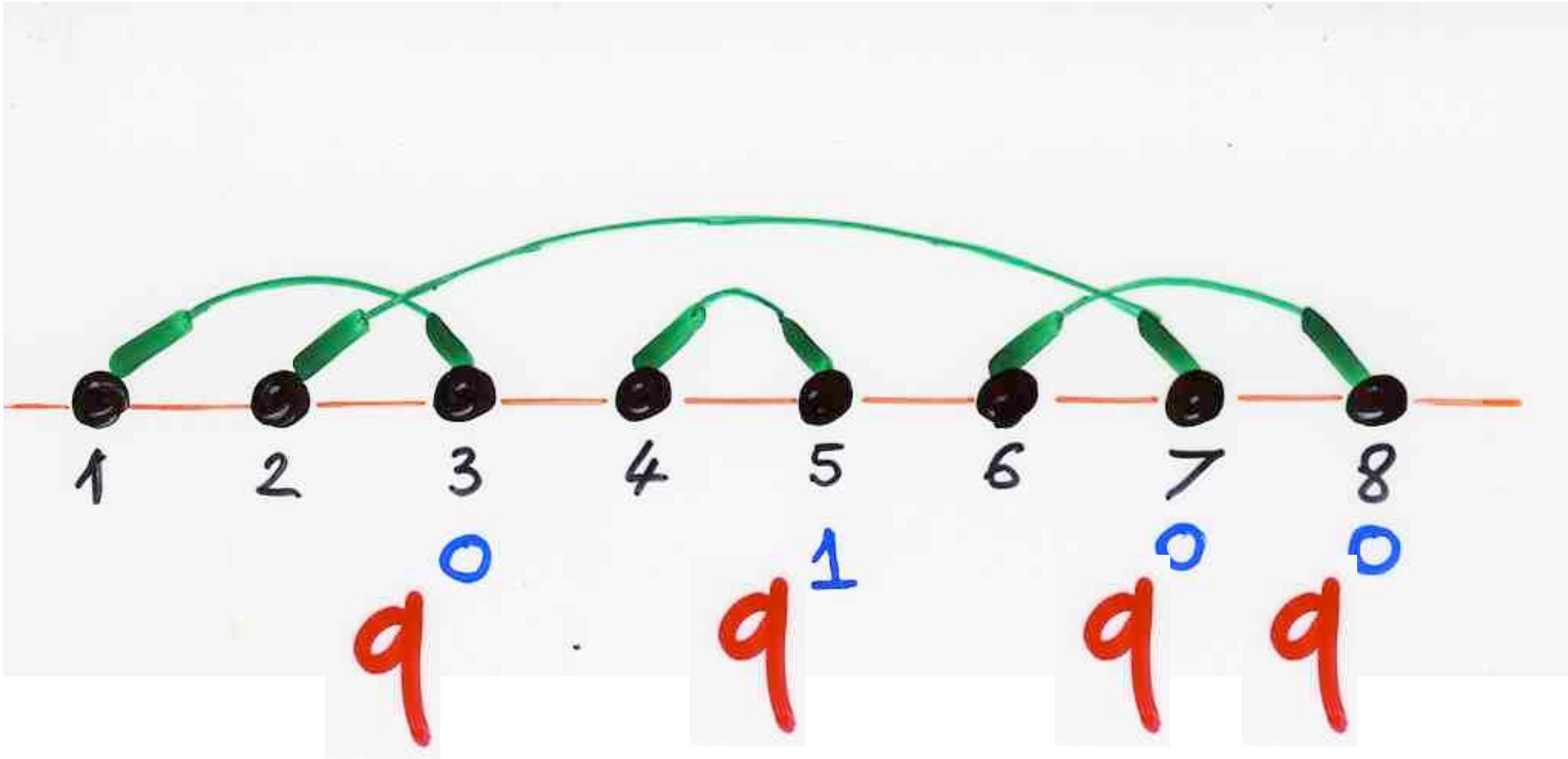

0 1 2 3

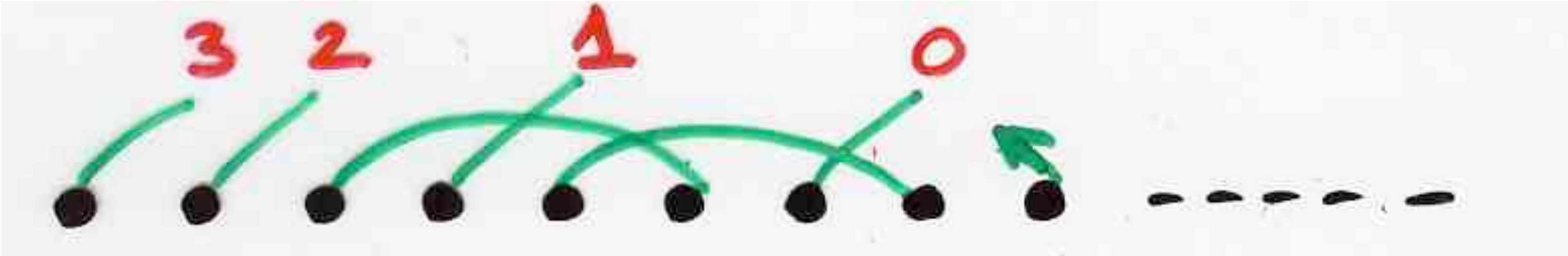

1 2 3 4 5 6 7 8
q^0
q^1

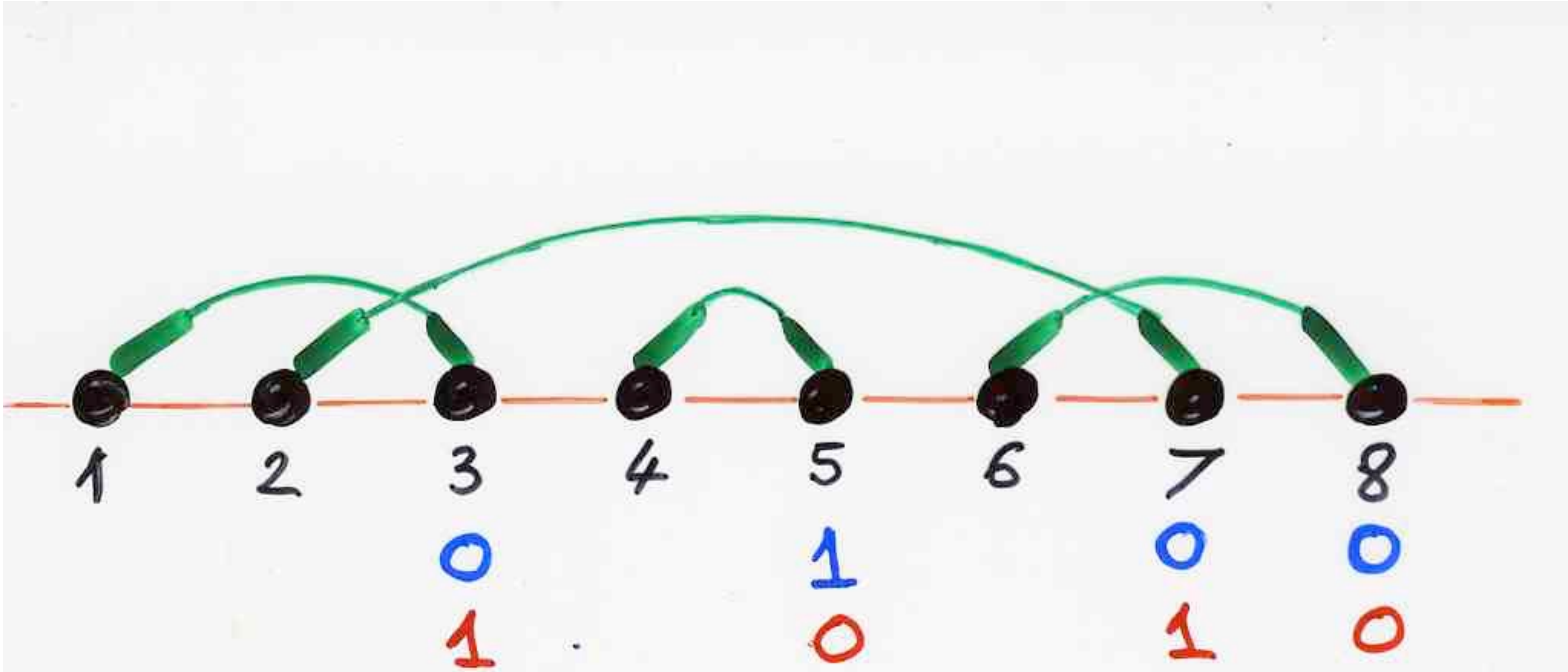

0 1 2 3

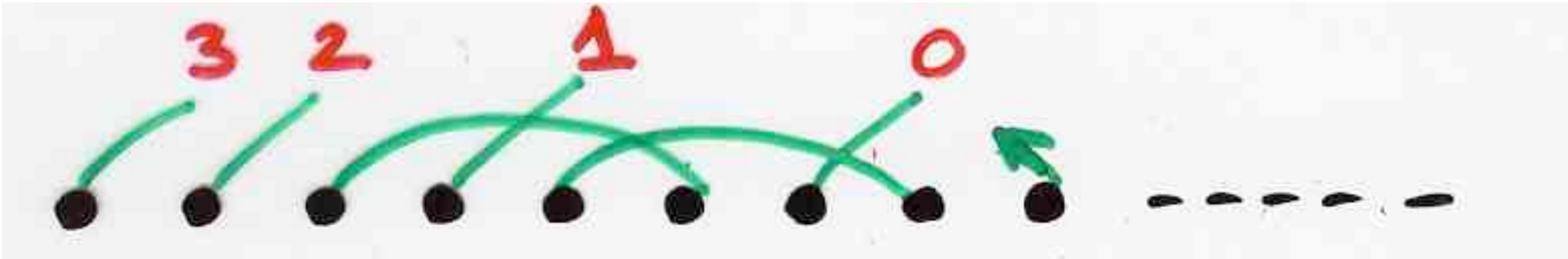

1 2 3 4 5 6 7 8
