Combinatorics and Physics

Chapter 7

The cellular ansatz

Ch7e
The general theory: Q-tableaux

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Quadratic algebra $\mathcal{Q}$

generators $\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

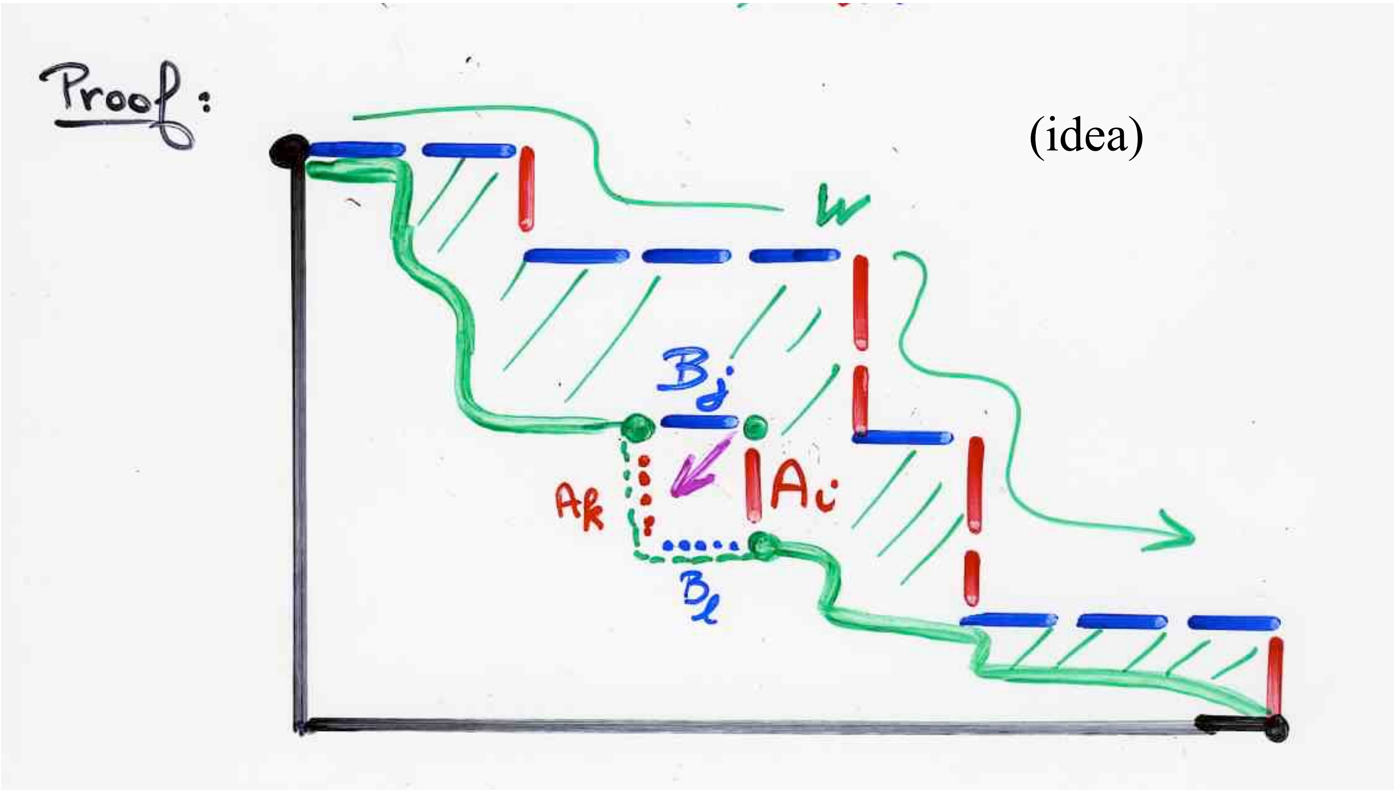
commutation relations

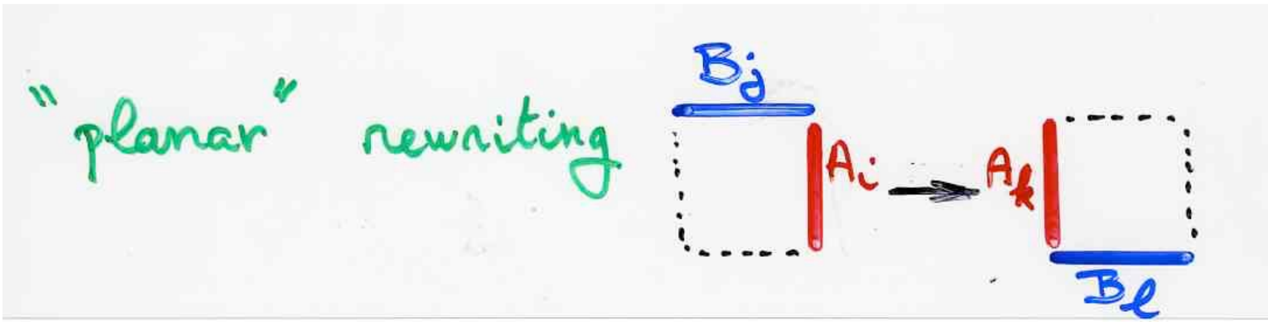
$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \quad \text{for every } \begin{array}{l} i \in I \\ j \in J \end{array}$$

<u>Lemma</u>. In $\mathcal{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

(no complete proof here)


Proof:
(idea)
w
B_j
A_k
A_i
B_l

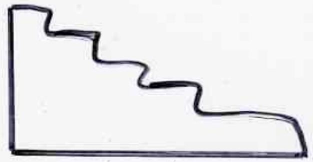

"planar" rewriting
B_j
A_i
A_k
B_l

complete Q-tableaux

complete

Def- Q-tableau

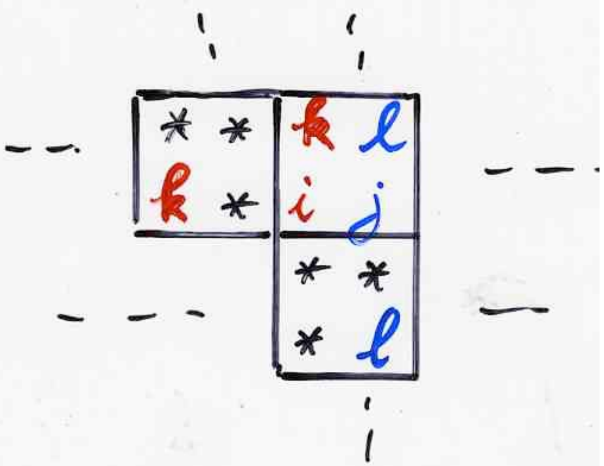
Ferrers diagram F



each cell $\alpha \in F$ labeled

$i, k \in I$
$j, l \in J$

with "compatibility" condition:



commutation relations

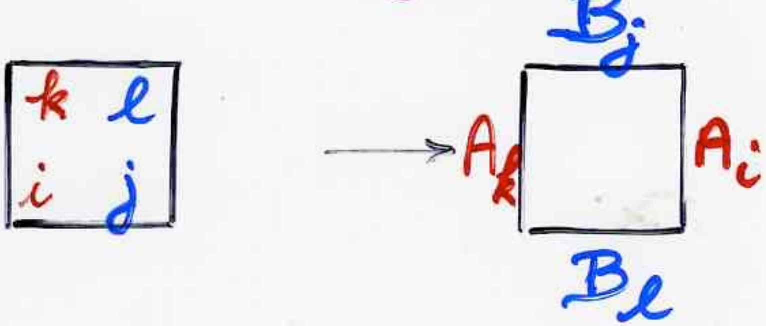
$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l$$

$i \in I$
$j \in J$

complete

Def- edge-labeling of a Q-tableau T

each cell α

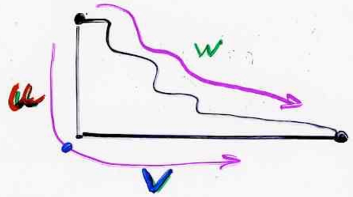


complete

Def. For T a Q-tableau

$uwb(T) \in (\mathcal{A} \cup \mathcal{B})^*$ upper word border

$lwb(T)$ lower word border



complete

Def. weight of a Q-tableau T

$$p(T) = \prod_{\substack{\text{cells} \\ \alpha \in F}} c_{ij}^{kl} \qquad \alpha = \boxed{\begin{matrix} k & l \\ i & j \end{matrix}}$$

Prop For any $w \in (\mathcal{A} \cup \mathcal{B})^*$, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

$$c(u, v; w) = \sum_{T} p(T)$$

T complete Q-tableau

$\left(\begin{matrix} uwb(T) = w \\ lwb(T) = uv \end{matrix} \right.$

Q-tableaux

S set of labels

$$\varphi : \left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\} = R \longrightarrow S$$

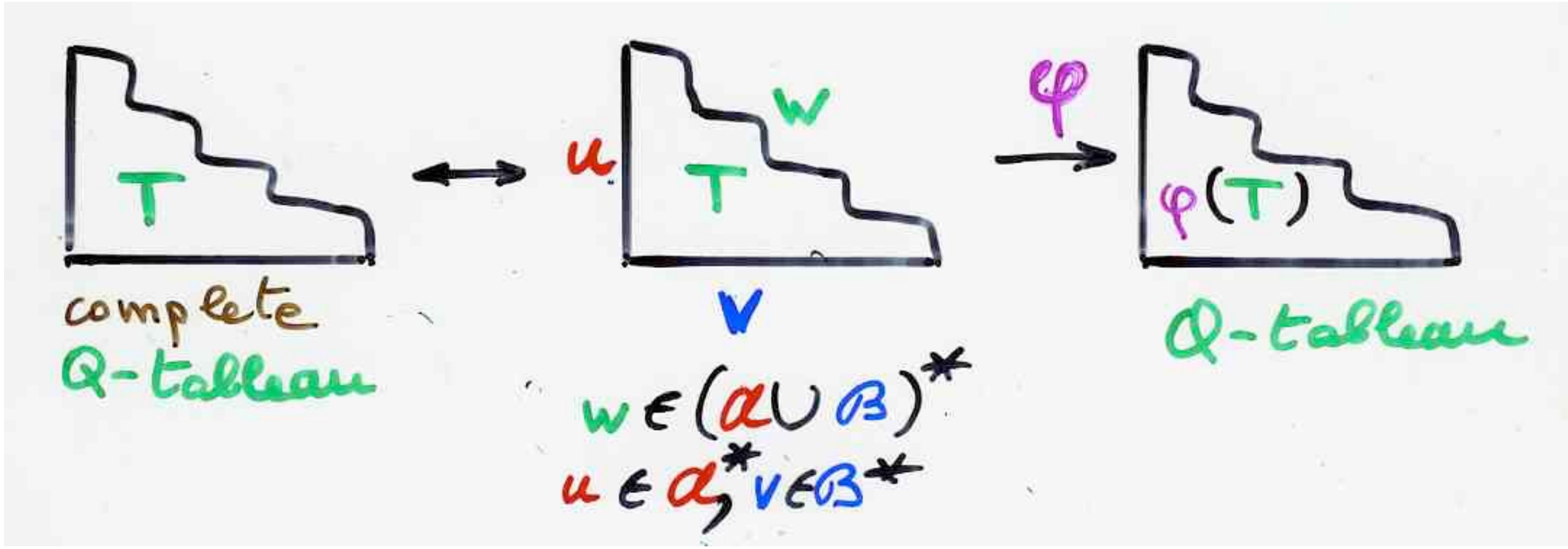
set of rewriting rules

$$B_j A_i \rightarrow c_{ij}^{kl} A_k B_l$$

such that: $\varphi\begin{pmatrix} k & l \\ i & j \end{pmatrix} = \varphi\begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix} \Rightarrow (i, j) \neq (i', j')$

