Combinatorics and Physics

Chapter 7

The cellular ansatz

Ch7f

The XYZ algebra and its Q-tableaux

IIT-Madras
9 March 2015

Xavier Viennot
CNRS, LaBRI, Bordeaux

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

orthogonal
polynomials

bijections

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

rooks placements

permutations

RSK

permutations ⟷ pairs of Tableaux Young

DE = qED + E + D

PASEP

alternative tableaux ⟷ permutations

Laguerre histories

dynamical systems in physics
stationary probabilities

Q-tableaux

quadratic algebra Q

ASM, (alternating sign matrices)

commutations
rewriting rules

planarization

planar
automata

"The cellular Ansatz"

combinatorial
objects
on a 2d lattice

representation
by operators

Physics

data structures
"histories"

orthogonal
polynomials

bijections

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

dynamical systems in physics
stationary probabilities

rooks placements

permutations ⟷ (RSK) pairs of Tableaux Young

alternative tableaux ⟷ permutations

Laguerre histories

quadratic algebra Q

Q-tableaux

the XYZ algebra

ASM, (alternating sign matrices)

FPL (Fully packed loops)

tilings, non-crossing paths

commutations
rewriting rules

planarization

planar
automata

The 8-vertex algebra
(or XYZ - algebra)
(or Z - algebra)

The quadratic algebra $\mathcal{Z}$

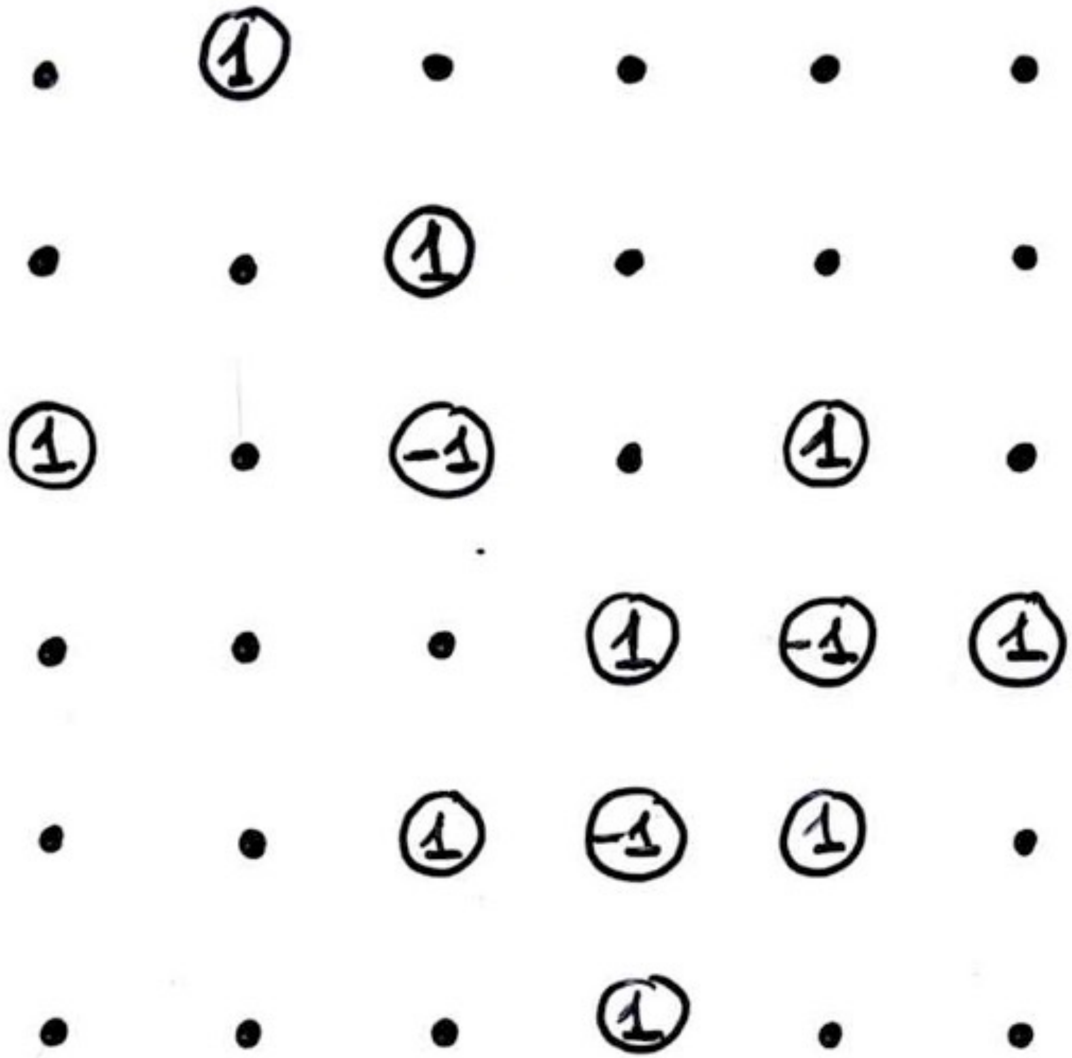
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet \end{cases}$$

alternating sign matrices (ASM)

ASM



Alternating sign matrices

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

The quadratic algebra Z

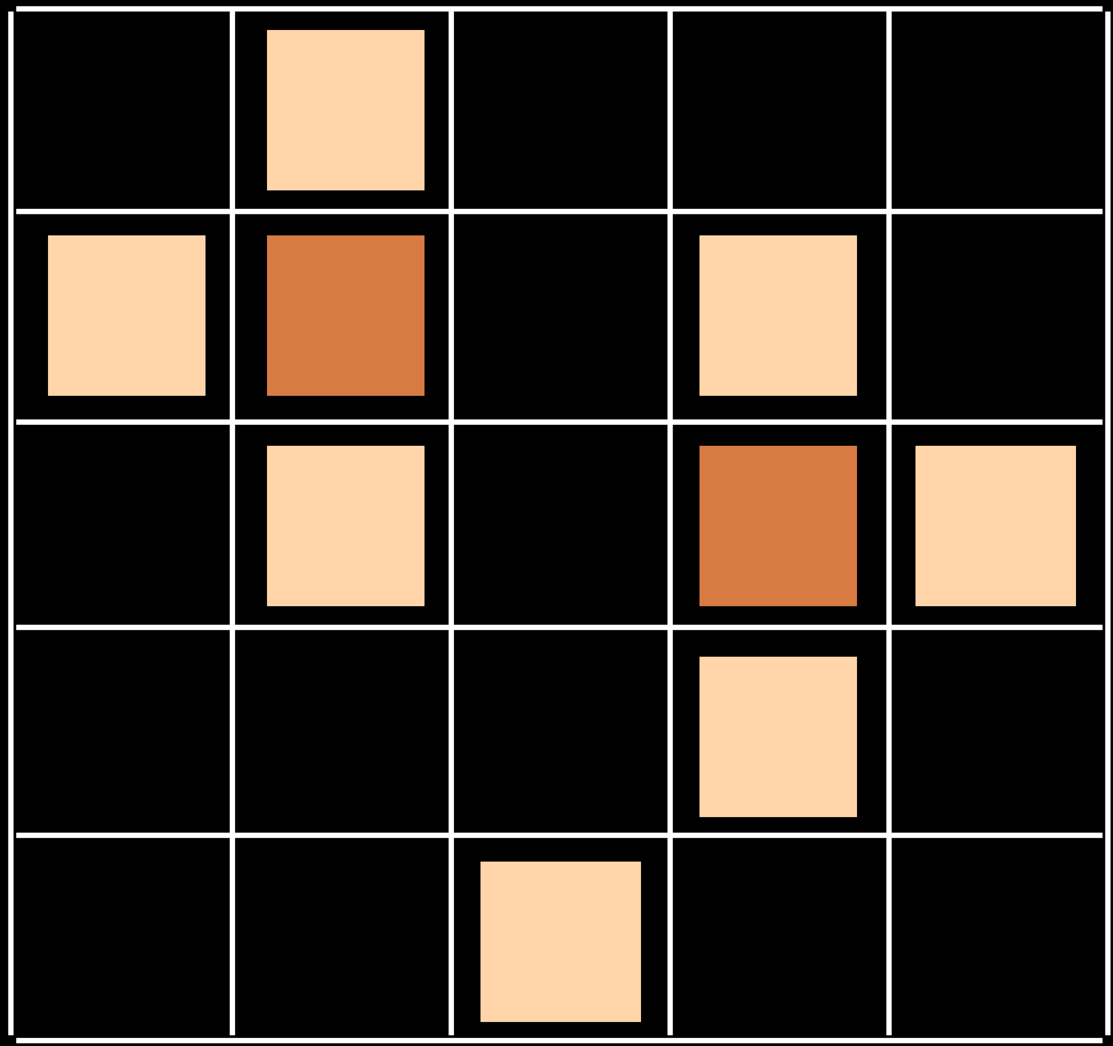
4 generators $B_\bullet \; A_\bullet \; B \; A$

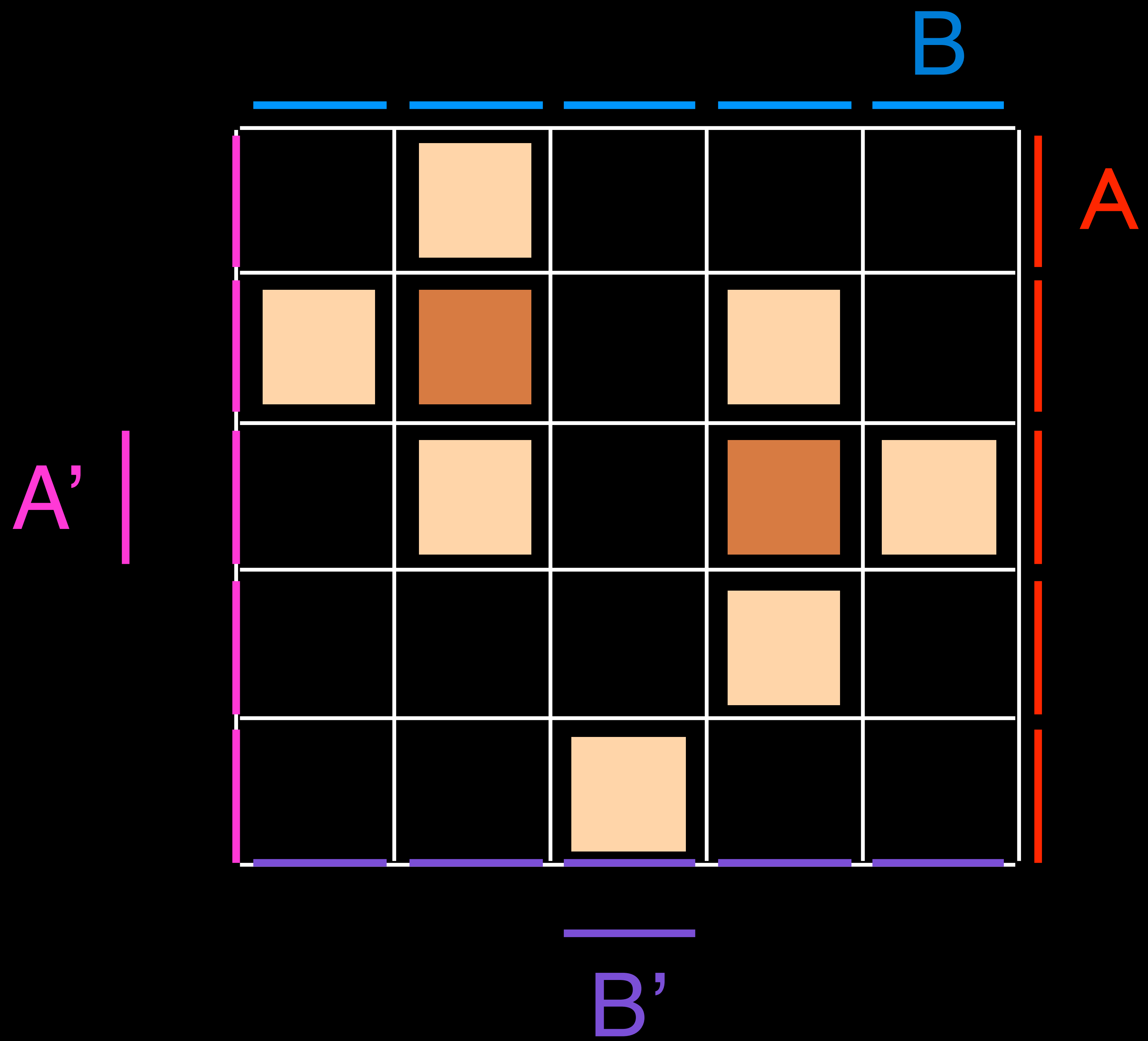
8 parameters $q_{....}$, $t_{....}$

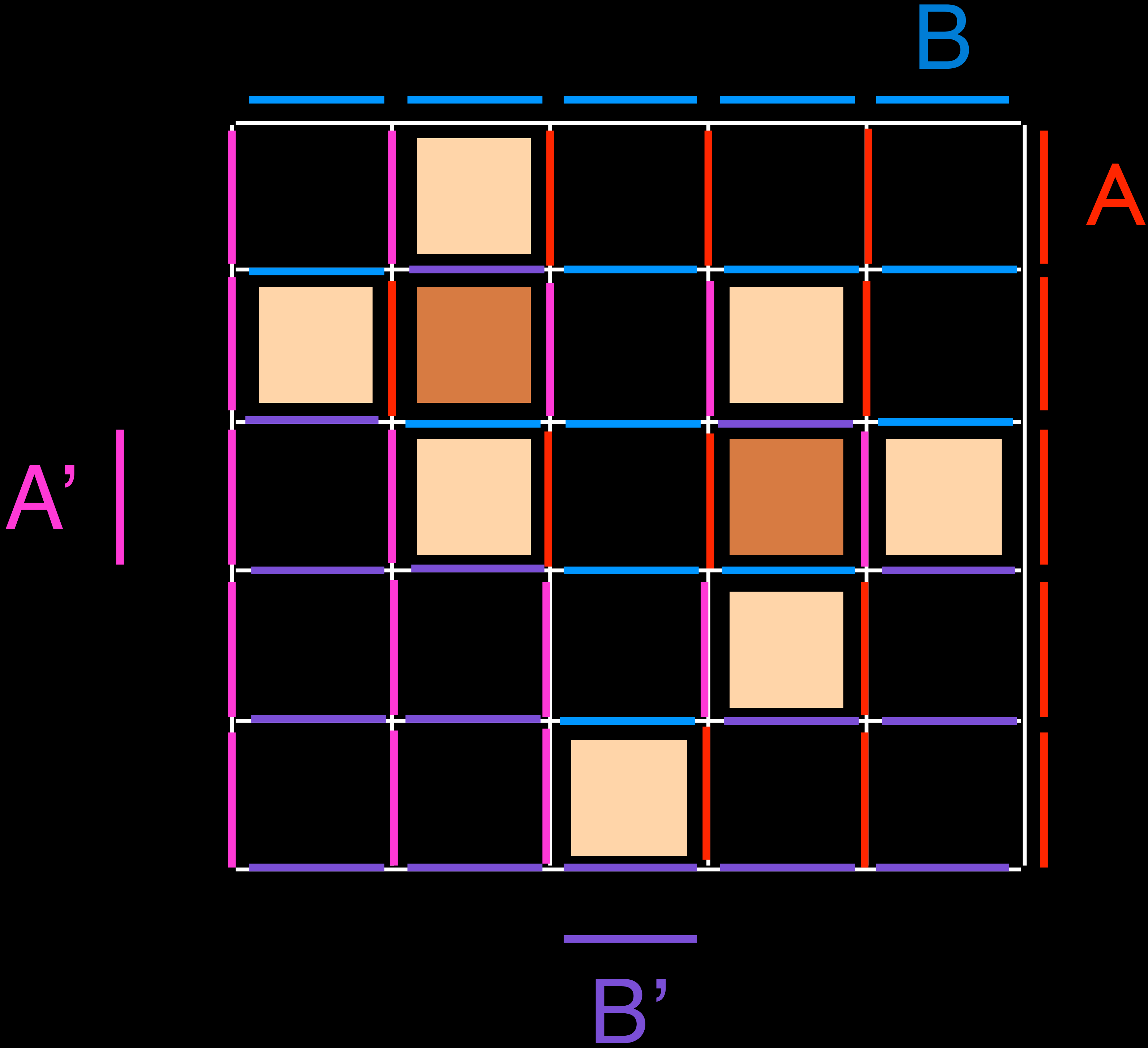
$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, AB_\bullet + \bigcirc\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, AB_\bullet \end{cases}$$

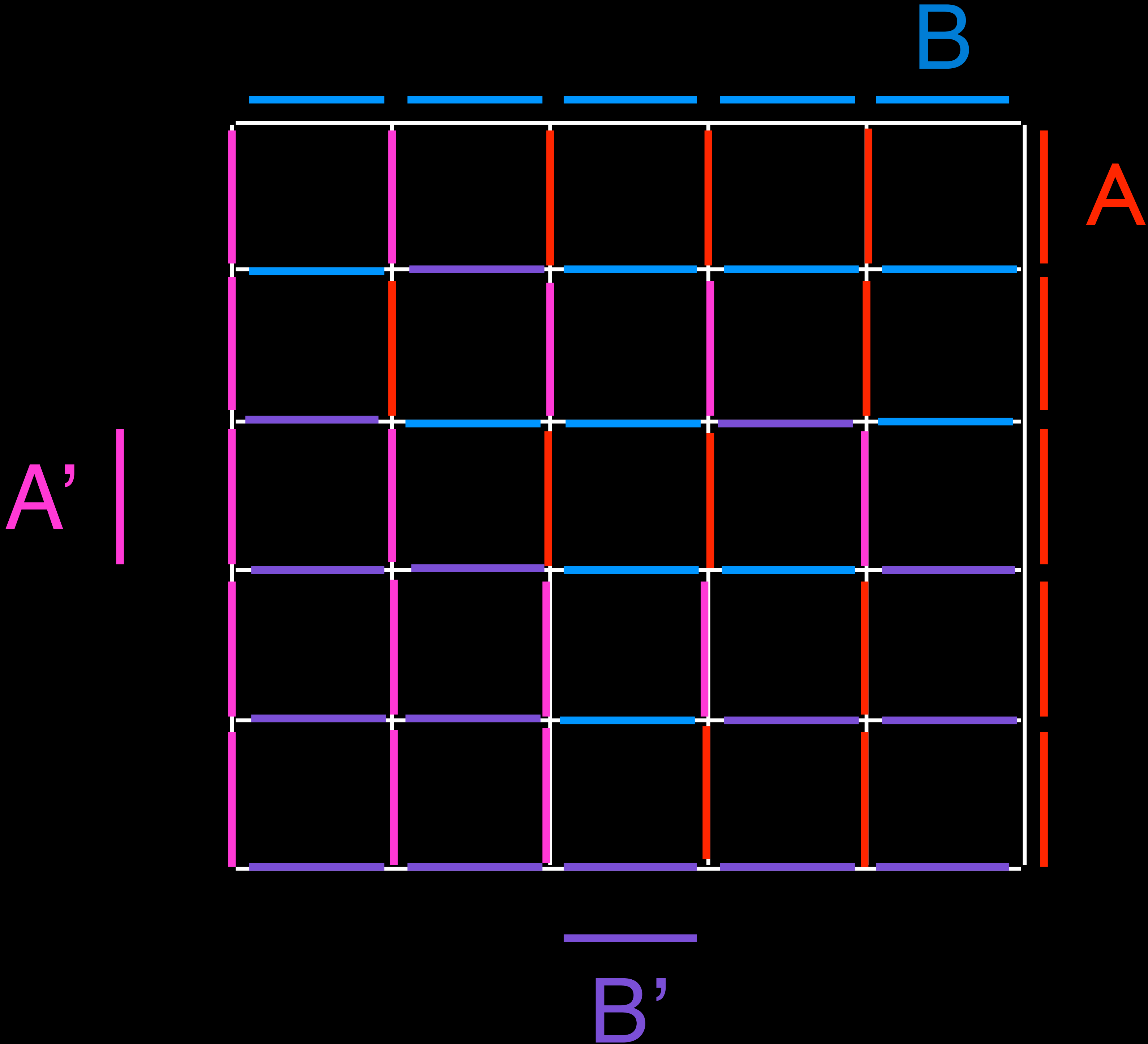
$$w = B^n A^n \qquad uv = A_\bullet^n B_\bullet^n$$

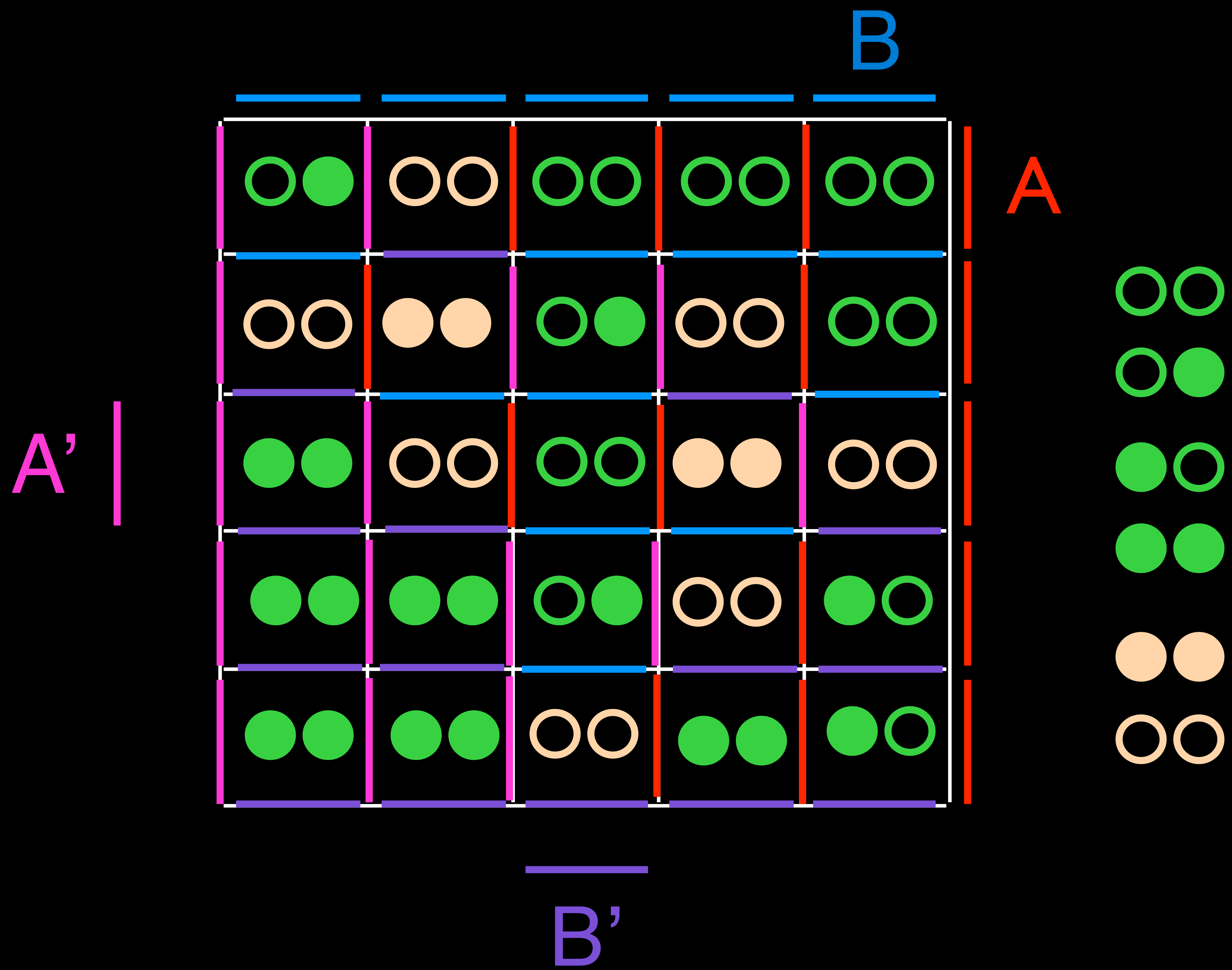
$c(u, v; w) =$ nb of ASM $n \times n$

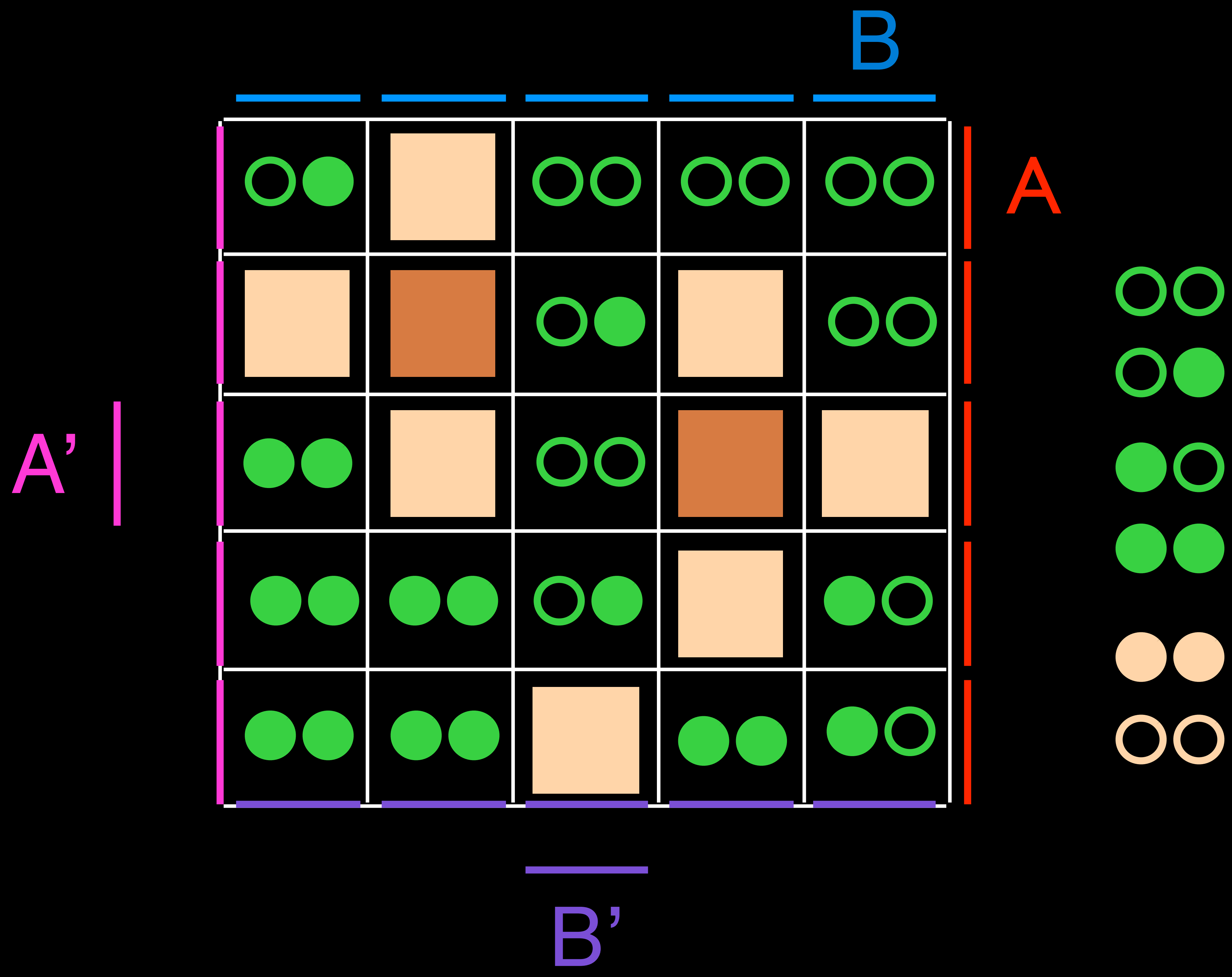



B
A
A'
B'