0 1 0
q q q

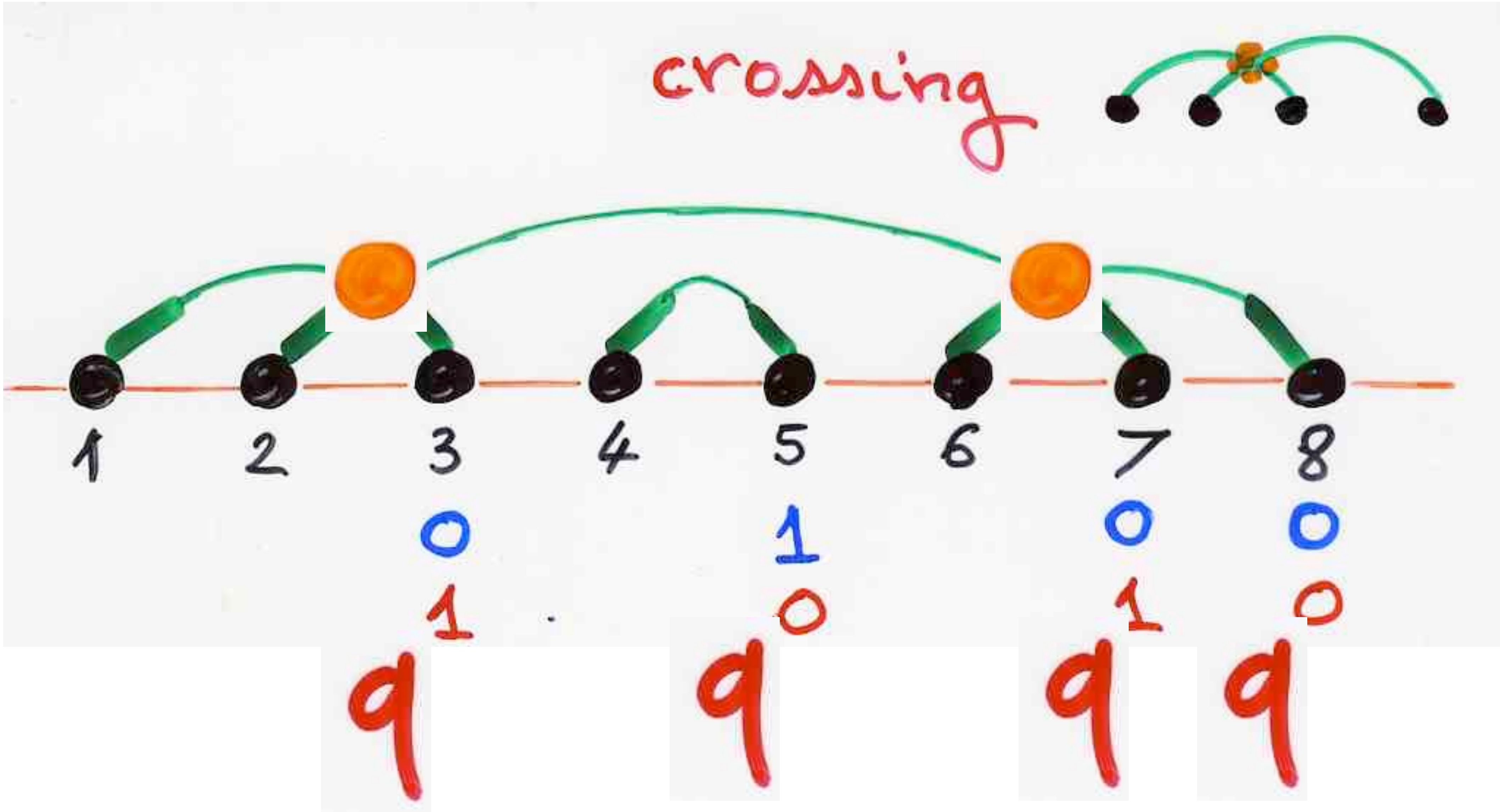

nesting


1 2 3 4 5 6 7 8
0 1 0 0
q q q q


3 2 1 0


1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0


3 2 1 0


crossing
1 2 3 4 5 6 7 8
0 1 0 0