<u>Def</u> Q-tableau

"image" by φ of a "complete Q-tableau"



w-compatible

w fixed

{set of Q-tableaux w-compatible}

$\updownarrow$ bijection

{set of complete Q-tableaux T with uwb(T) = w} example 1:

the PASEP algebra

$$DE = qED + E + D$$

PASEP algebra

$$DE = qED + EI_h + I_vD$$

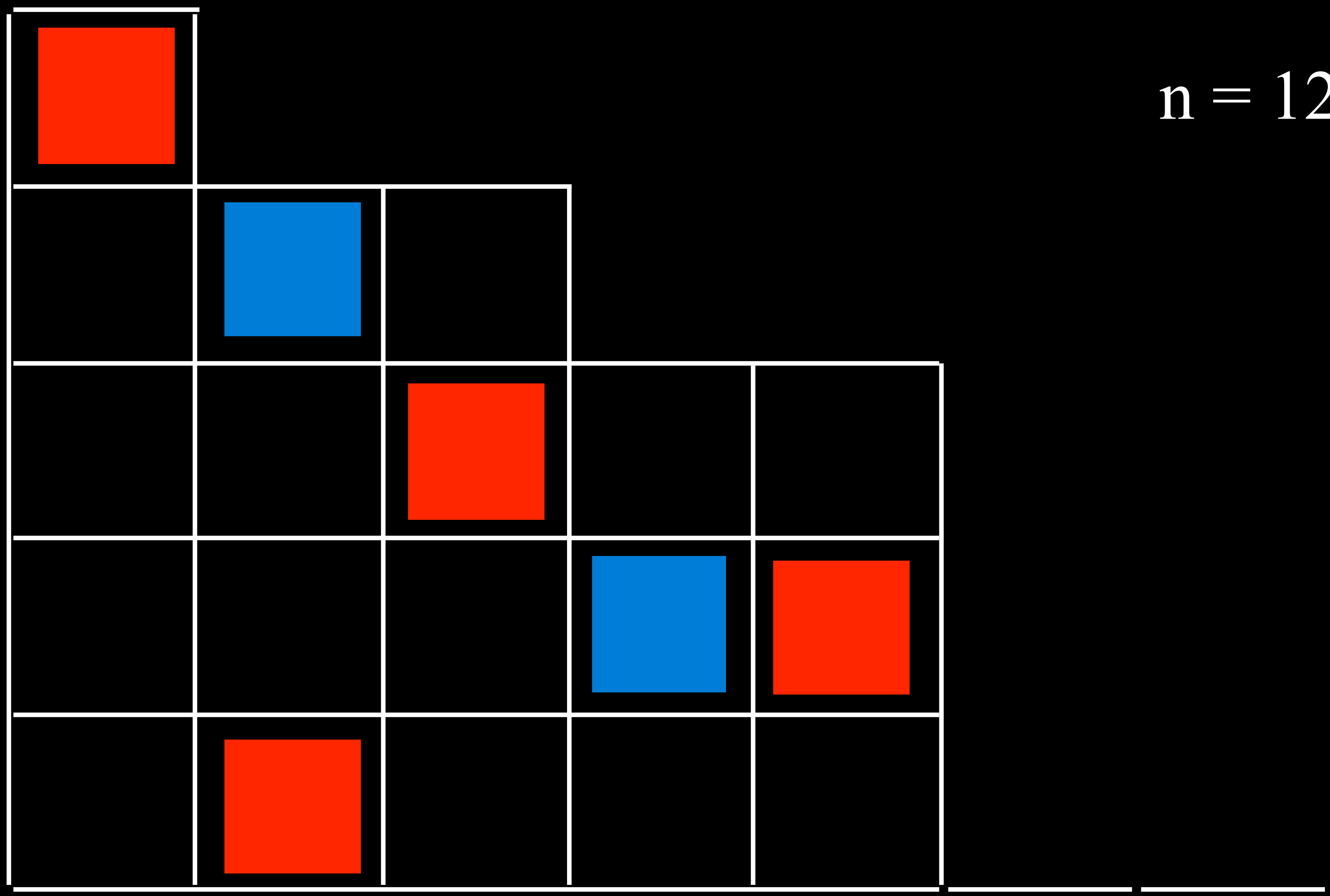
$$DI_v = I_vD$$

$$I_hE = EI_h$$

$$I_hI_v = I_vI_h$$

alternative tableau

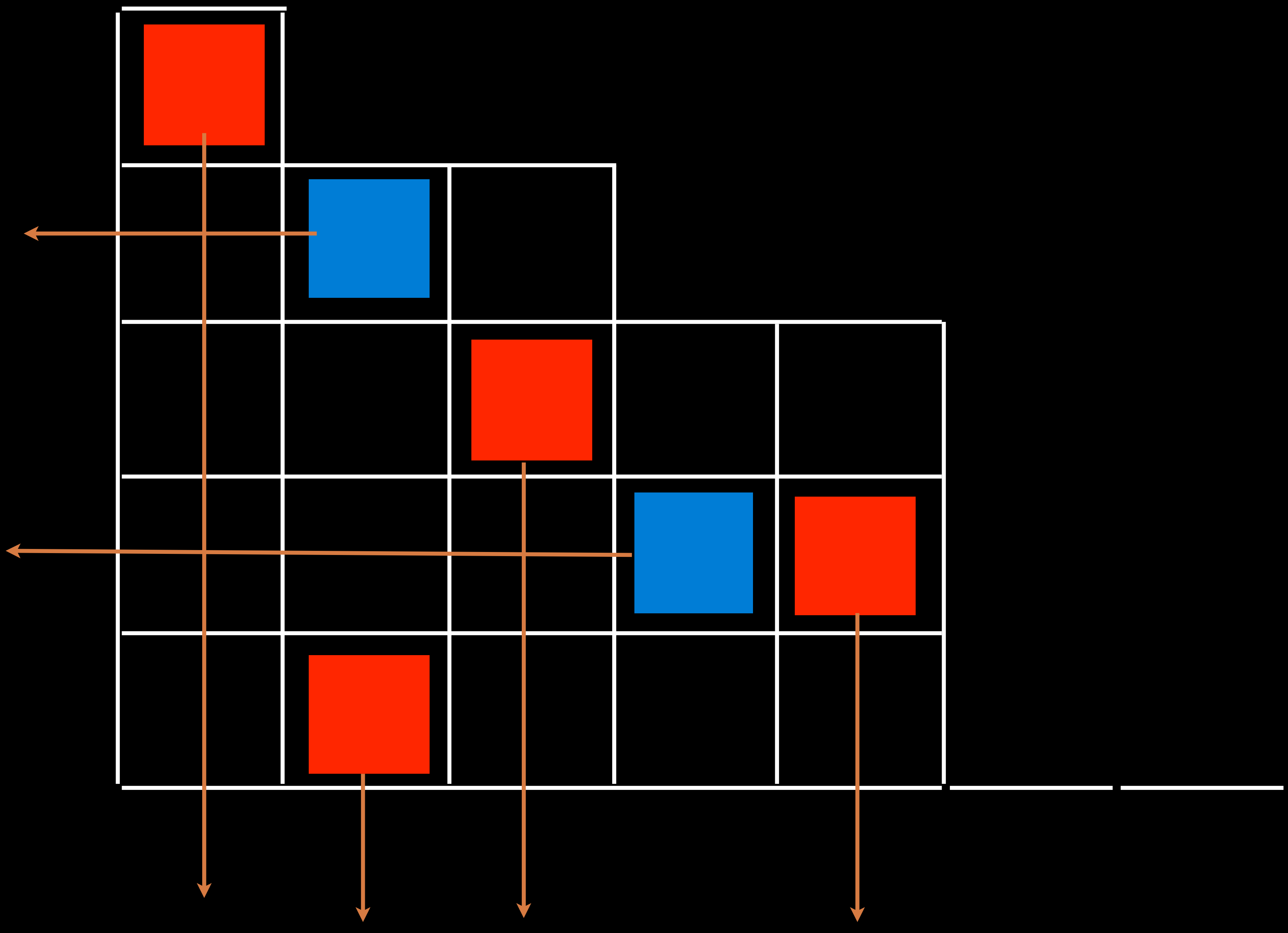
$n = 12$

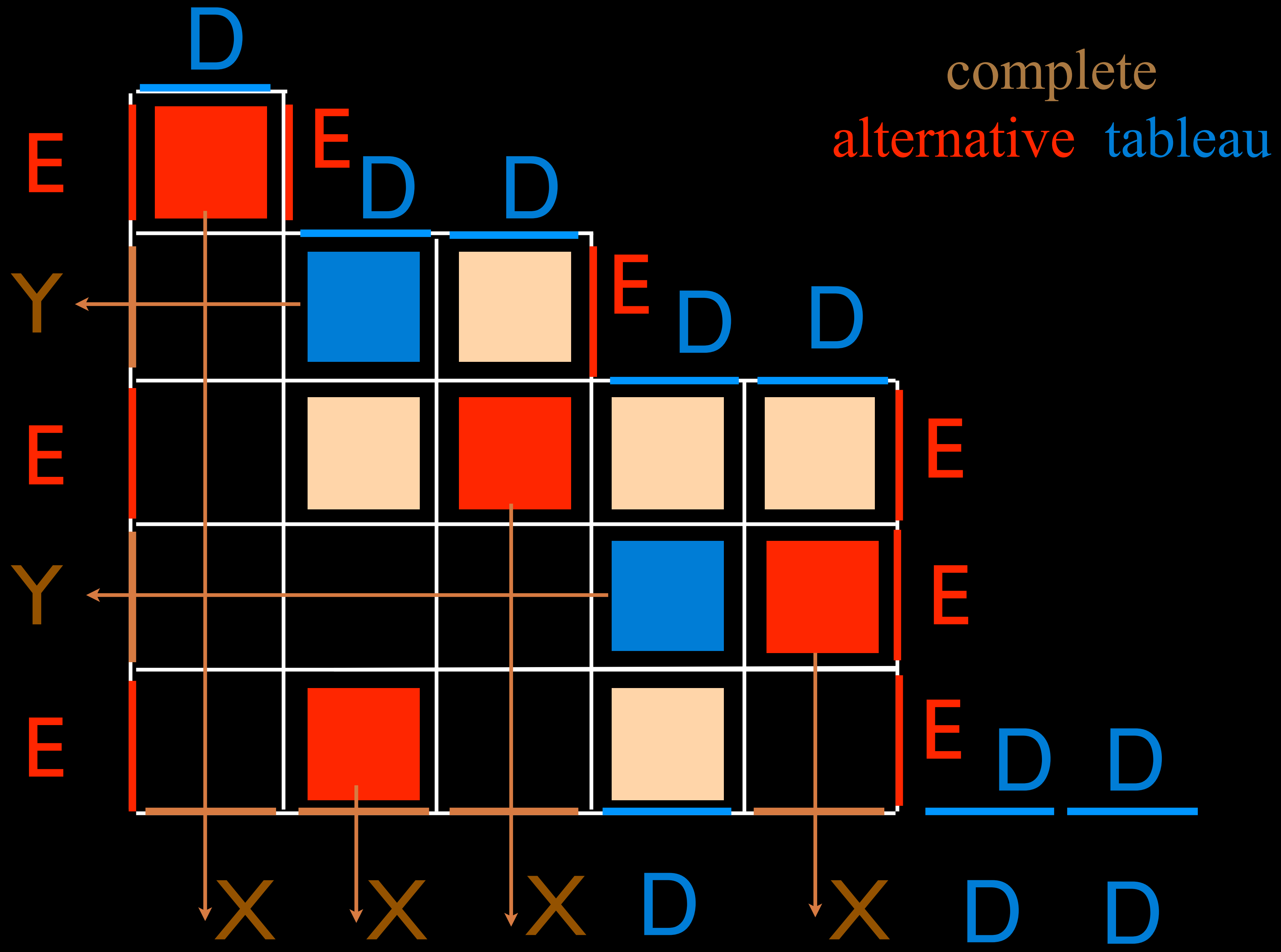


