Combinatorics and Physics

Chapter 7

The cellular ansatz

Ch7d

From a representation of the PASEP algebra
to a bijection permutations — alternating tableaux

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combinatorial representation of the operators E and D

PASEP algebra

$$DE = qED + E + D$$

V vector space generated by B basis

B alternating words two letters {○, ●}

(no occurence of ○○ or ●●)

4 operators A, S, J, K

V vector space

<u>notations</u> - A operator $V \longrightarrow V$ (linear map)

$v \in V$ $\quad \langle v | A = A(v)$

B basis of V, $v_0 \in B$

$v | v_0 \rangle =$ coeff. of v on $v_0 \in B$

$\langle v_0 | A | v_0 \rangle$

4 operators $A, S, J, K,$ $\quad u \in B$

$$\langle u | A = \sum_{\text{letter } \circ \text{ of } u} v$$, v obtained by: $\circ \to \circ\bullet\circ$

$$\langle u | S = \sum_{\circ \text{ of } u} v$$ v obtained by: $\circ \to \bullet$ (and $\bullet\bullet \to \bullet \quad \bullet\bullet\bullet \to \bullet$)

$$\langle u | J = \sum_{\circ \text{ of } u} v$$ v, $\circ \to \bullet\circ$ (and $\bullet\bullet \to \bullet$)

$$\langle u | K = \sum_{\circ \text{ of } u} v$$ v, $\circ \to \circ\bullet$ (and $\bullet\bullet \to \bullet$)

$$\circ\bullet\circ\bullet \,|\, S = \bullet\circ\bullet + \circ\bullet$$

$$AS = SA + J + K$$

$$AK = KA + A$$

$$JS = SJ + S$$

$$JK = KJ$$

$$D = A + J$$

$$E = S + K$$

$$DE = (A+J)(S+K)$$

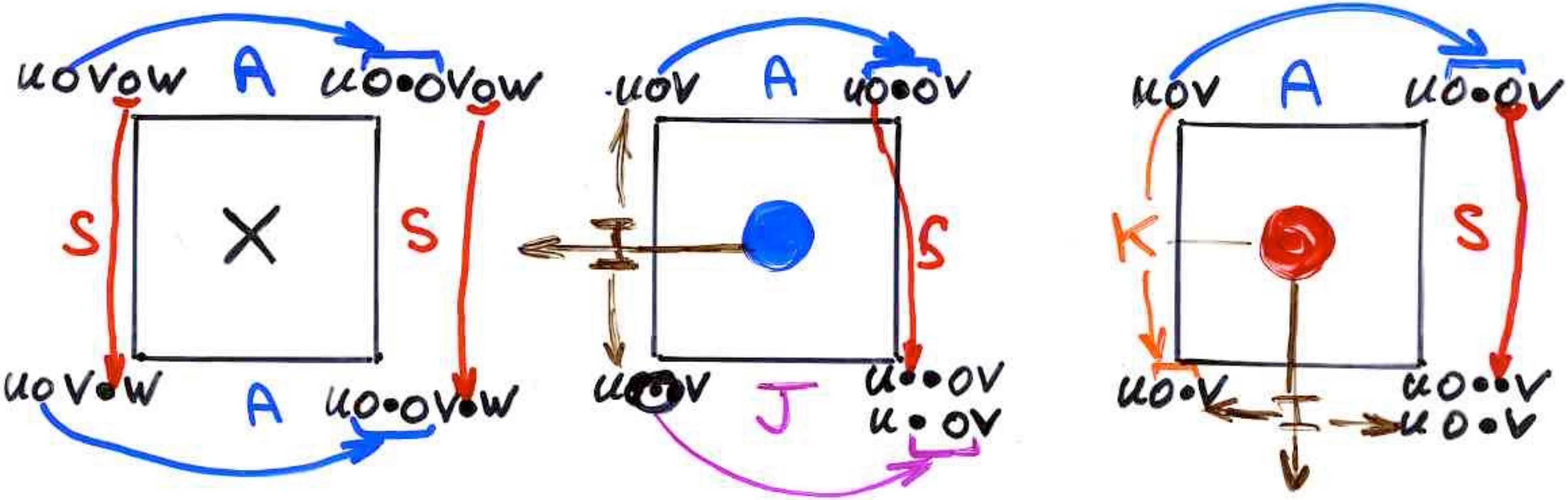
$$= AS + AK + JS + JK$$

$$= \underbrace{(SA + KA + SJ + KJ)}_{(S+K)(A+J)} + \underbrace{J + K + A + S}_{E + D}$$