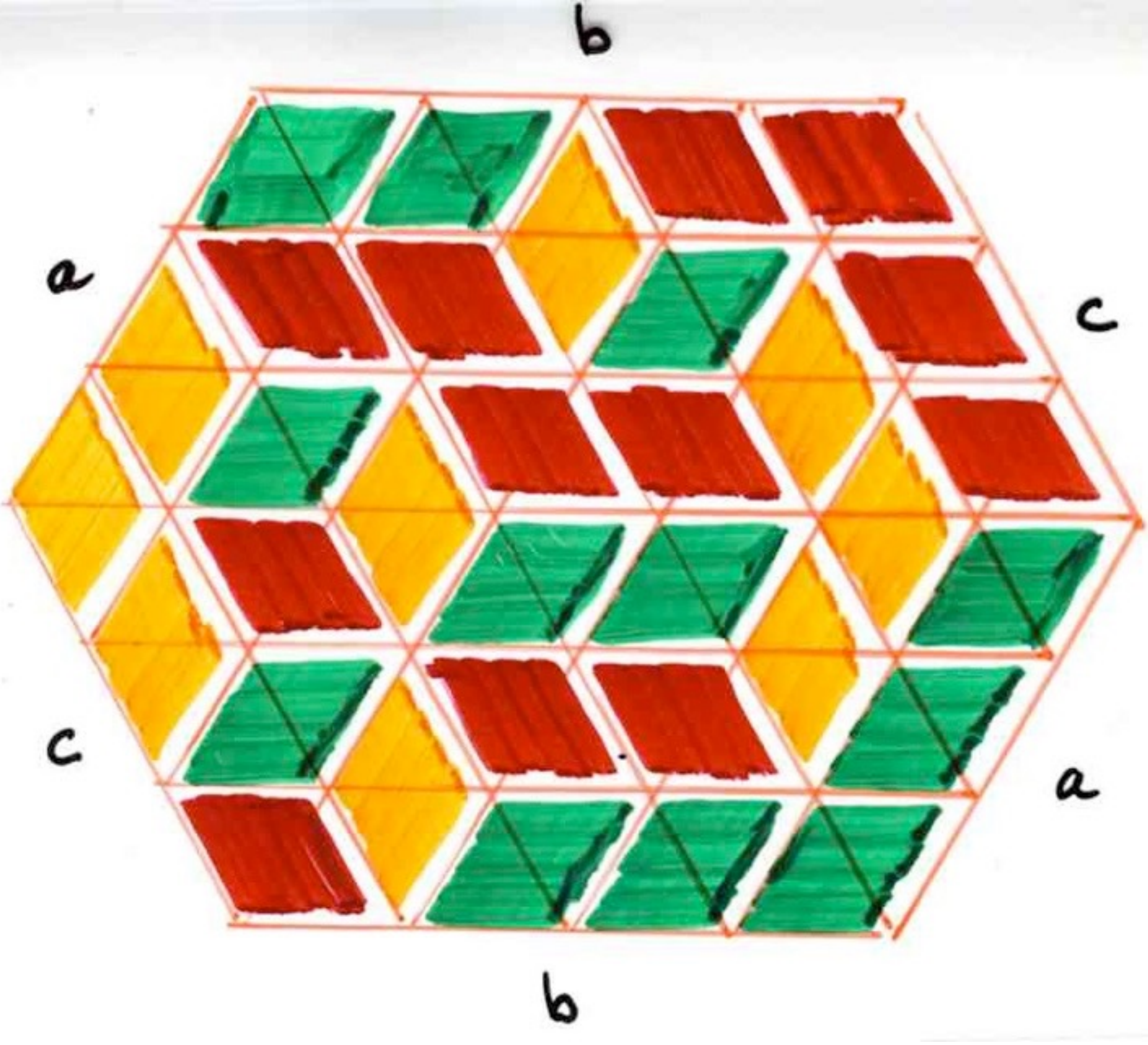

B
A
A'
B'

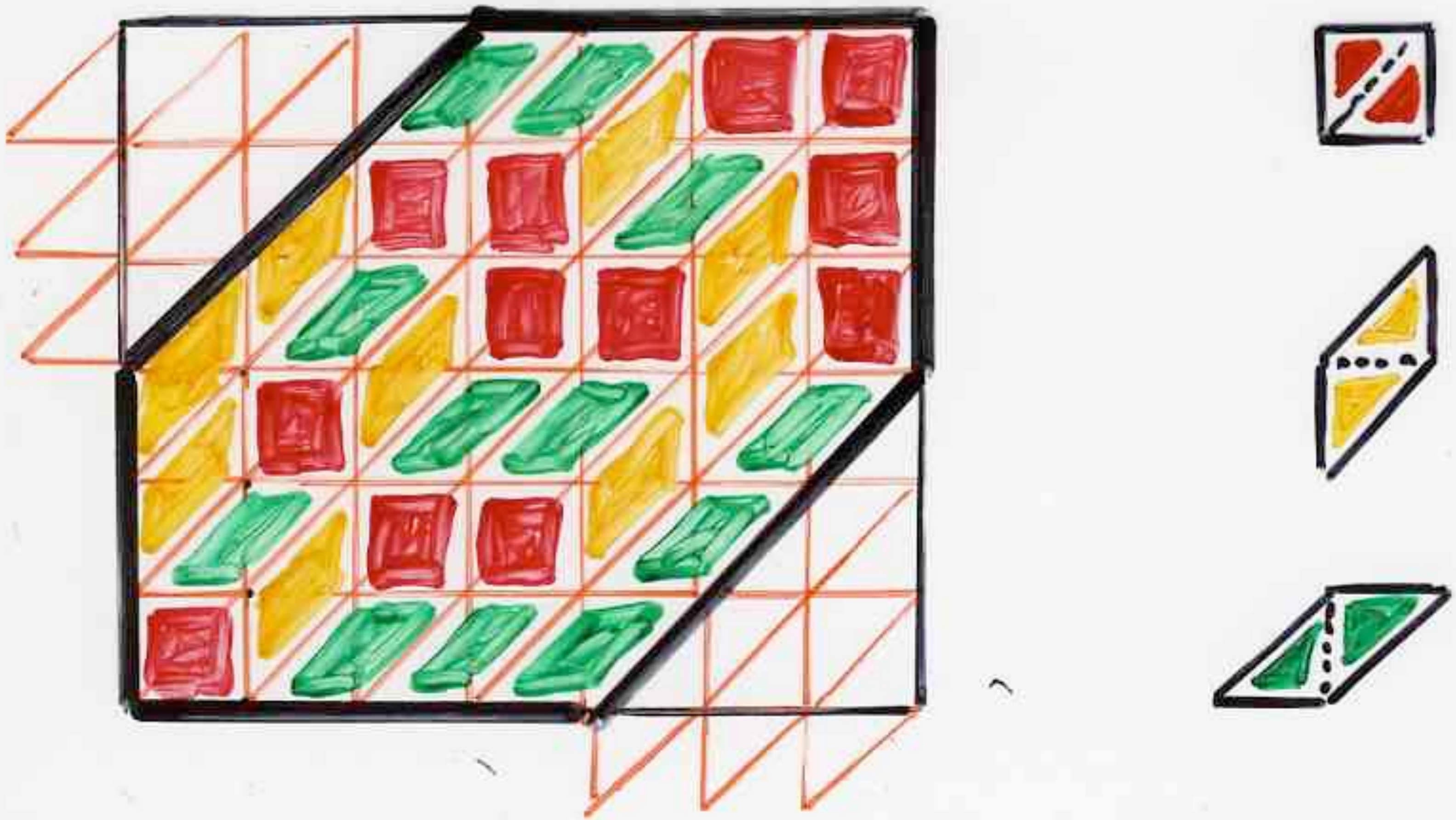

B
A
A'
B'

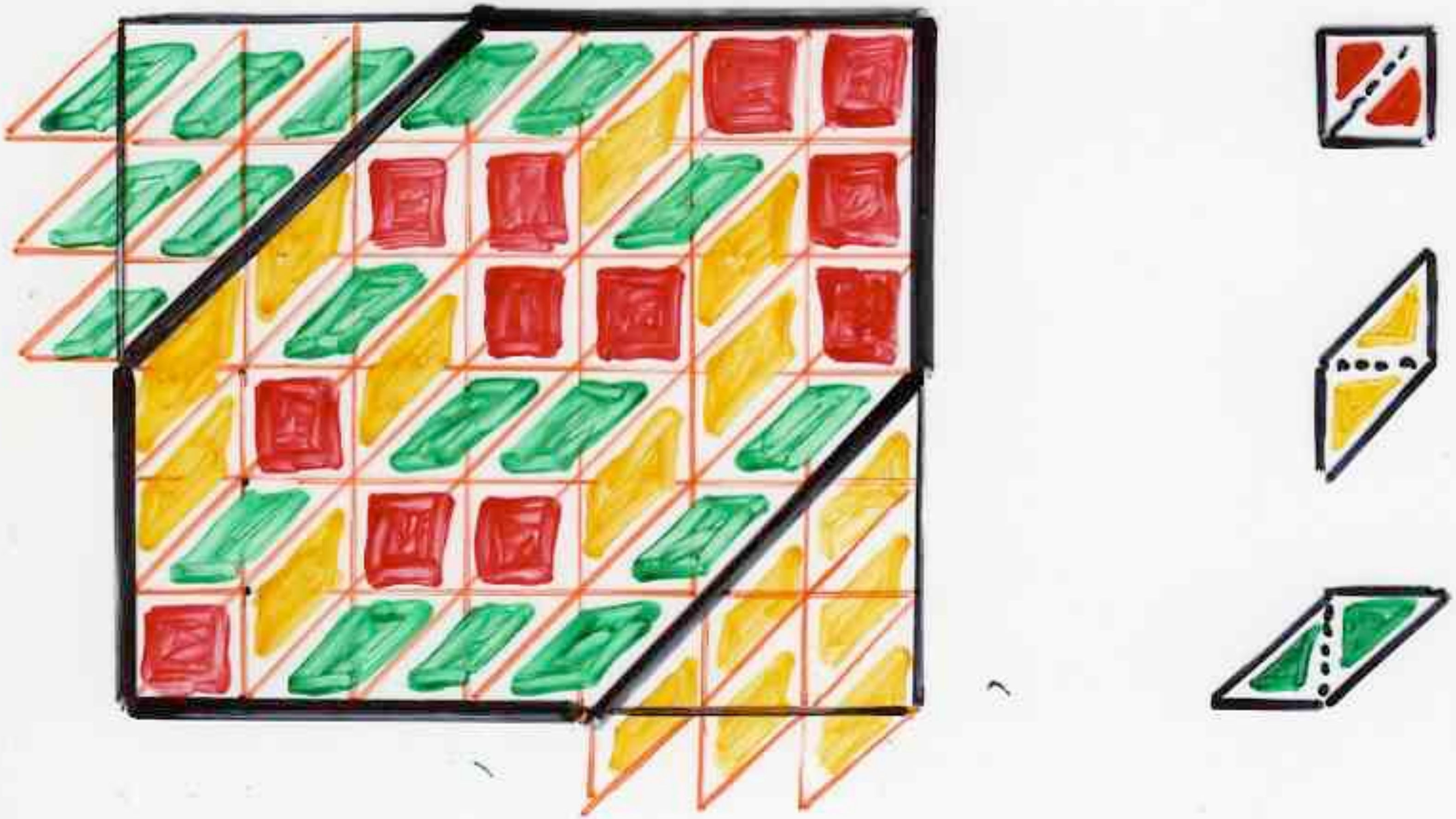

B
A
A'
B'

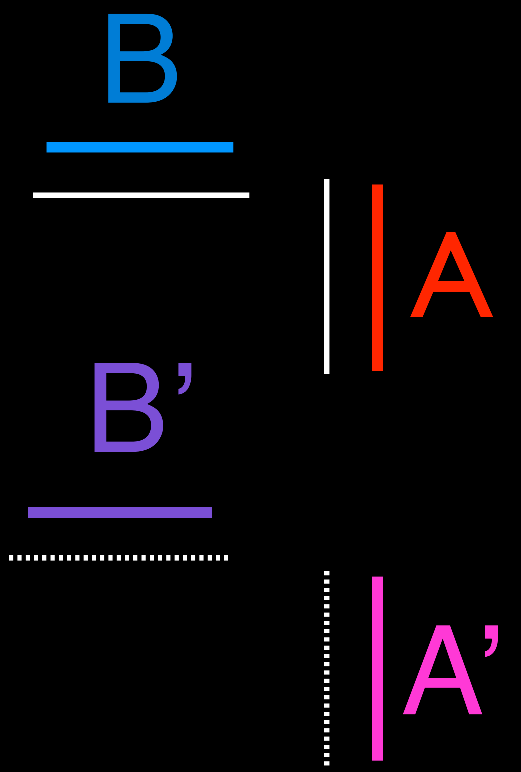

B
A
A'
B'

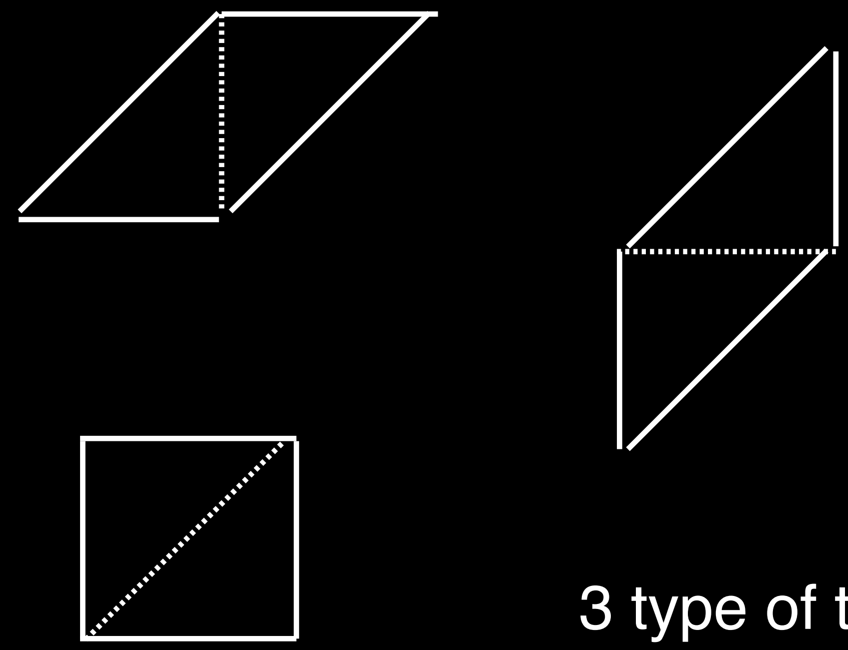
rhombus tilings


b
a
c
c
a
b









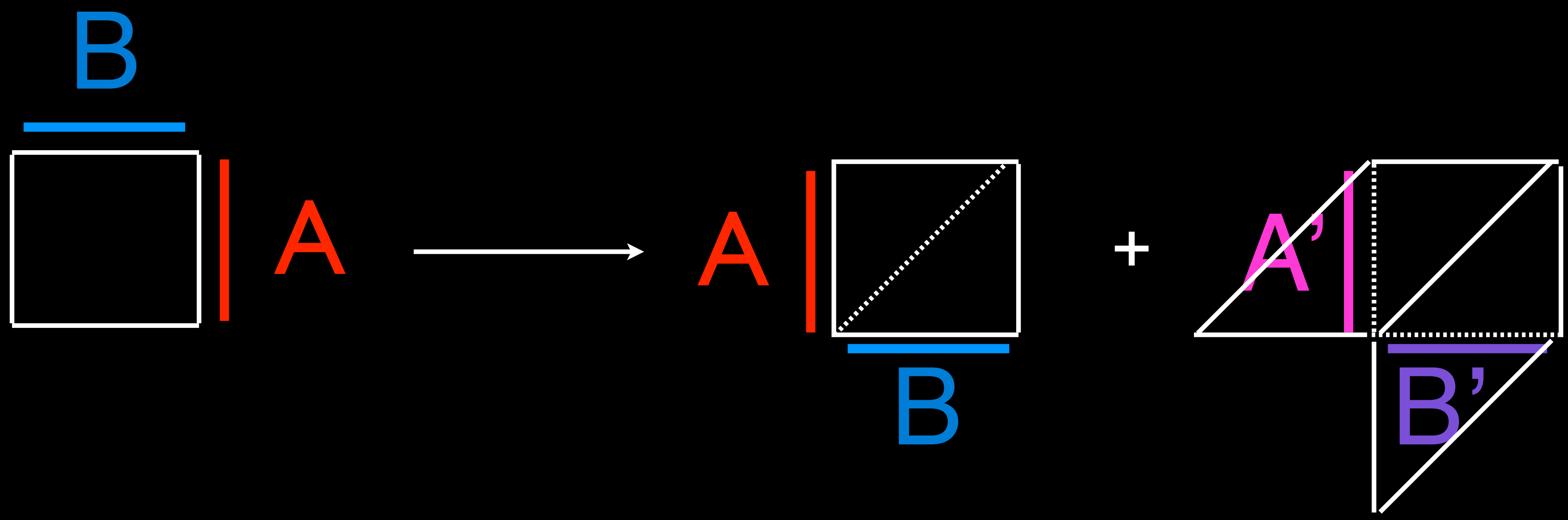
3 type of tiles

border of a tile

inside a tile

coding of the edges
for tilings
of the triangular lattice

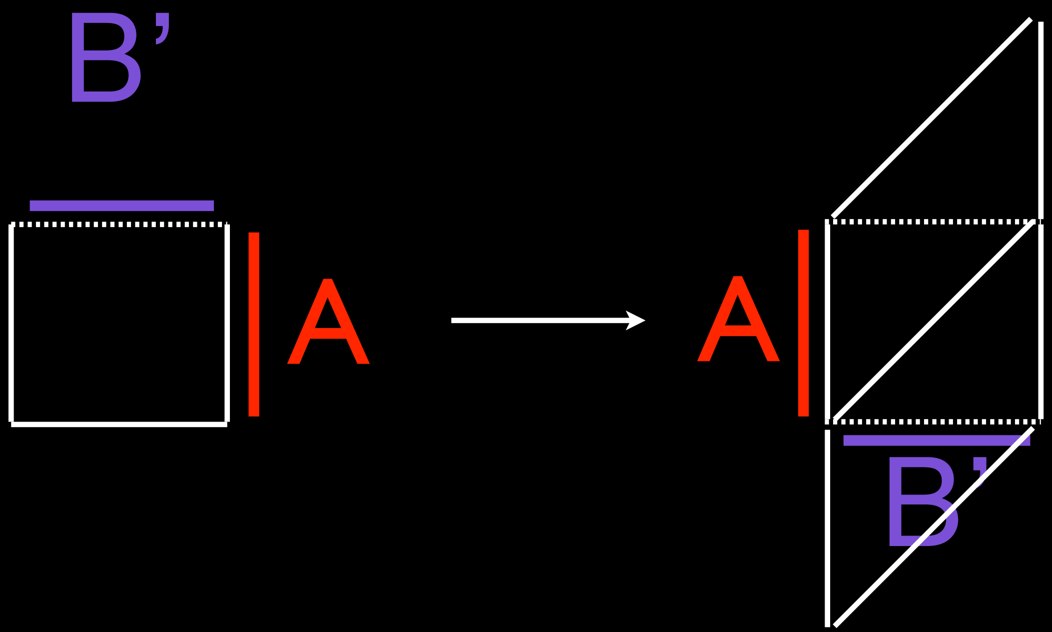
“rewriting rules” for tilings of the triangular lattice



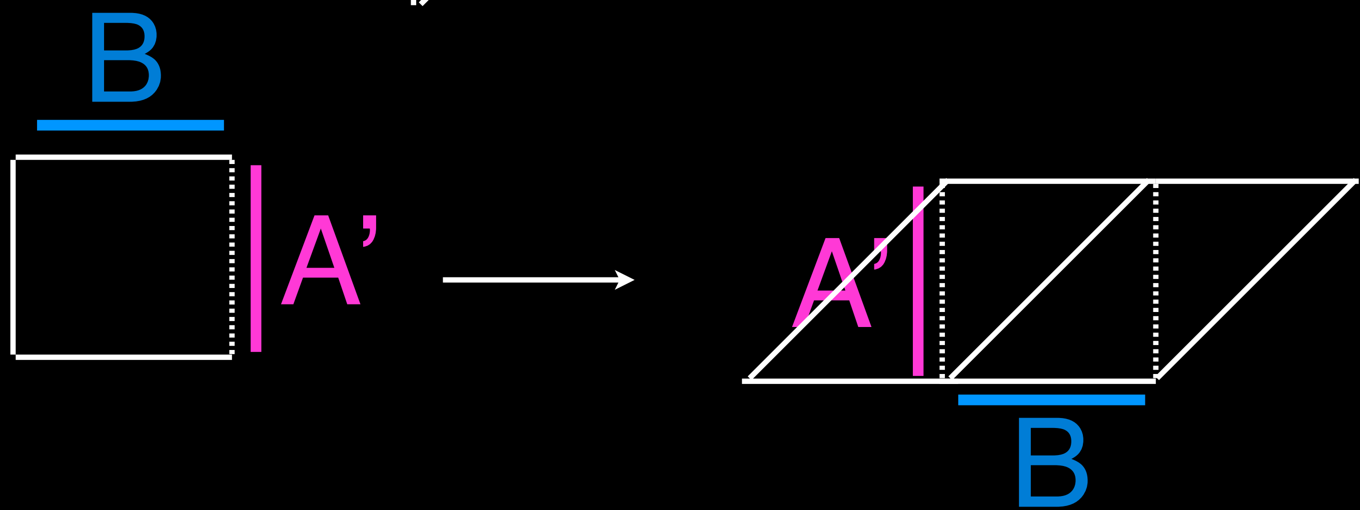
$$BA = AB + A'B'$$

$$B'A' = AB$$

“rewriting rules” for tilings of the triangular lattice



$$B' A = A B'$$



$$B A' = A' B$$

"rewriting rules" for tilings of the triangular lattice

BA = AB + A'B'

B'A' = AB

B' A = A B'

B A' = A' B

same as for ASM , except the rewriting rule

B' A' ⟶ A' B' is forbidden

$$\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases} \qquad \text{(ASM)}$$