1 0 1 0
q q q q

Sheffer orthogonal polynomials

orthogonal

polynomials

(binomial type)

Scheffer type

$$\sum P_n(x) \frac{t^n}{n!} = g(t) e^{x f(t)}$$

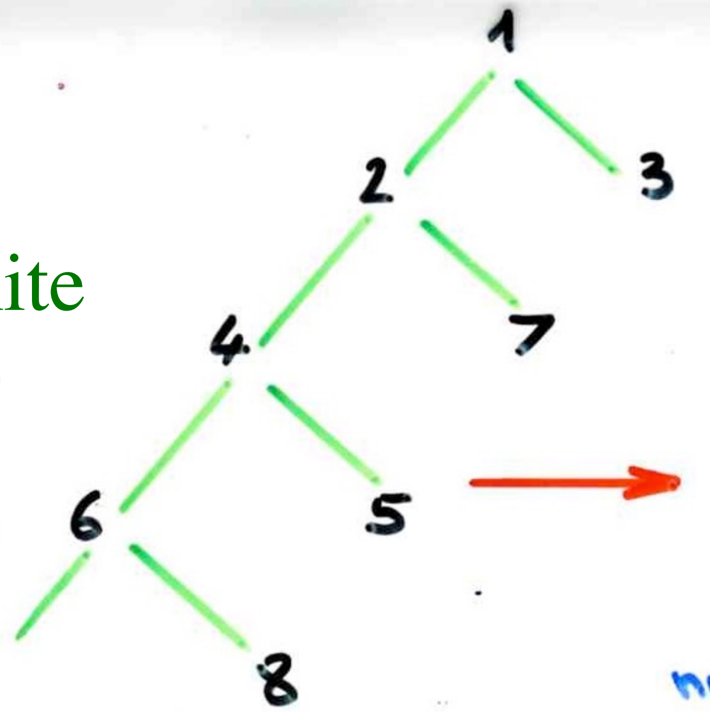
orthogonal
polynomials

(binomial type)
Sheffer type

$$\sum P_n(x)\frac{t^n}{n!} = g(t)e^{xf(t)}$$

- Hermite H_n
- Laguerre $L_n^{(\alpha)}$
- Charlier $C_n^{(a)}$
- Meixner I $M_n^{I\,(\alpha,\beta)}$
- Meixner II $M_n^{II\,(\delta,\eta)}$