complete

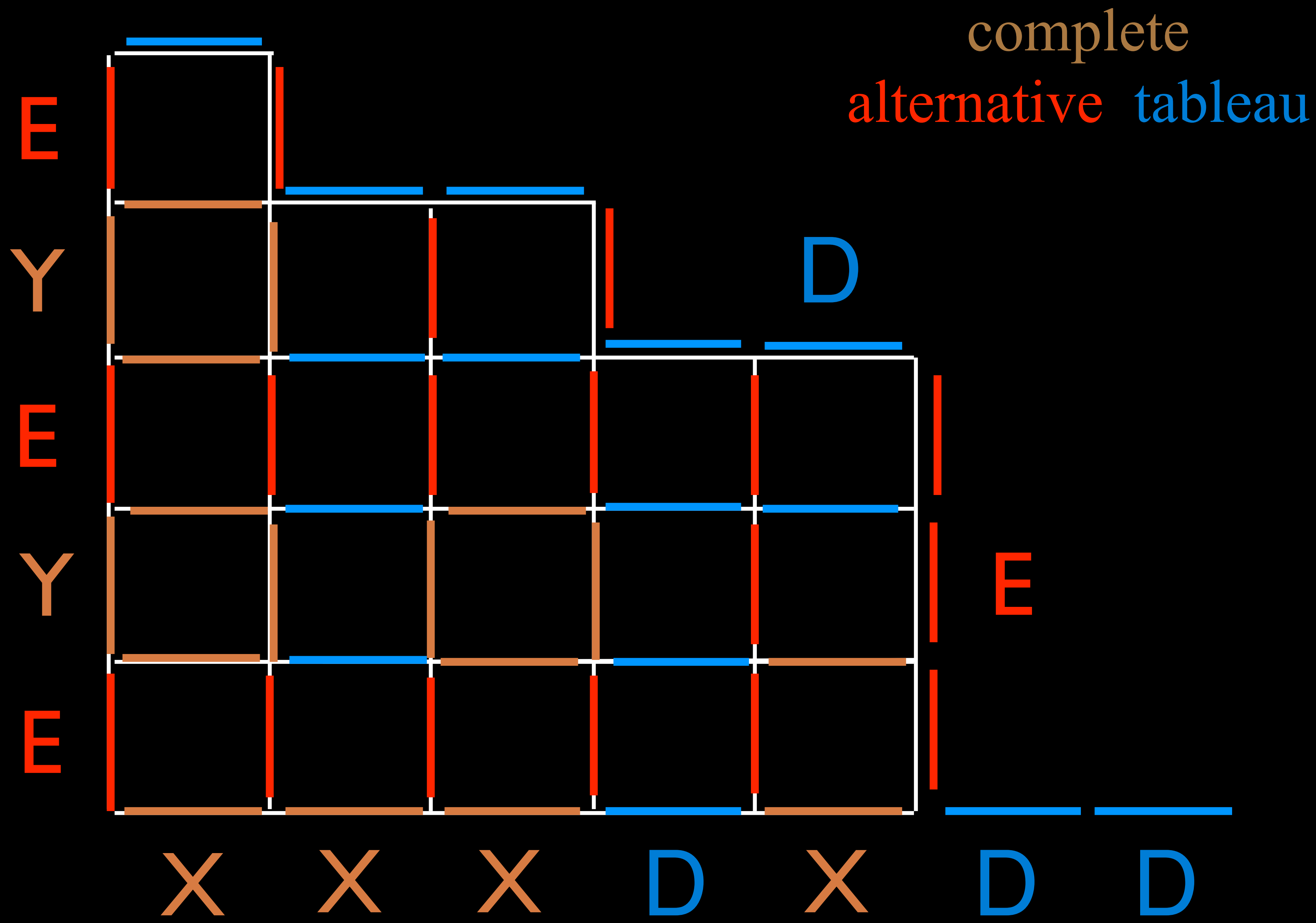
Q-tableau ⟷ alternative tableaux

Q PASEP algebra

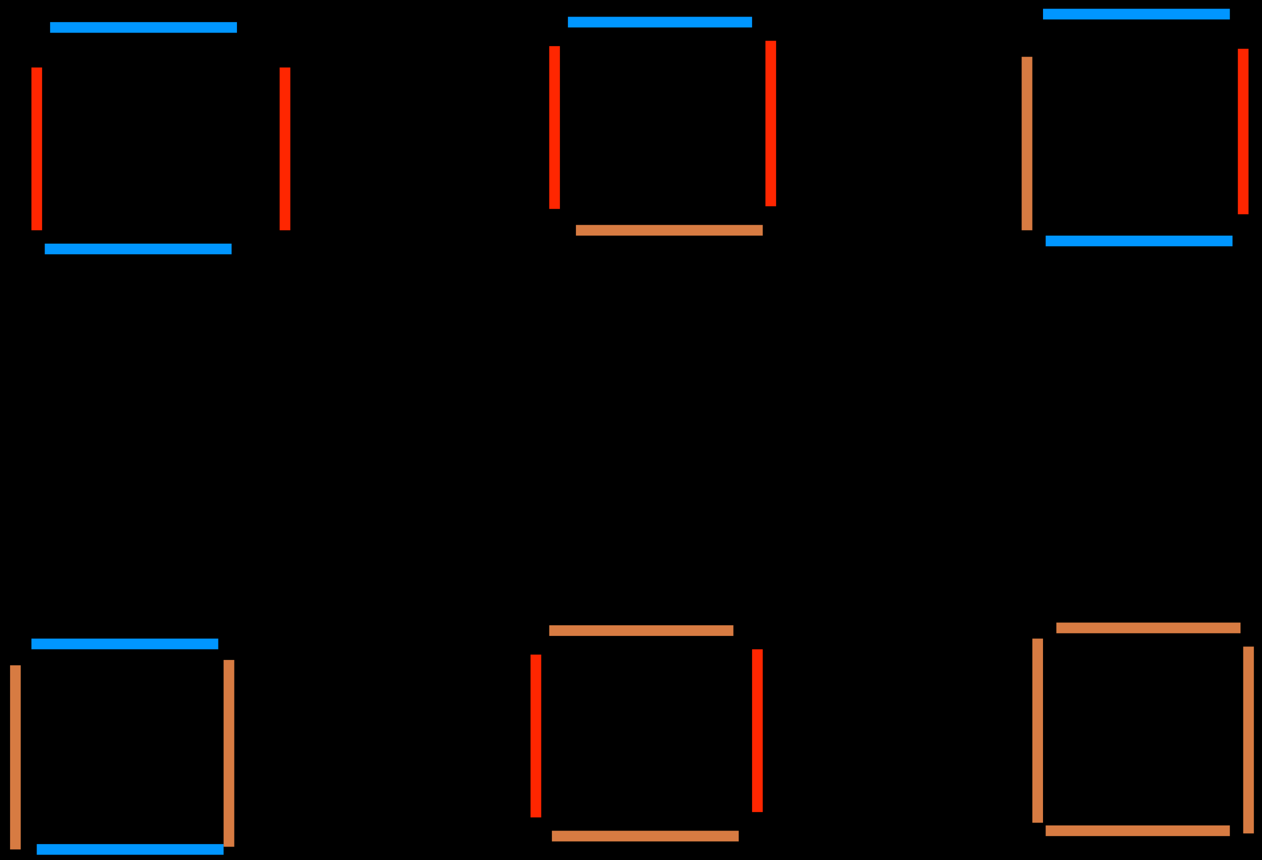




complete
alternative tableau
D
E
E D D
Y
E
E D D
E
Y
E
E
E D D
X X X D X D D

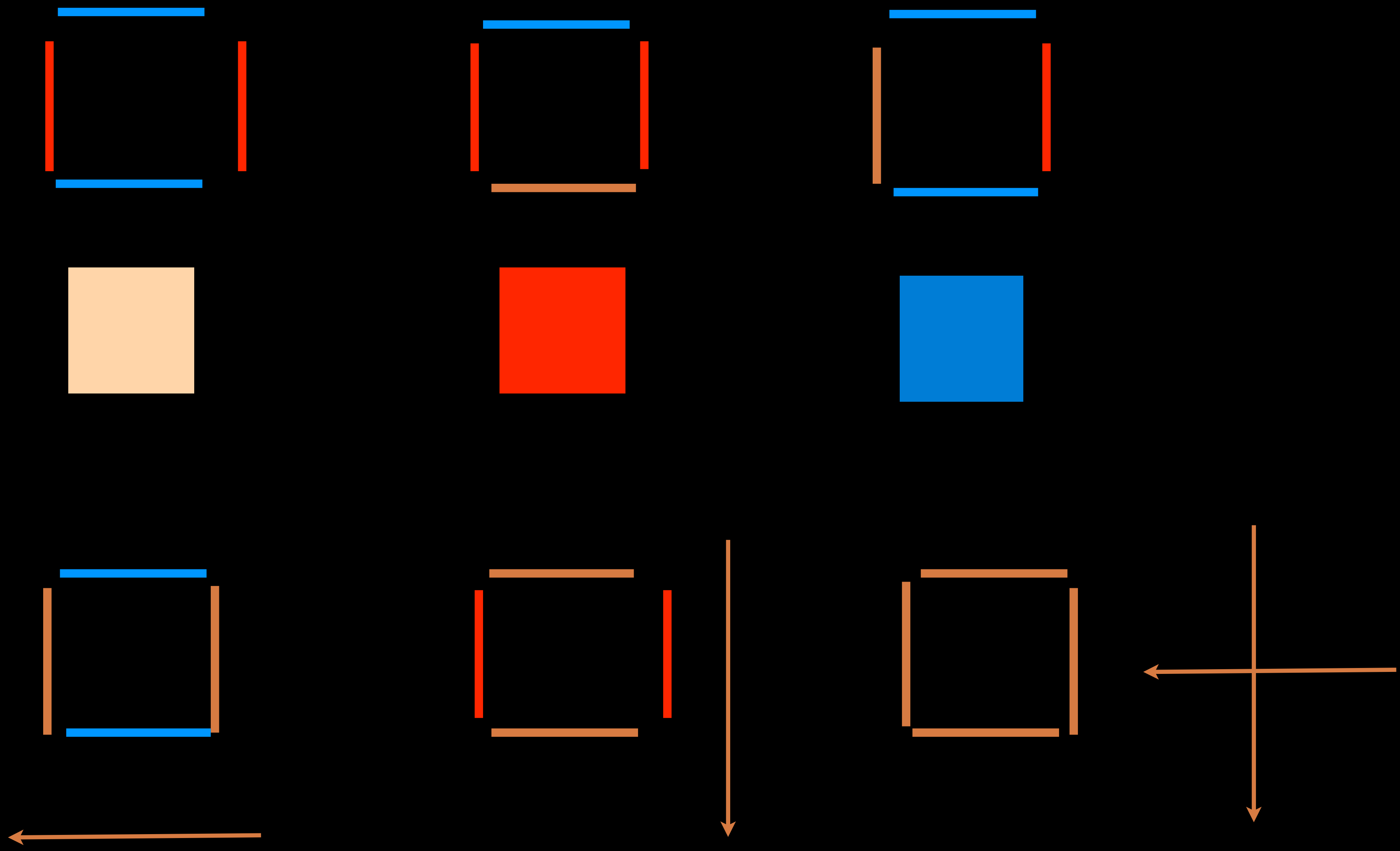

complete
alternative tableau
E
Y
E
Y
E
D
E
X X X D X D D

complete
alternative tableau



complete

alternative tableau



example 2

Heisenberg
operators
U, D

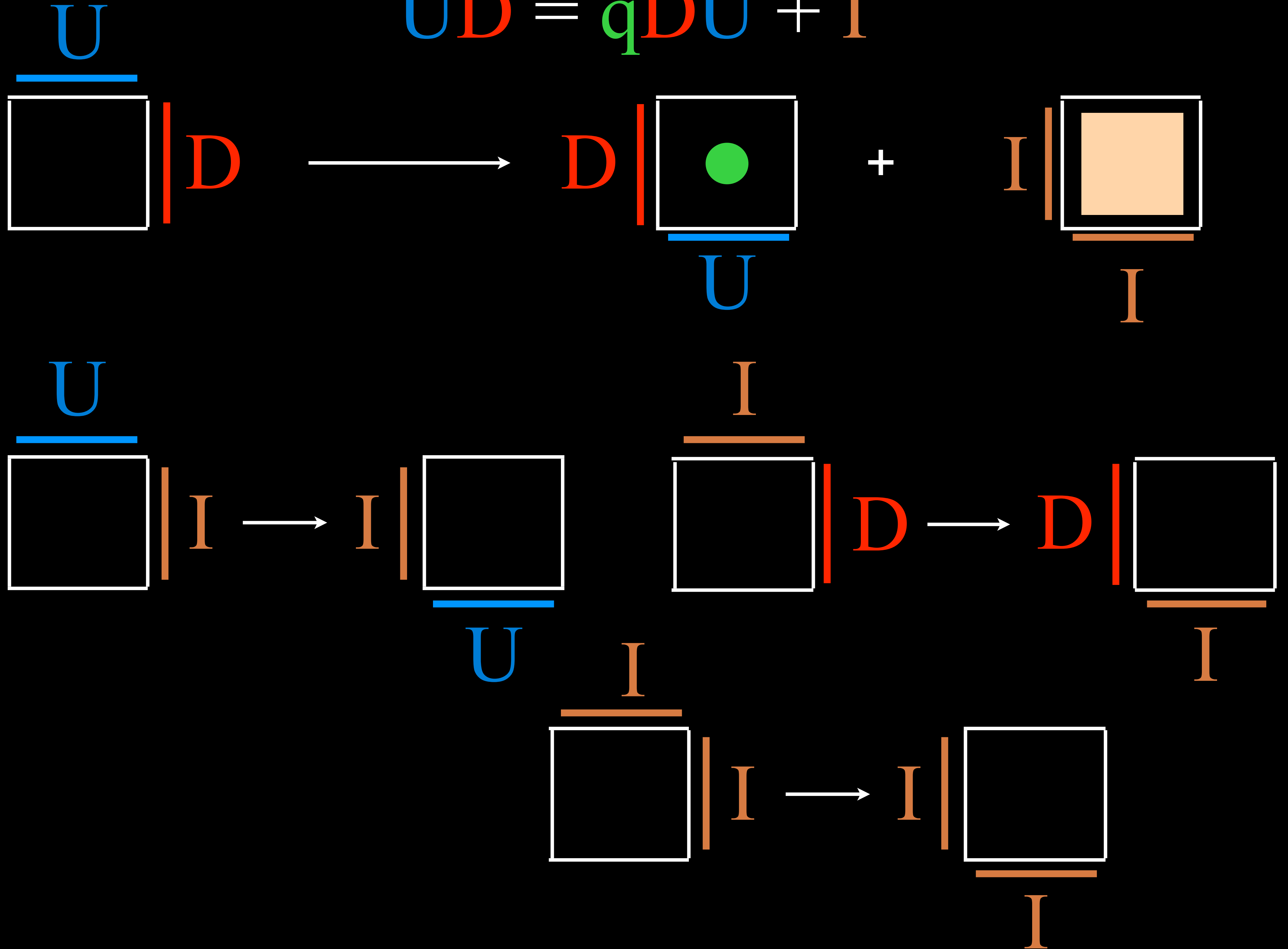