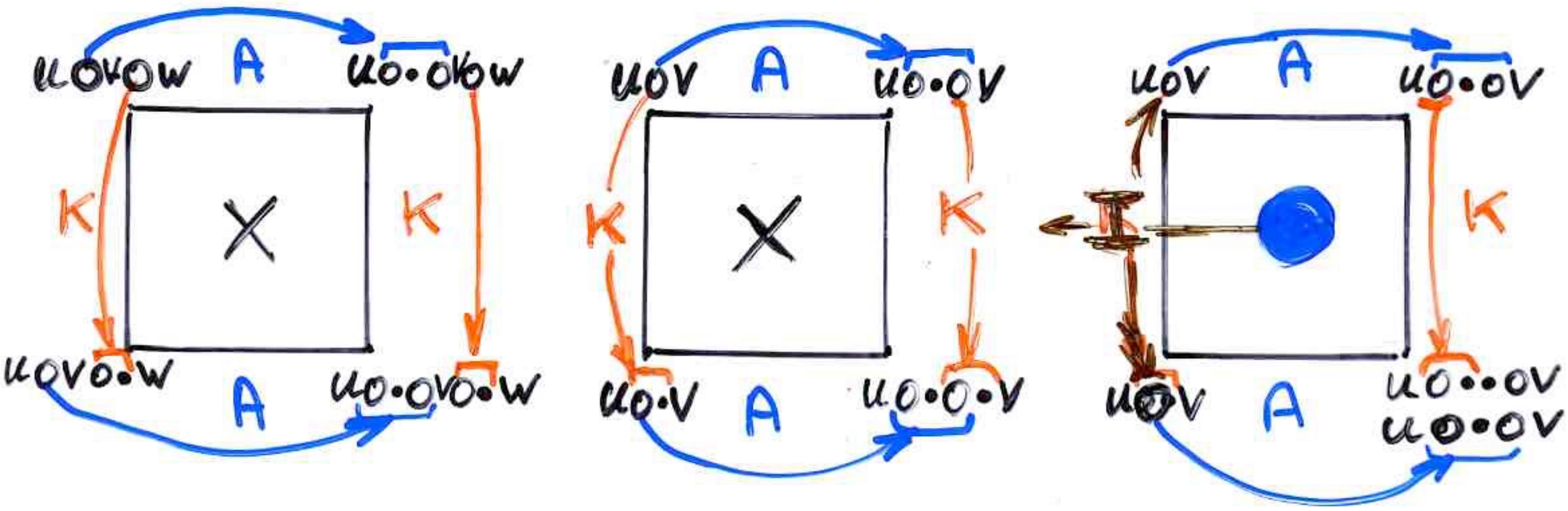
$$\underbrace{(S+K)(A+J)}_{ED}$$

$$DE = ED + E + D$$

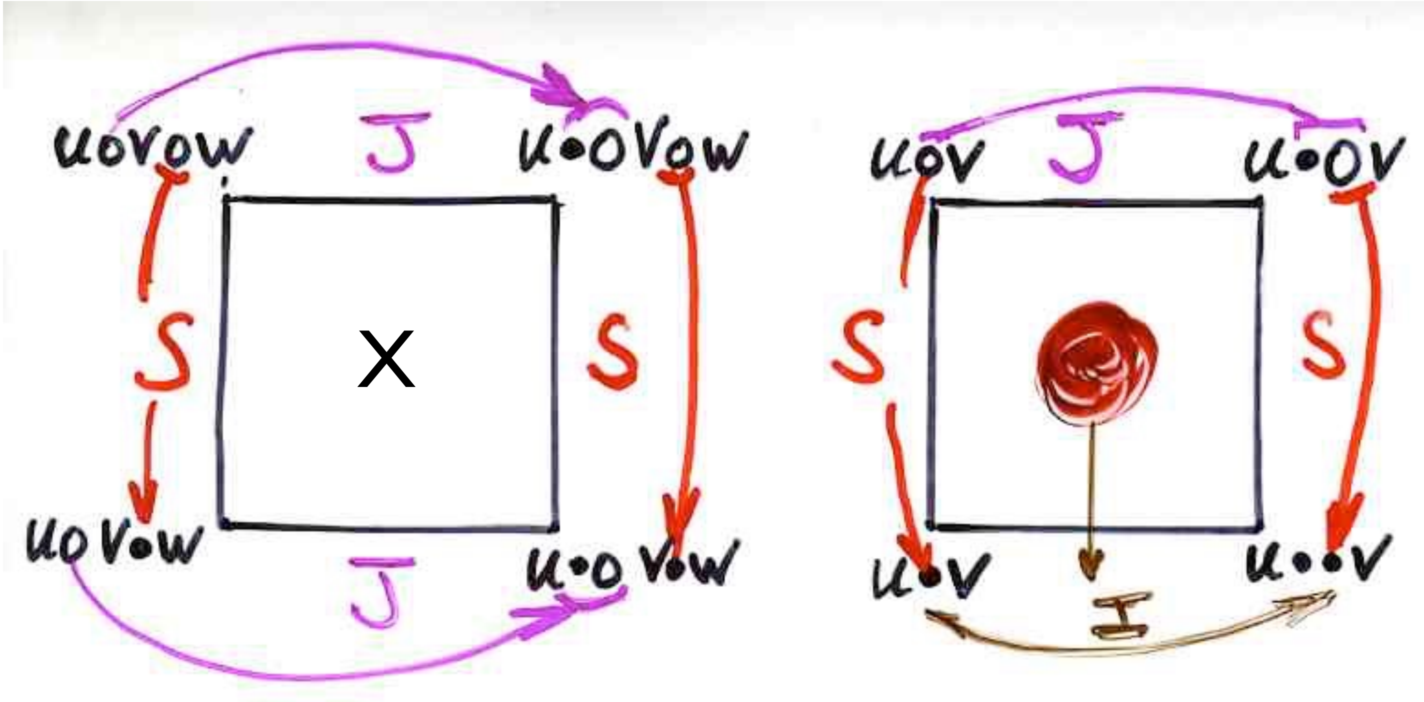
commutation diagrams



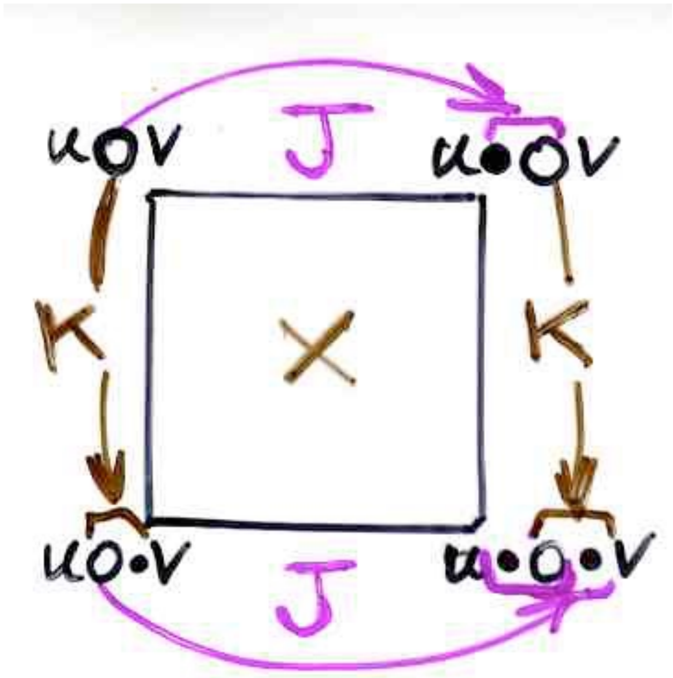
$$AS = SA + I_v J + K I_h$$



$$AK = KA + I_V A$$



$$JS = SJ + SI_h$$



$$JK = KJ$$

$$AS = SA + I_v J + K I_h$$

$$AK = KA + I_v A$$

$$JS = SJ + S I_h$$

$$JK = KJ$$

$$A I_v = I_v A$$

$$J I_v = I_v J$$

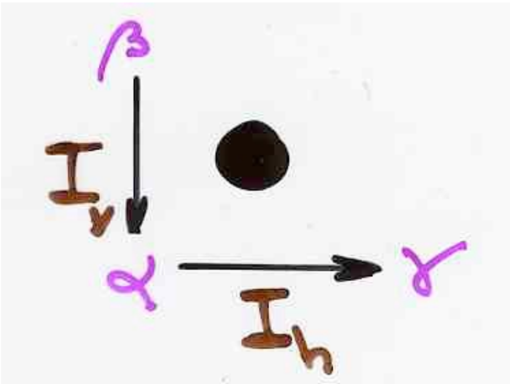
$$I_h S = S I_h$$

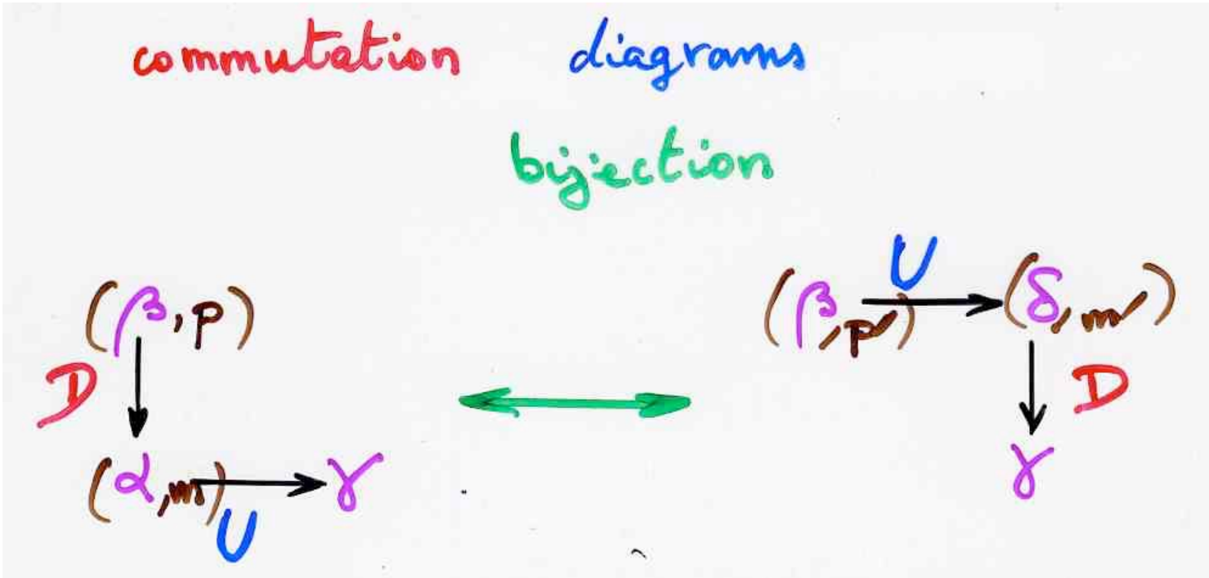
$$I_h K = K I_h$$

commutation diagrams bijections

analogy with commutation diagrams bijection for the representation of the Weyl-Heilenberg algebra (Ch 7b)

$$UD = DU + I_v I_h$$





p, m, p', m' "positions"

in $\beta, \alpha, \beta, \delta$ resp.