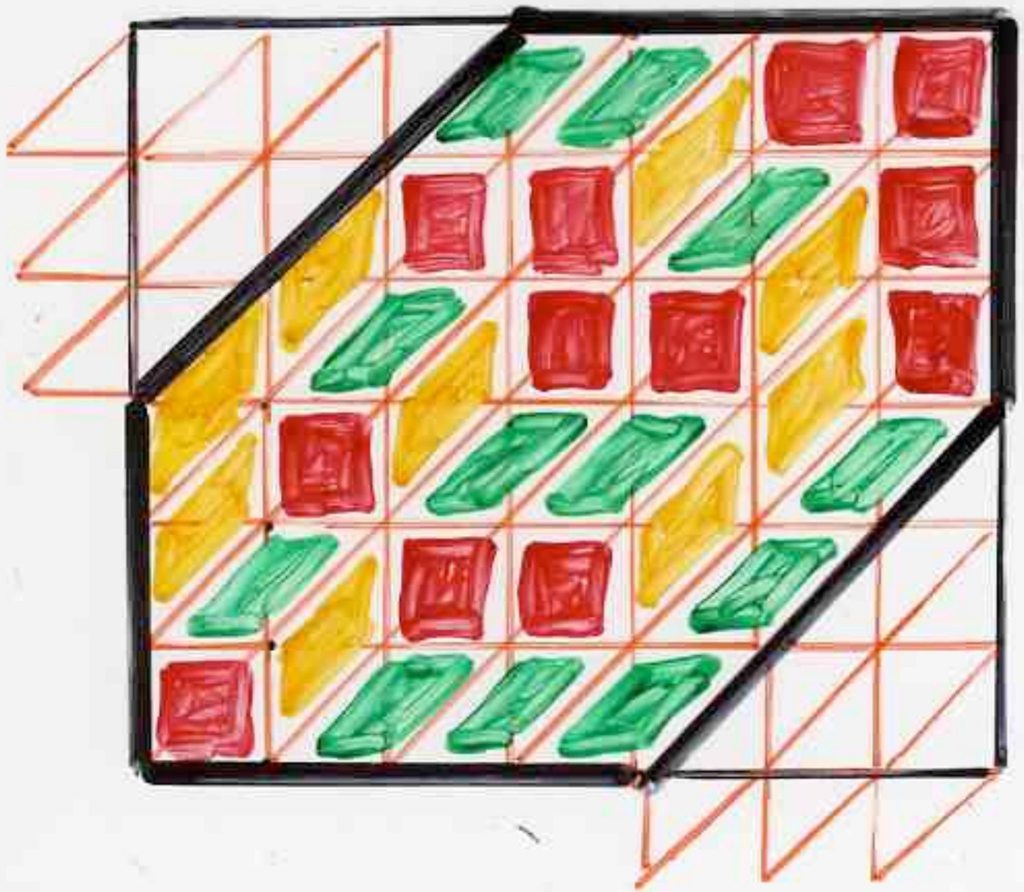
Rhombus tilings

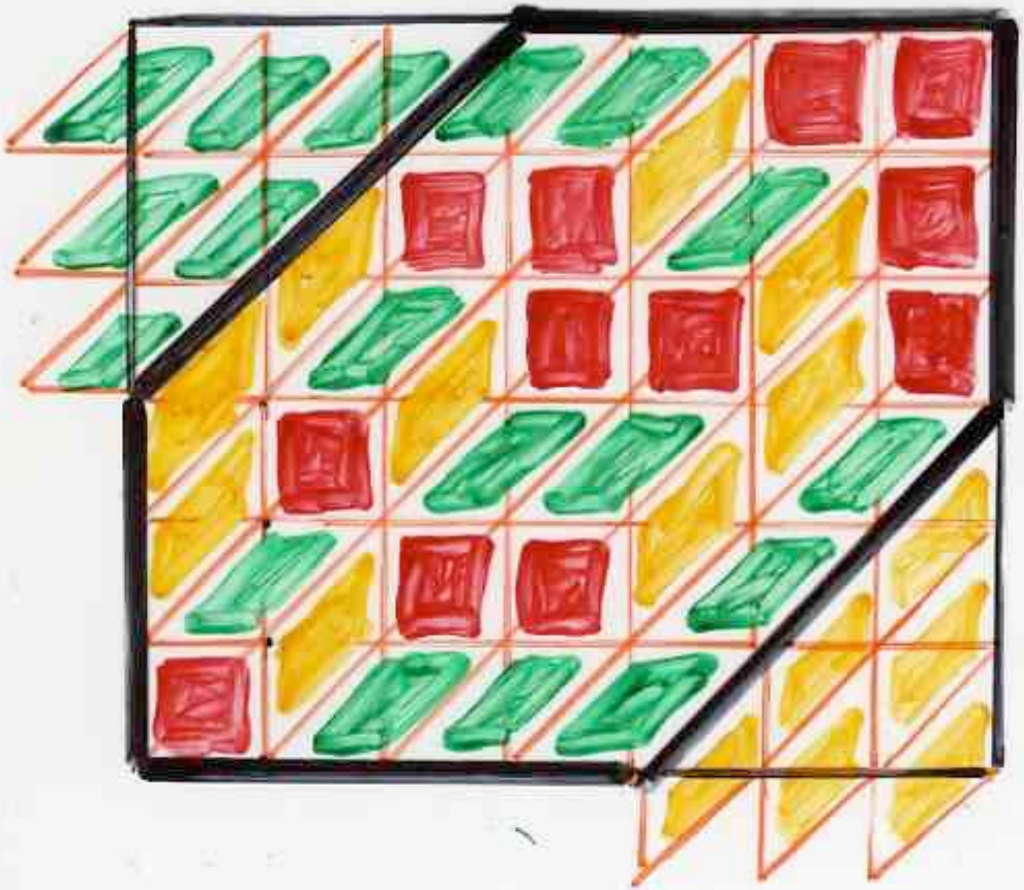
The quadratic algebra $\mathbb{Z}$

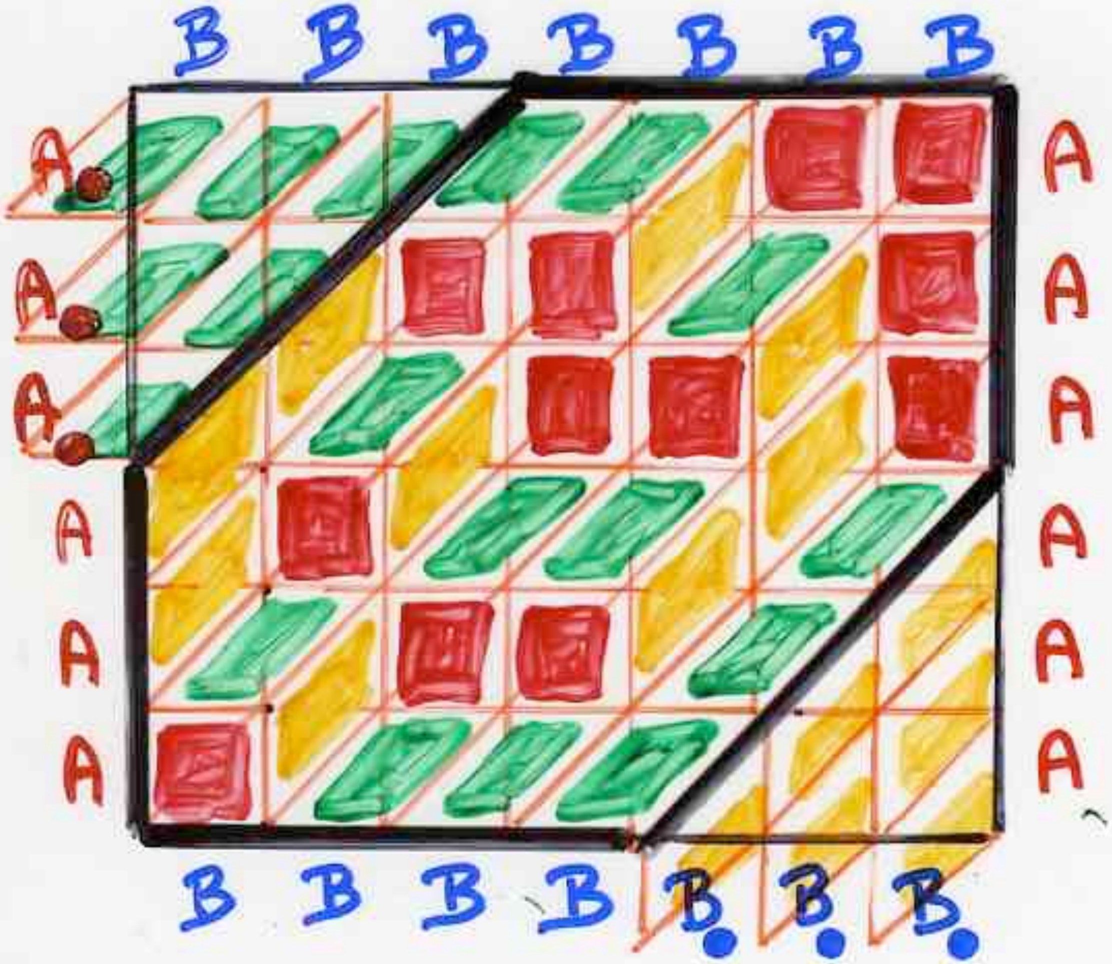
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = 0\; A_\bullet\, B_\bullet + t_{\bullet\bullet}\, A\,B \\ B_\bullet\, A = q_{\bullet\circ}\, A\, B_\bullet + 0\; A_\bullet\, B \\ B\, A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + 0\; A\, B_\bullet \end{cases}$$

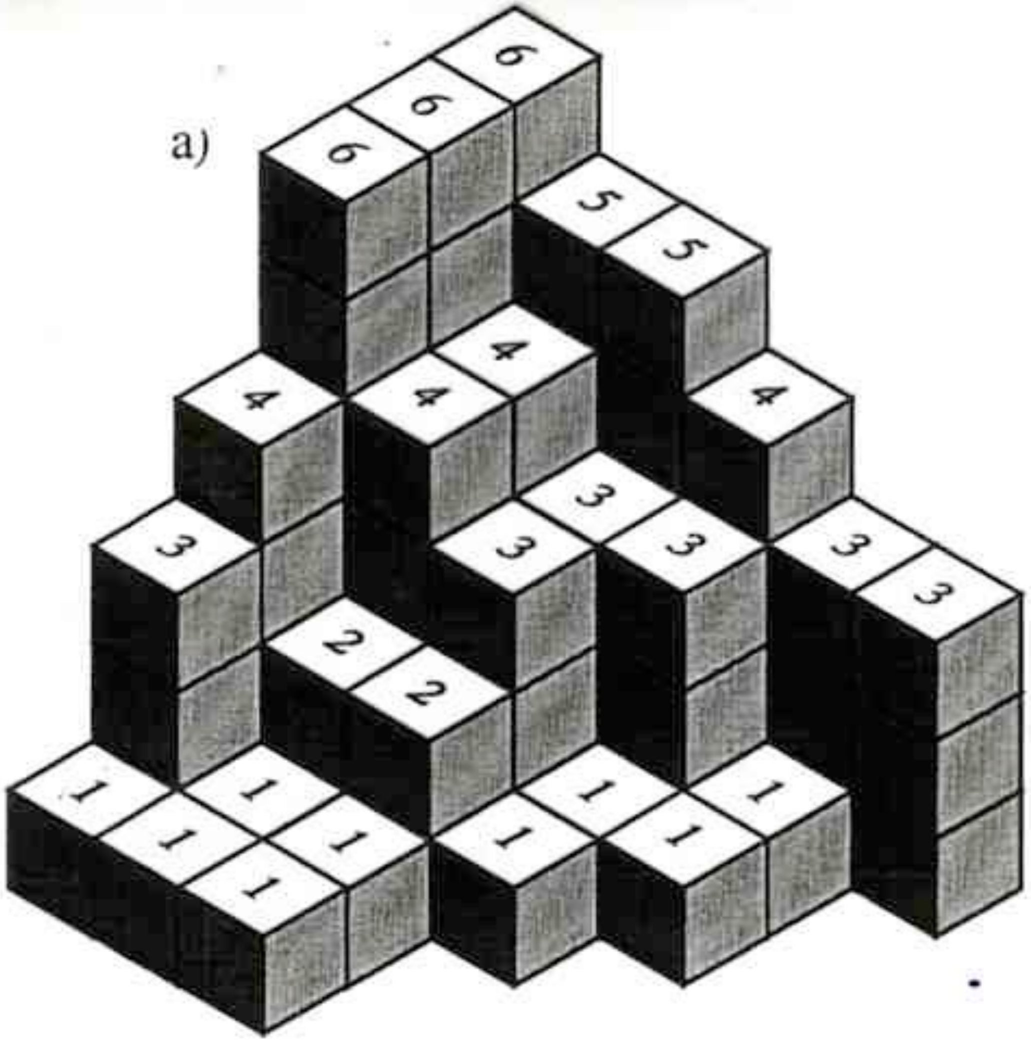



B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

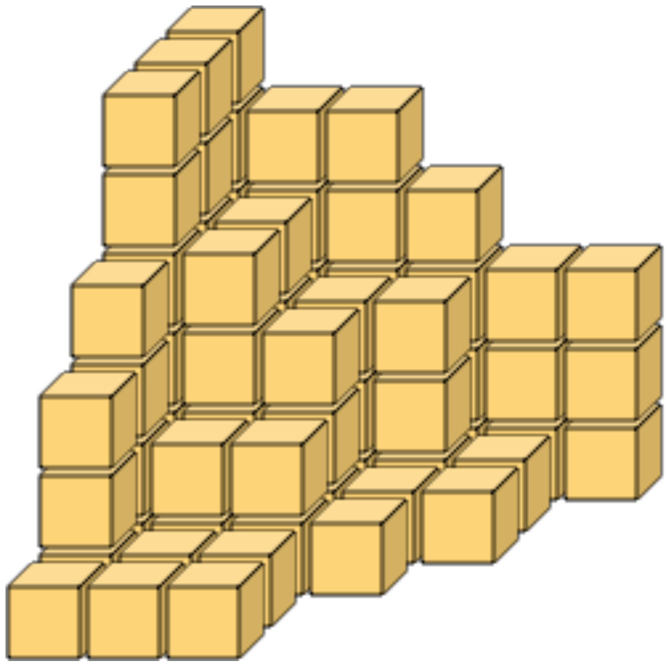
plane partitions

a)



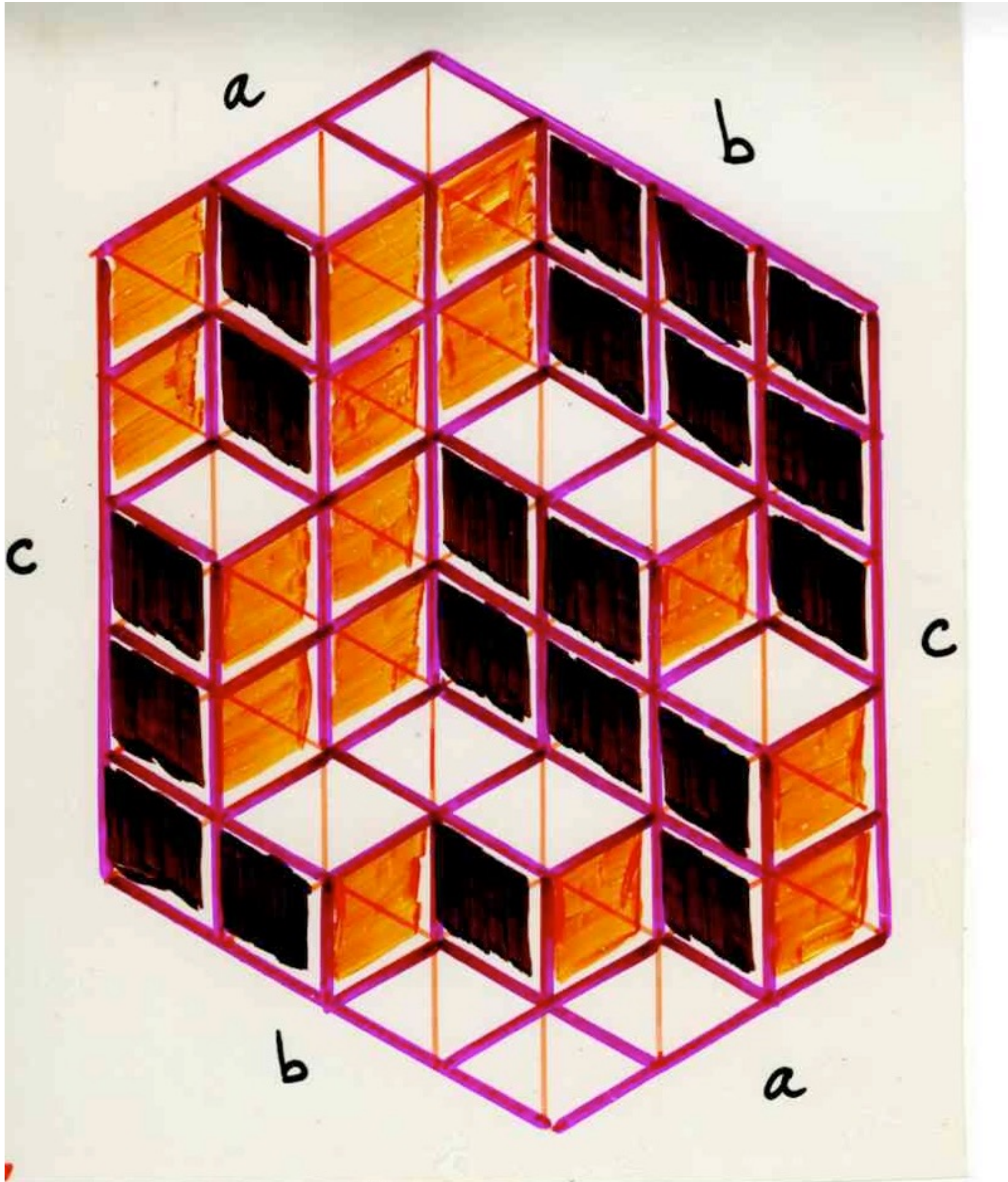
b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



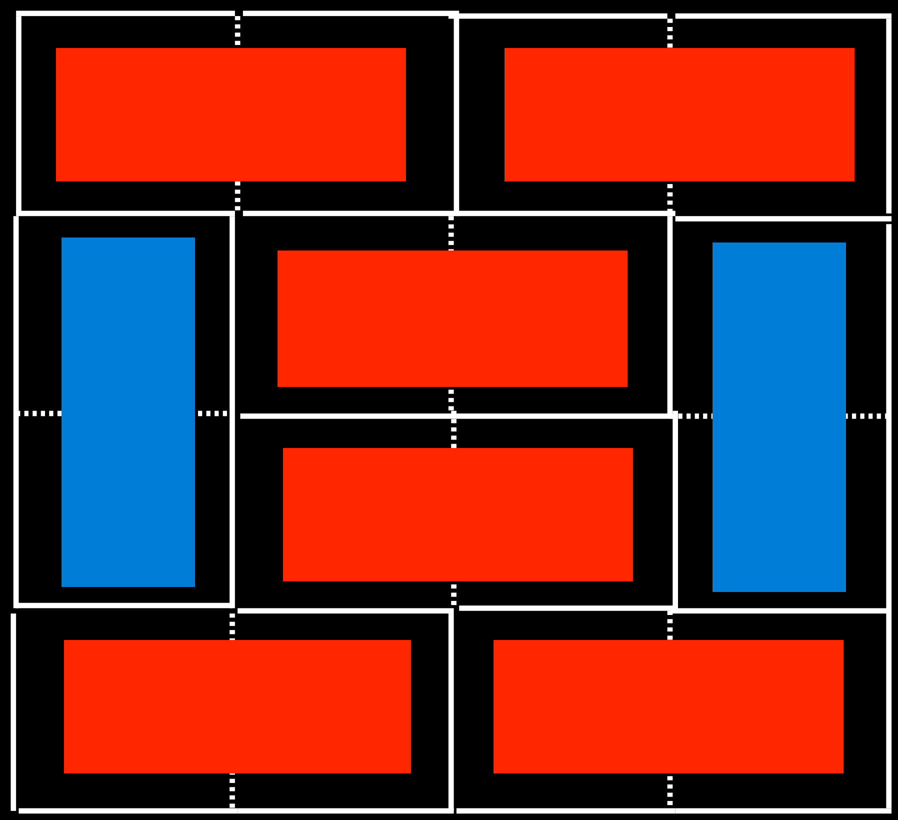
example:
plane
partitions
in a box

(MacMahon
formula)



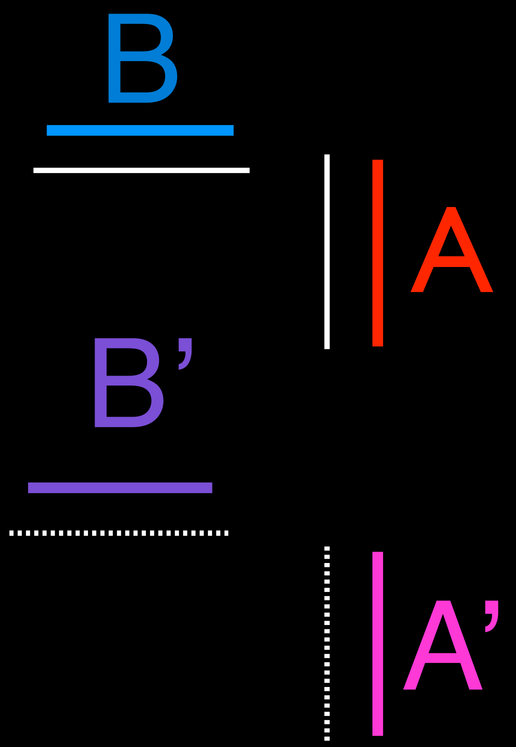
$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

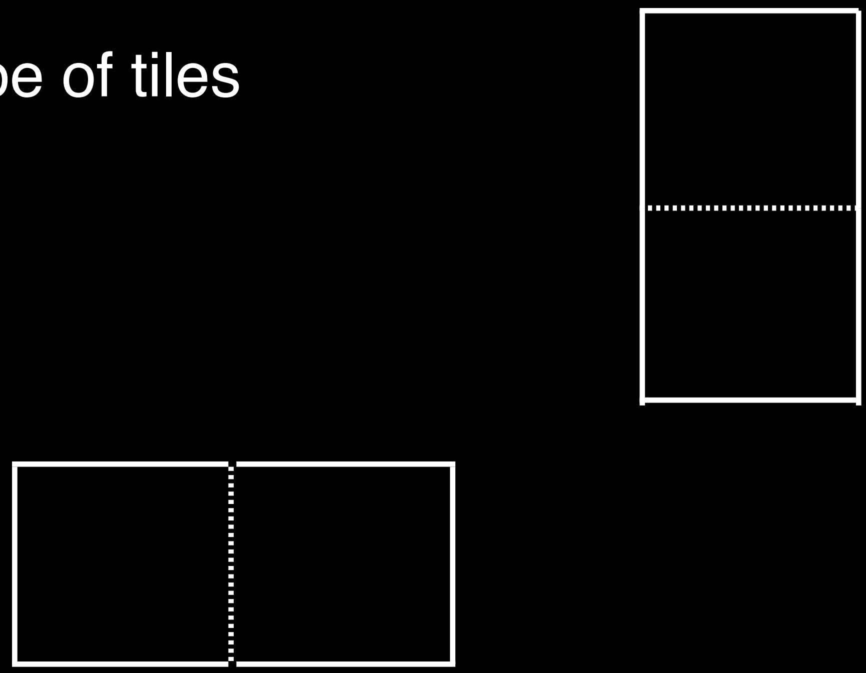
dimers tiling
on a square lattice



a tiling
on the
square lattice

2 type of tiles



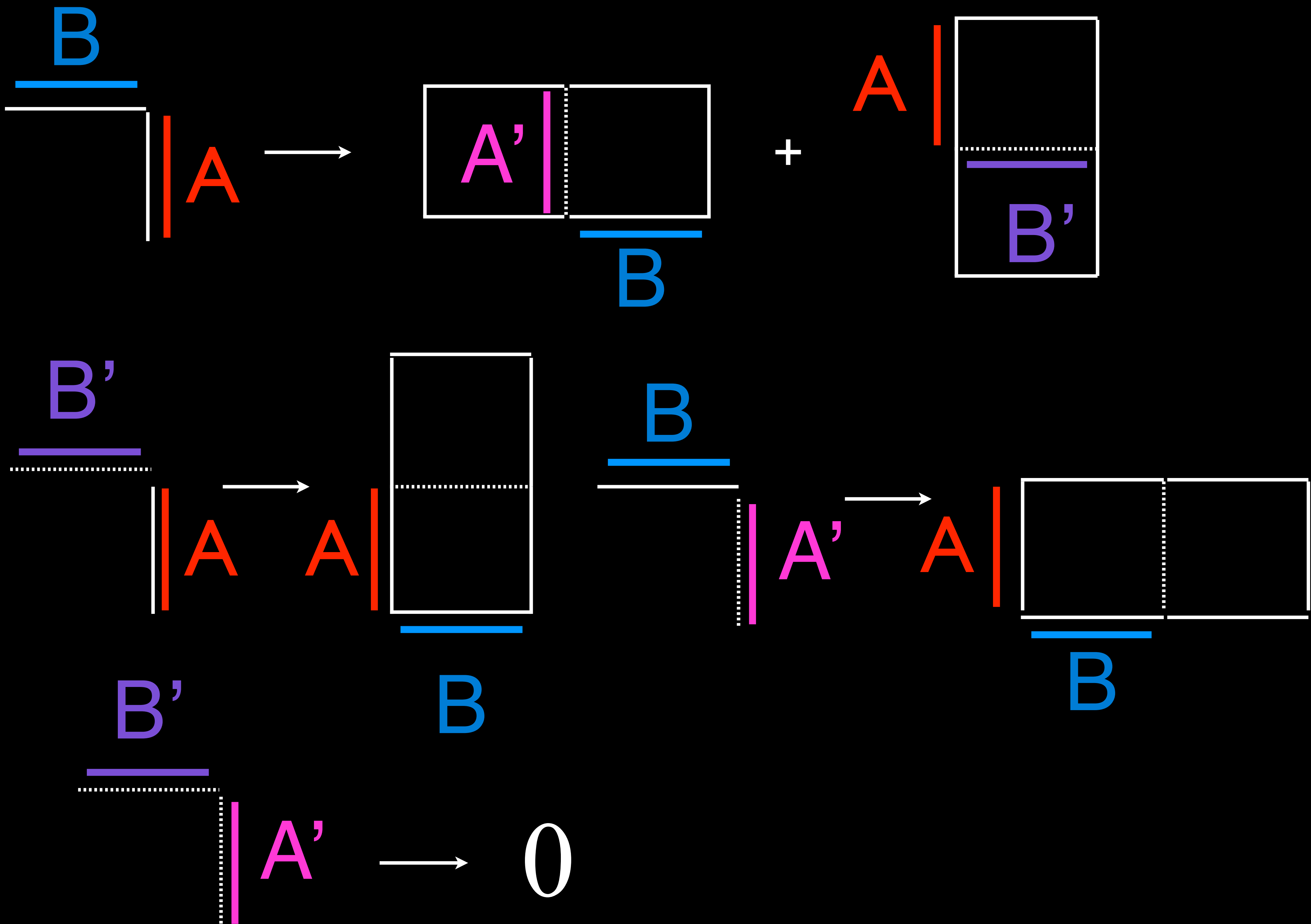


coding of the edges
for tilings
on the square lattice

border of a tile

inside a tile

“rewriting rules” for tilings (square lattice)