$$UD = DU + I$$

creation and annihilation opeators

quantum mechanics

normal ordering

$$UD = qDU + I$$



$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

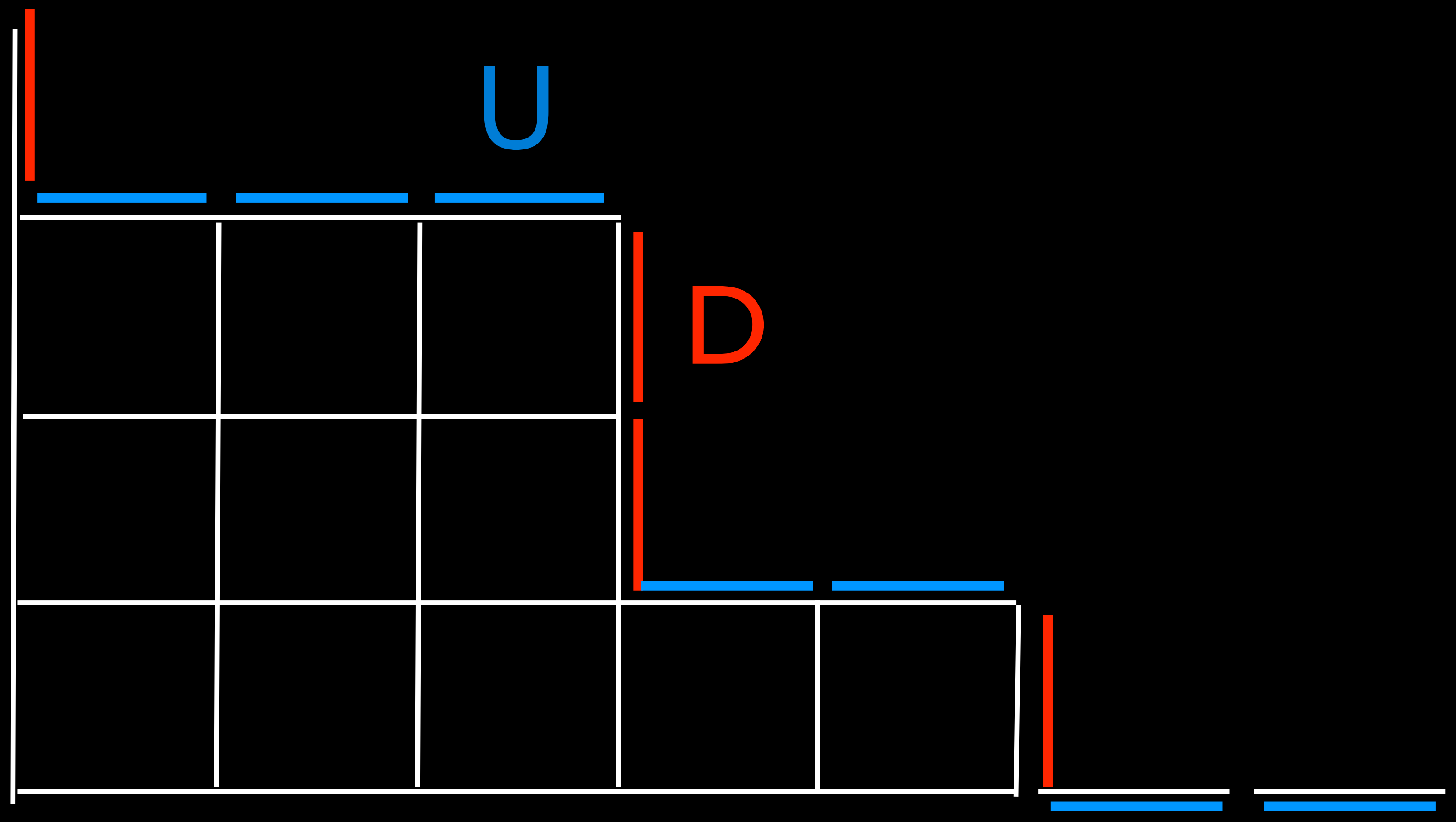
quadratic algebra

4 generators U, D, I_v, I_h

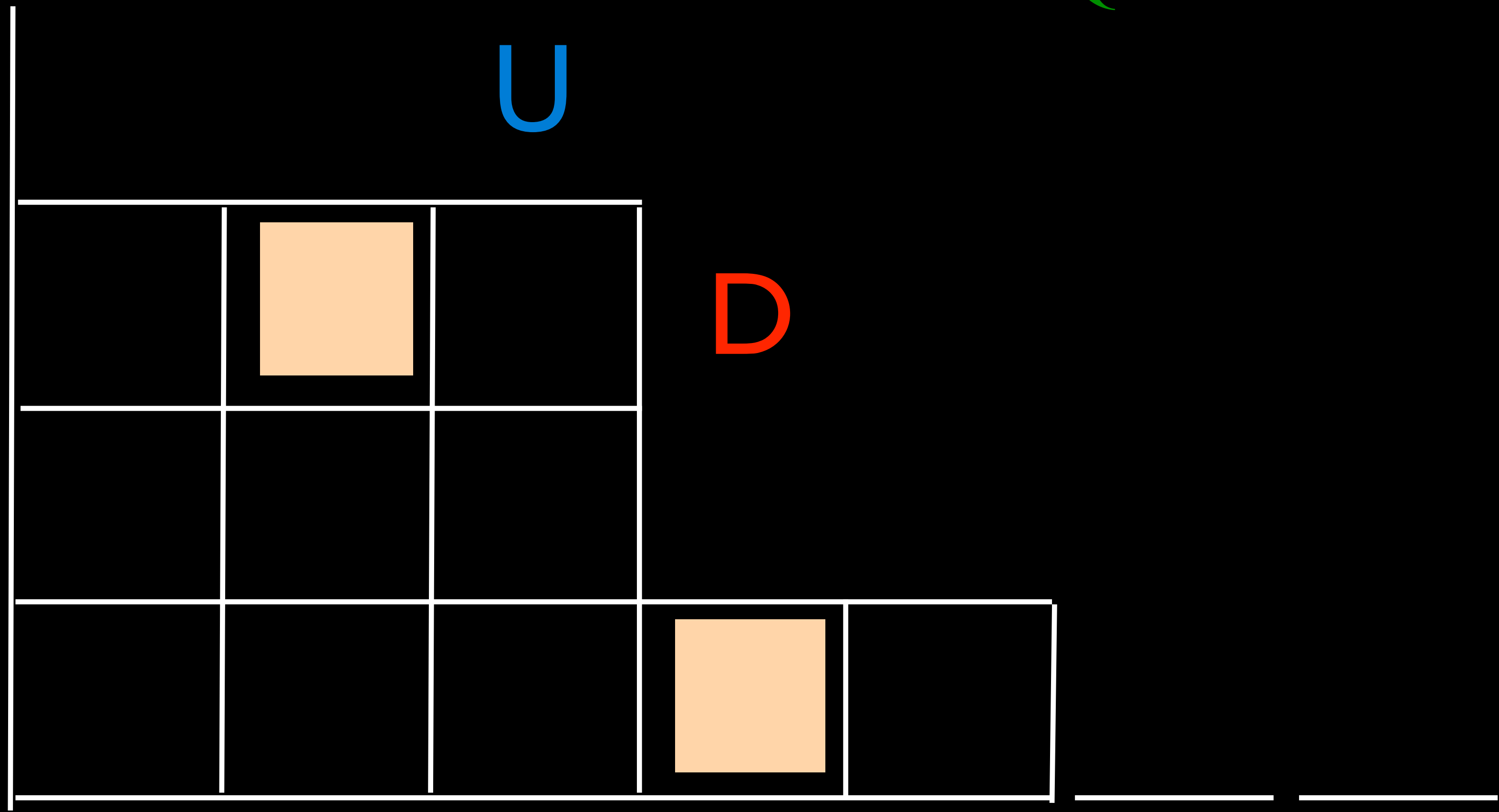
4 relations

$$\begin{cases} UD \rightarrow DU \\ U I_v \rightarrow I_v U \\ I_h D \rightarrow D I_h \\ I_h I_v \rightarrow I_v I_h \end{cases} \qquad UD \rightarrow I_v I_h$$