Hermite



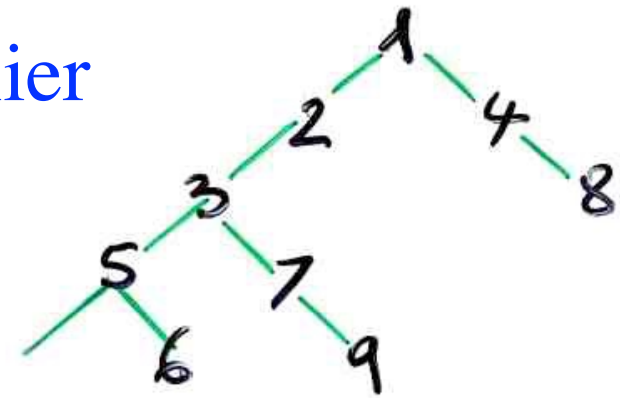
Involution

$$\sigma = (1\,3)(2\,7)(4\,5)(6\,8)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 5 & 4 & 8 & 2 & 6 \end{pmatrix}$$

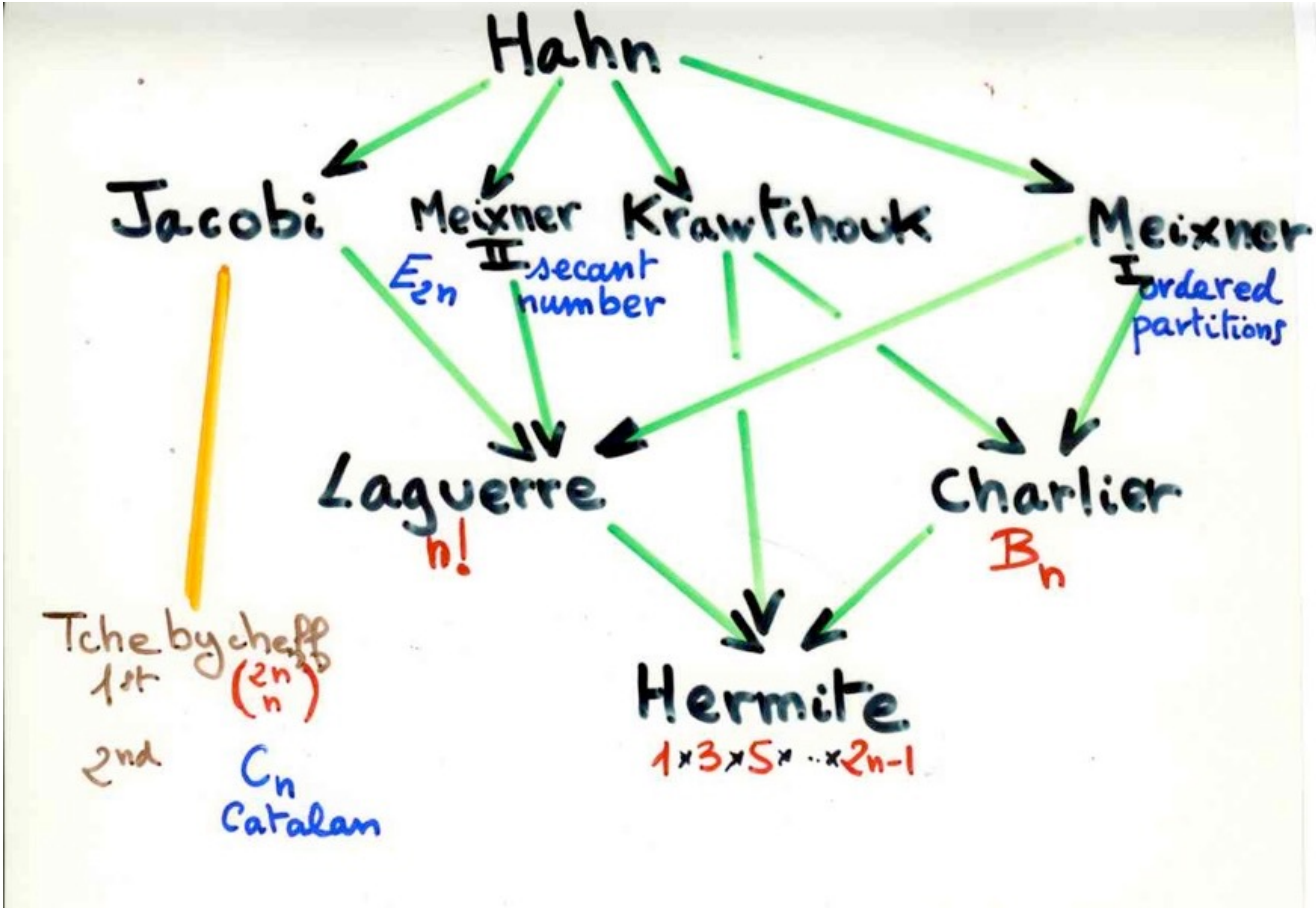
no fixed points

Charlier



$\{1, 4, 8\}$
$\{2\}$
$\{3, 7, 9\}$
$\{5, 6\}$

Askey-Wilson



$$\mathrm{tg}(t) = \sum_{n \geq 0} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}$$