the bijection

permutations -- alternative tableaux

416978352

●

● 1 ●

●1●2 ●

●1●3●2

41● 3 ●2

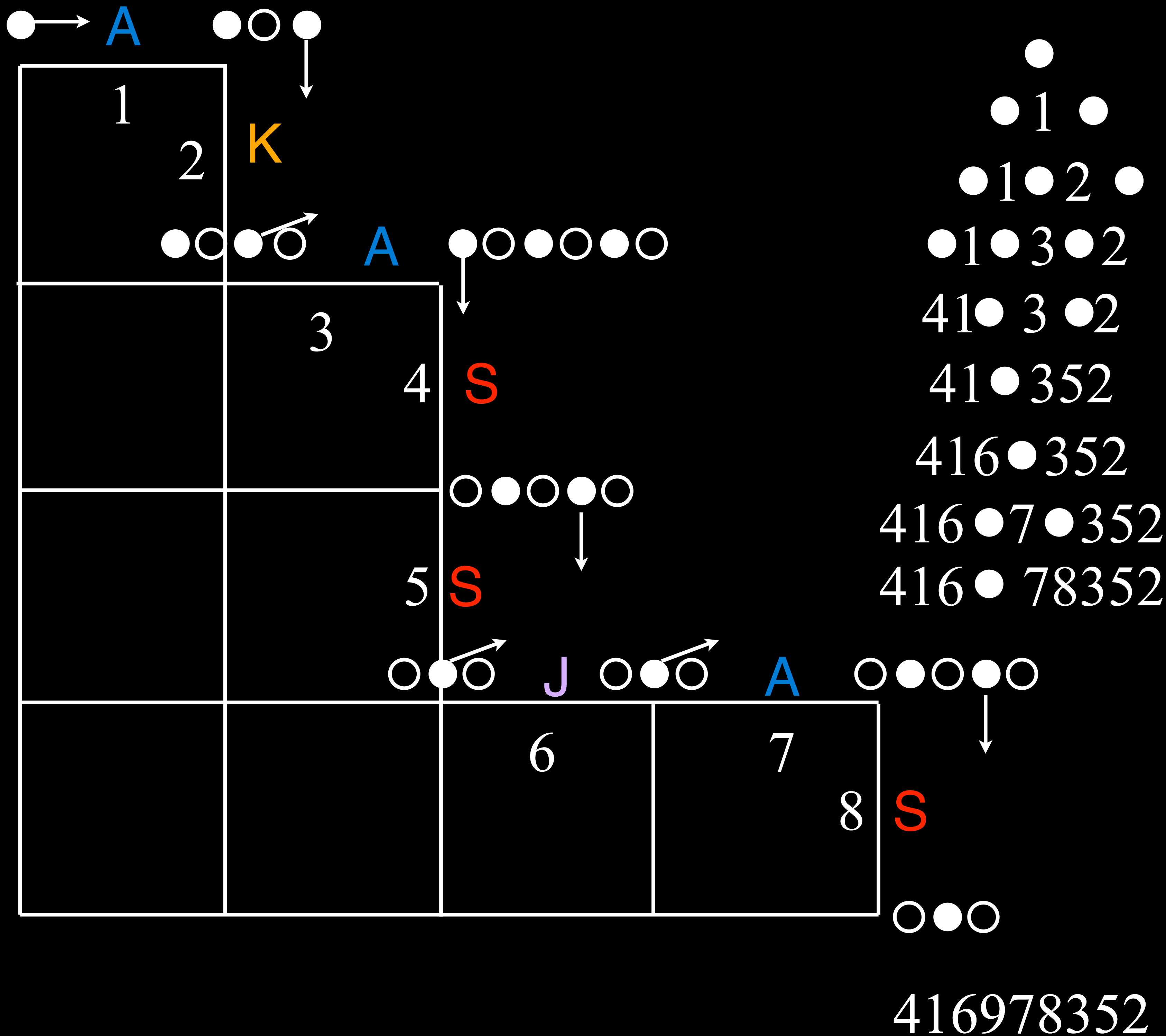
41●352

416●352

416●7●352

416● 78352

416978352


A
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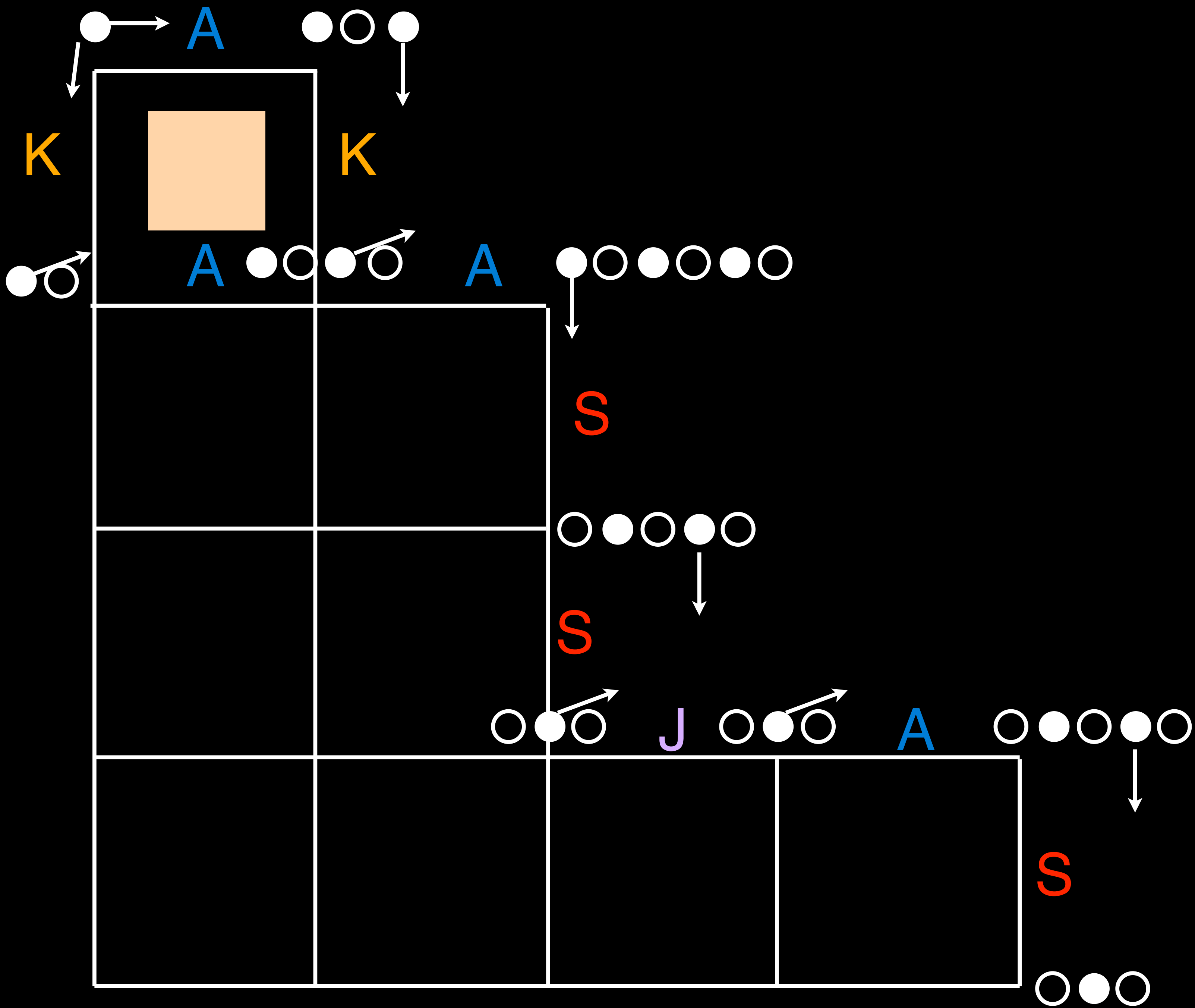
1 3 2
41 3 2
41 352
416 352
416 7 352
416 78352
416978352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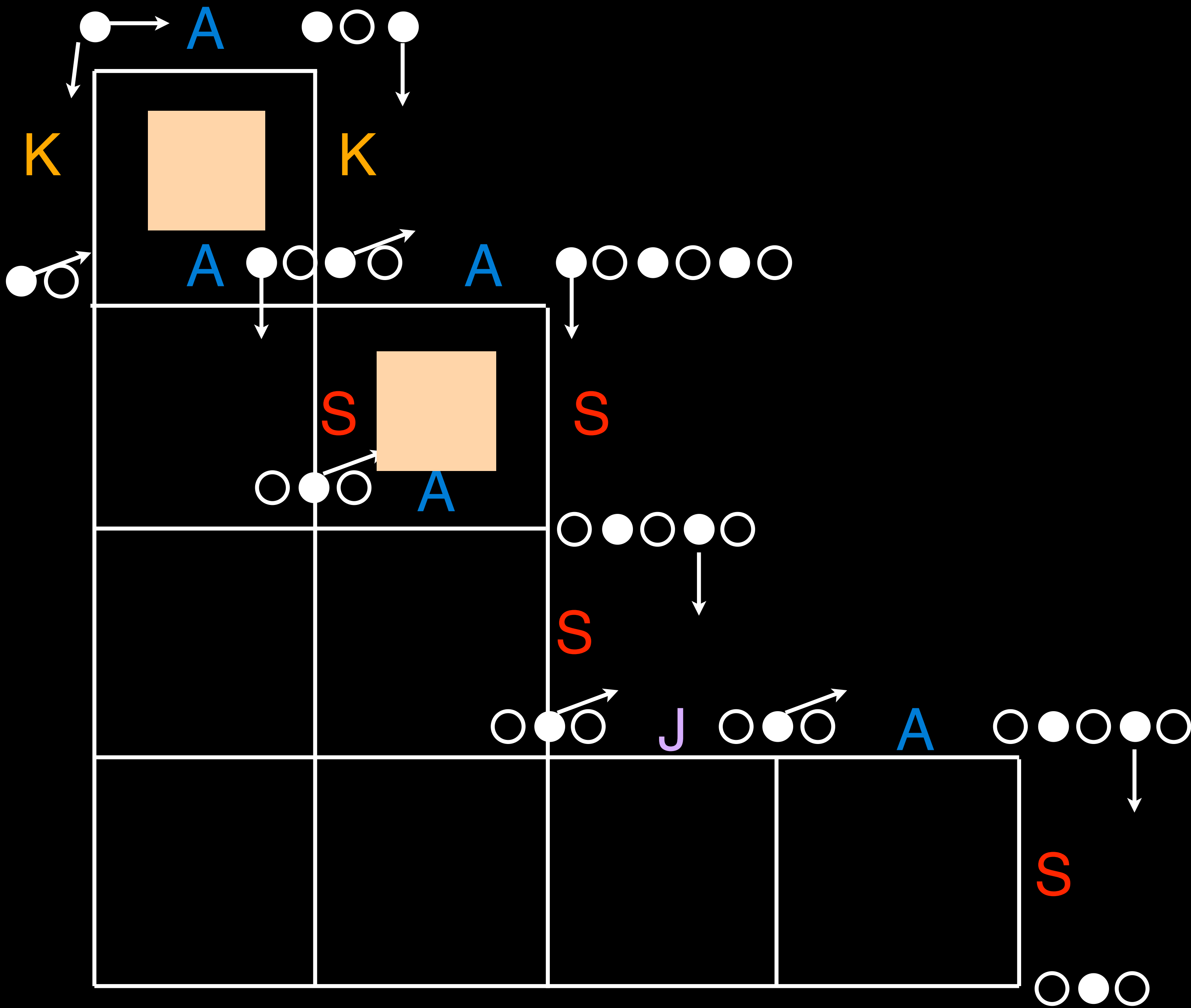
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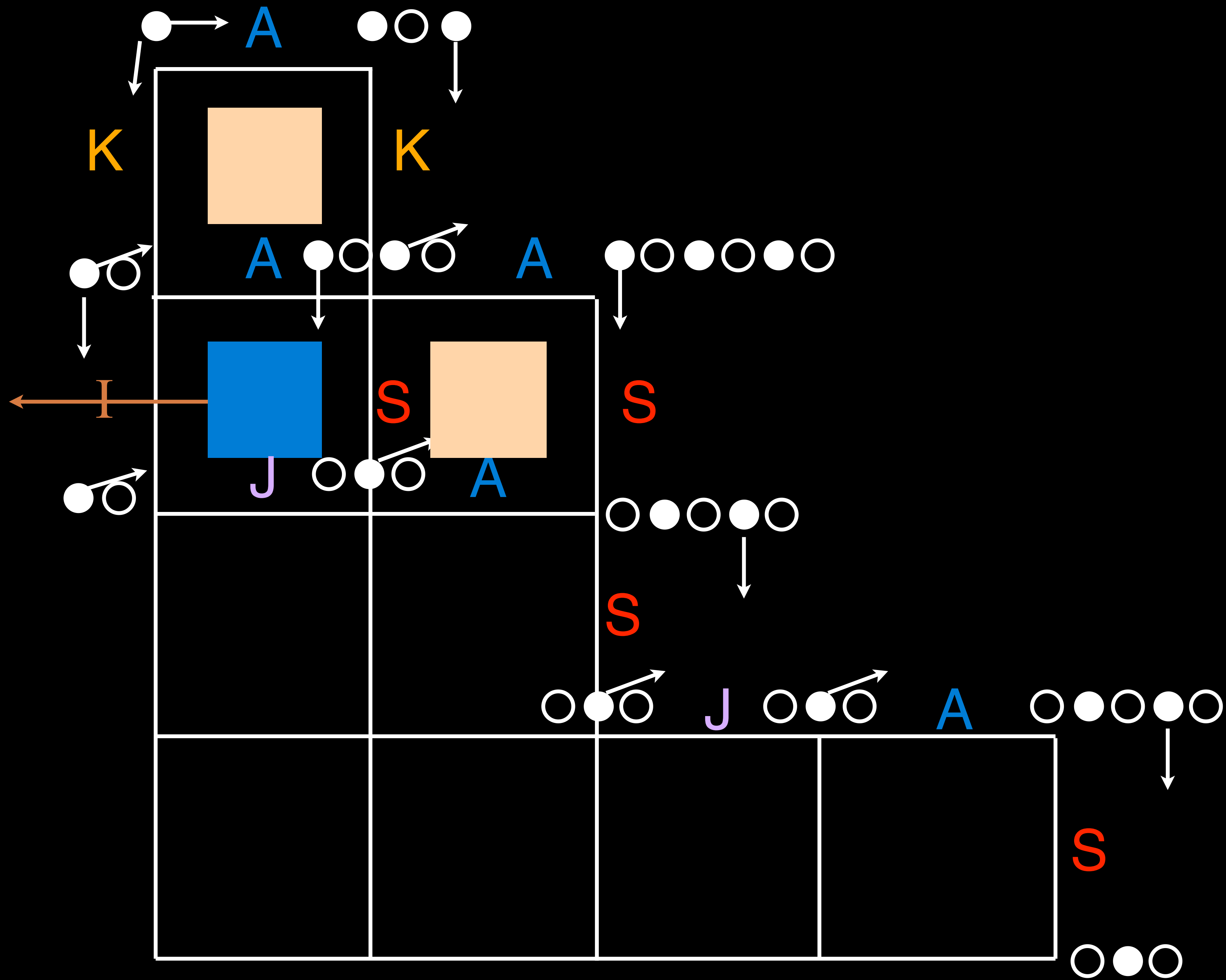

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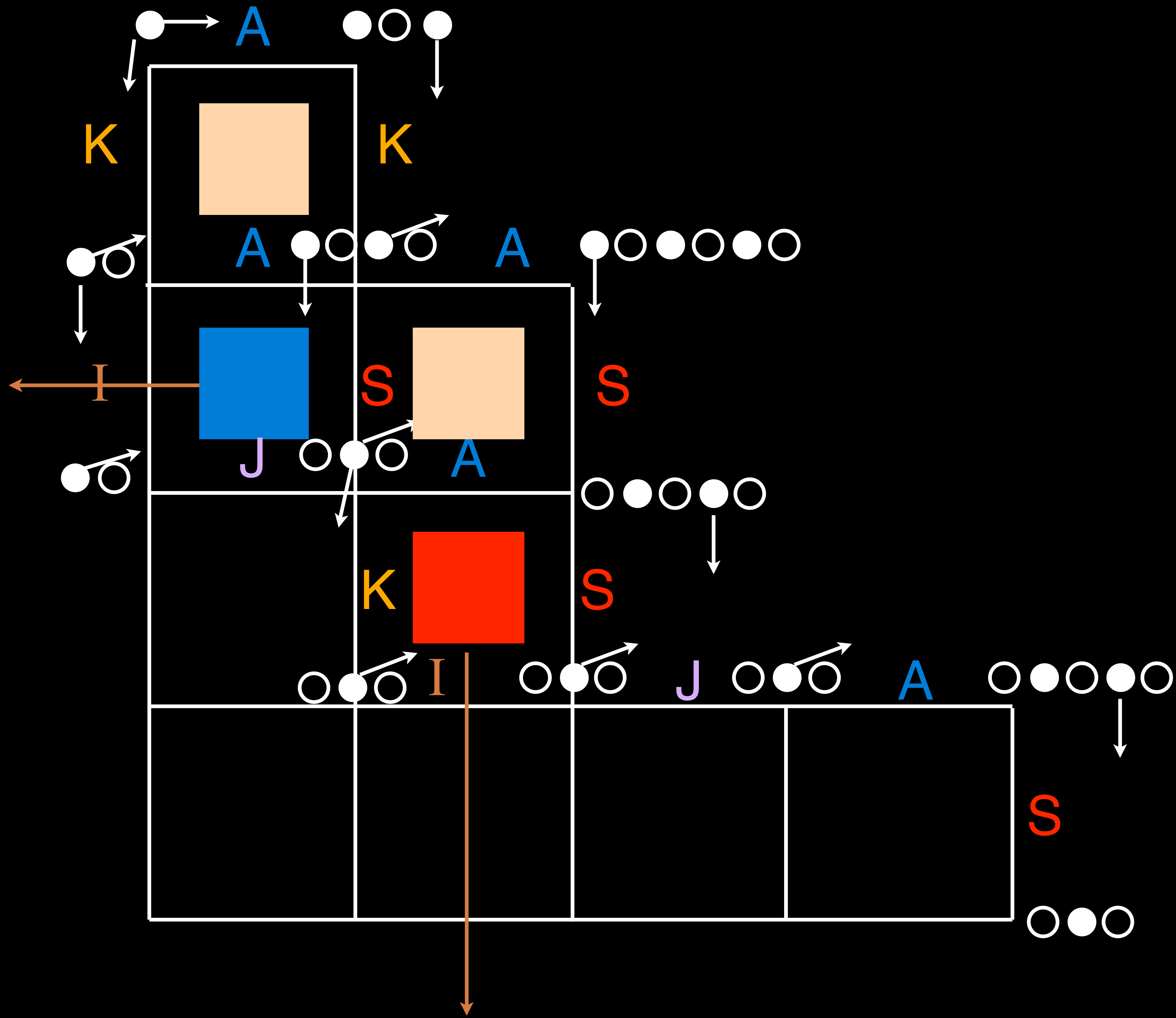