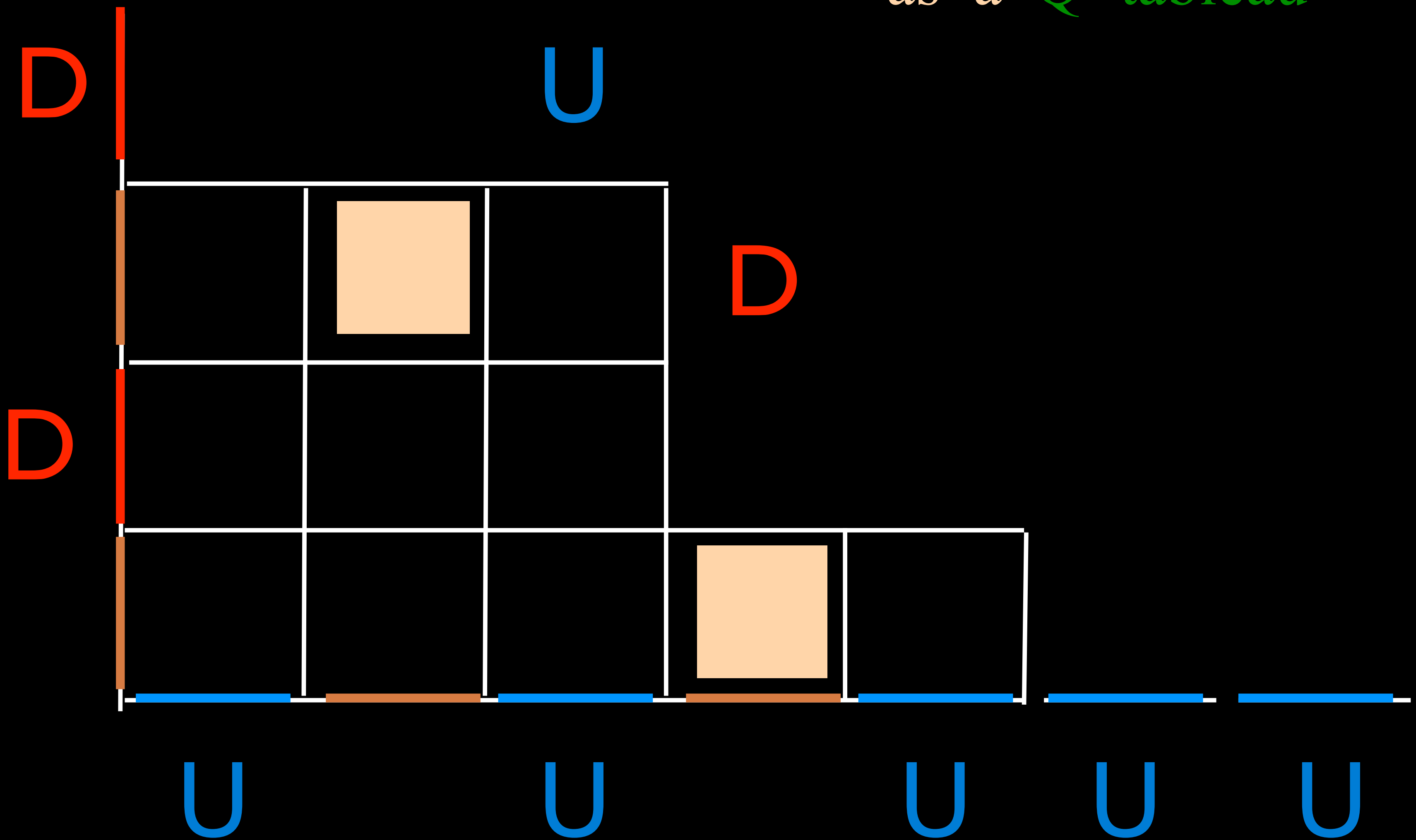
rewriting rules


U
D

rook placement
as a Q- tableau

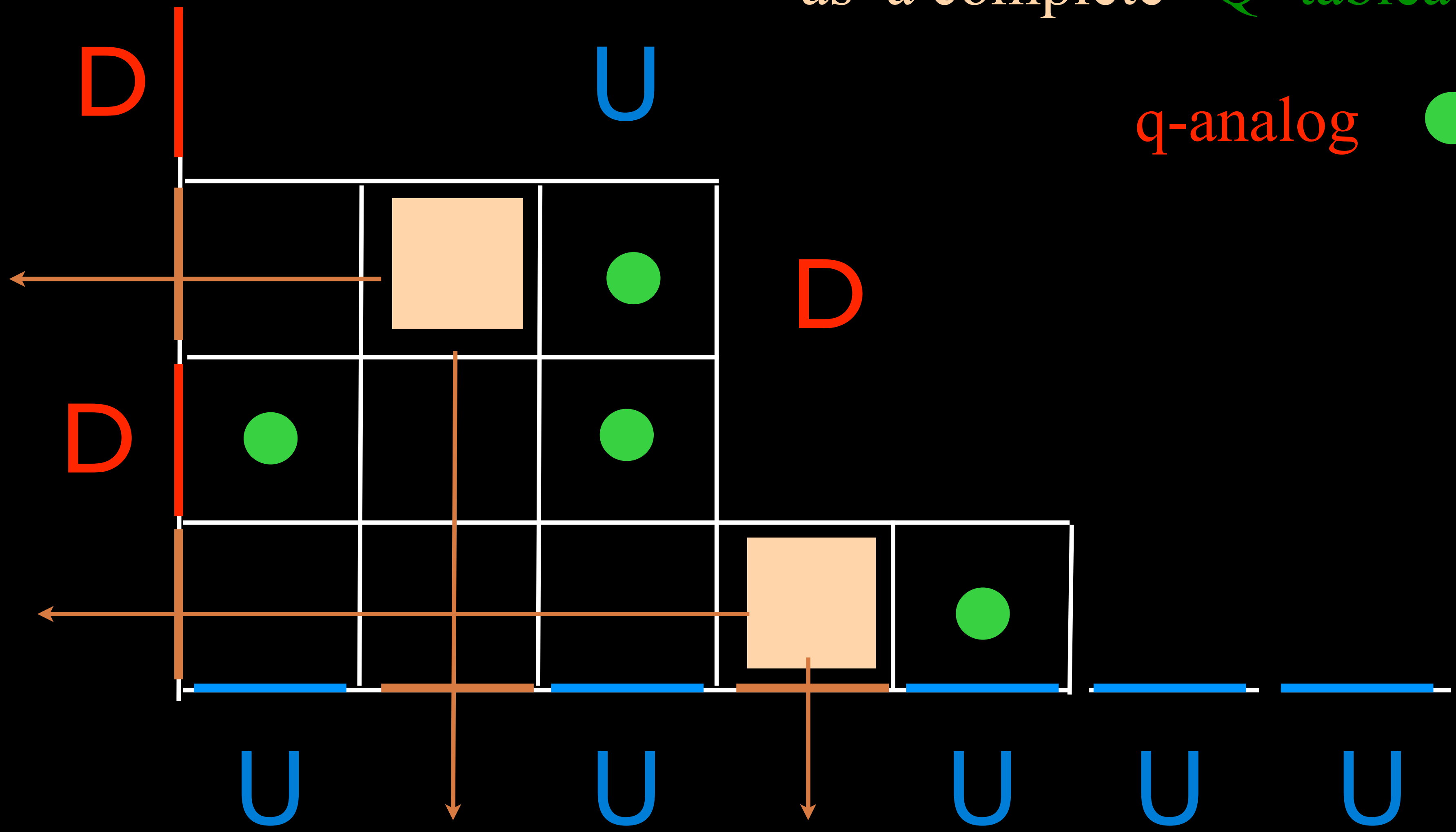


rook placement as a Q- tableau



rook placement
as a complete Q- tableau