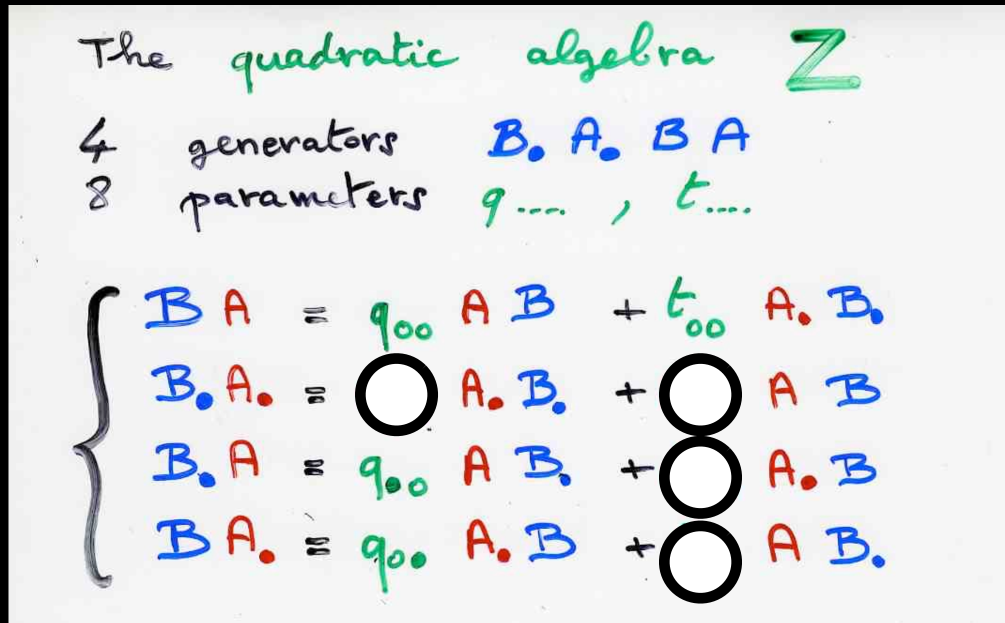
operators and commutations for tilings (square lattice)

$$B\,A = A'\,B + A\,B'$$

$$B'\,A' = 0$$

$$B'\,A = A\,B$$

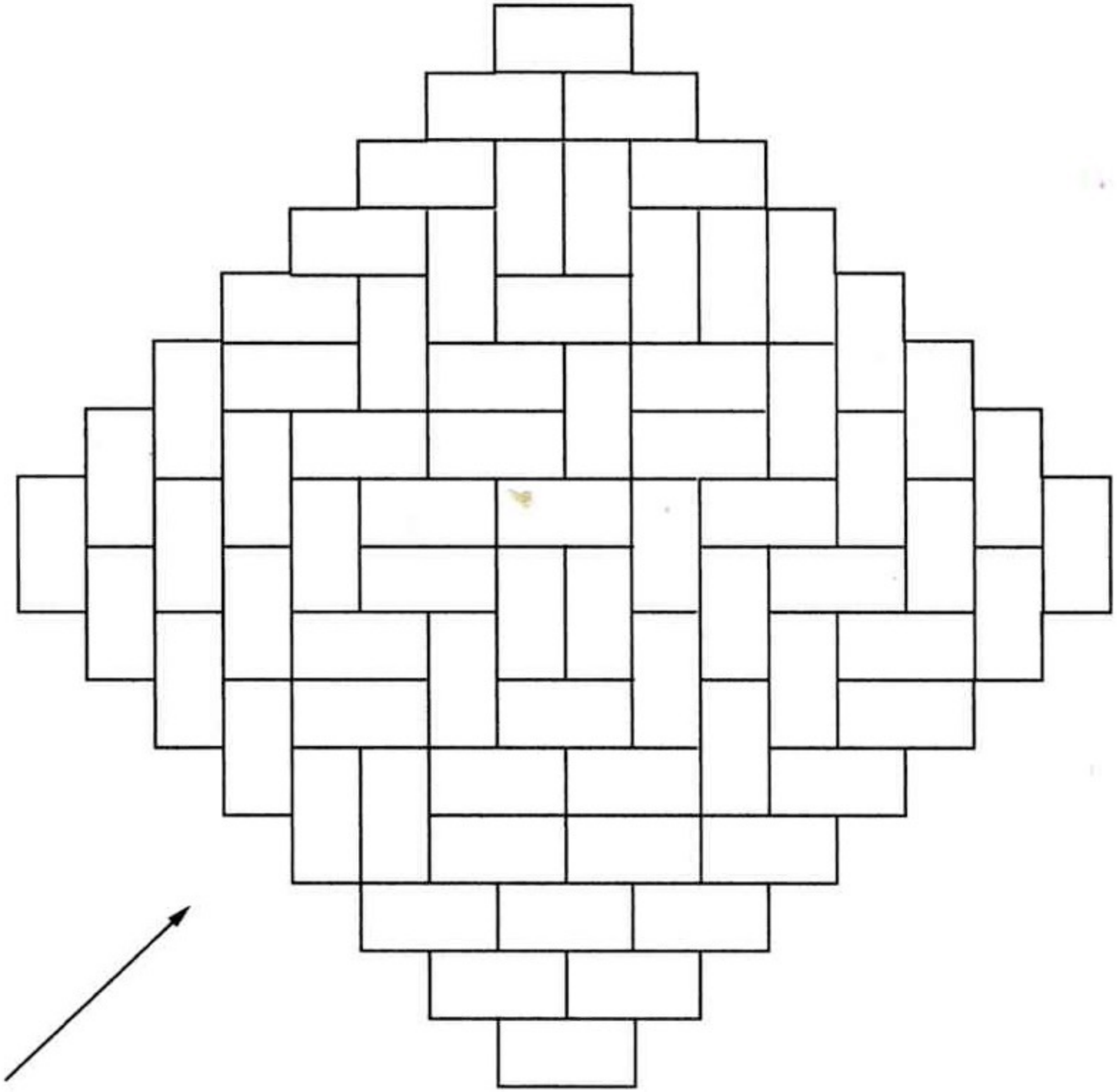
$$B\,A' = A\,B$$



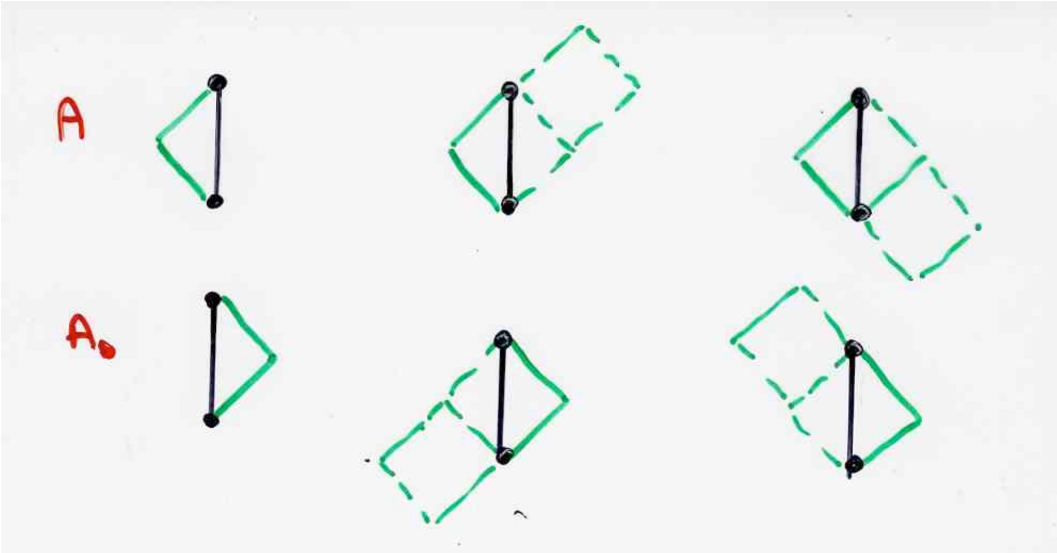
exercice: tiling of a square lattice with rectangular bars

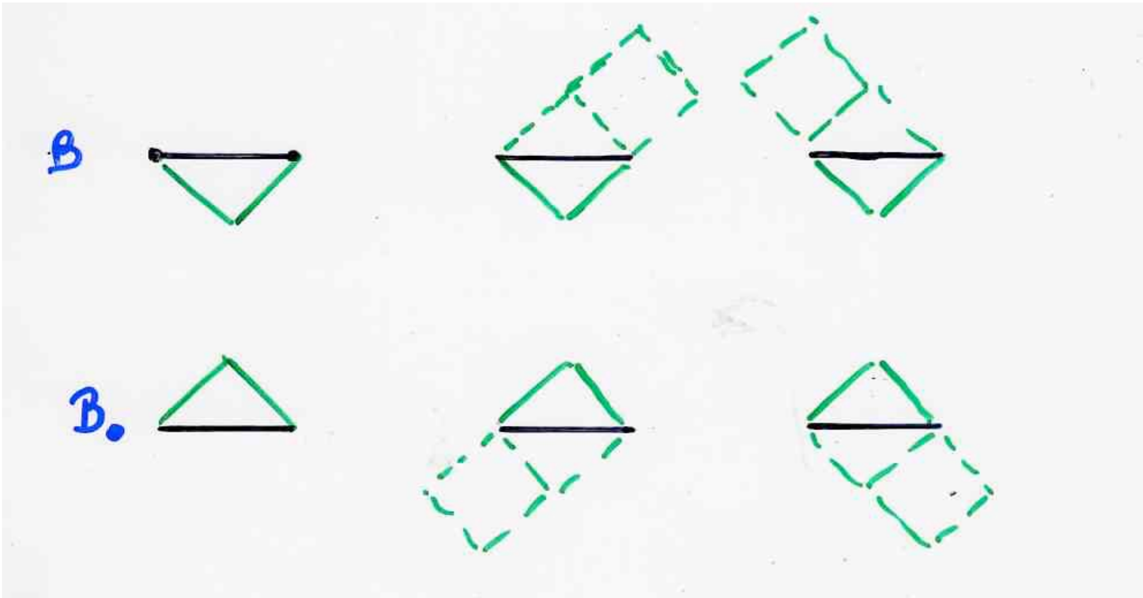
Aztec tilings

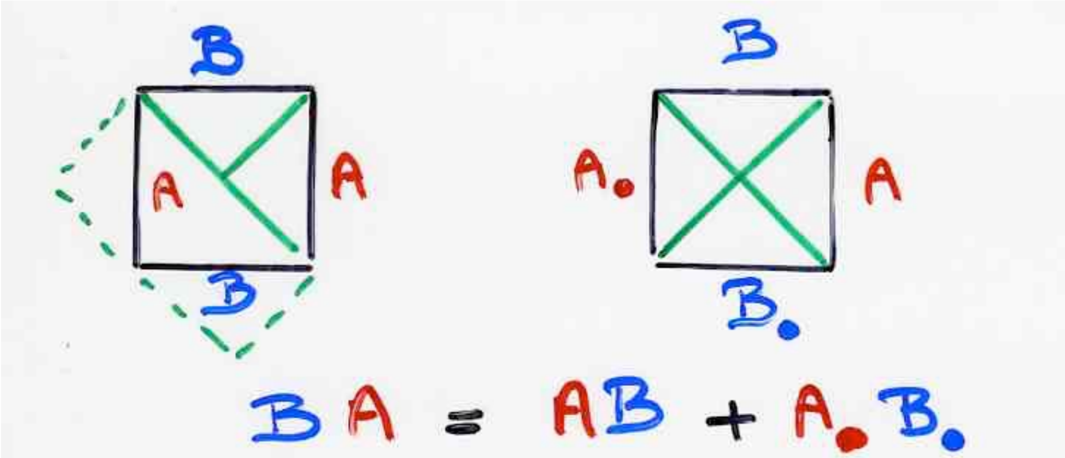
$2^{n(n-1)/2}$

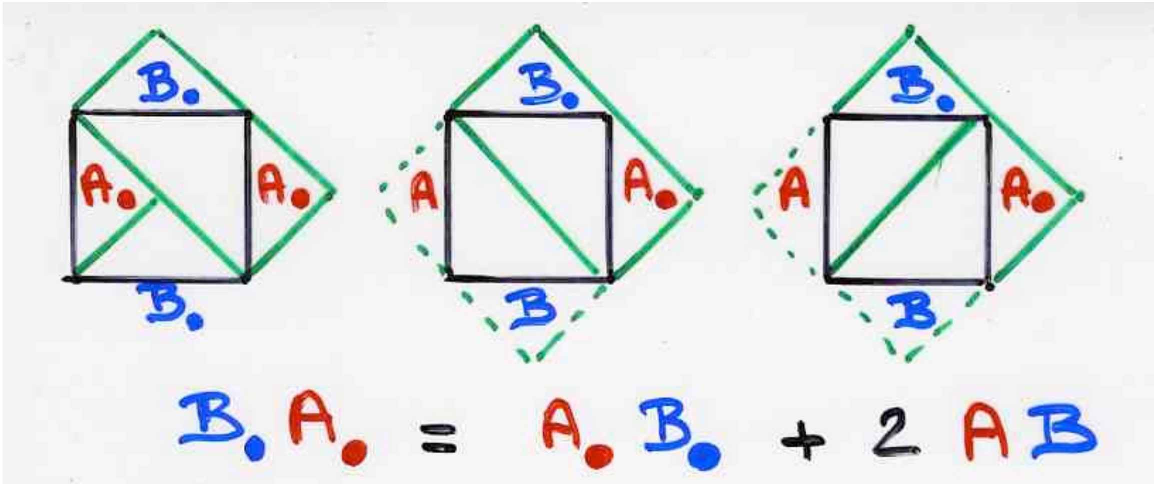


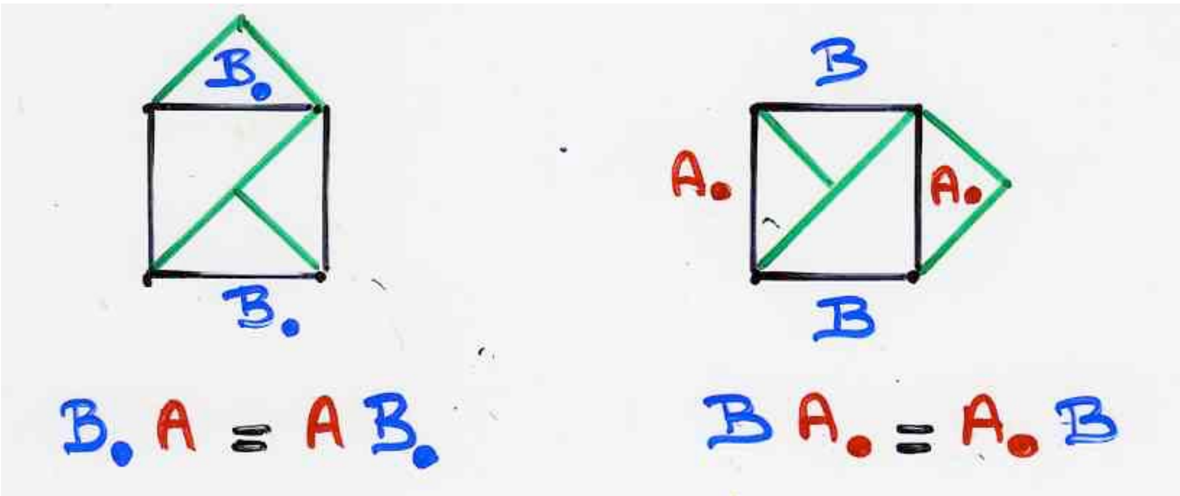
Elkies,
Kuperberg,
Larsen,
Propp
(1992)


A
A•


B
B•


B
A A
B
B
A• A
B•
B A = A B + A• B•


B• B• B•
A• A• A A• A A•
B• B B
B• A• = A• B• + 2 A B


B• B
A• A•
B• B
B• A = A B•
B A• = A• B

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0 \qquad \text{(ASM)}$$

$$t_{\bullet\bullet} = 2 \qquad \text{(nb of } -1 \text{ in ASM)}$$

The quadratic algebra Z

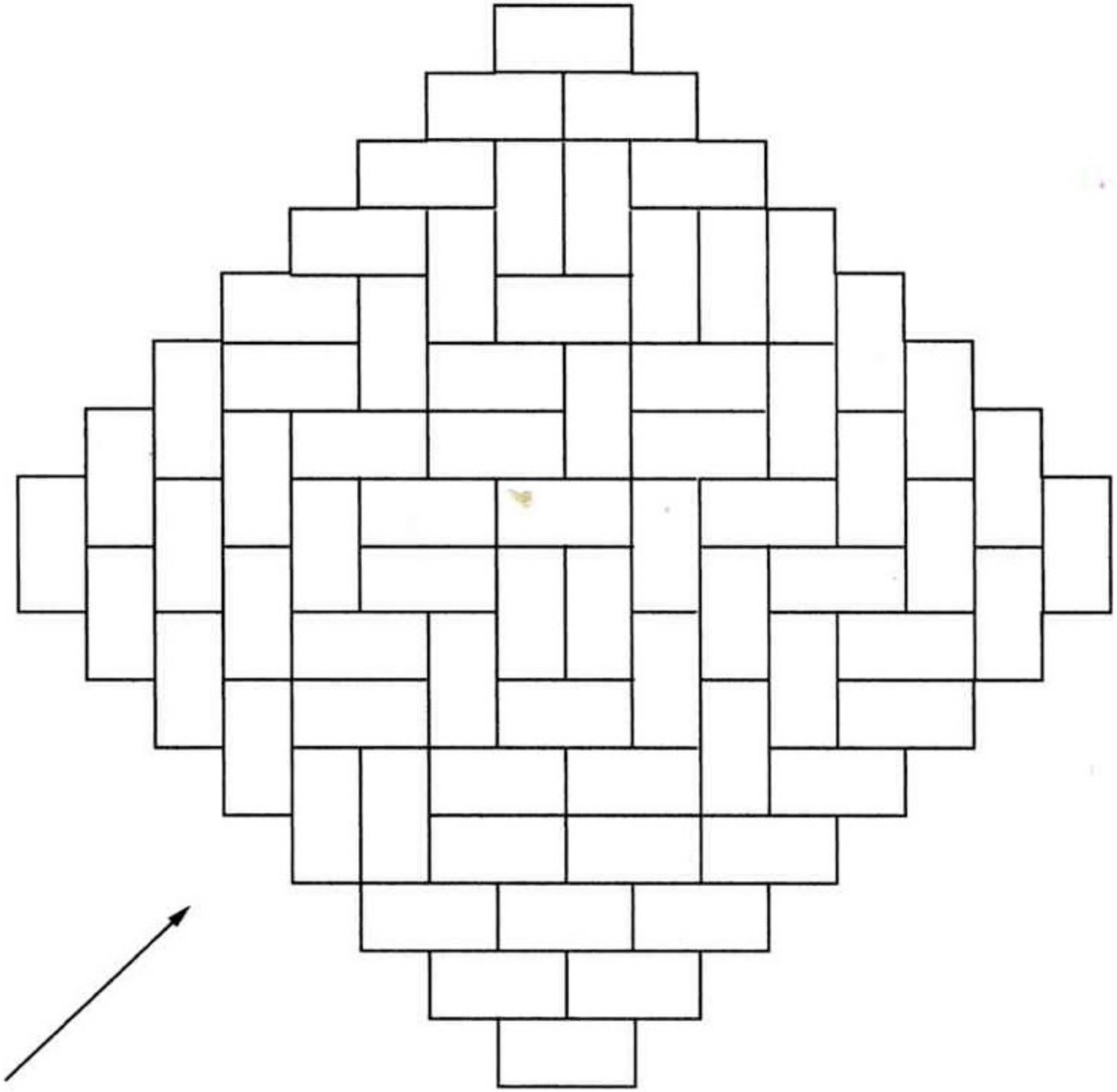
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdot\cdot\cdot\cdot}$, $t_{\cdot\cdot\cdot\cdot}$

$$\begin{cases} B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet\, B_\bullet \\ B_\bullet\, A_\bullet = q_{\bullet\bullet}\, A_\bullet\, B_\bullet + 2\, A\,B \\ B_\bullet\, A = q_{\bullet\circ}\, A\, B_\bullet + 0\, A_\bullet\, B \\ B\, A_\bullet = q_{\circ\bullet}\, A_\bullet\, B + 0\, A\, B_\bullet \end{cases}$$

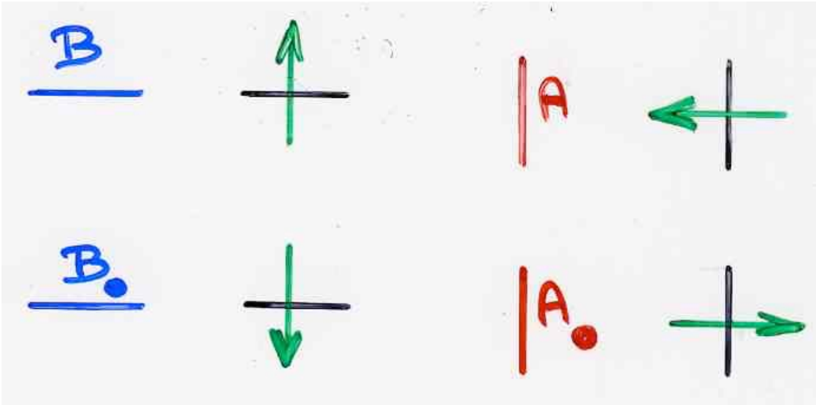
$2^{n(n-1)/2}$

$A_n(2)$

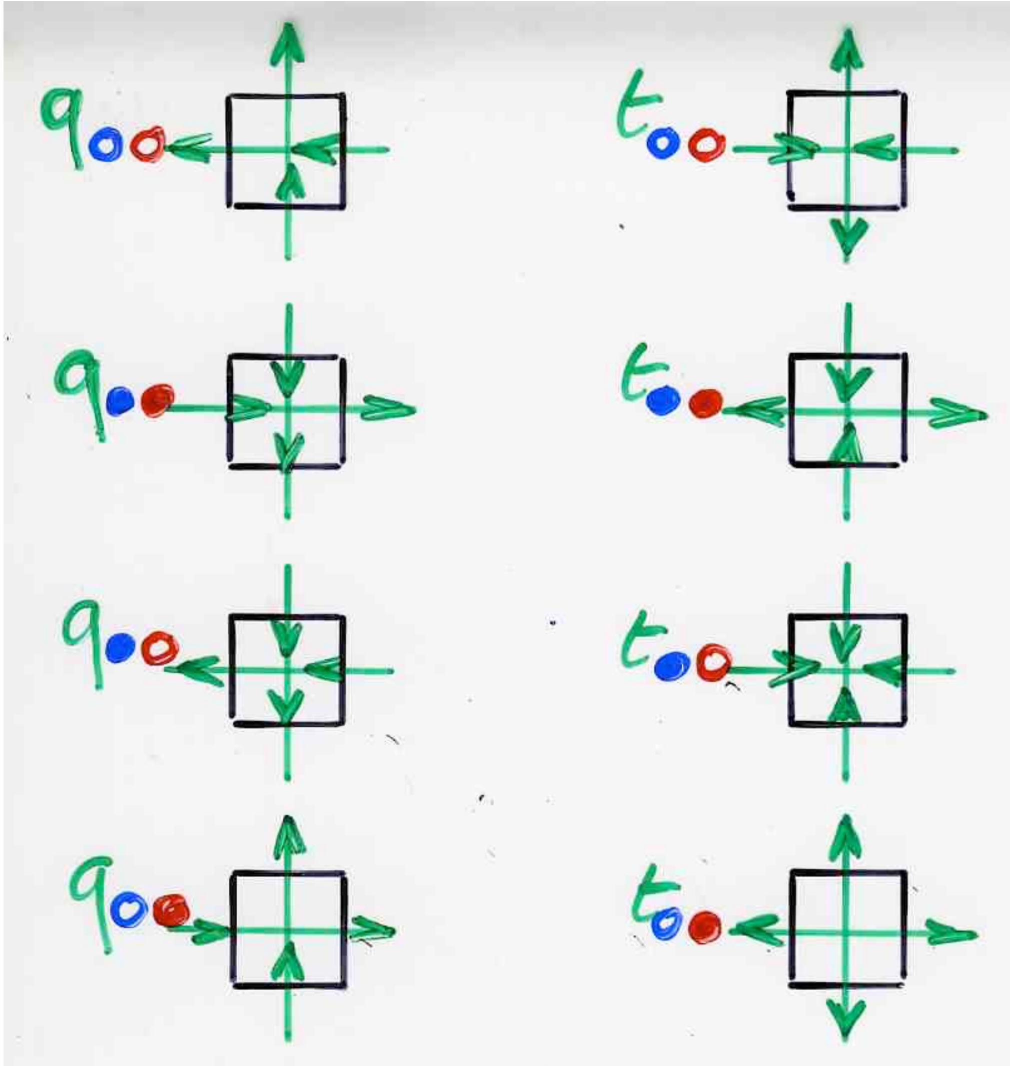


Elkies,
Kuperberg,
Larsen,
Propp
(1992)

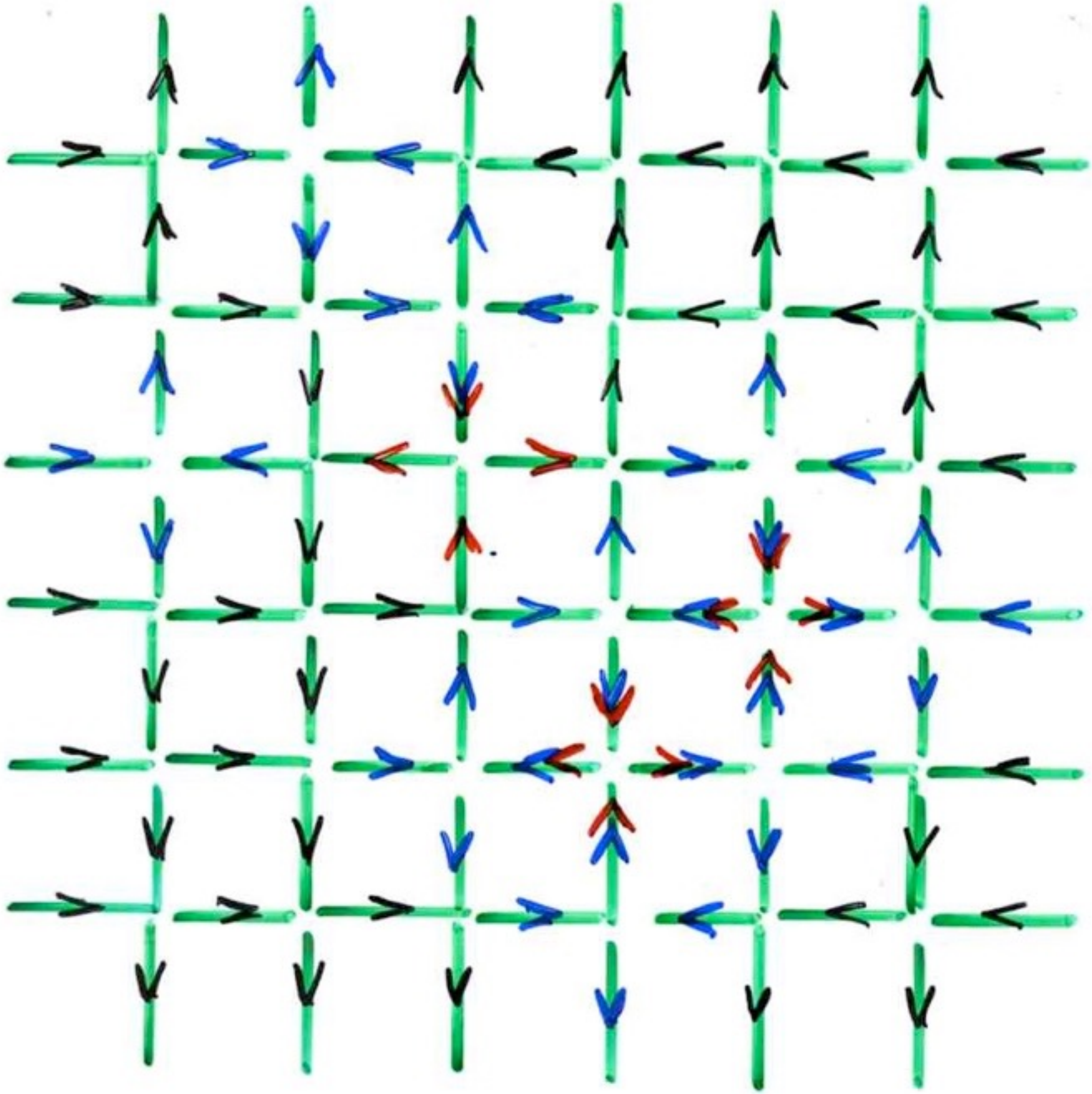
geometric interpretation of XYZ-tableaux