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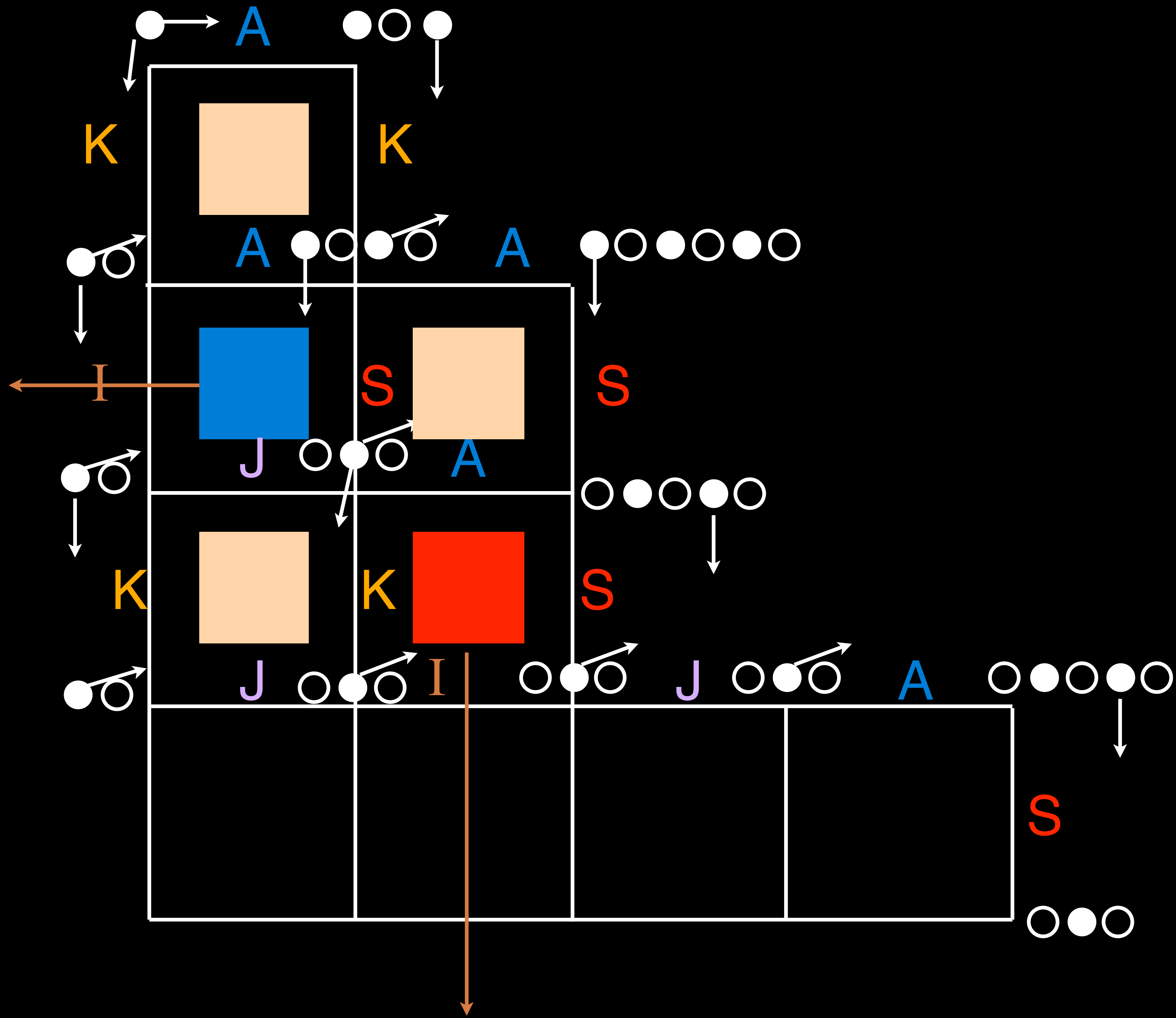

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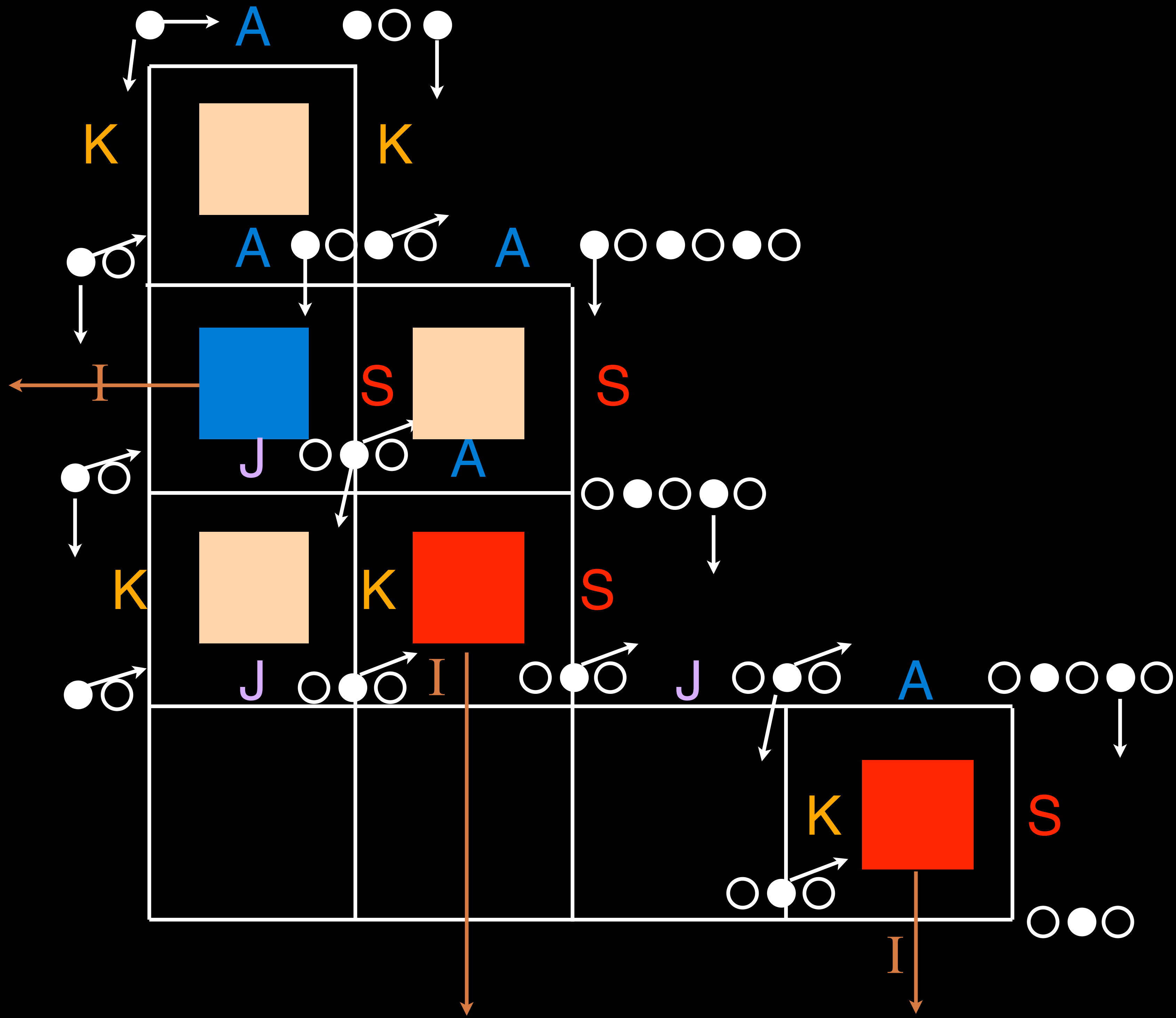