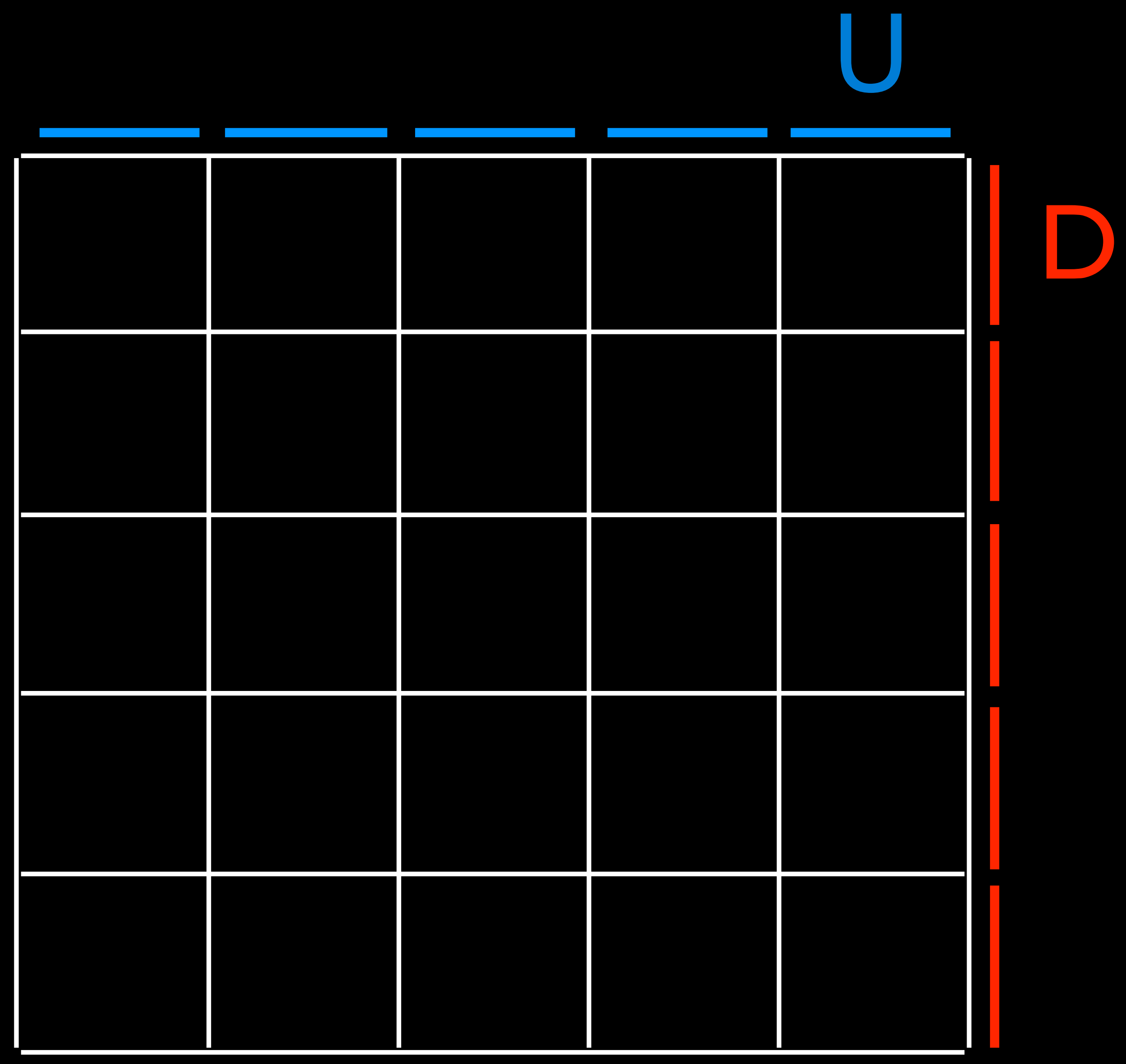
q-analog

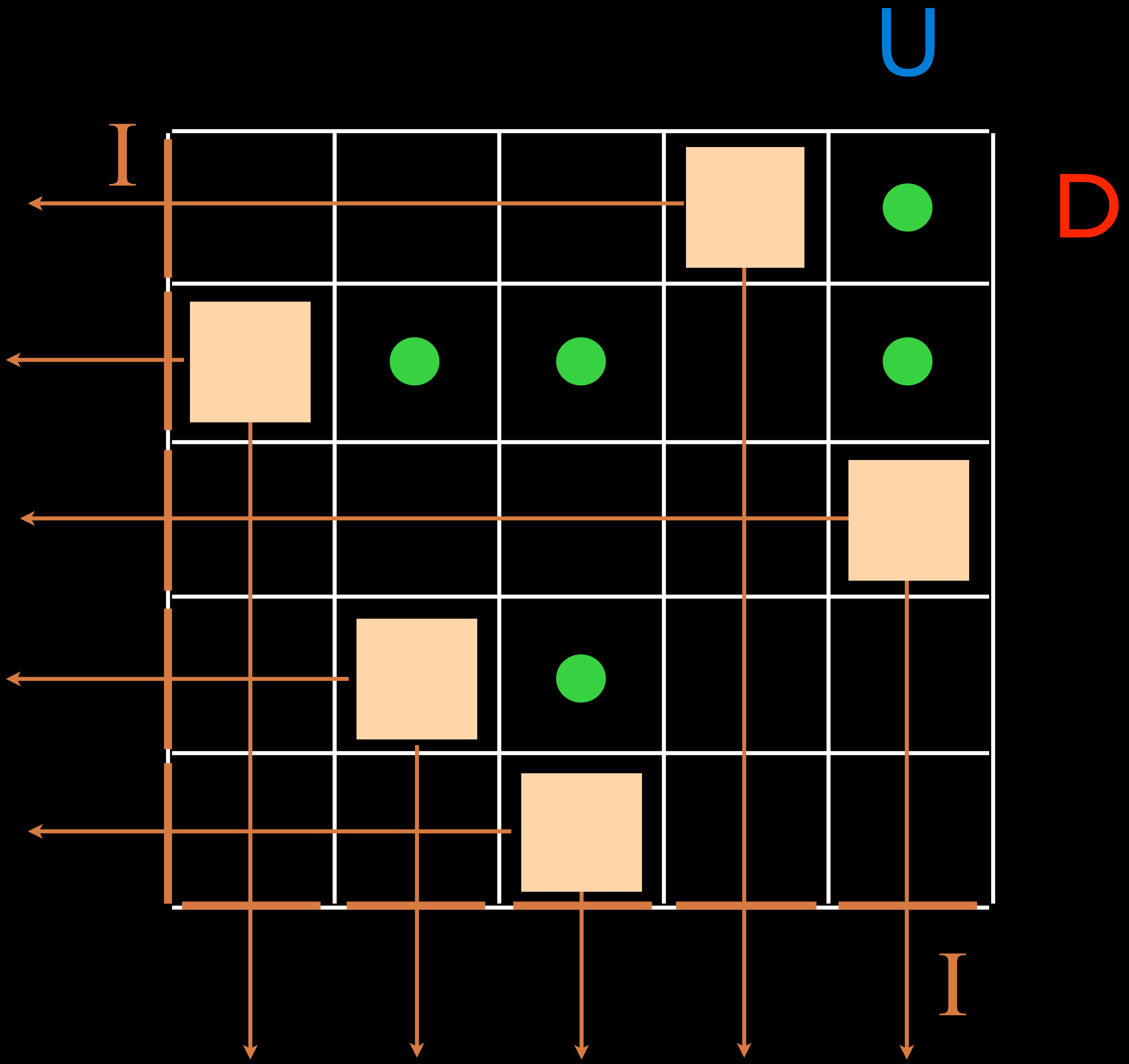


$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$


U
D



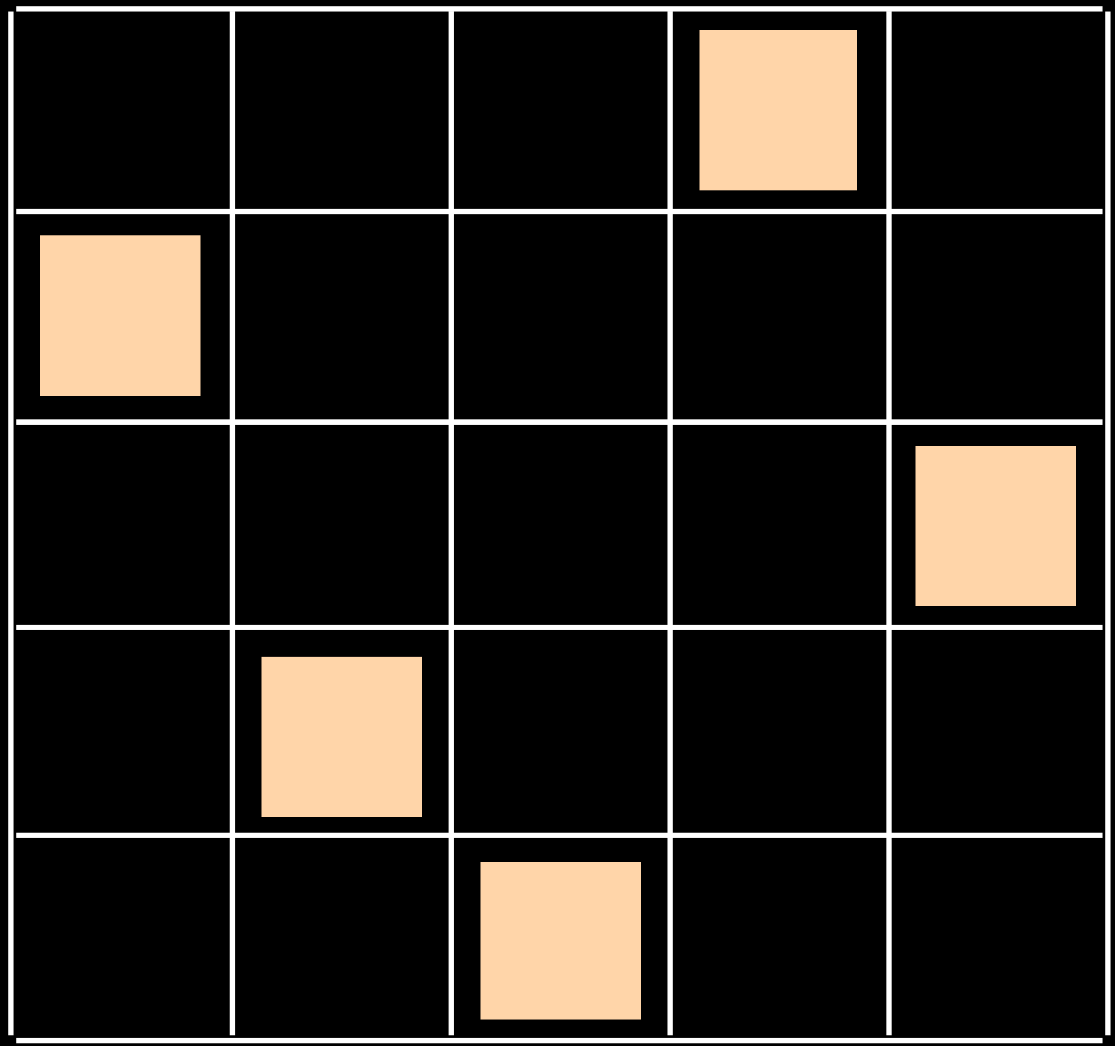
complete
Q- tableau
(5 labels)