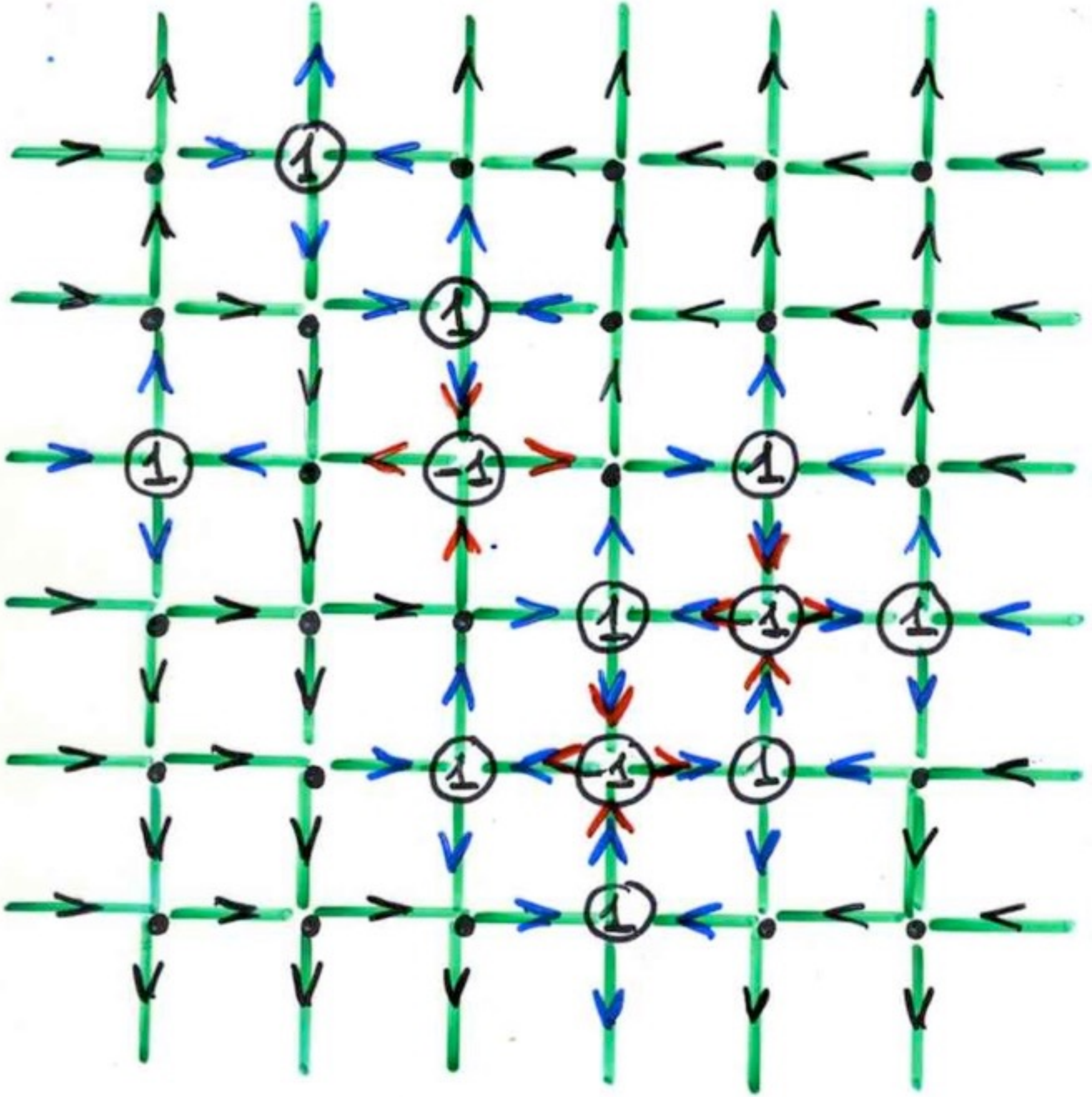
8 - vertex model



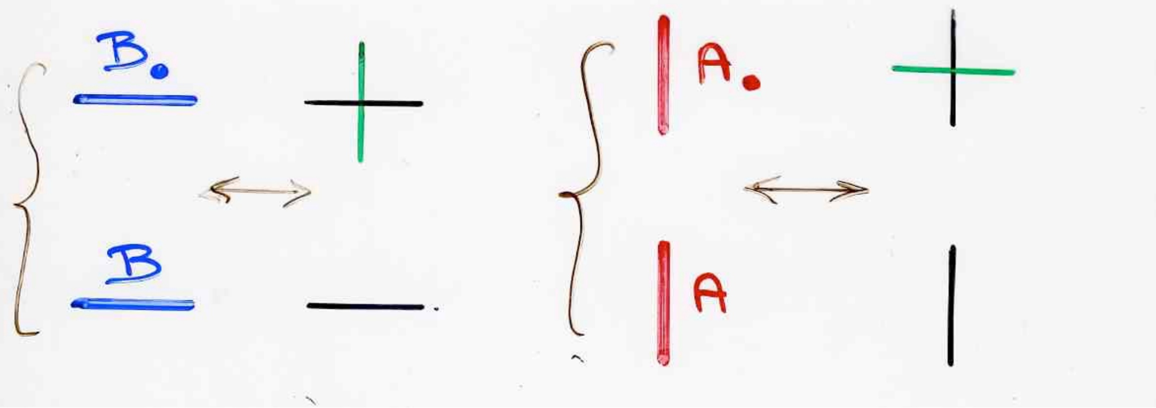
The 6-vertex model

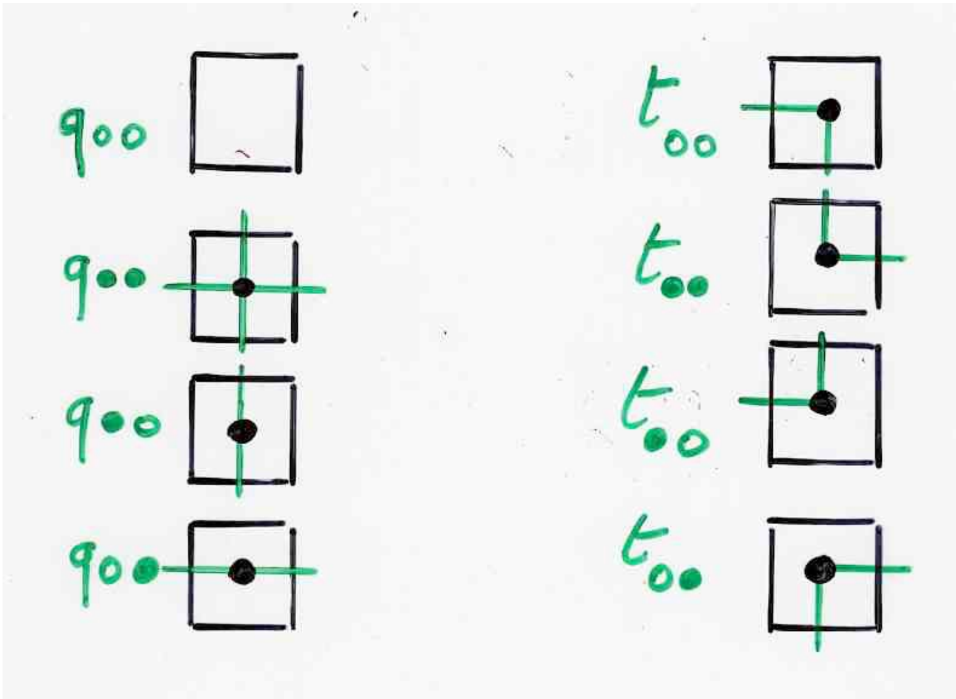


The
6-vertex
model

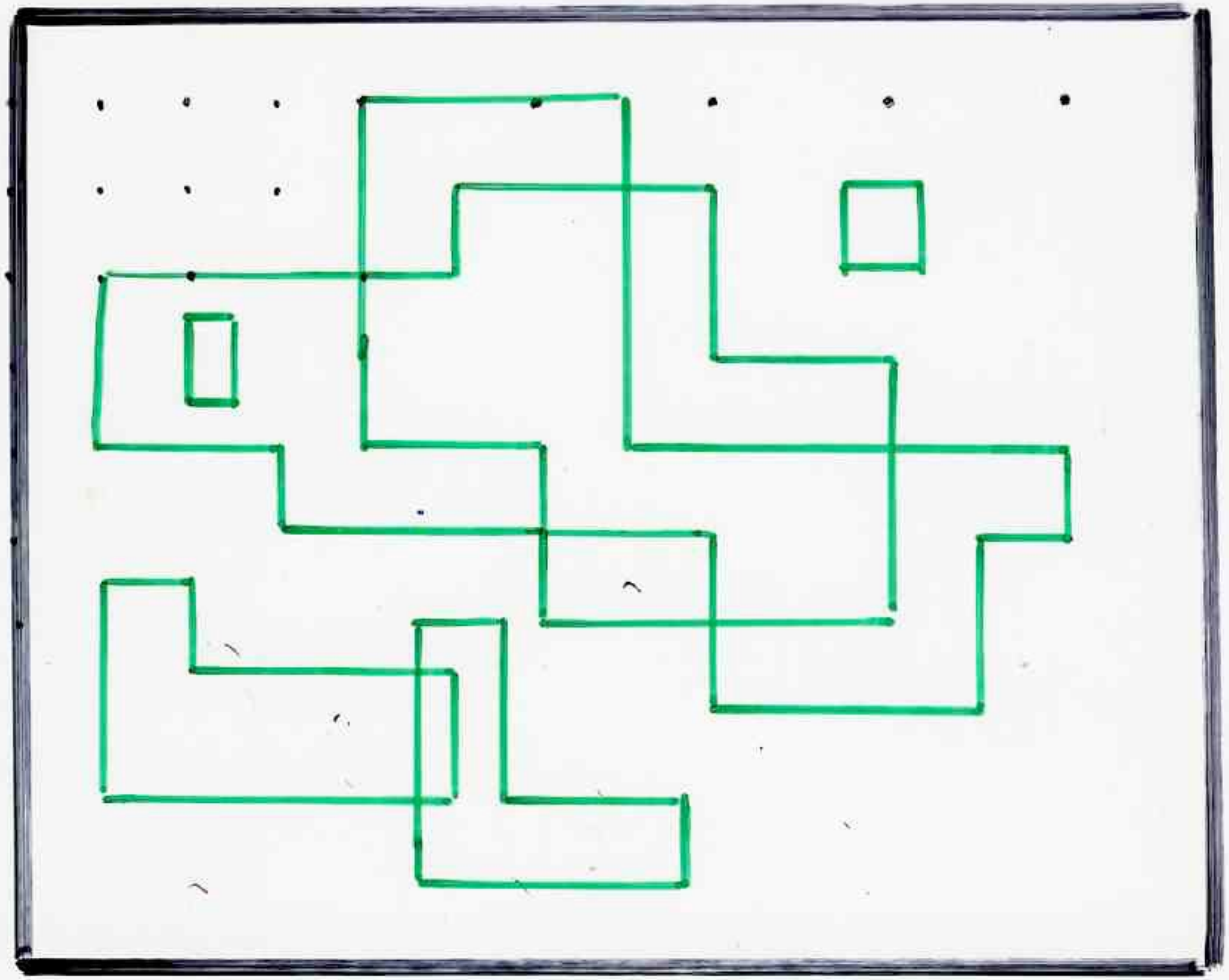


geometric interpretations of Z-tableaux





8 - vertex model

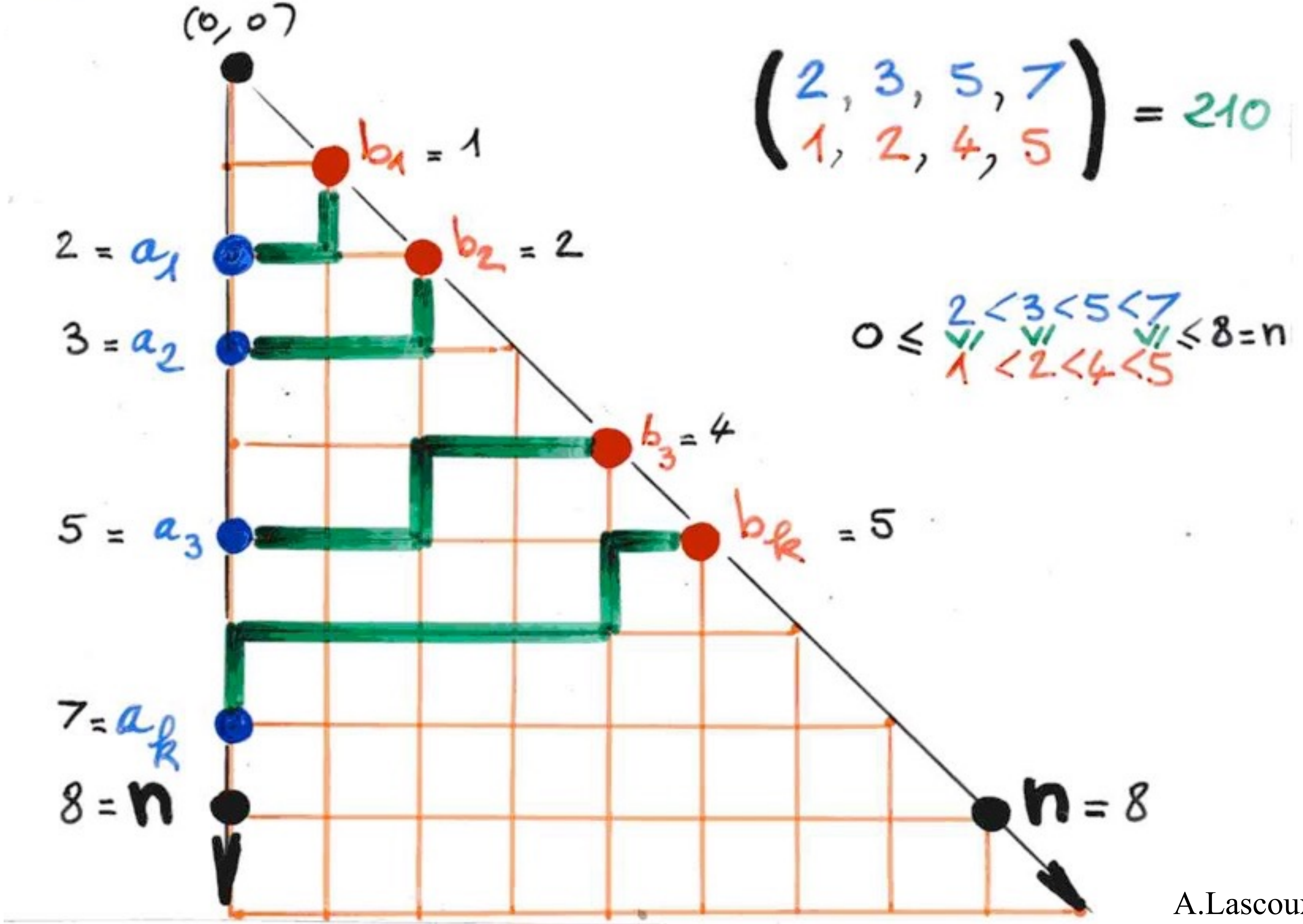


"closed" graph

Ising model

$$w = B^m A^n$$
$$uv = A^n B^m$$

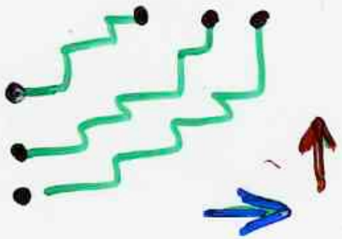
non-intersecting paths



A.Lascoux

example: binomial determinant

I.Gessel, X.G.V., 1985



$$\begin{cases} t_{\circ\circ} = 0 \\ q_{\bullet\bullet} = t_{\bullet\bullet} = 0 \end{cases}$$

$A \leftrightarrow A_\bullet$

exchanging

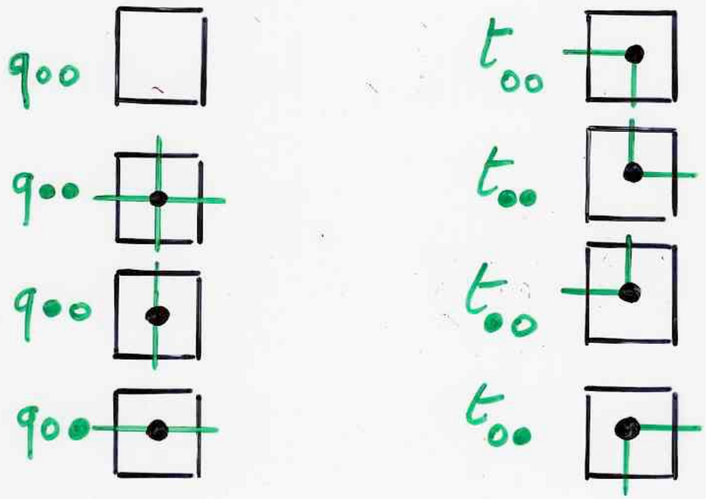
$$\begin{cases} t_{\circ\bullet} = 0 \\ q_{\bullet\circ} = t_{\bullet\circ} = 0 \end{cases}$$

The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{....}$, $t_{....}$

$$\begin{cases} BA = q_{\circ\circ}\, AB + \bigcirc\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc\, A_\bullet B_\bullet + \bigcirc\, AB \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + t_{\bullet\circ}\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A B_\bullet \end{cases}$$



The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{....}$, $t_{....}$

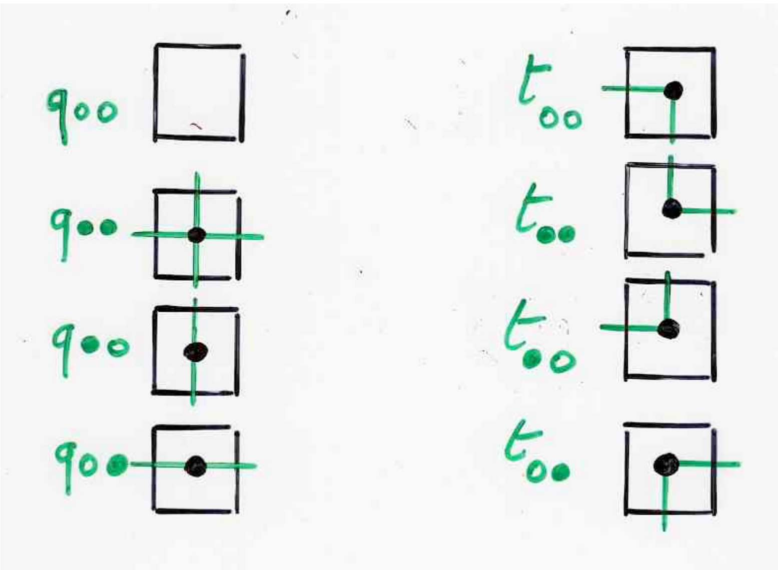
$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = \bigcirc\, A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$

non intersecting paths

$$\begin{cases} q_{\bullet\bullet} = 0 \\ t_{\bullet\circ} = t_{\circ\bullet} = 0 \end{cases}$$

(ASM)

(osc. paths)



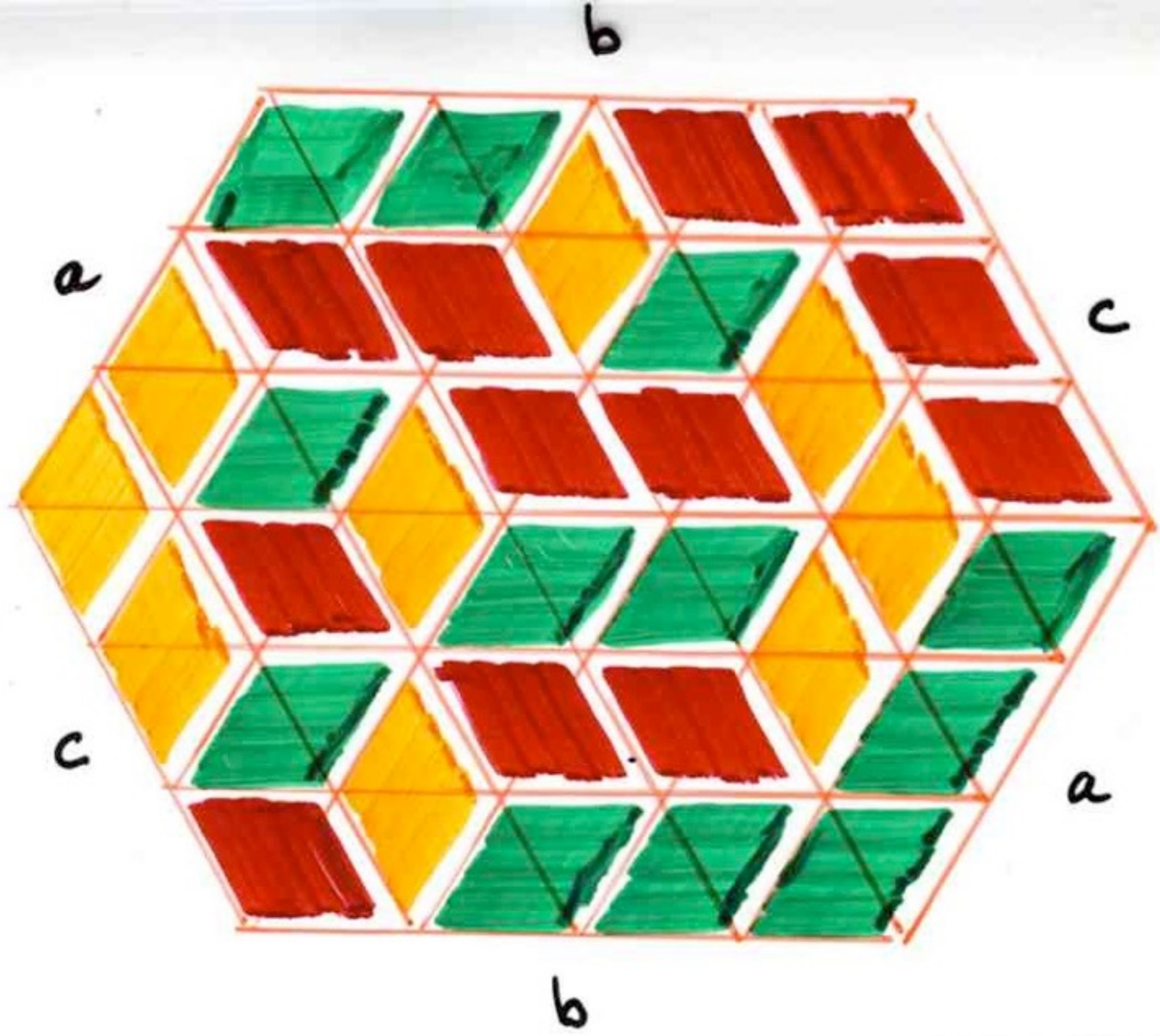
The quadratic algebra Z

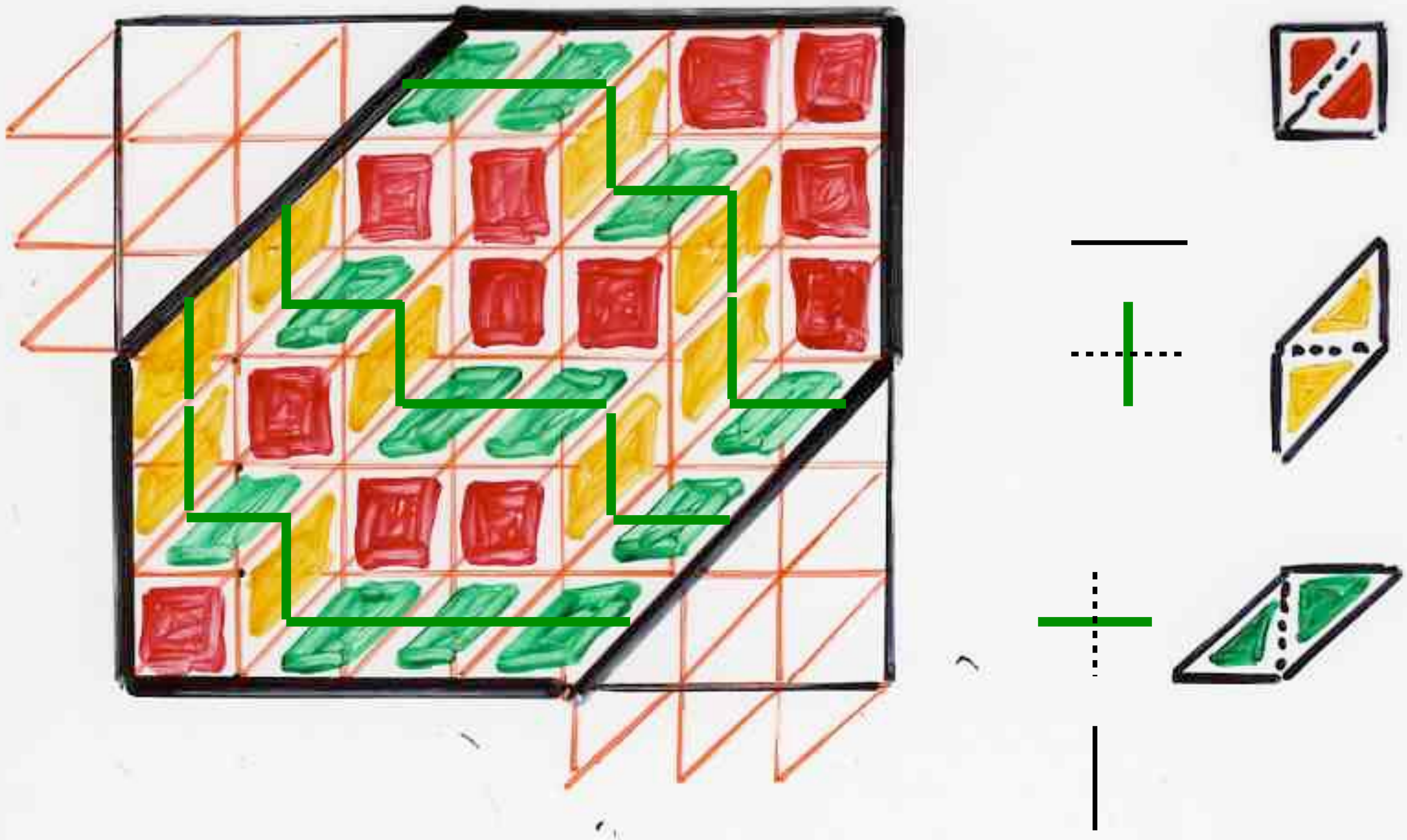
4 generators $B_\bullet \; A_\bullet \; B \; A$