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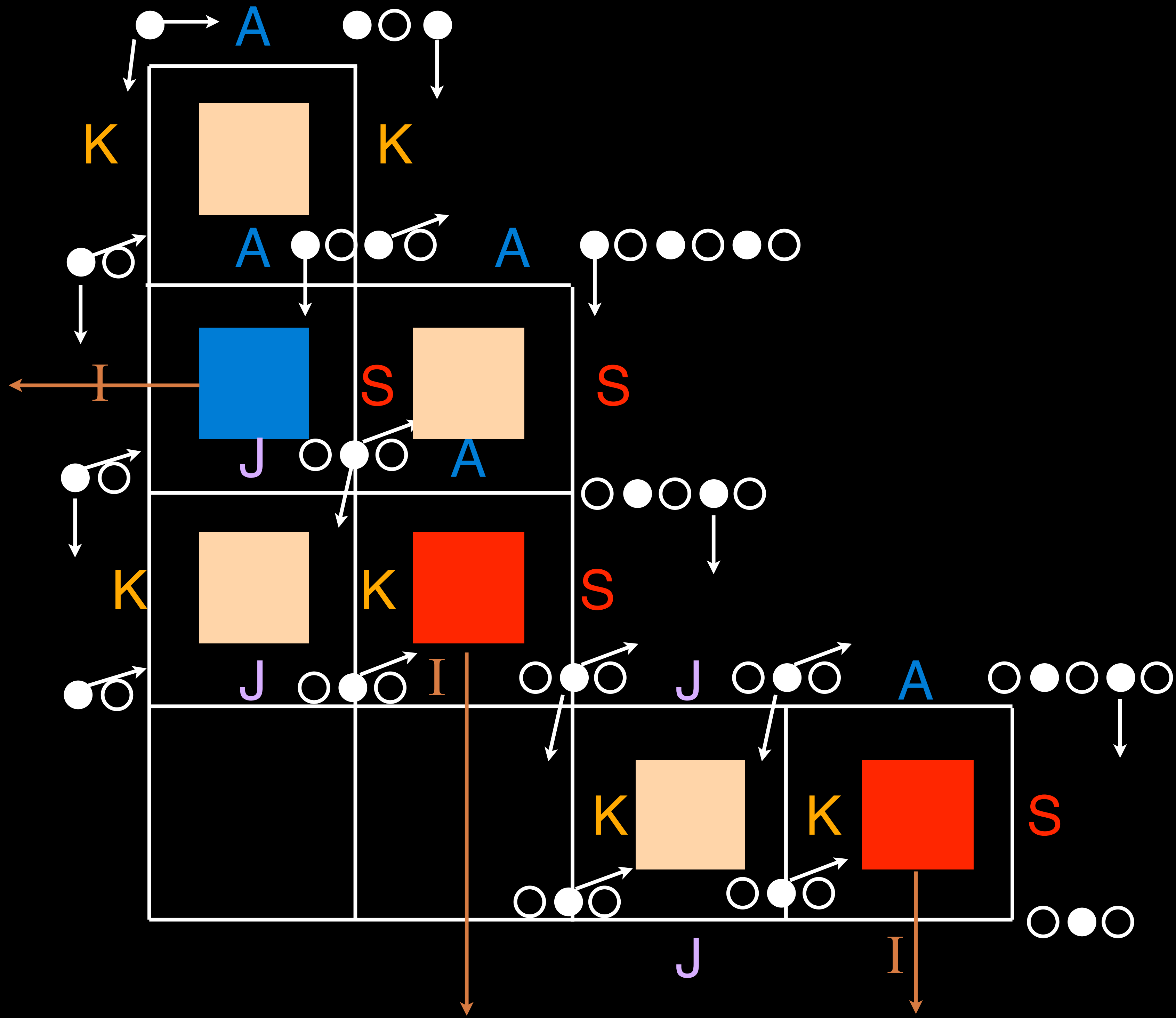
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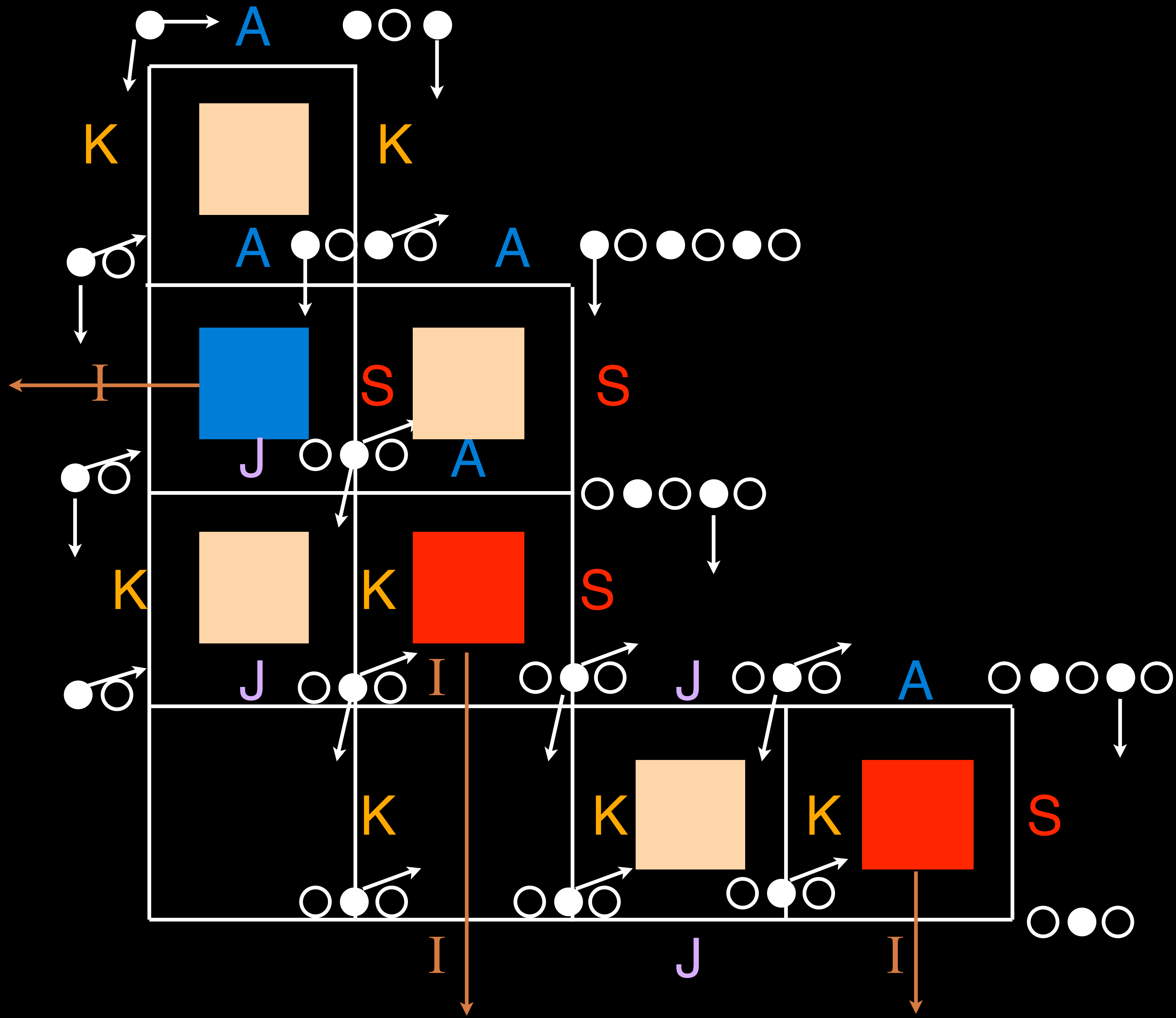
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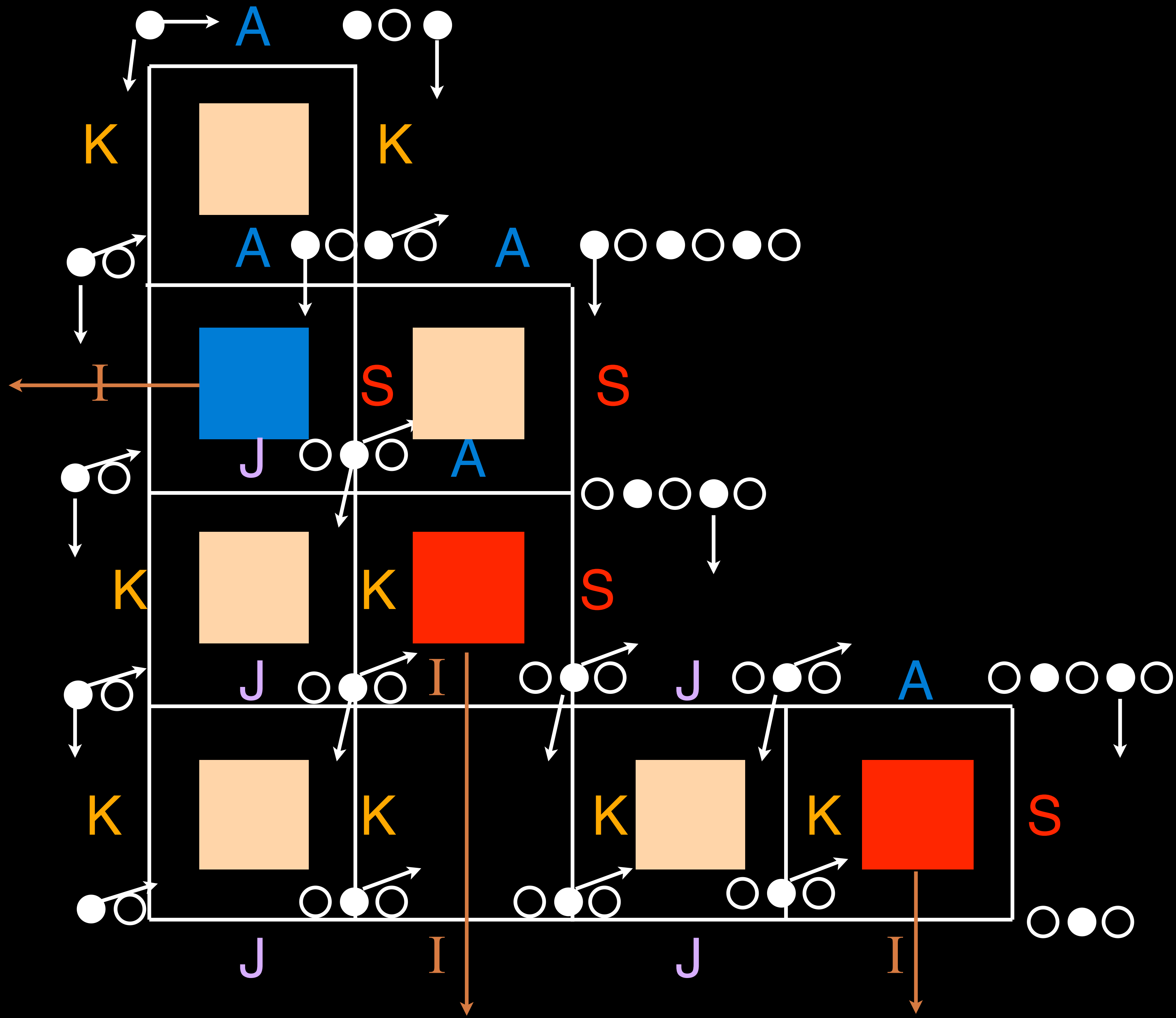

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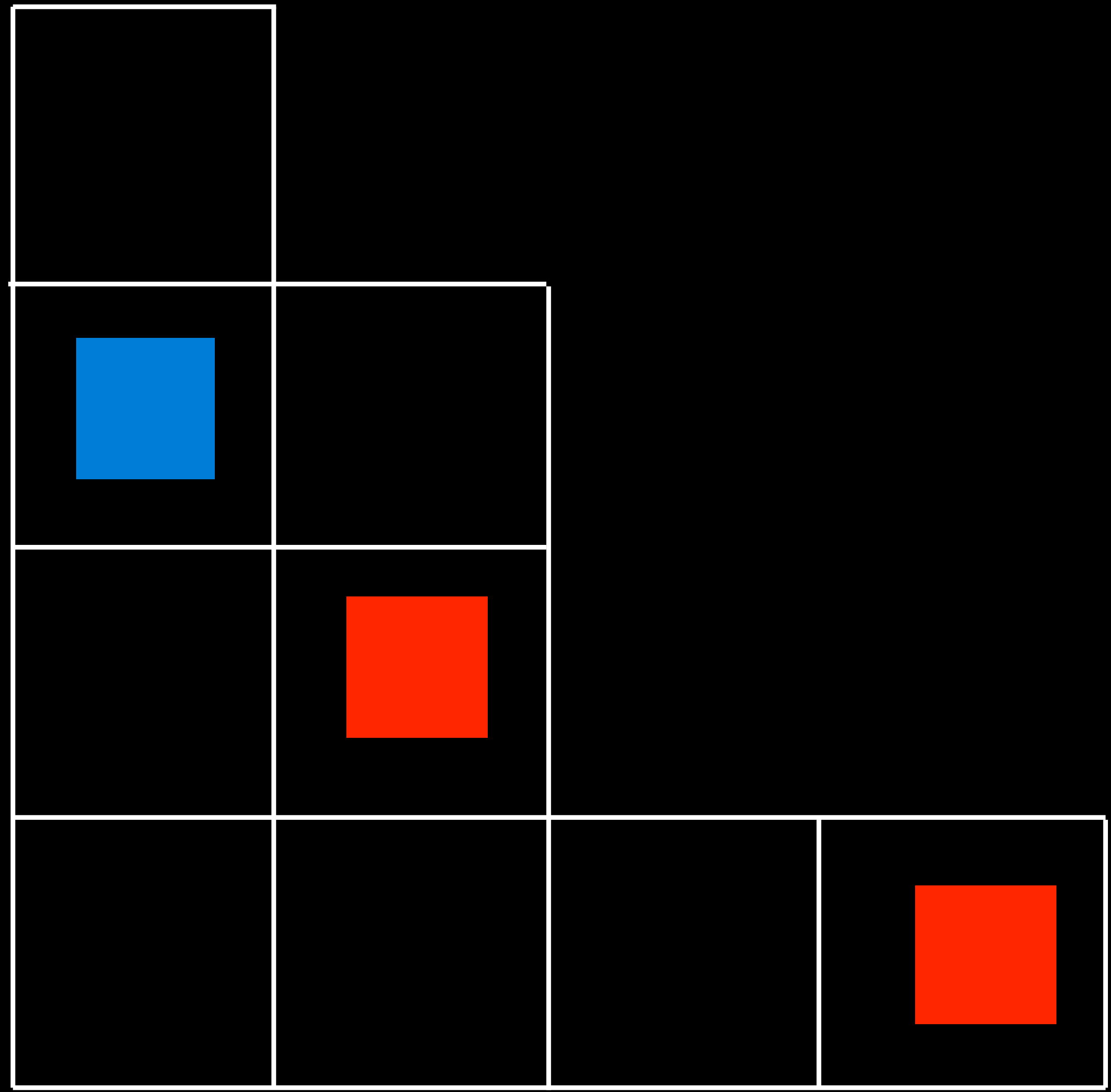