$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

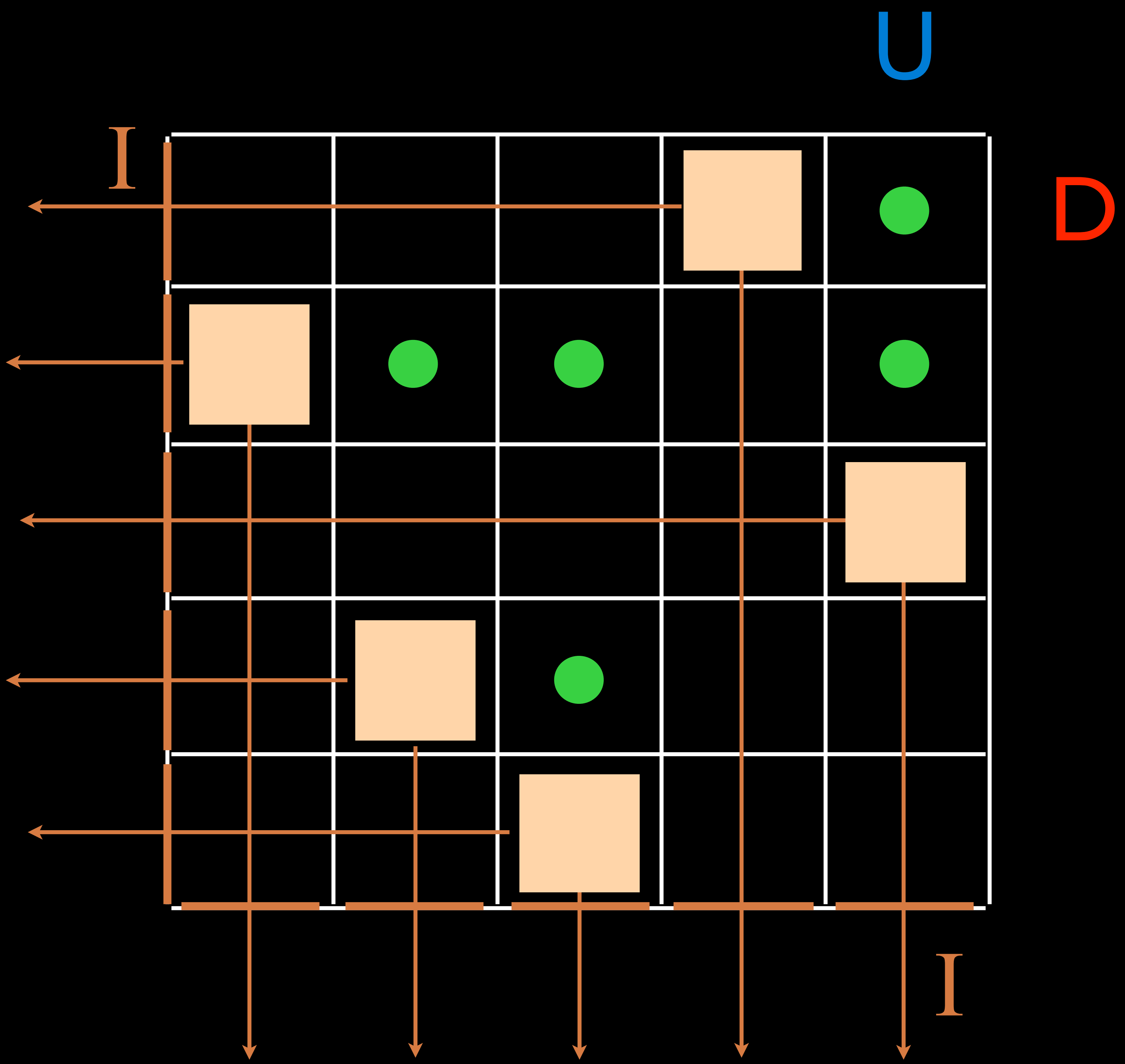
quadratic algebra

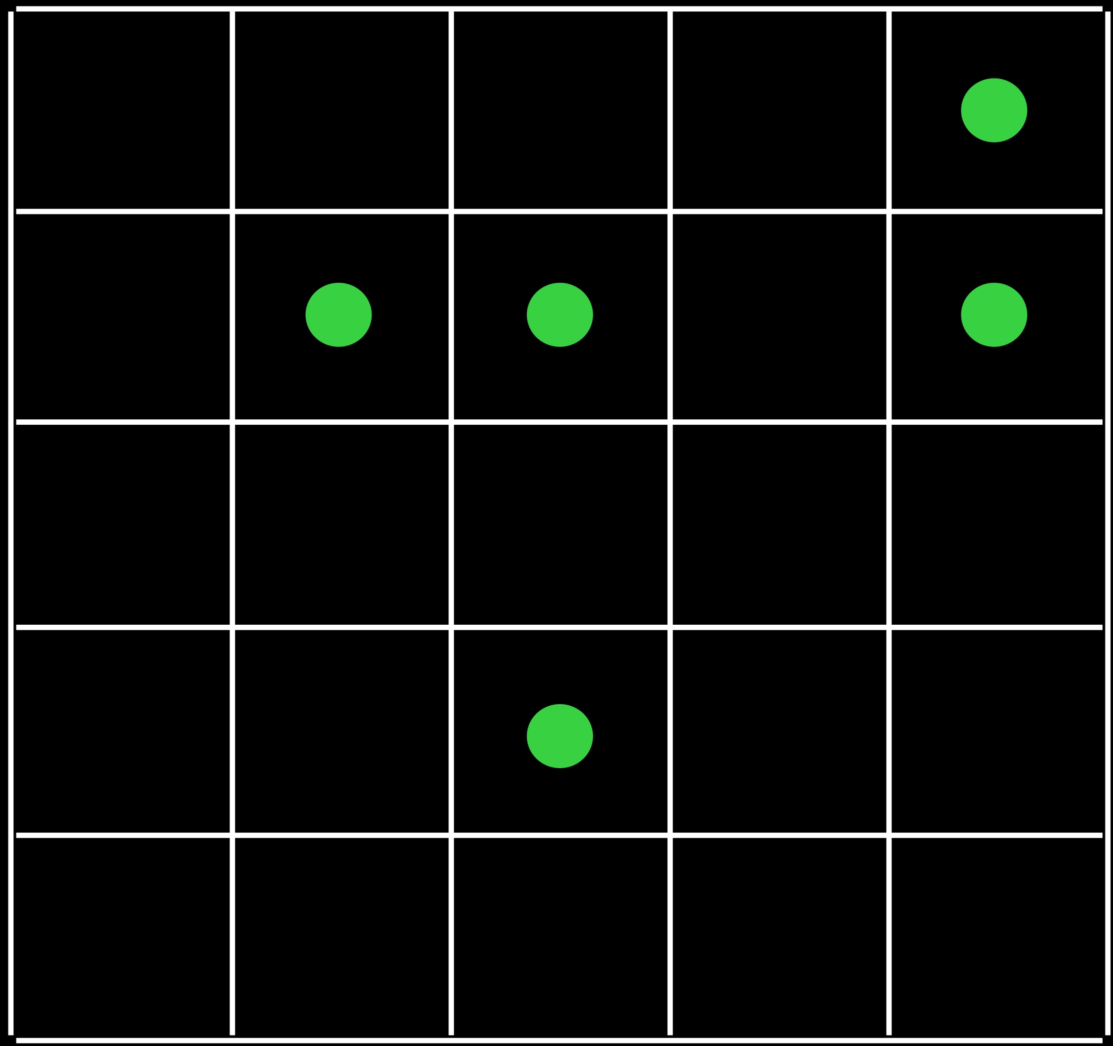
4 generators U, D, I_v, I_h

4 relations



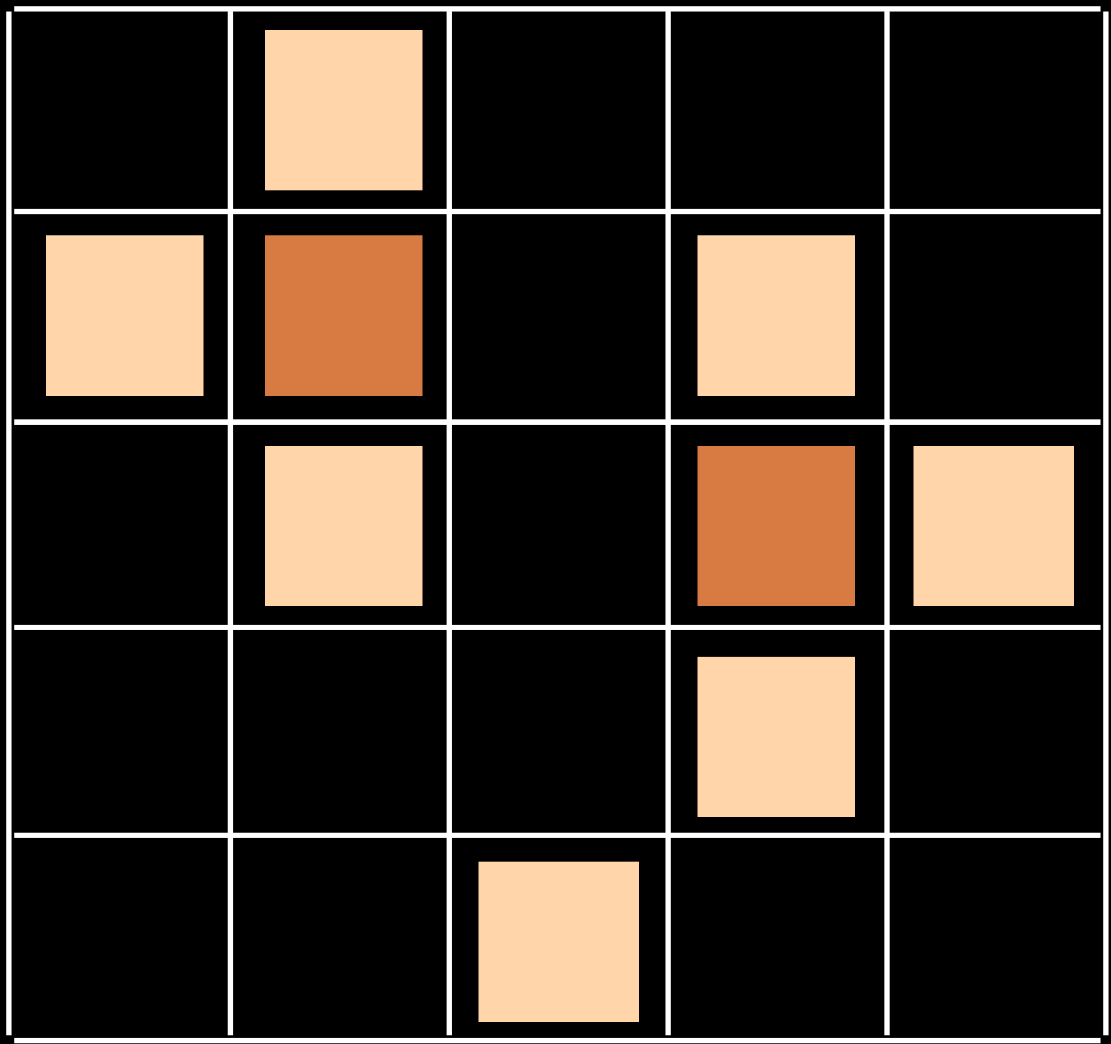
permutation as a Q- tableau


U
D
I
I



another Q-tableau:
Rothe diagram of a permutation

ASM



$A, A', B, B',$