$$\frac{1}{\cos(t)} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

E_{2n} $\{1, 5, 61, 1385, \dots\}$

nombres sécant (d'Euler)

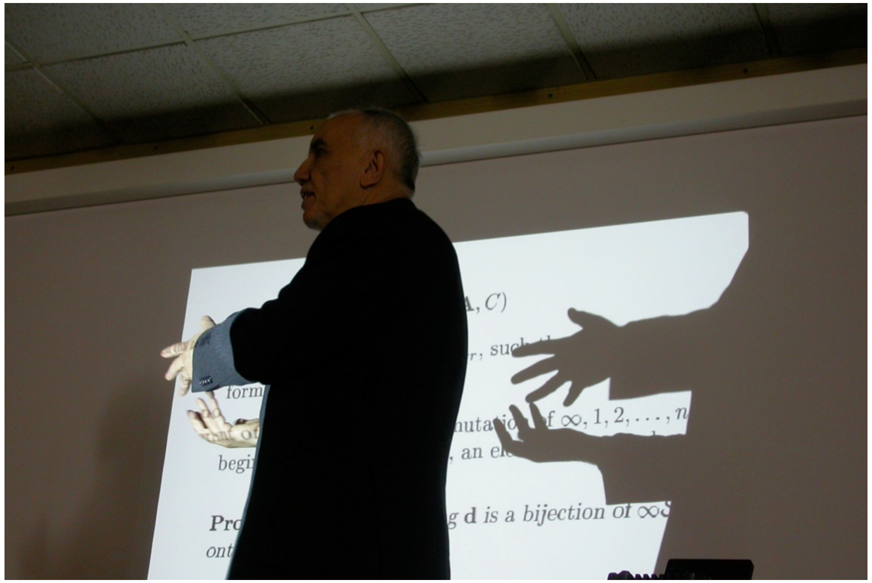
T_{2n+1} $\{1, 2, 16, 272, 7936, \dots\}$

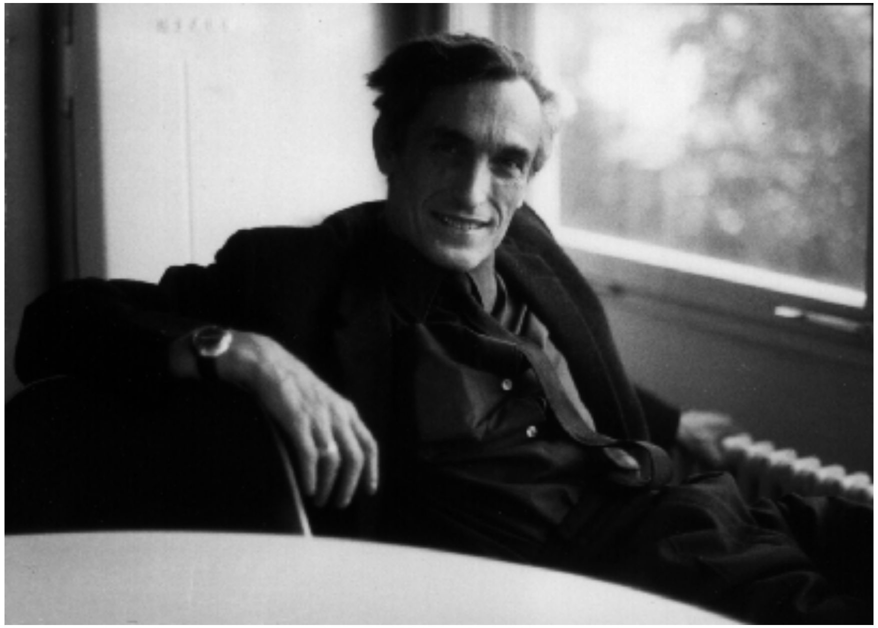
nombres tangents

Permutations alternantes

D. André (1880)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 2 & 9 & 7 & 8 & 4 & 5 & 1 & 3 \end{pmatrix}$$





D. Foata
M.P. Schützenberger

"Théorie géométrique des polynômes Eulériens" (1970)

$$\int_0^\infty e^{-t}\tan(tu)\,dt = \cfrac{1}{1-\cfrac{1.2\,u^2}{1-\cfrac{2.3\,u^2}{1-\cfrac{3.4\,u^2}{\cfrac{\cdots}{1-\cfrac{k(k+1)u^2}{\cdots}}}}}}$$