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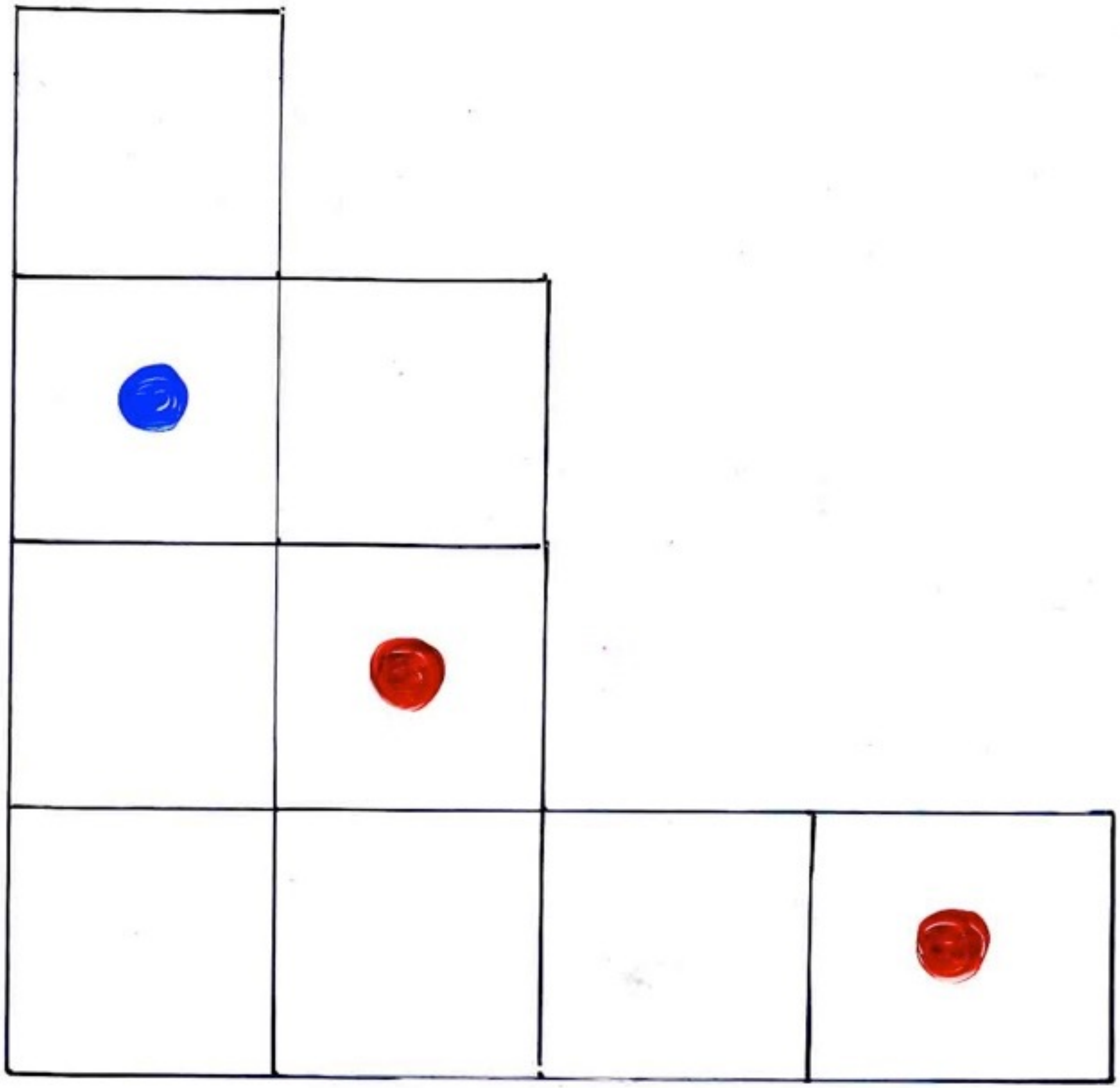
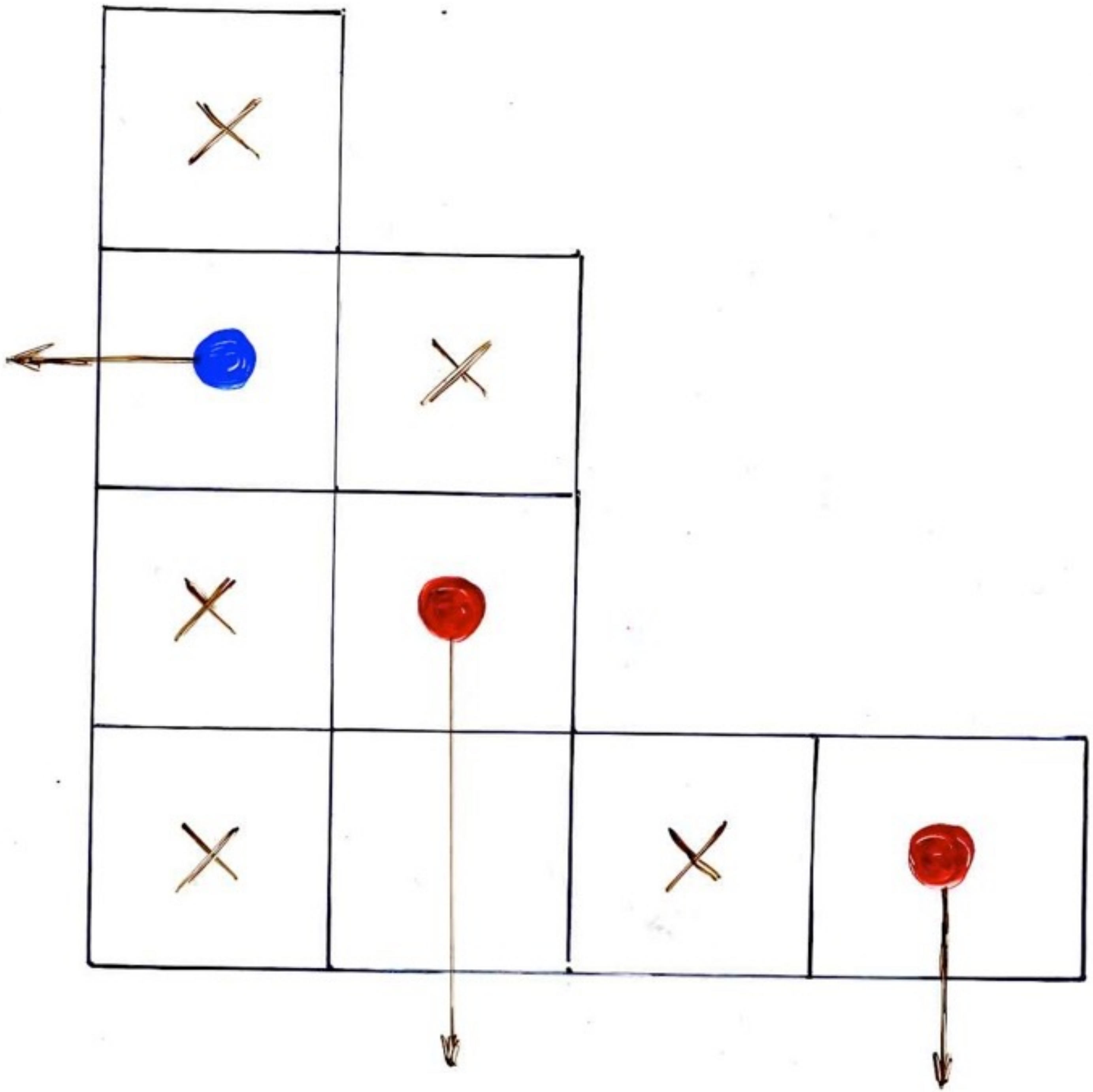
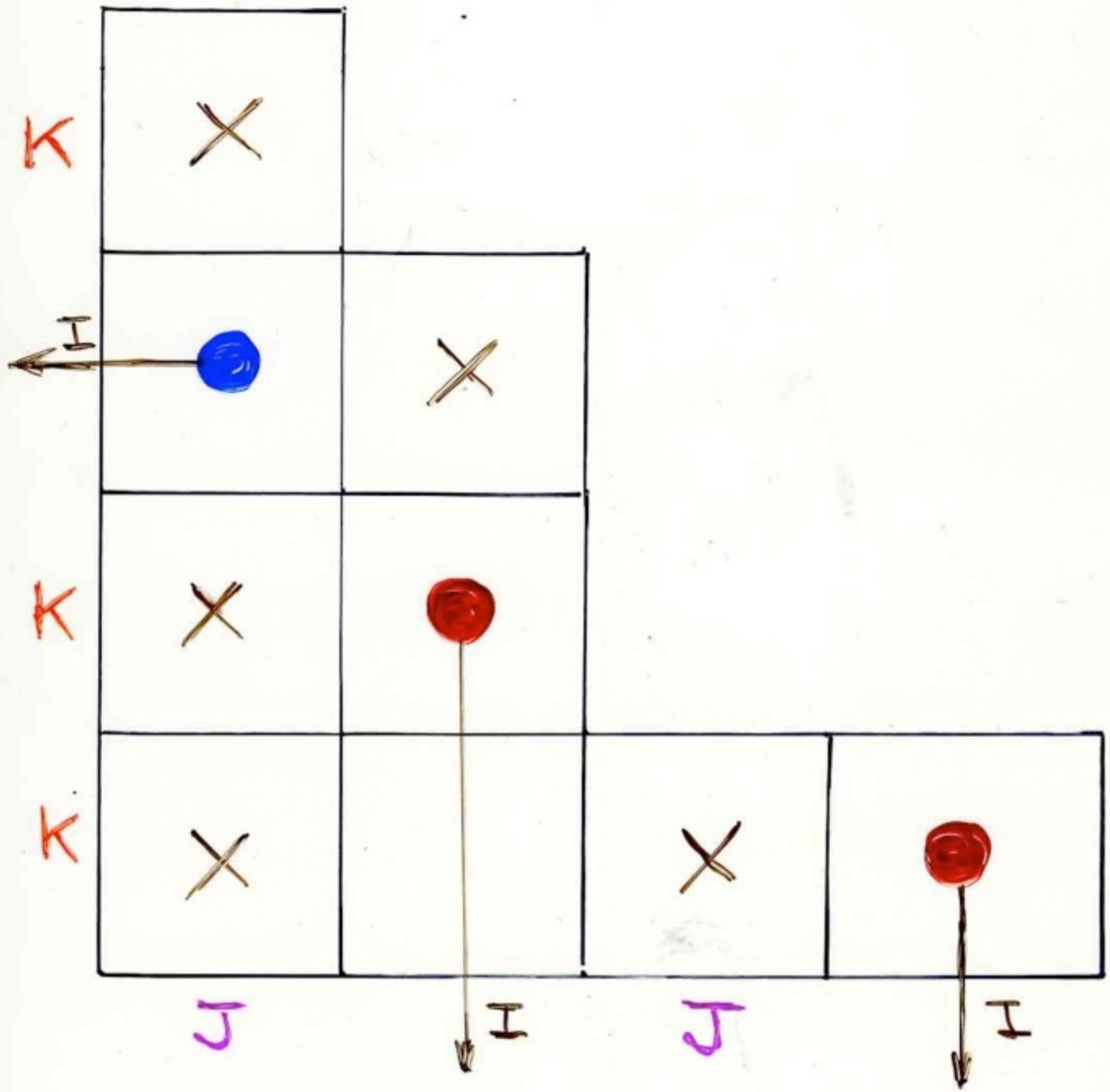
the inverse bijection