commutations

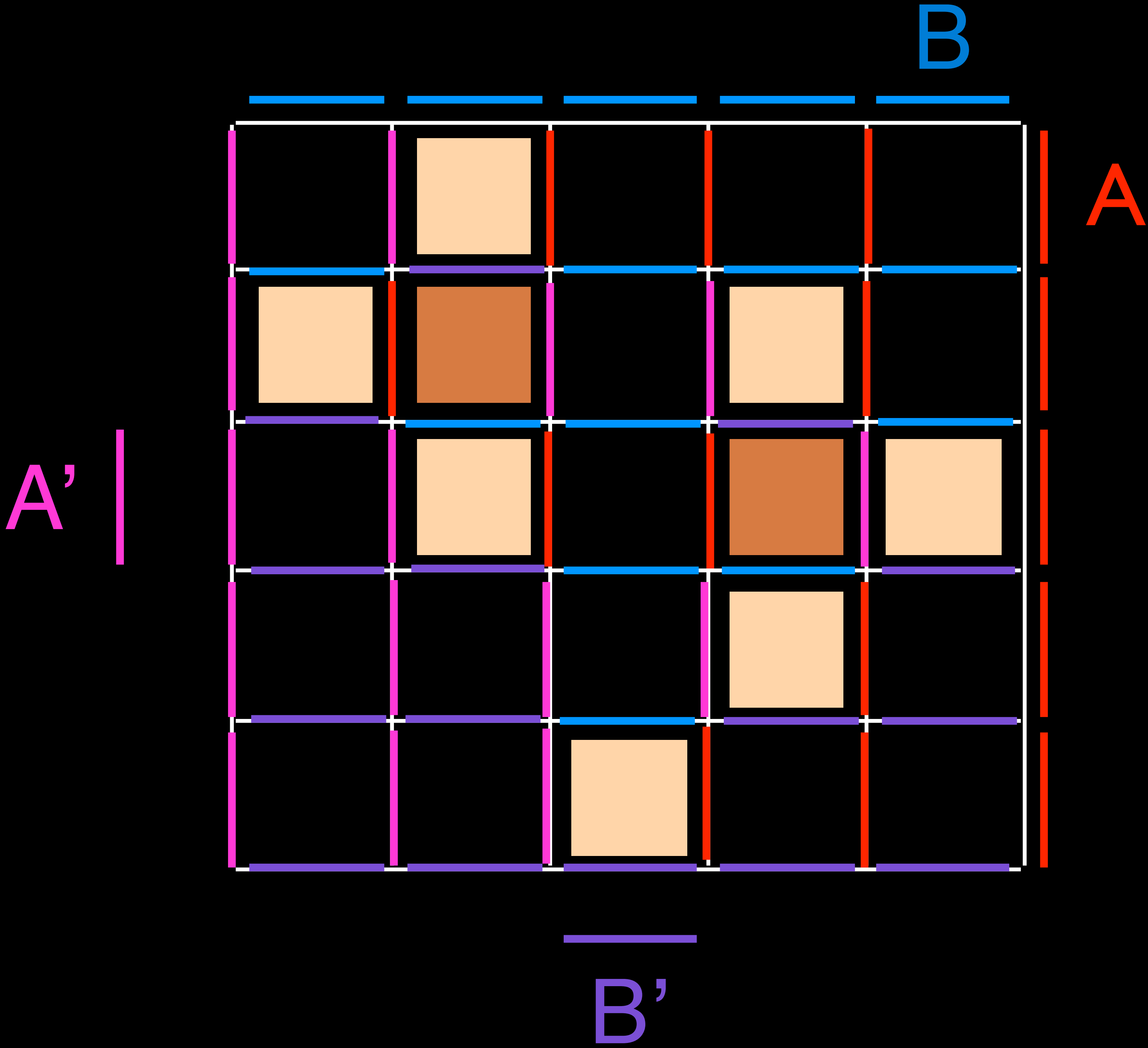
$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

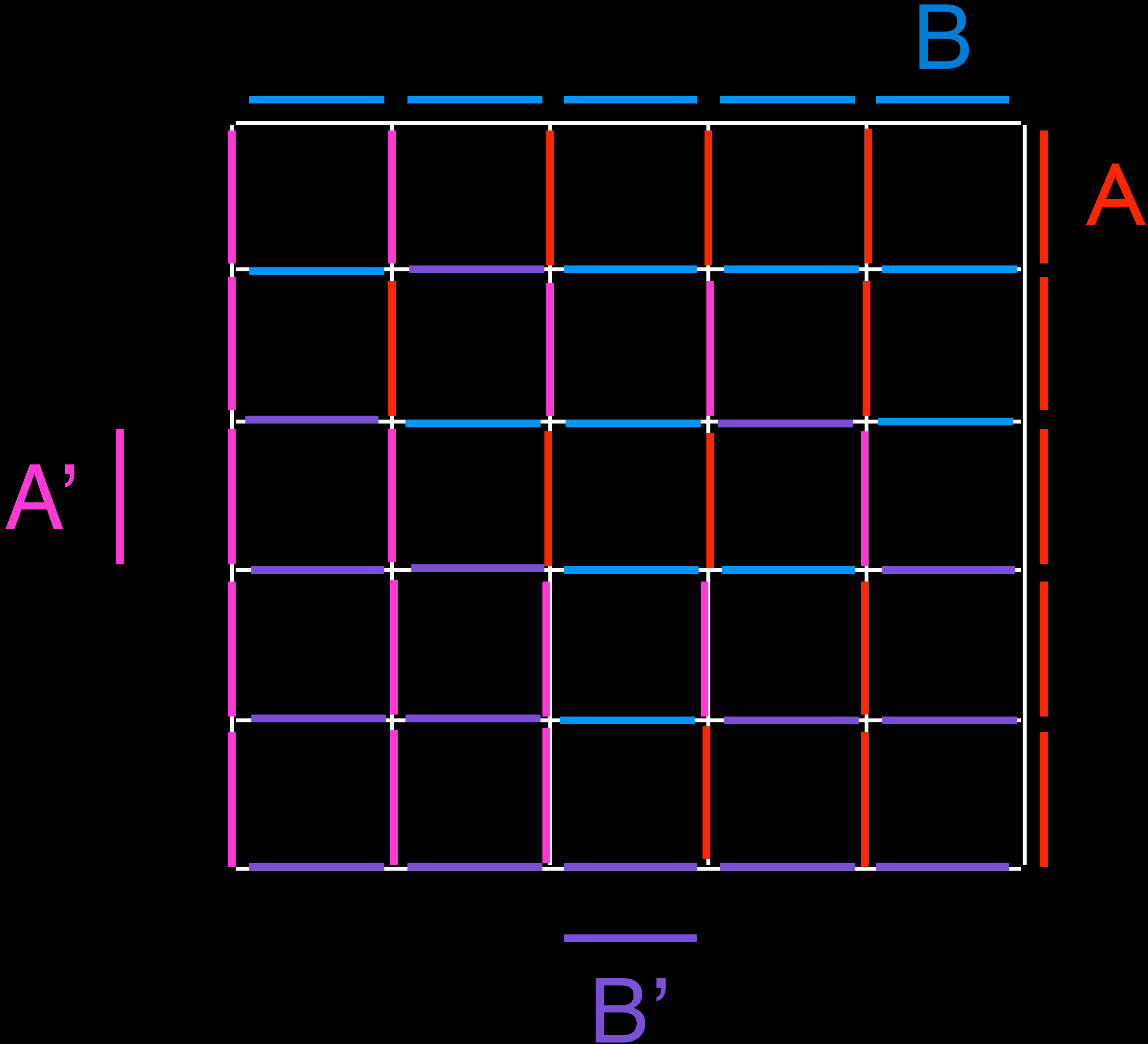
$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

Prop. For $w = B^n A^n$

$u = A'^n$, $v = B'^n$

$c(u, v; w)$ = the number of $n \times n$ ASM (alternating sign matrices)


B
A
A'
B'



B
A
A’
B’