Laplace transform

Polynômes	$b_k = b'_k + b''_k$	$\lambda_k = a_{k-1} c_k$	Moments
Tchebycheff unitaires $U_n(x)$ $T_n(x)$	0 0	$1/4$ $1/4$ $\lambda_0 = 1/2$	$\frac{1}{4^n} C_n$ Catalan $\frac{1}{4^n} \binom{2n}{n}$
Laguerre $L_n^1(x)$ $L_n^\alpha(x)$			$(n+1)!$ $(\alpha+1)\dots(\alpha+n) = (\alpha+1)_n$
Hermite $H_n(x)$			$\mu_{2n} = 1.3\dots(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^a(x)$			$\sum S(n,k)\, a^k$
Meixner I $\hat{m}_n(x; \beta, c)$ Kreweras $\beta = 1$ $c = 1/2$			$\dfrac{\sum_{\sigma \in \mathfrak{S}_n} \beta^{s(\sigma)} c^{1+d(\sigma)}}{(1-c)^n}$ $= (1-c)^\beta \sum_{k \geq 0} k^n c^k \dfrac{(\beta)_k}{k!}$
Meixner II $M_n(x; \delta, \eta)$ $\delta = 0$ $\eta = 1$			$\delta^n \sum_{\sigma \in \mathfrak{S}_n} \eta^{s(\sigma)} \left(1 + \frac{1}{\delta^2}\right)^{p(\sigma)}$ E_{2n} Sécant

Polynômes	$b_k = b'_k + b''_k$	$\lambda_k = a_{k-1} c_k$	Moments
Tchebycheff unitaires $U_n(x)$	0	1/4	$\frac{1}{4^n} C_n$ Catalan
$T_n(x)$	0	1/4 $\lambda_0 = 1/2$	$\frac{1}{4^n} \binom{2n}{n}$
Laguerre $L_n^1(x)$	$2k+2$	$k(k+1)$	$(n+1)!$
$L_n^{\alpha}(x)$	$2k+\alpha+1$	$k(k+\alpha)$	$(\alpha+1)\dots(\alpha+n) = (\alpha+1)_n$
Hermite $H_n(x)$	0	k	$\mu_{2n} = 1.3\dots(2n-1)$ $\mu_{2n+1} = 0$
Charlier $C_n^{a}(x)$	$k+a$	ak	$\sum S(n,k)\, a^k$
Meixner I $\hat{m}_n(x;\beta,c)$	$\frac{(1+c)k+\beta c}{1-c}$	$\frac{c\,k(k-1+\beta)}{(1-c)^2}$	$\frac{\sum_{\sigma\in\mathfrak{S}_n} \beta^{s(\sigma)} c^{1+d(\sigma)}}{(1-c)^n}$
Kreweras $\beta=1$ $c=1/2$	$3k+1$	$2k^2$	$=(1-c)^{\beta} \sum_{k\geq 0} k^n c^k \frac{(\beta)_k}{k!}$
Meixner II $M_n(x;\delta,\eta)$	$(2k+\eta)\delta$	$(\delta^2+1)k(k-1+\eta)$	$\delta^n \sum_{\sigma\in\mathfrak{S}_n} \eta^{s(\sigma)} \left(1+\frac{1}{\delta^2}\right)^{p(\sigma)}$
$\delta=0$ $\eta=1$	0	k^2	E_{2n} Sécant

www.xavierviennot.org

see the page: «livres»

more on X.G.V. website

X.G.V.: Une théorie combinatoire des polynômes orthogonaux
Notes de cours, 217p., LACIM, UQAM, Montréal, 1984 (french)

also on the page «petite école»,
a series of lectures with slides given at LaBRI, Bordeaux, in 2006/2007 (mixture of french and english)