permutations --- alternative tableaux



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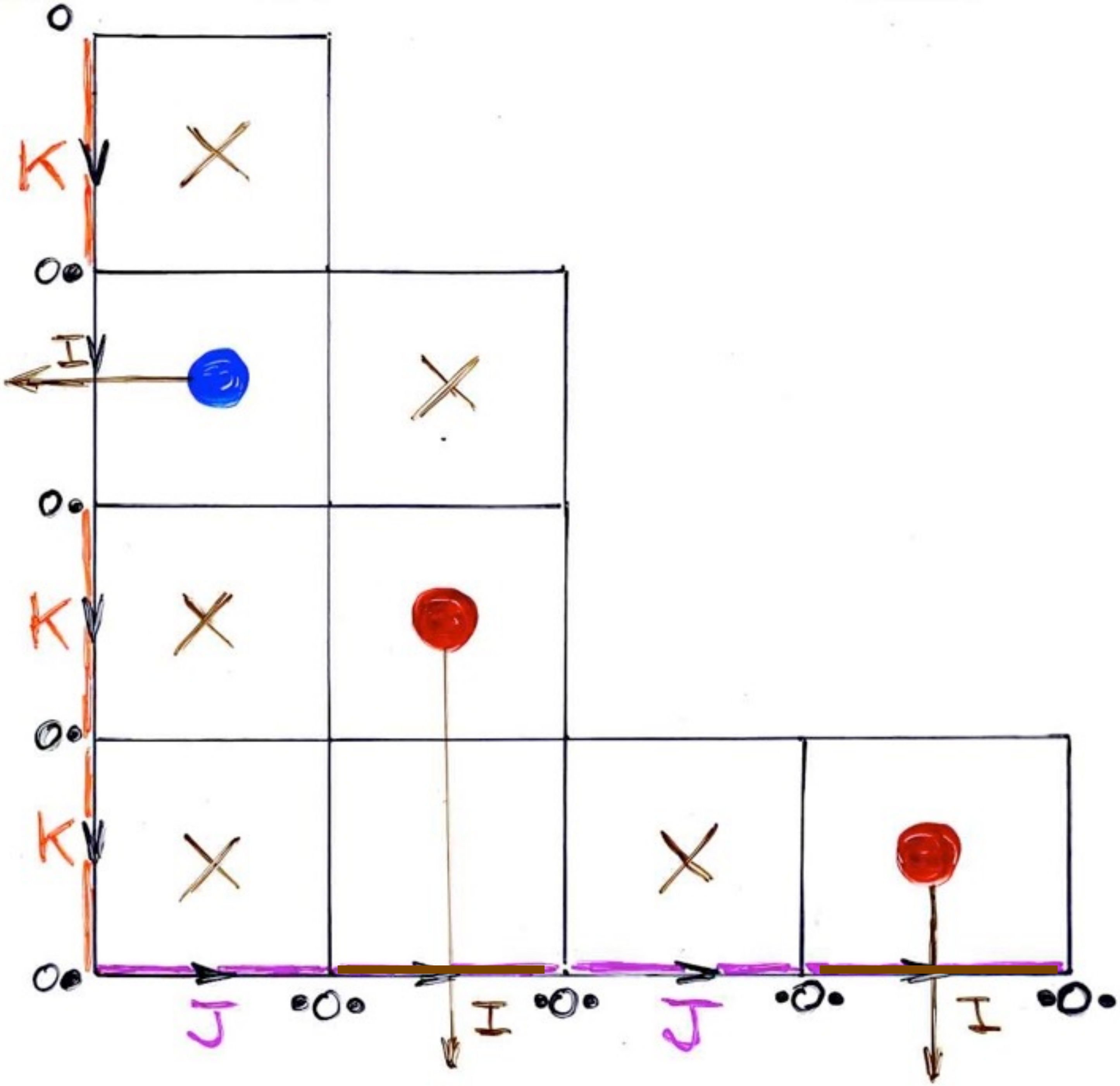

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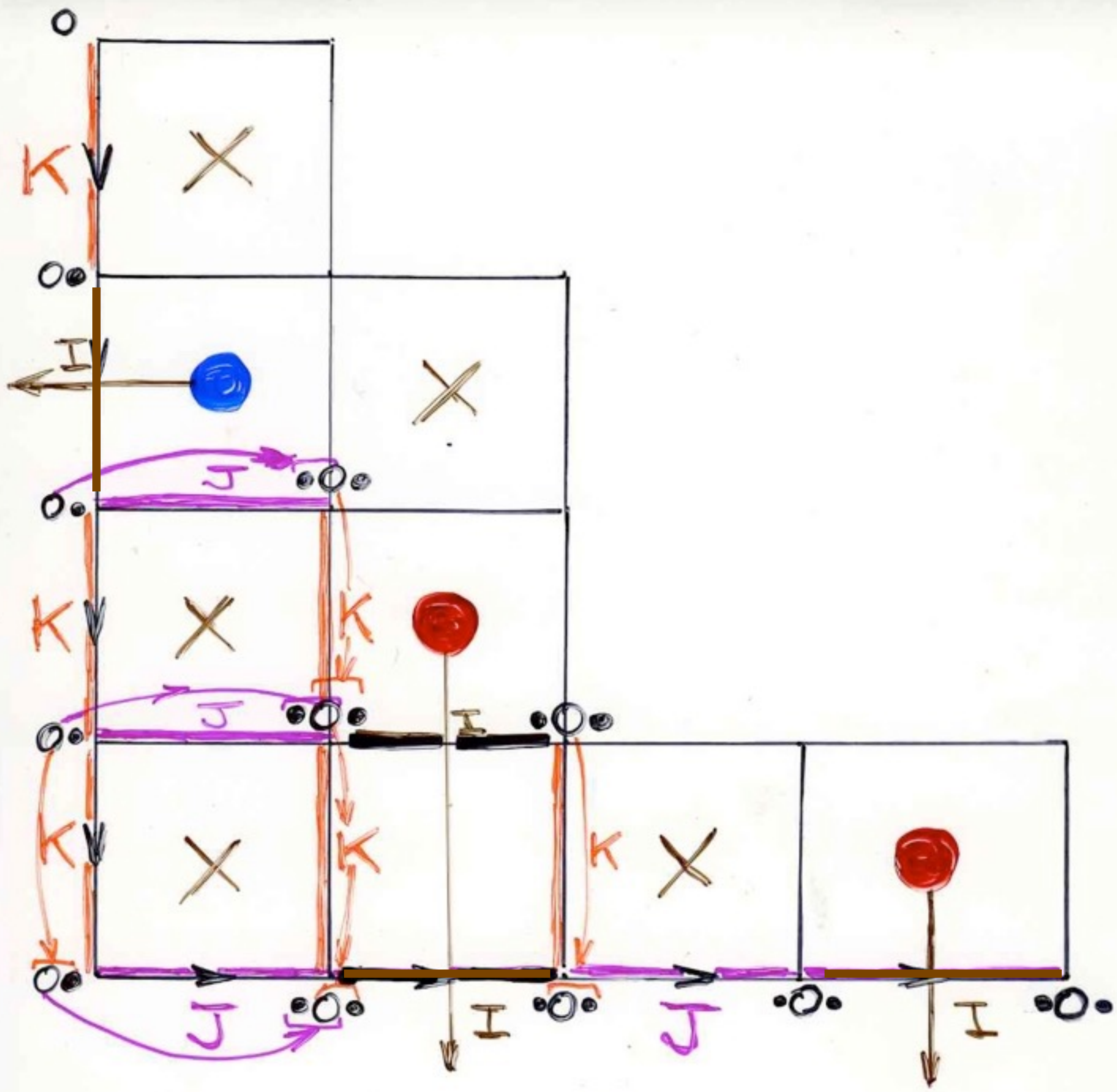
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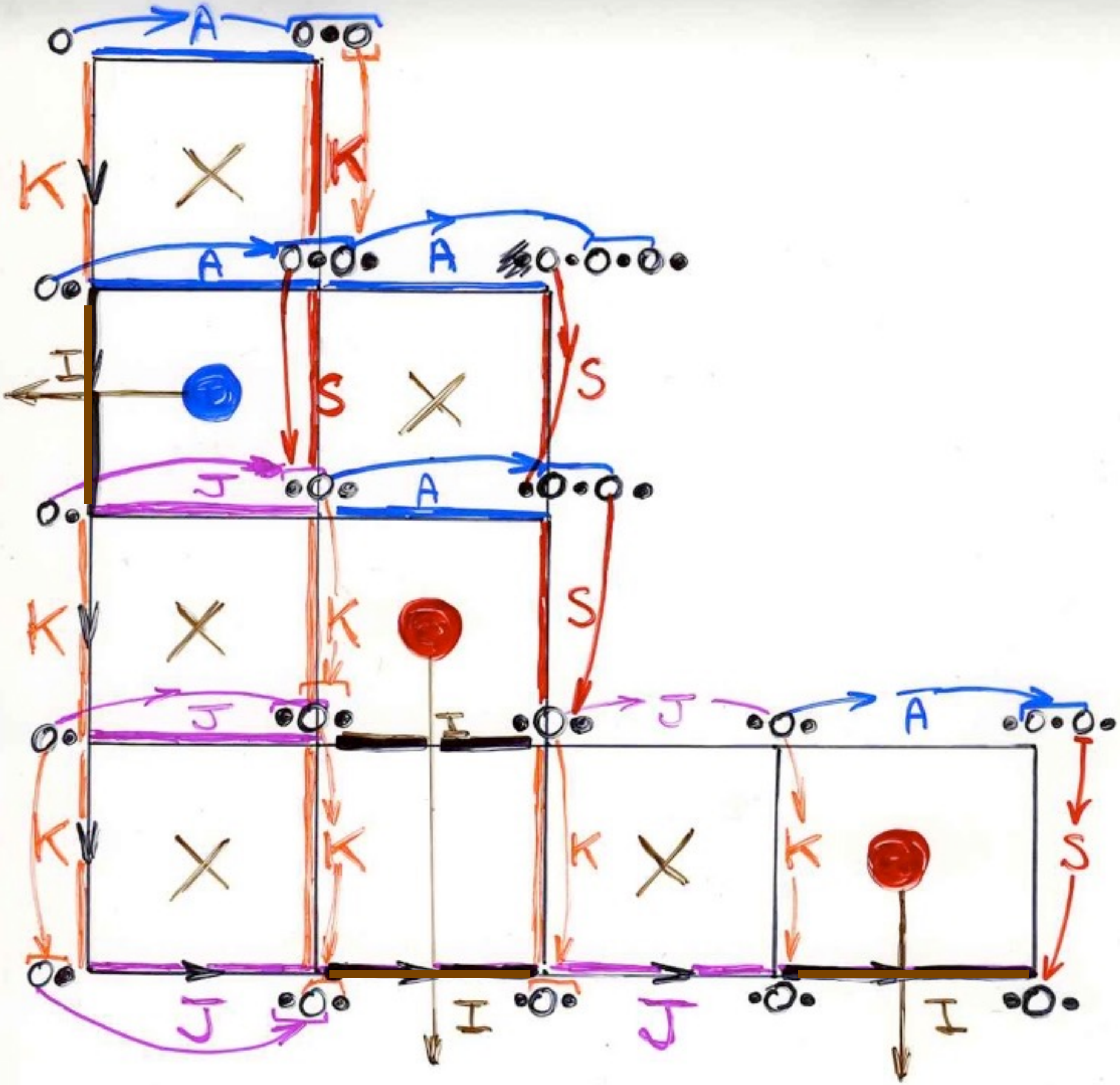
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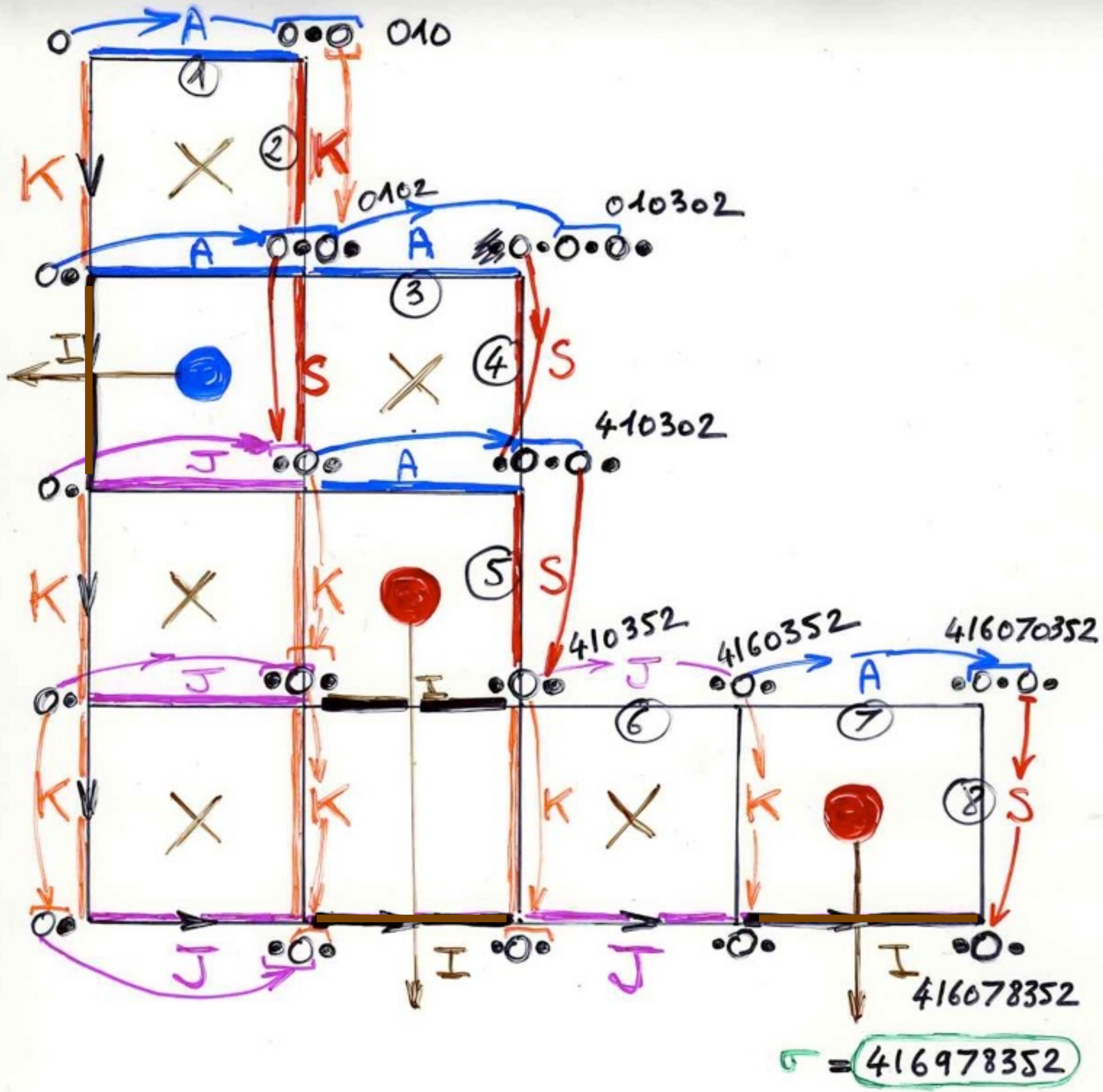
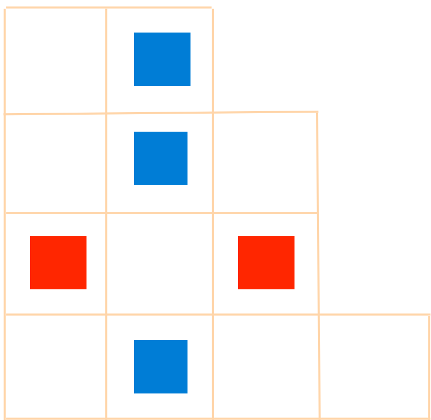