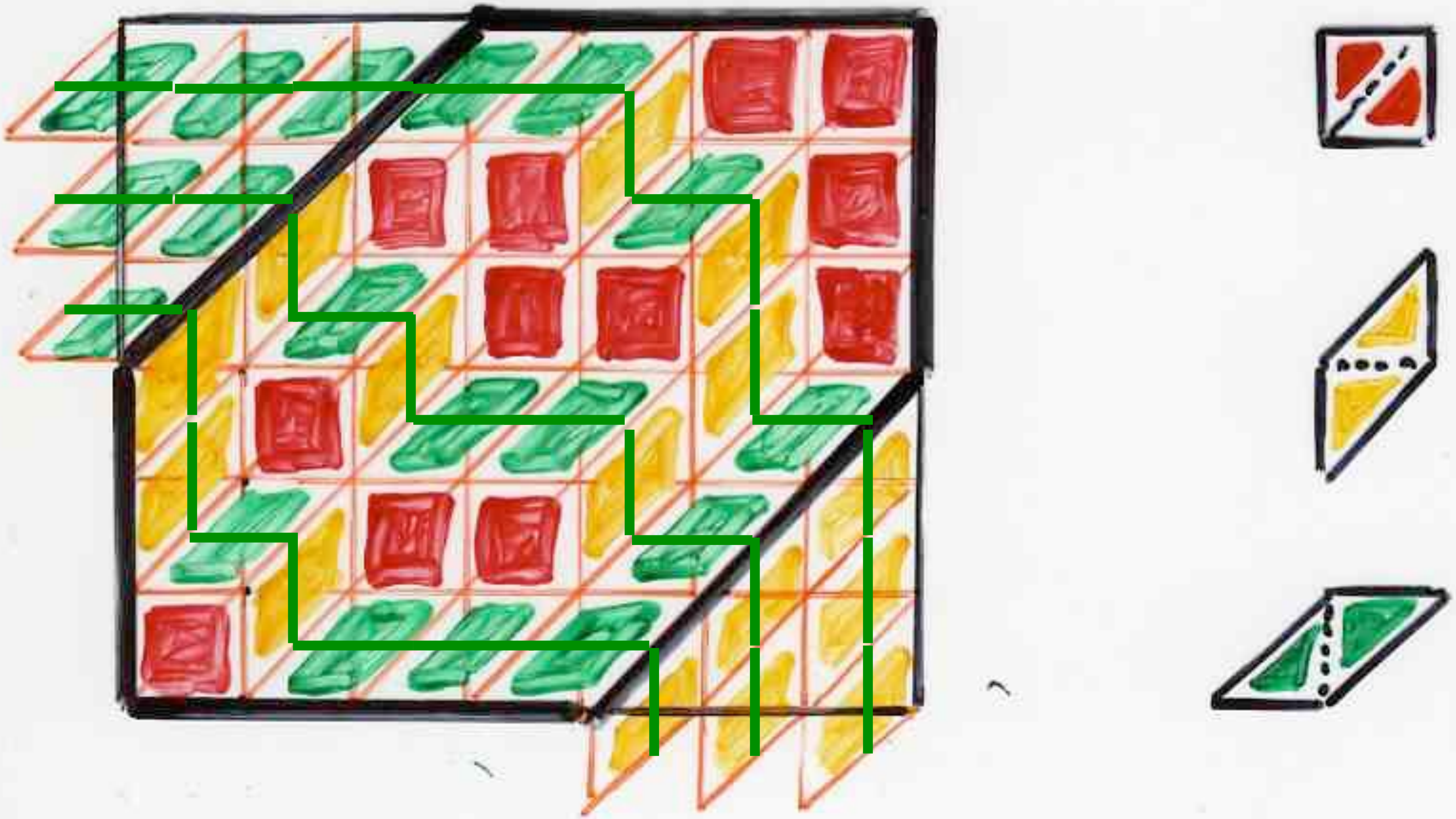
8 parameters $q_{\dots}$, $t_{\dots}$

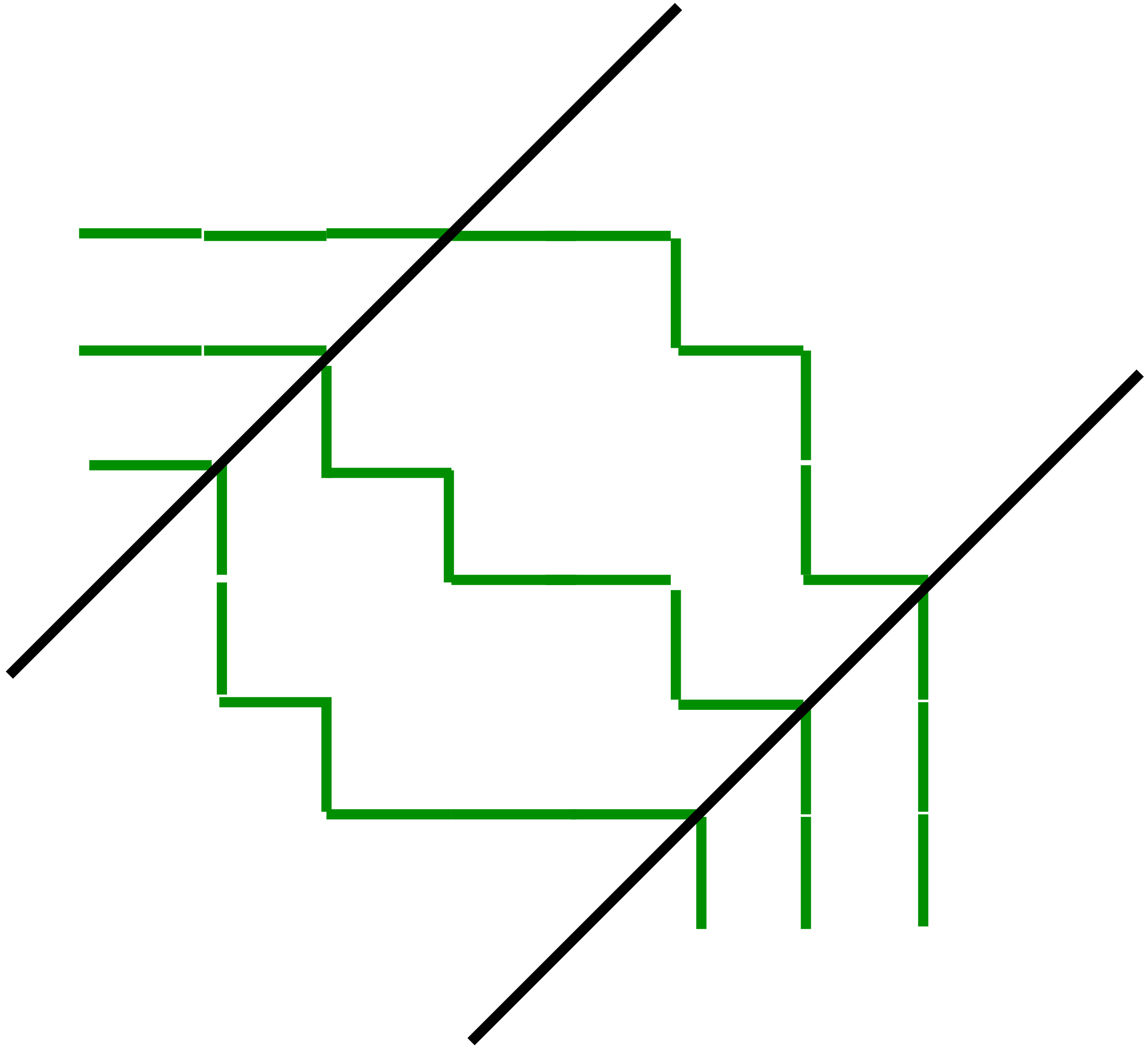
$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, AB_\bullet + \bigcirc\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, AB_\bullet \end{cases}$$

bijection
rhombus tilings
non-intersecting paths


b
a
c
c
a
b

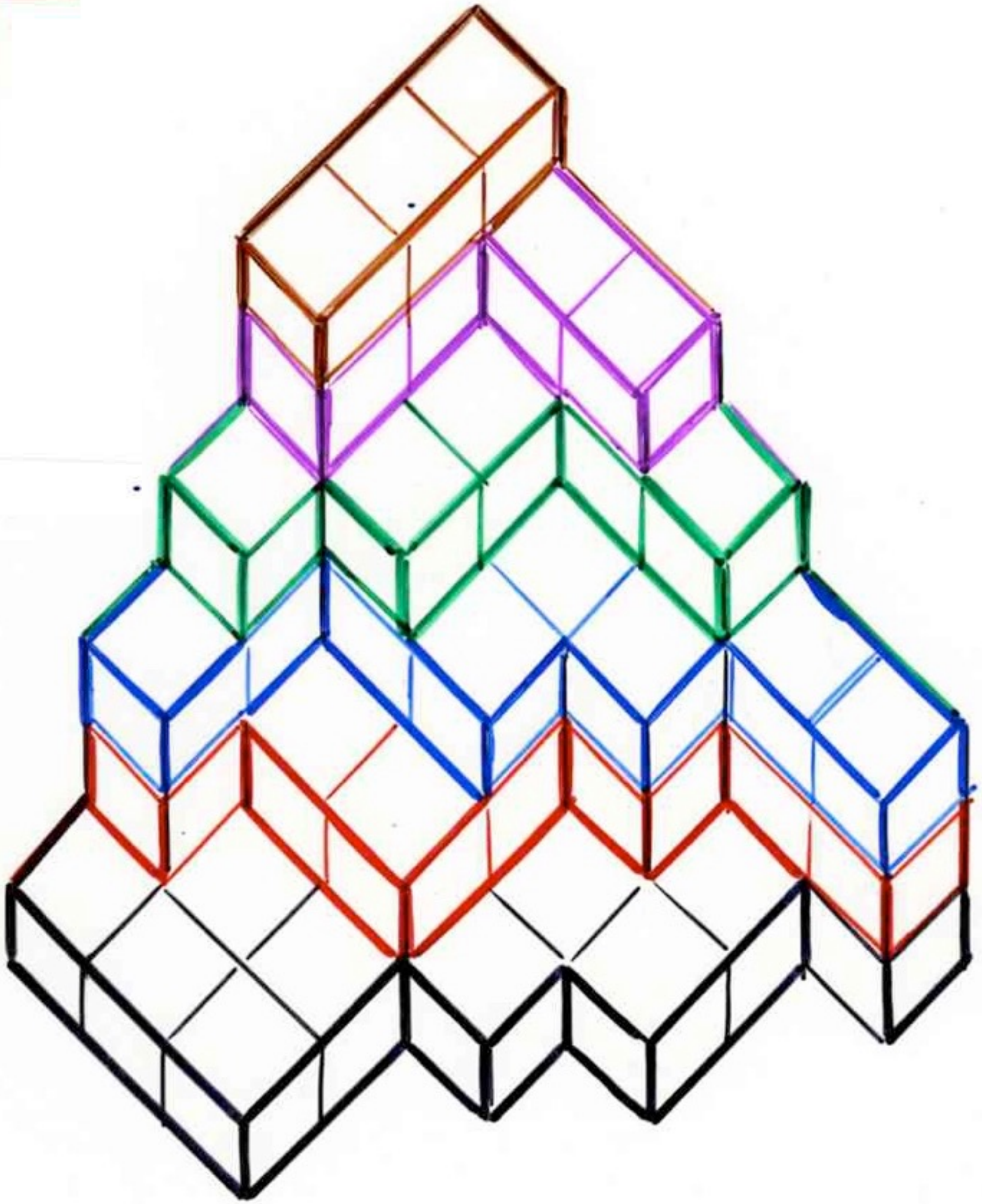


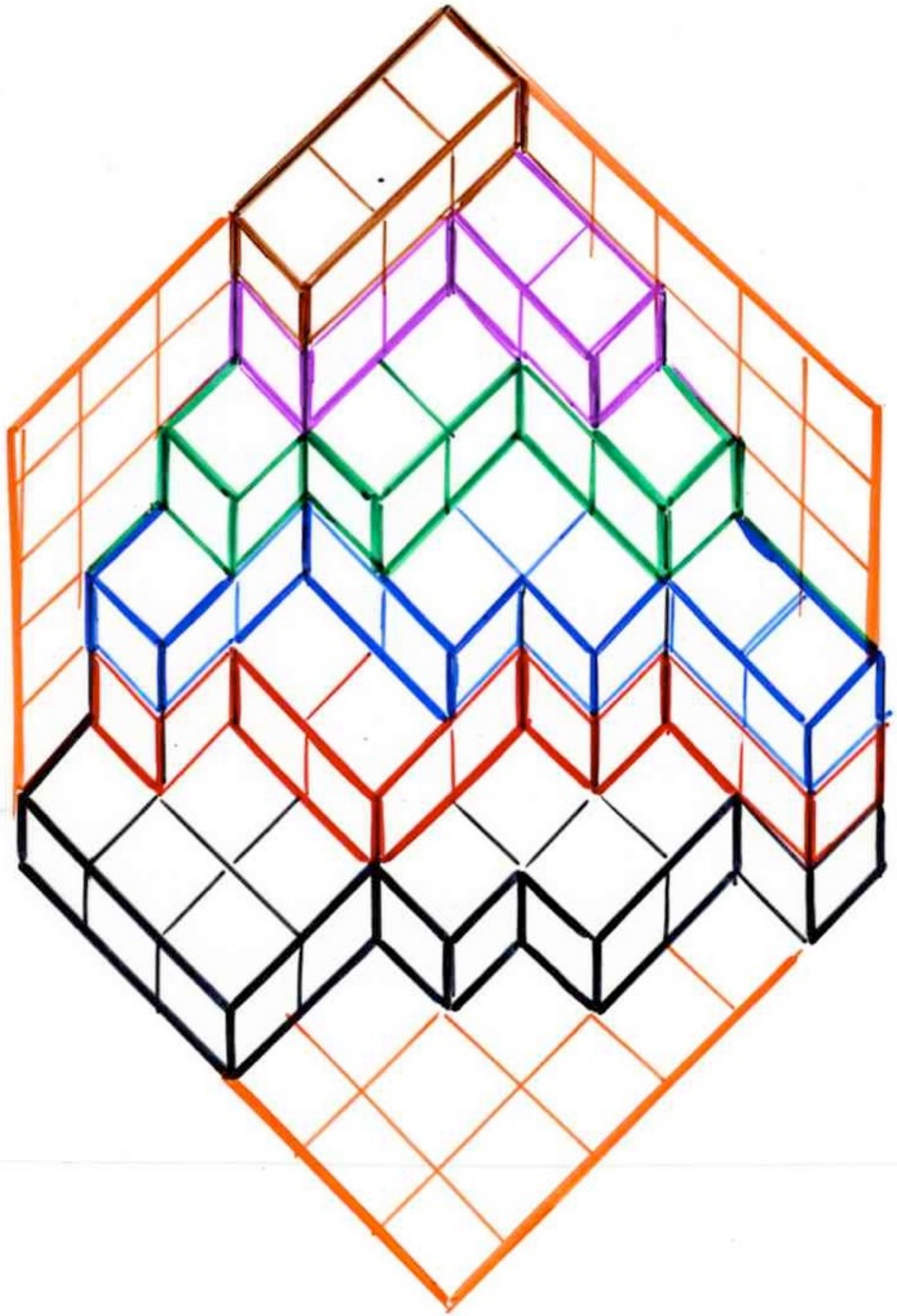


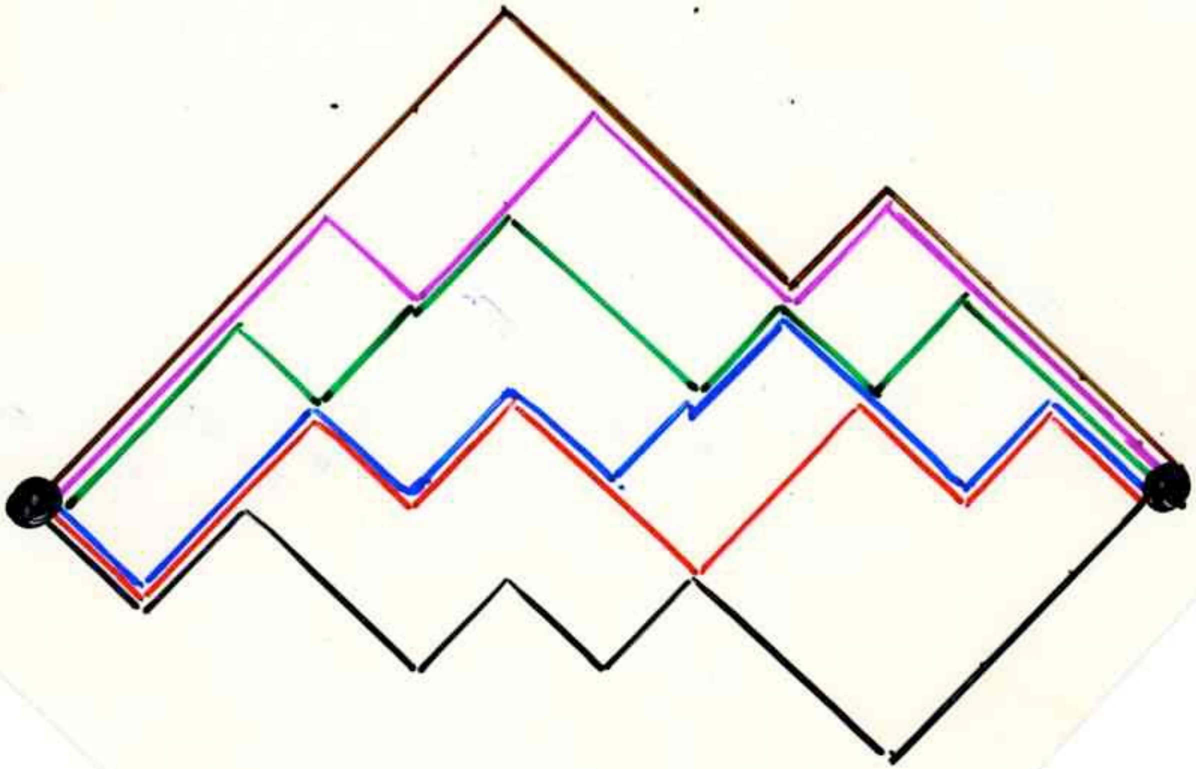


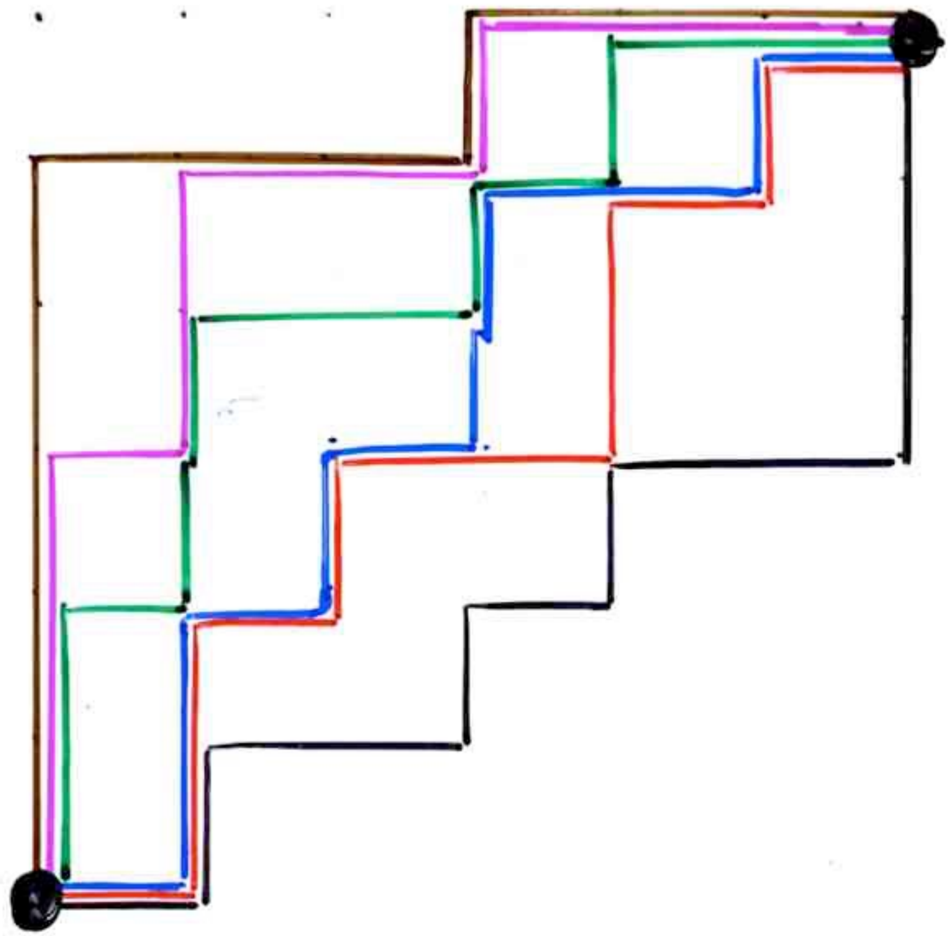
bijection
plane partitions
non-intersecting paths

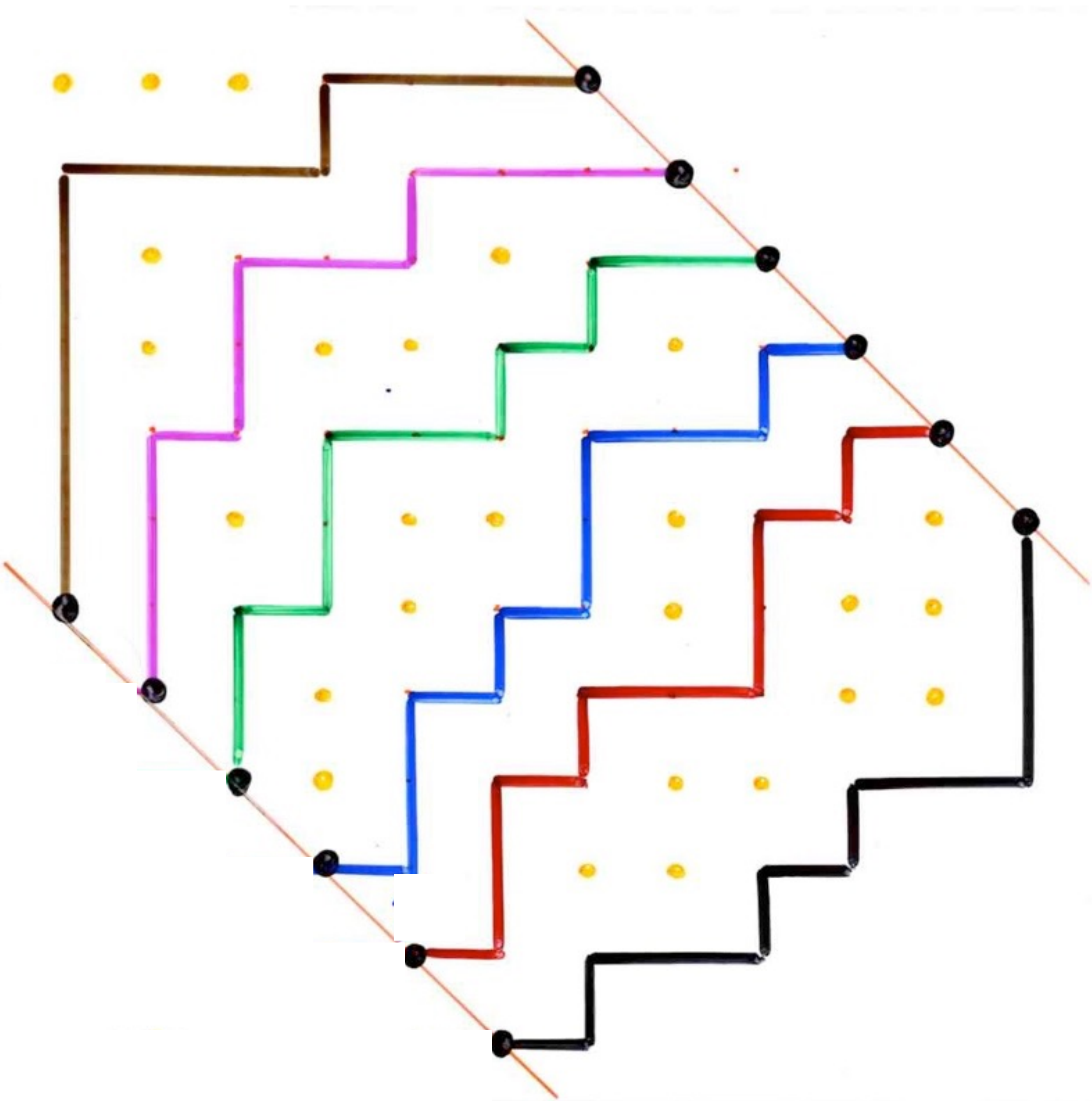
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

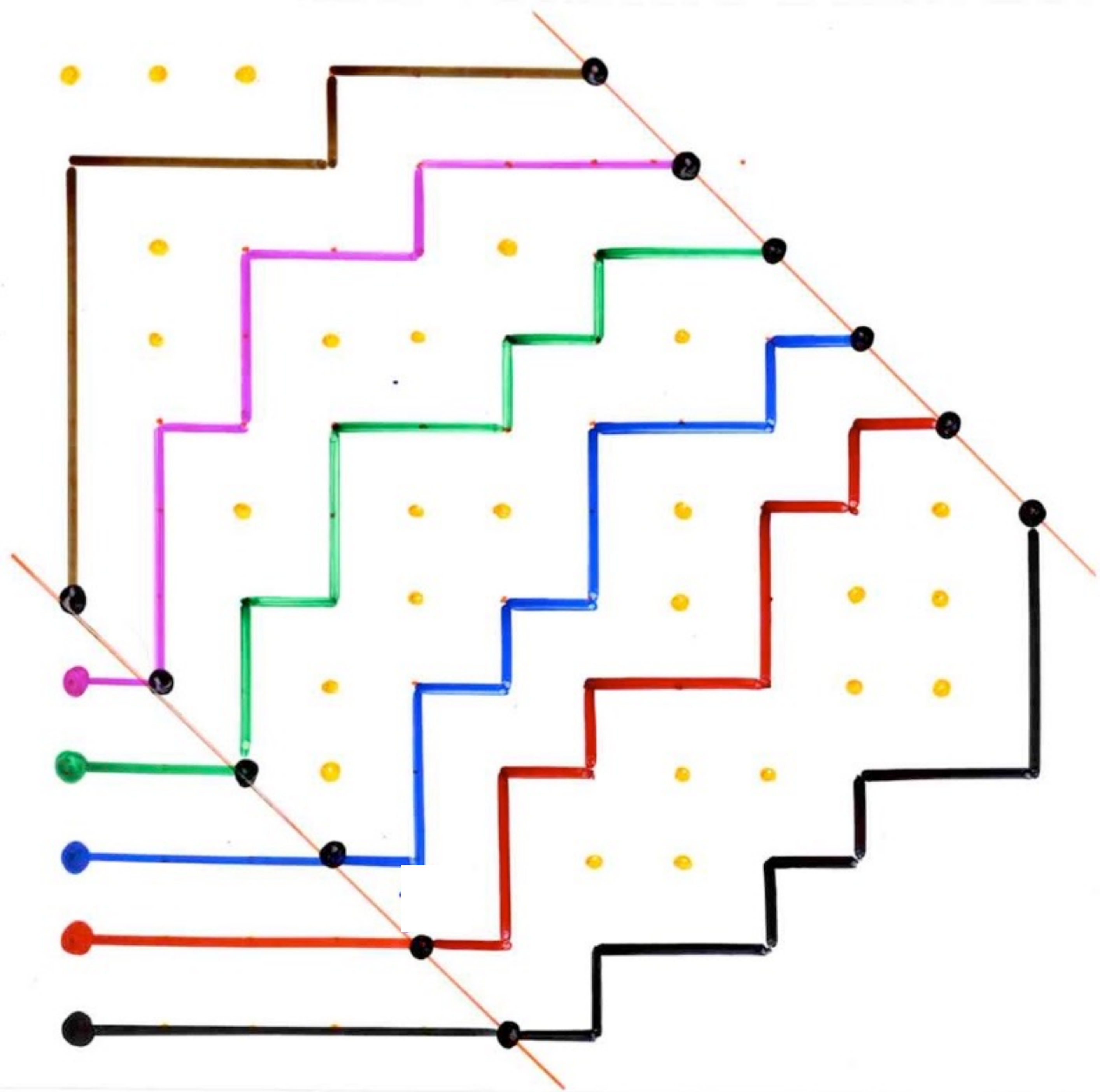




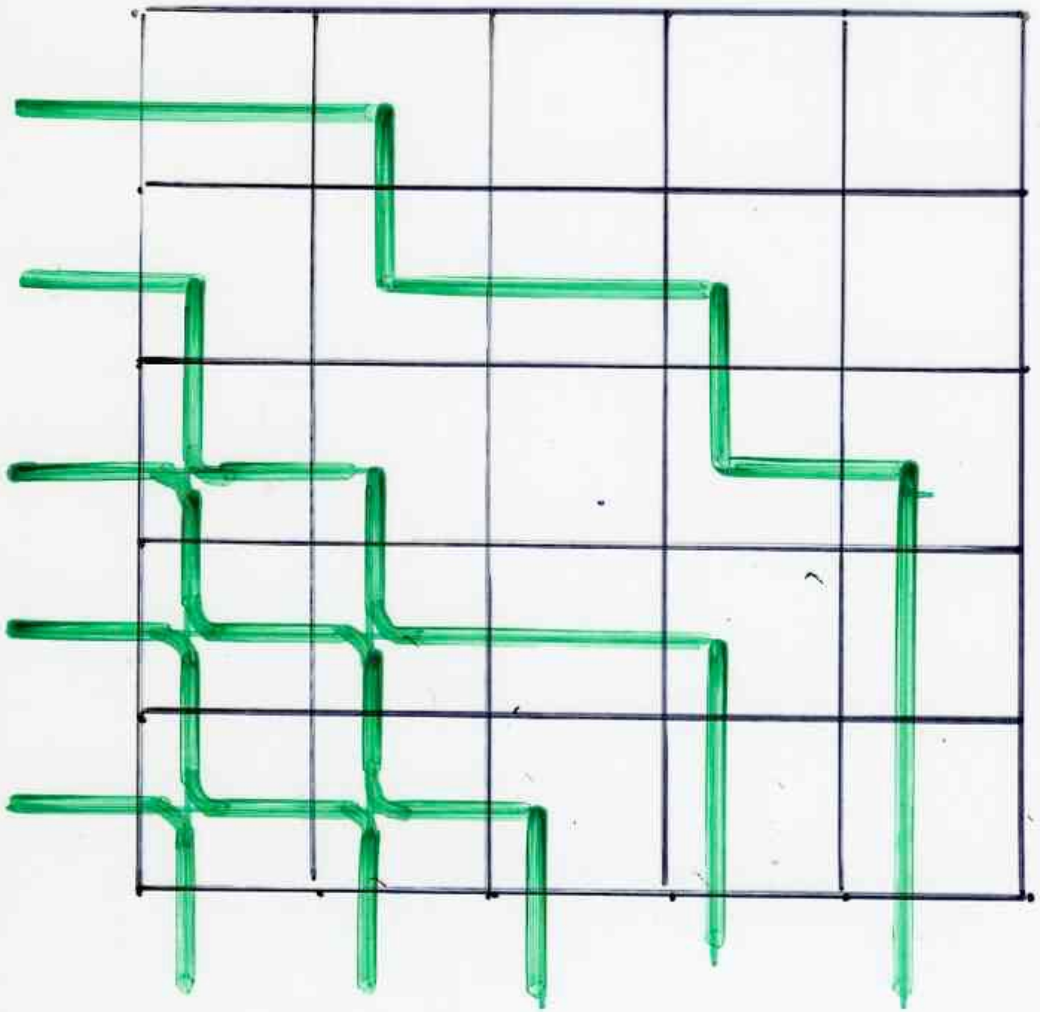








osculating paths



osculating paths

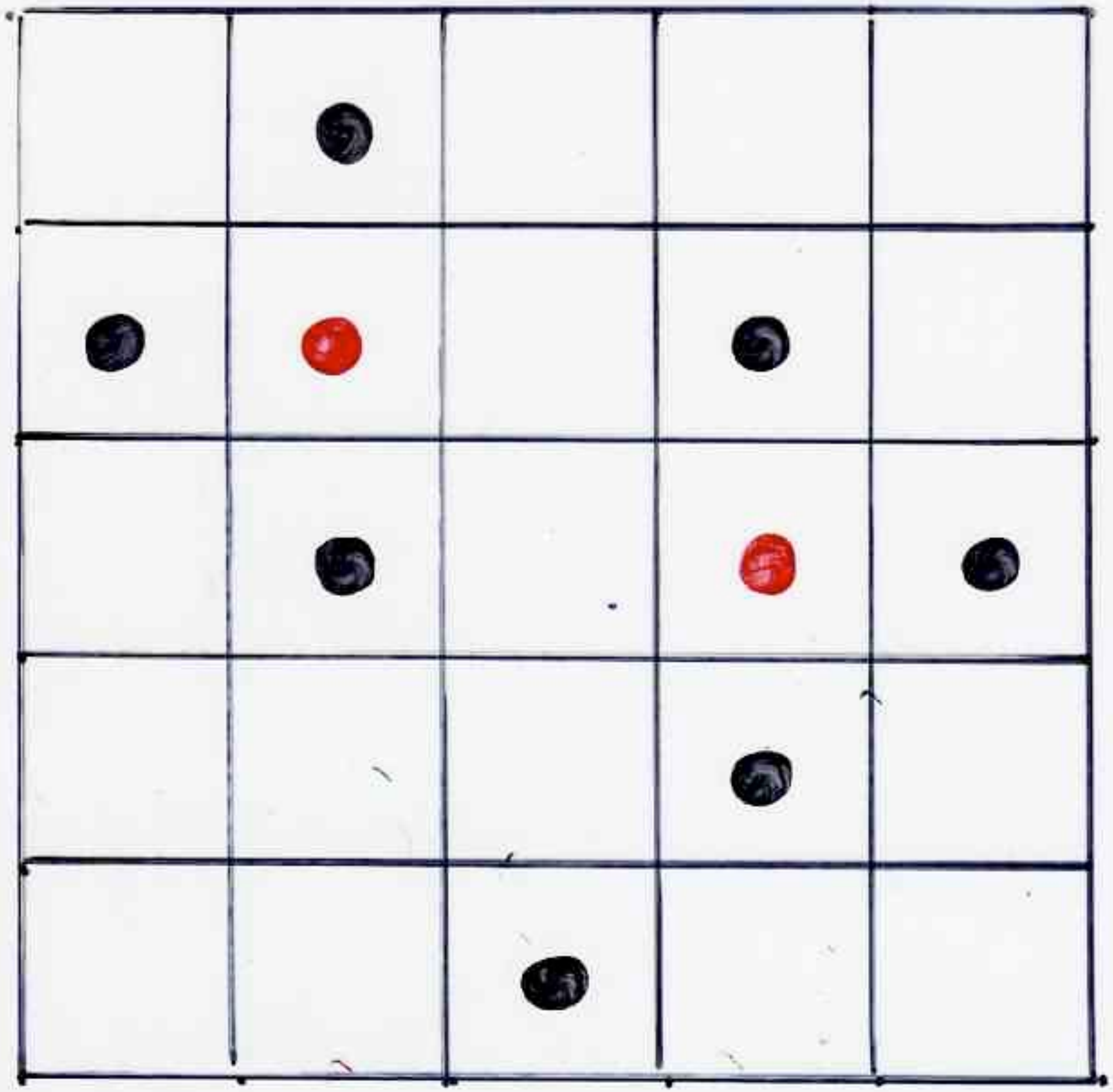
$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

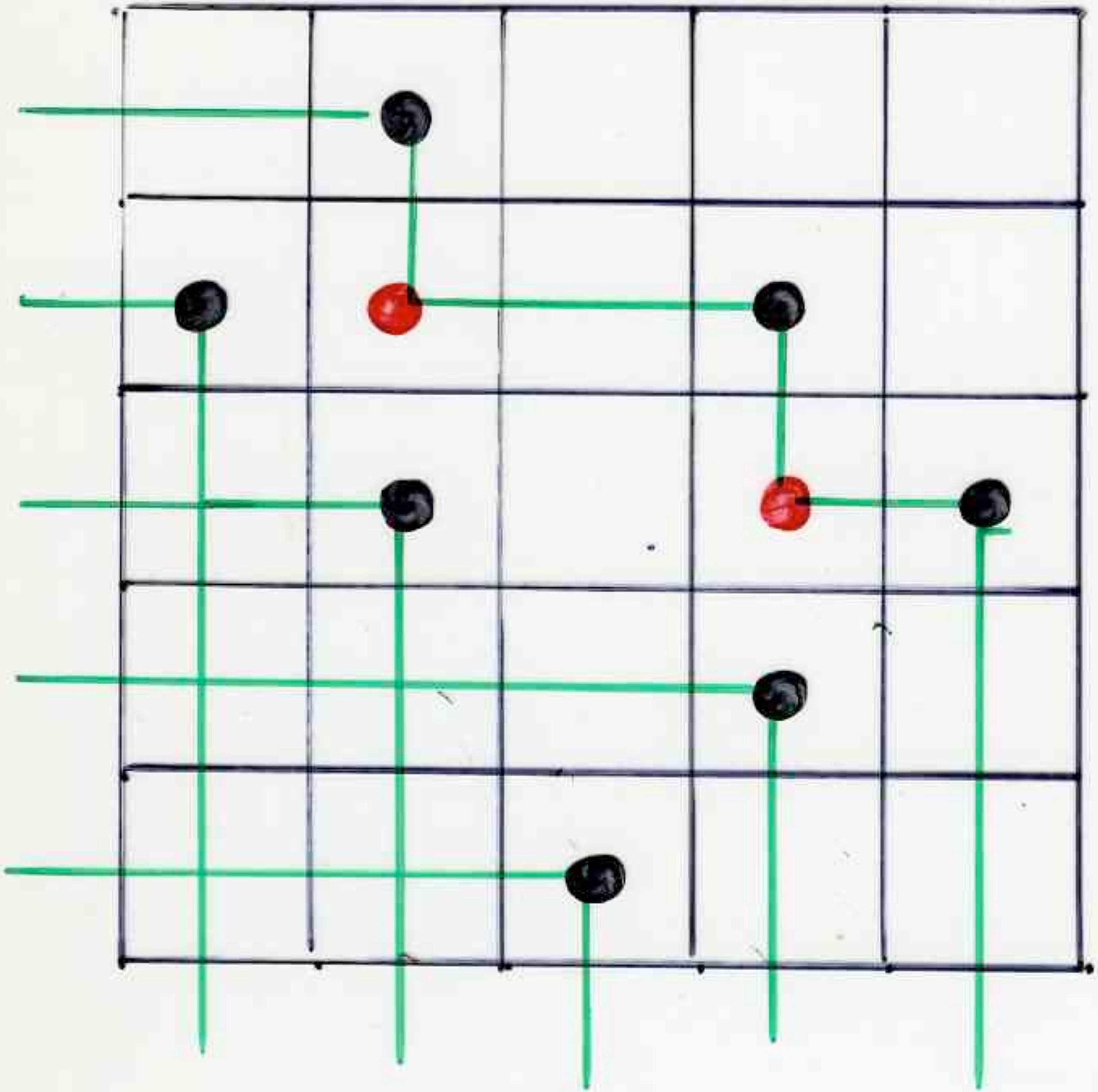
The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

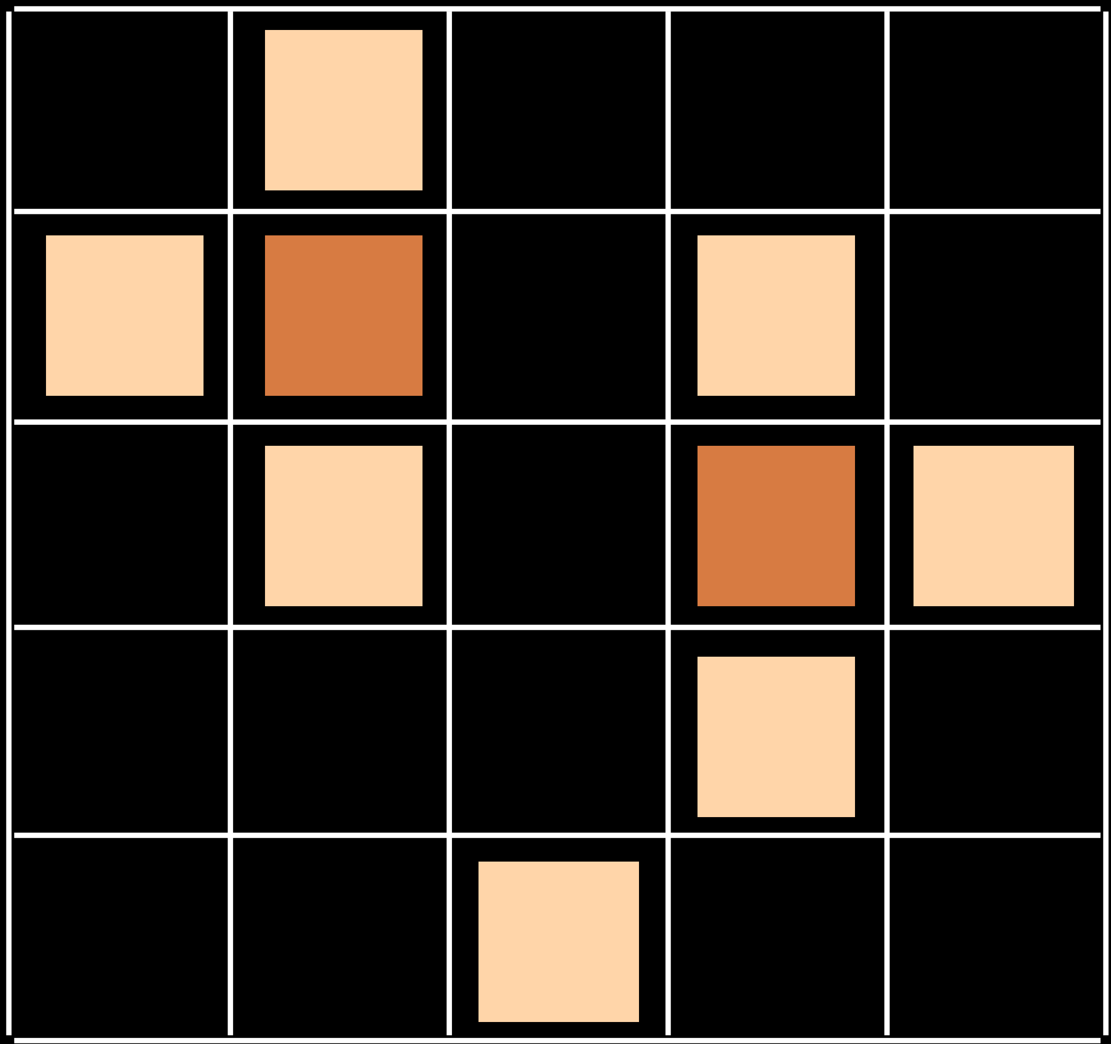
8 parameters $q_{\dots}$, $t_{\dots}$

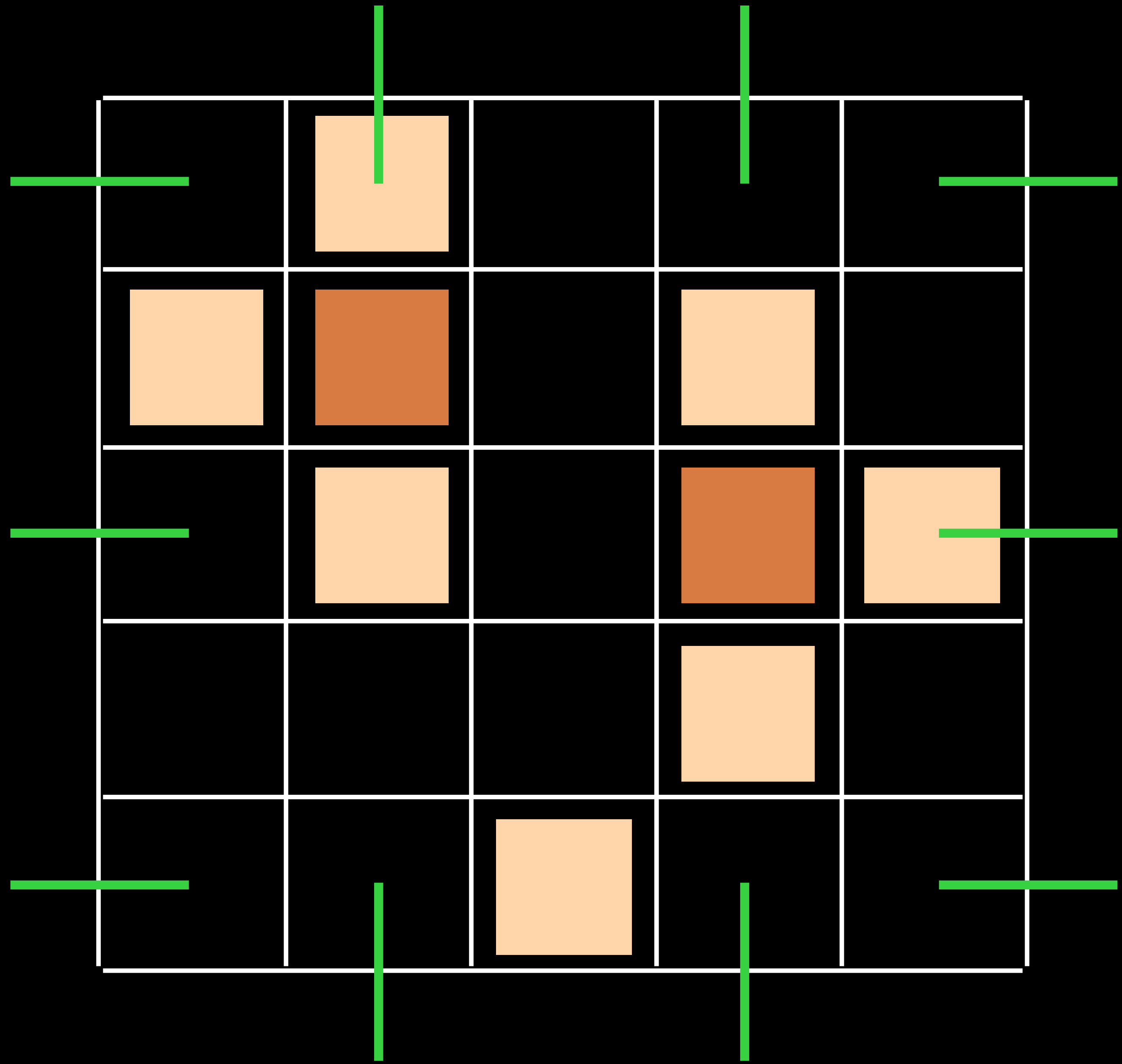
$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, AB_\bullet + \bigcirc\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, AB_\bullet \end{cases}$$