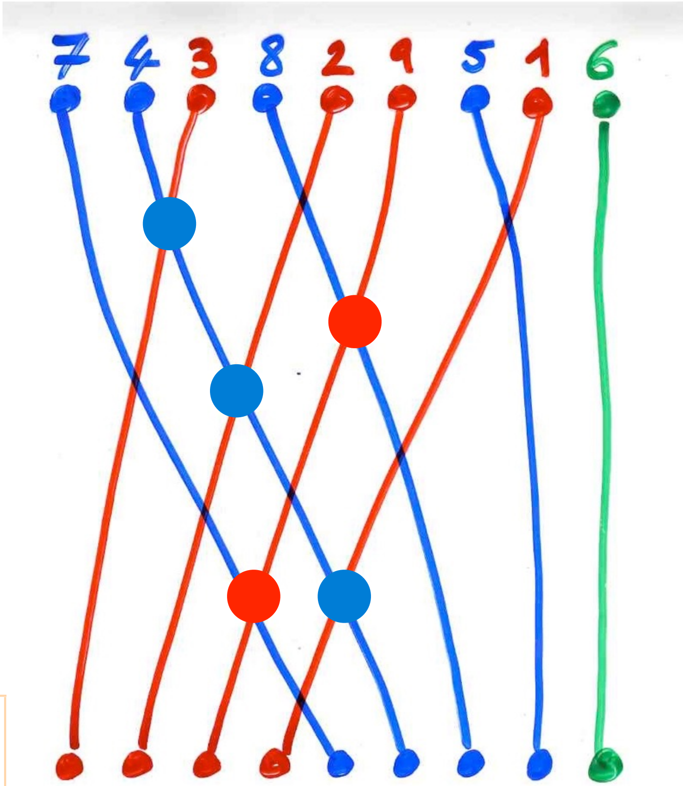
010
0102
010302
410302
410352
4160352
416070352
416078352
σ = 416978352

two bijections
one theorem

bijection alternative tableaux size n
permutations on (n+1)
with the exchange-fusion algorithm (Ch 5)

"exchange-
fusion"
algorithm





Here from a representation
of the PASEP algebra

bijection alternative tableaux size n
Laguerre histories of length n

bijection permutations (Ch6)
Laguerre histories of length n

Prop.

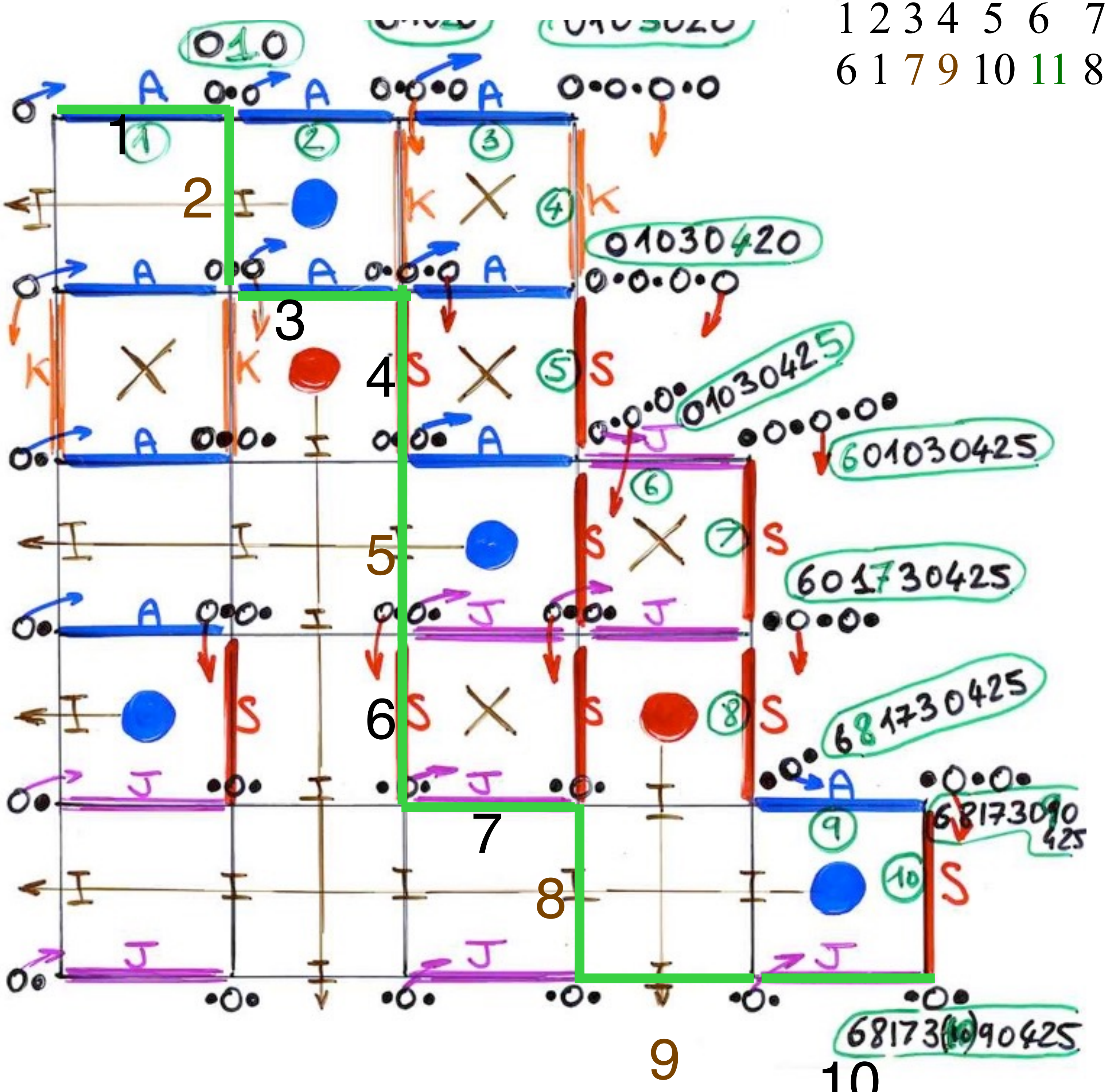
T alternative tableau → σ "exchange-fusion" inverse algorithm

T alternative tableau → τ "local" algorithm from $DE = ED + E + D$

$$\sigma = \tau^{-1}$$

$\sigma = 6\ 8\ 1\ 7\ 3\ (10)\ 9\ (11)\ 4\ 2\ 5$

1	2	3	4	5	6	7	8	9	10	11	
6	1	7	9	10	11	8	5	3	4	2	= S



"The cellular Ansatz"

combinatorial objects on a 2d lattice

representation by operators

Physics

data structures "histories"

orthogonal polynomials

bijections

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

rooks placements

permutations ⟷ RSK pairs of Tableaux Young

alternative tableaux ⟷ permutations

Laguerre histories

DE = qED + E + D

PASEP

dynamical systems in physics
stationary probabilities

Q-tableaux

quadratic algebra Q

commutations
rewriting rules

planarization