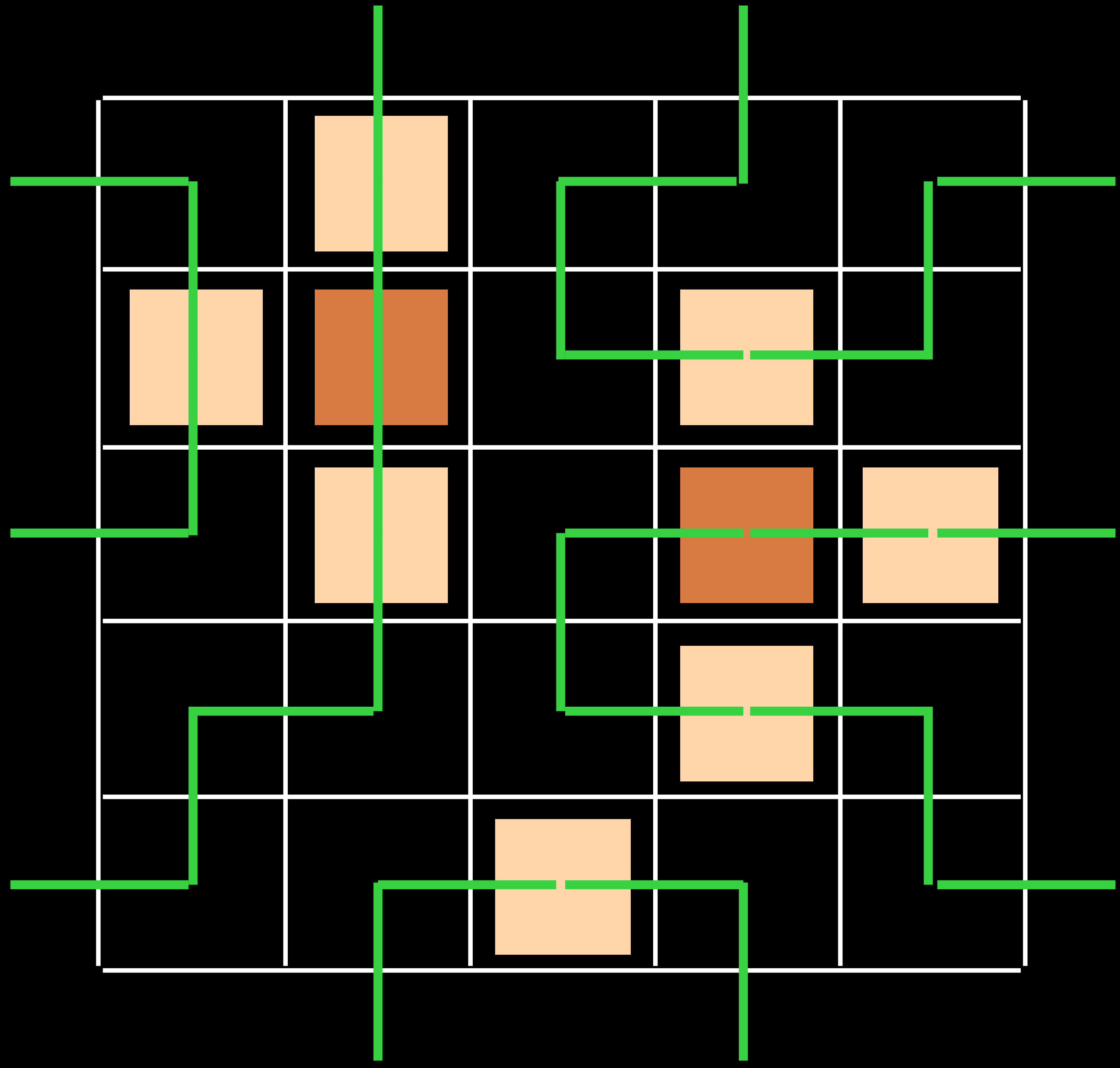




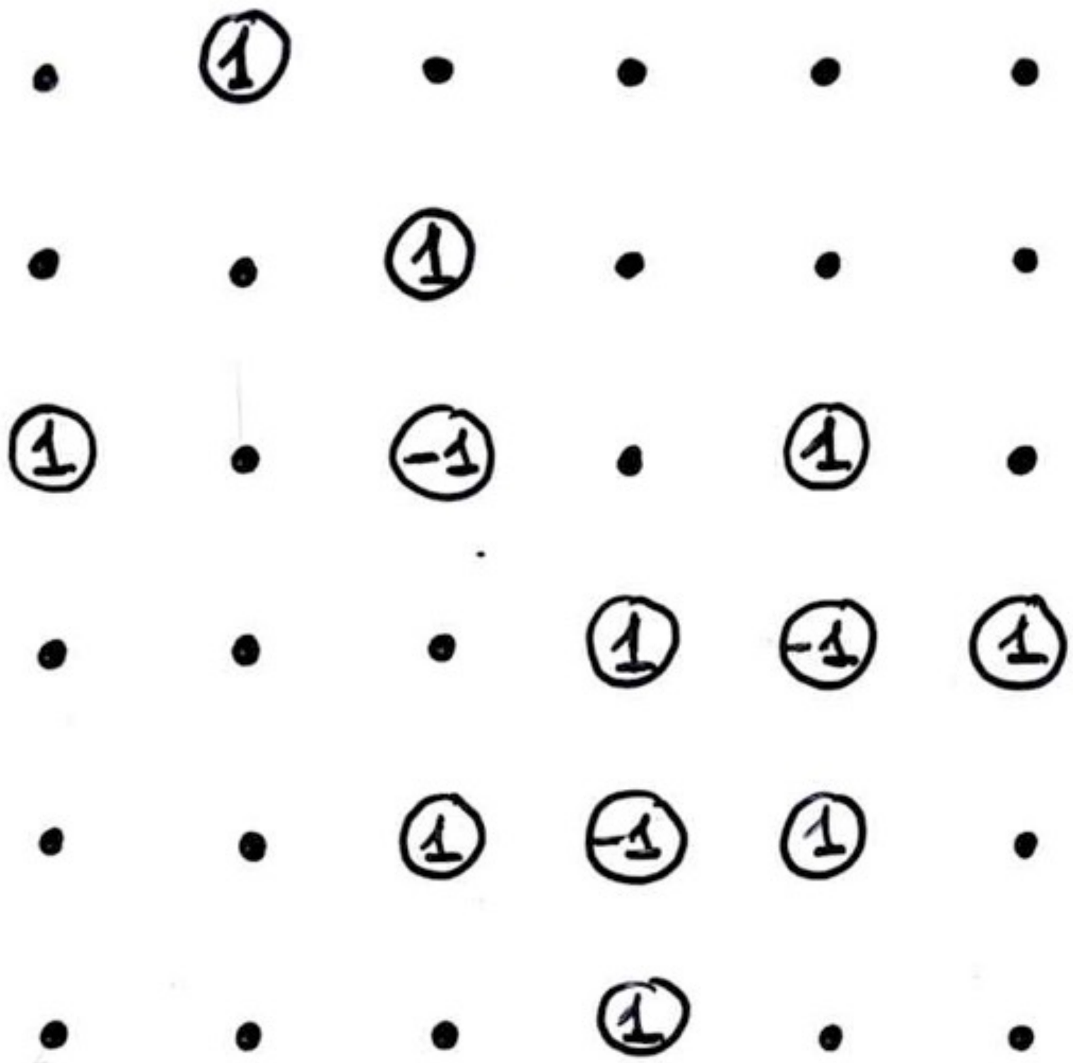
FPL
fully packed loops



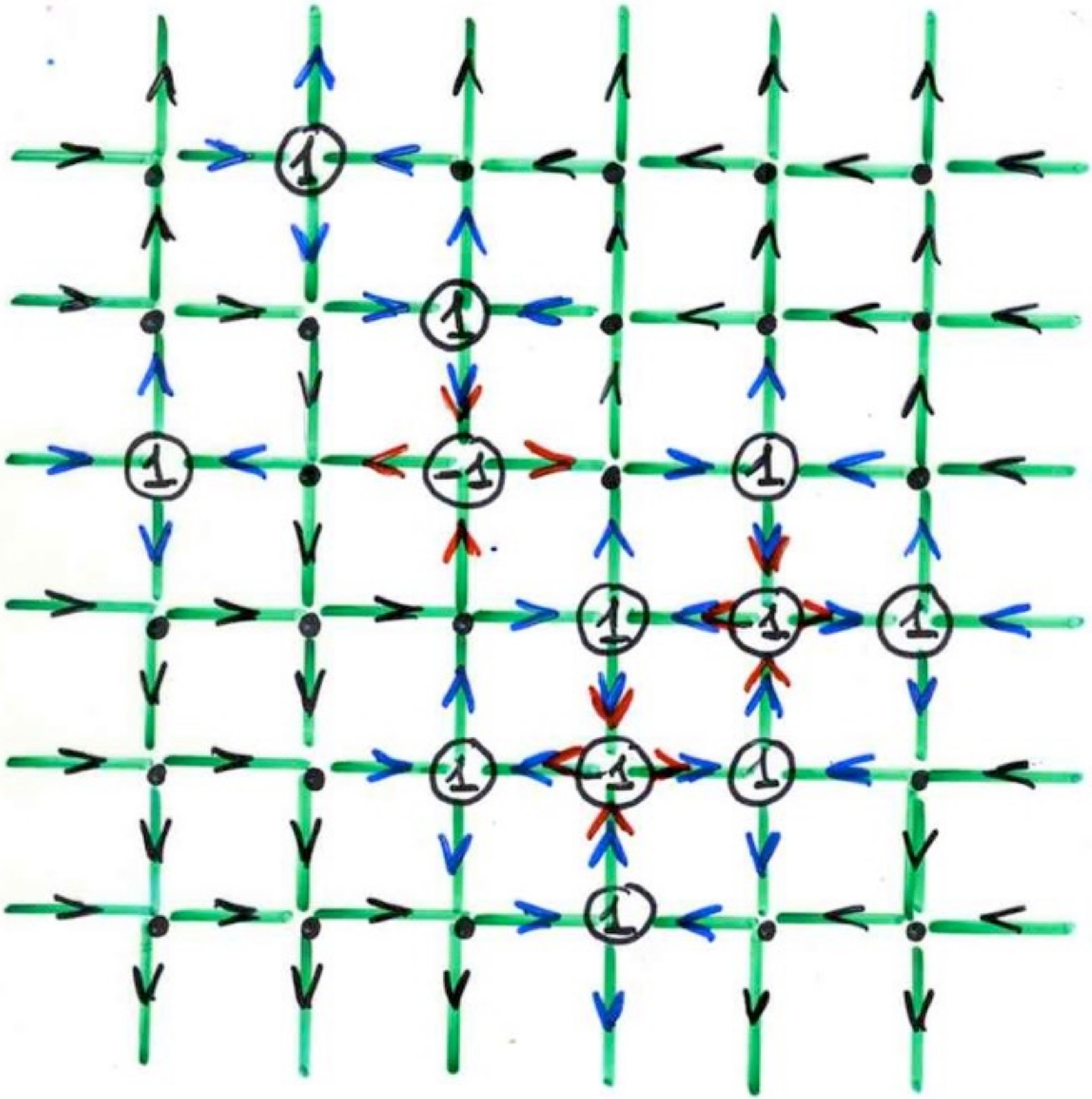


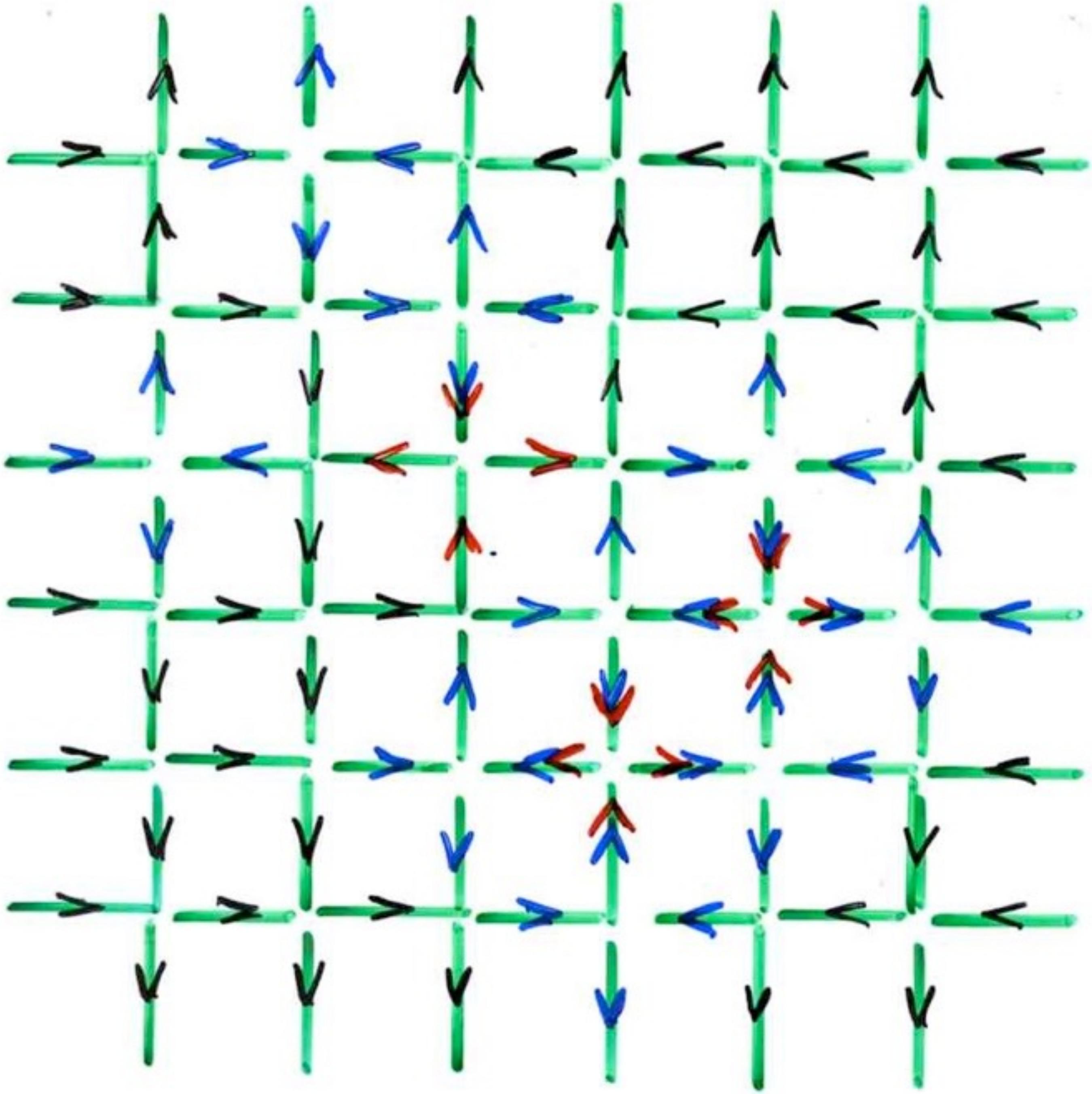


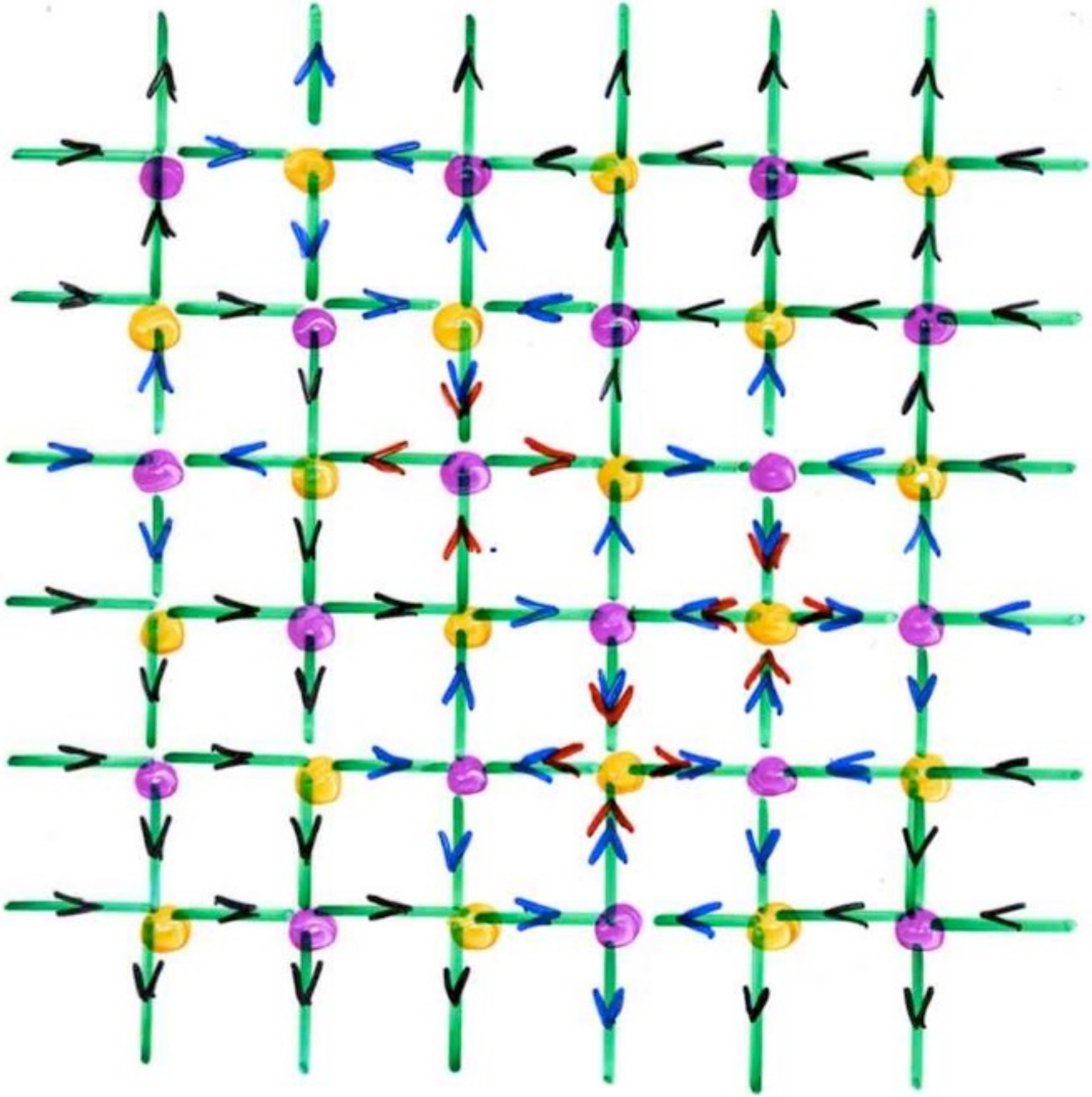
The
bijection
AMS
FPL

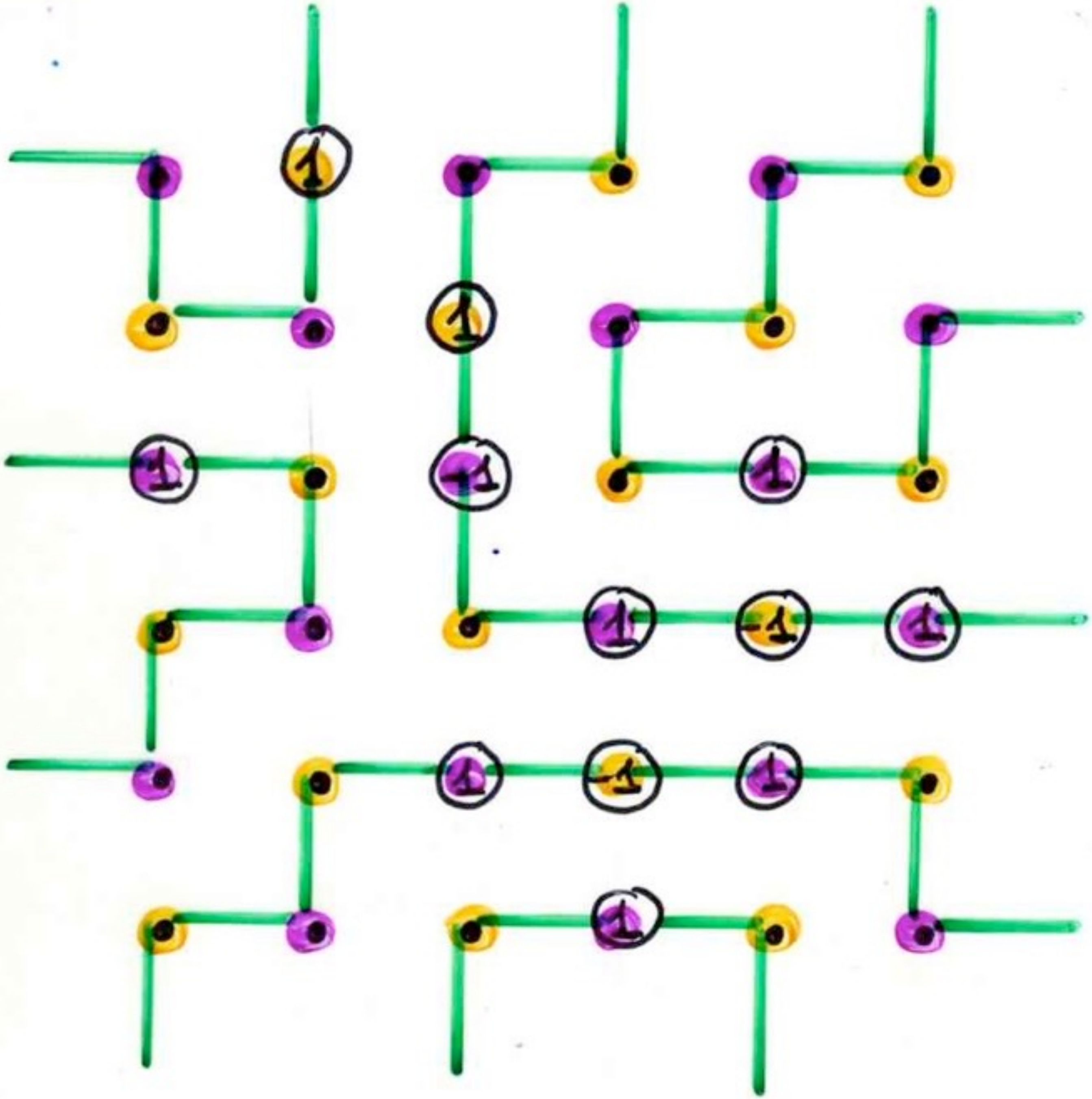


The 6-vertex model

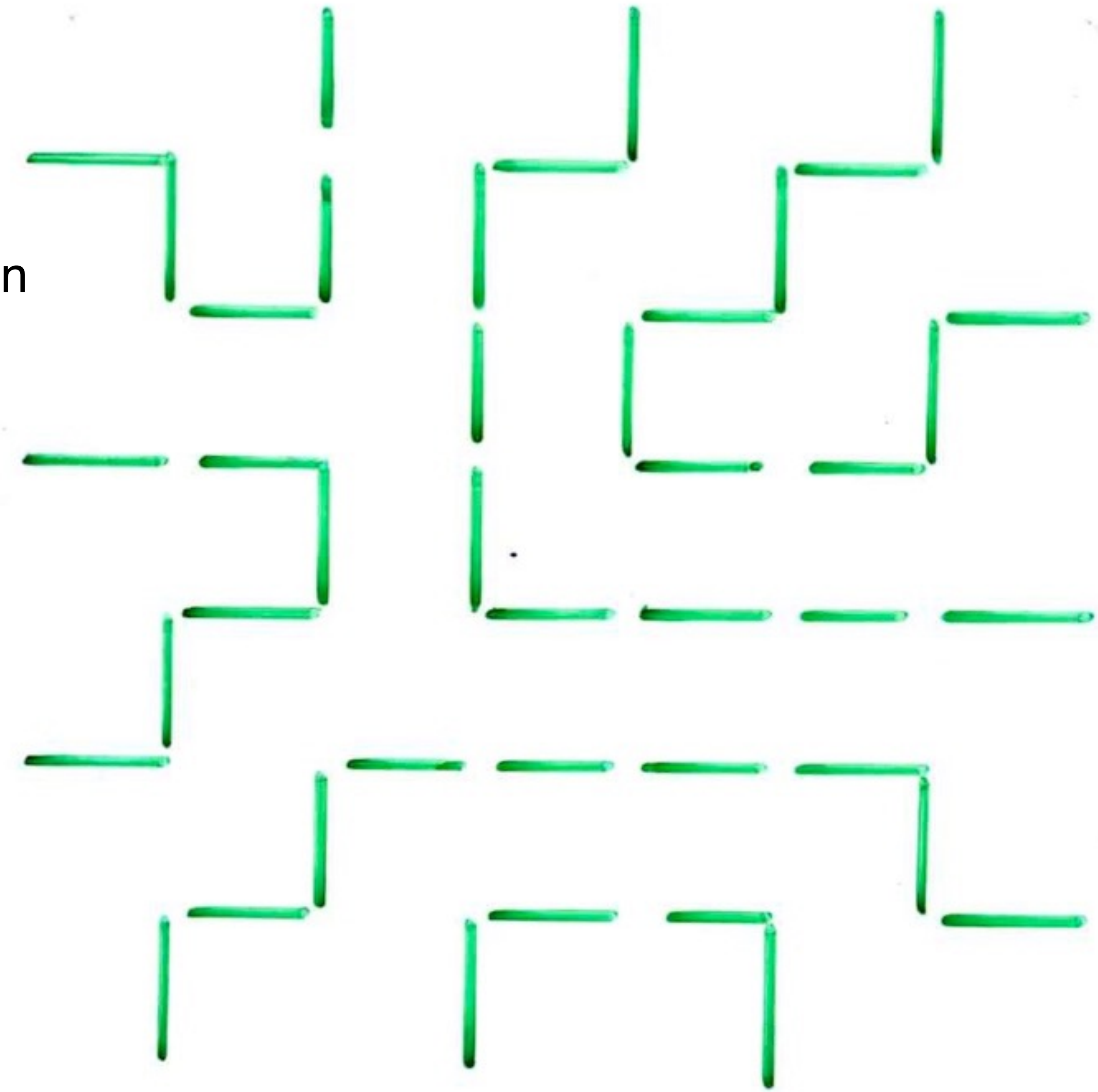








FPL
“Fully
Packed
Loop”
configuration

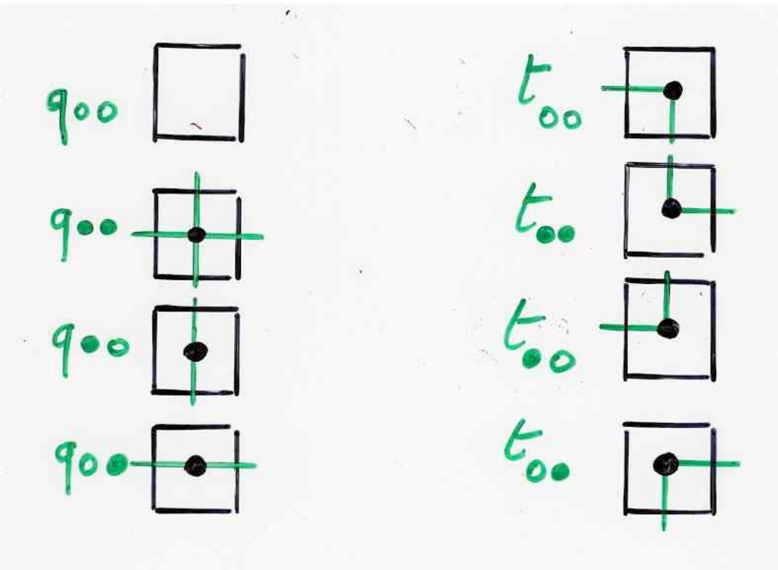


The quadratic algebra $\mathbb{Z}$

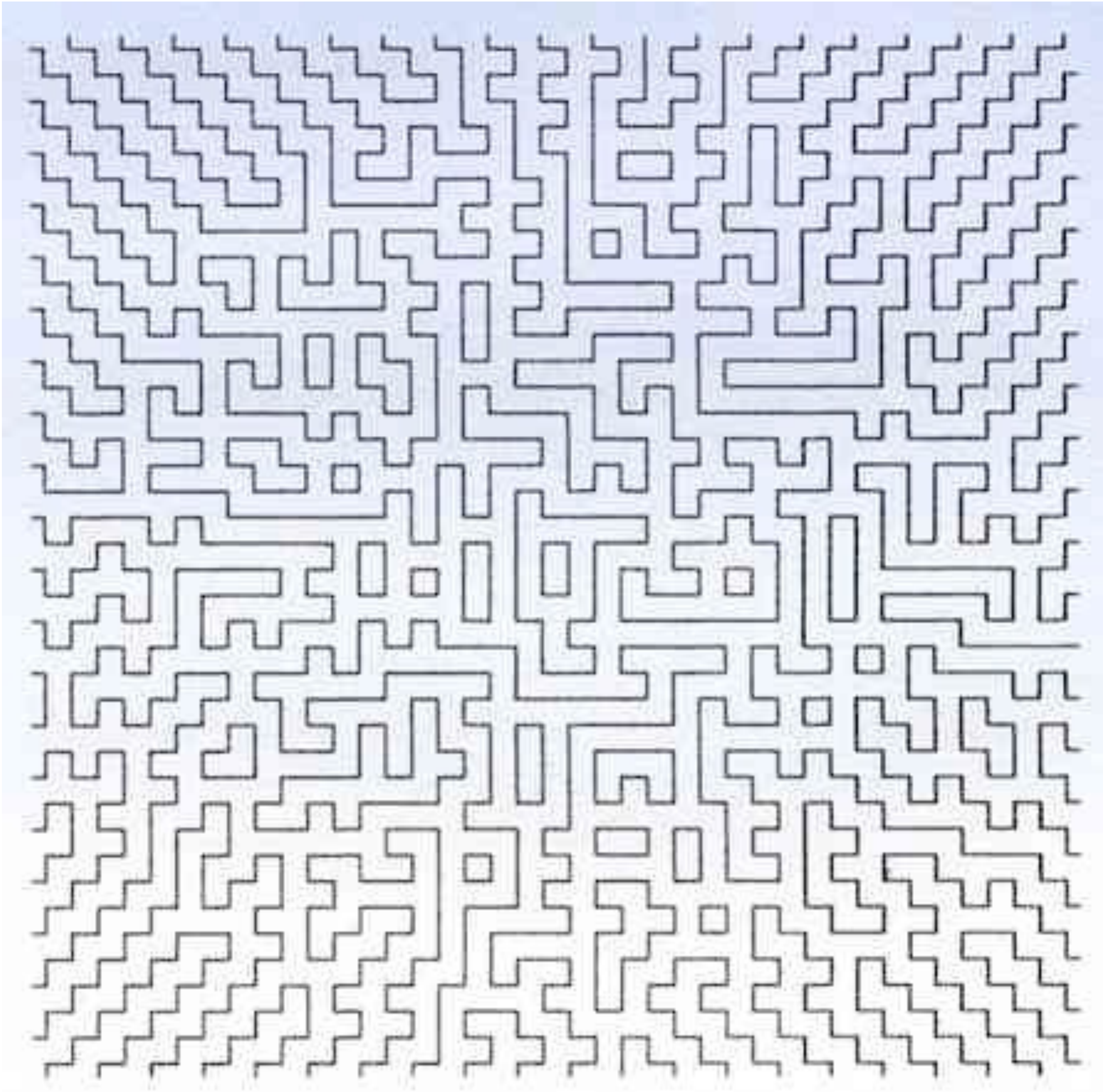
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = 0\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = 0\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, AB_\bullet + t_{\bullet\circ}\, A_\bullet B \\ BA_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, AB_\bullet \end{cases}$$



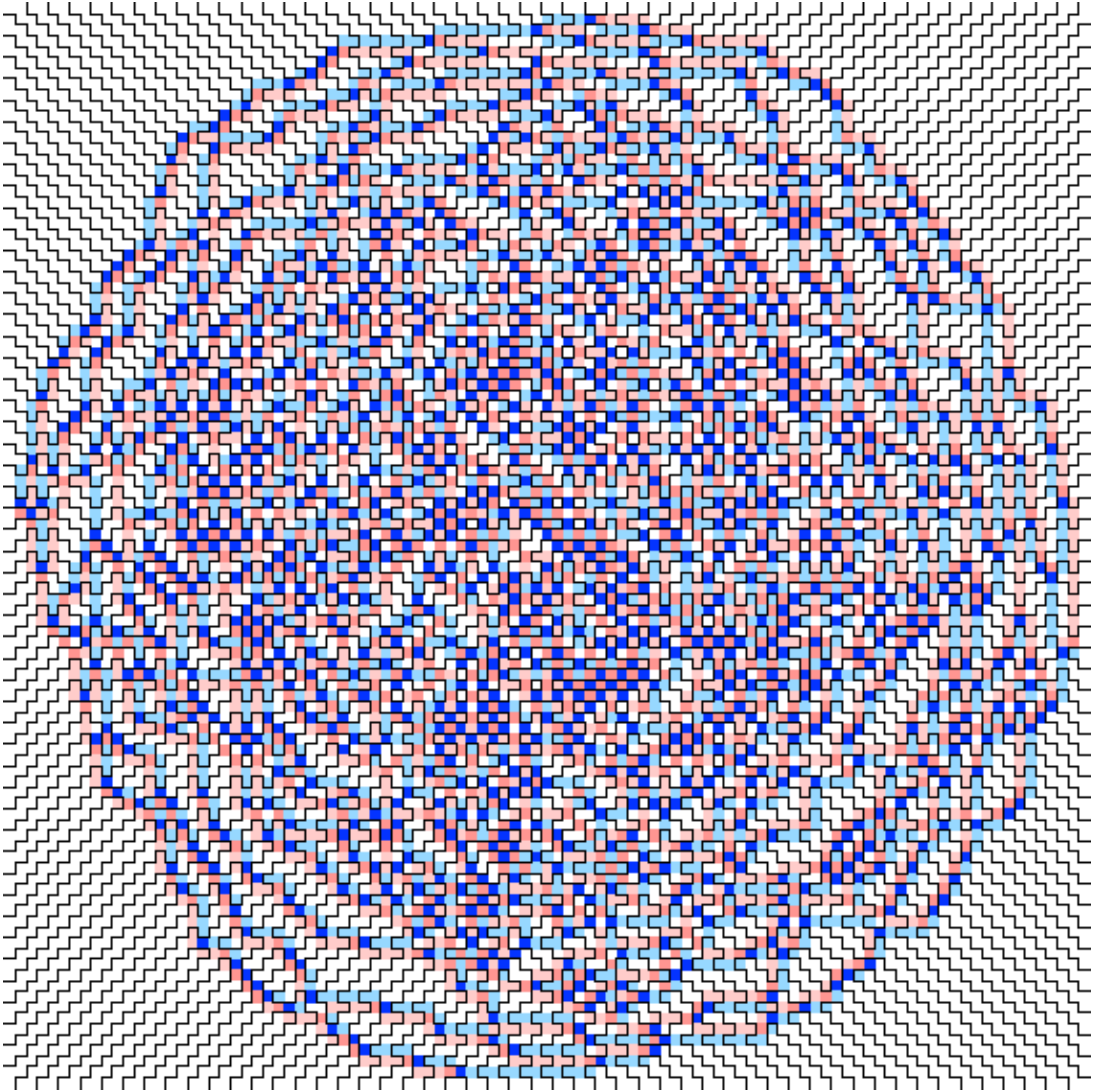
random
FPL



random

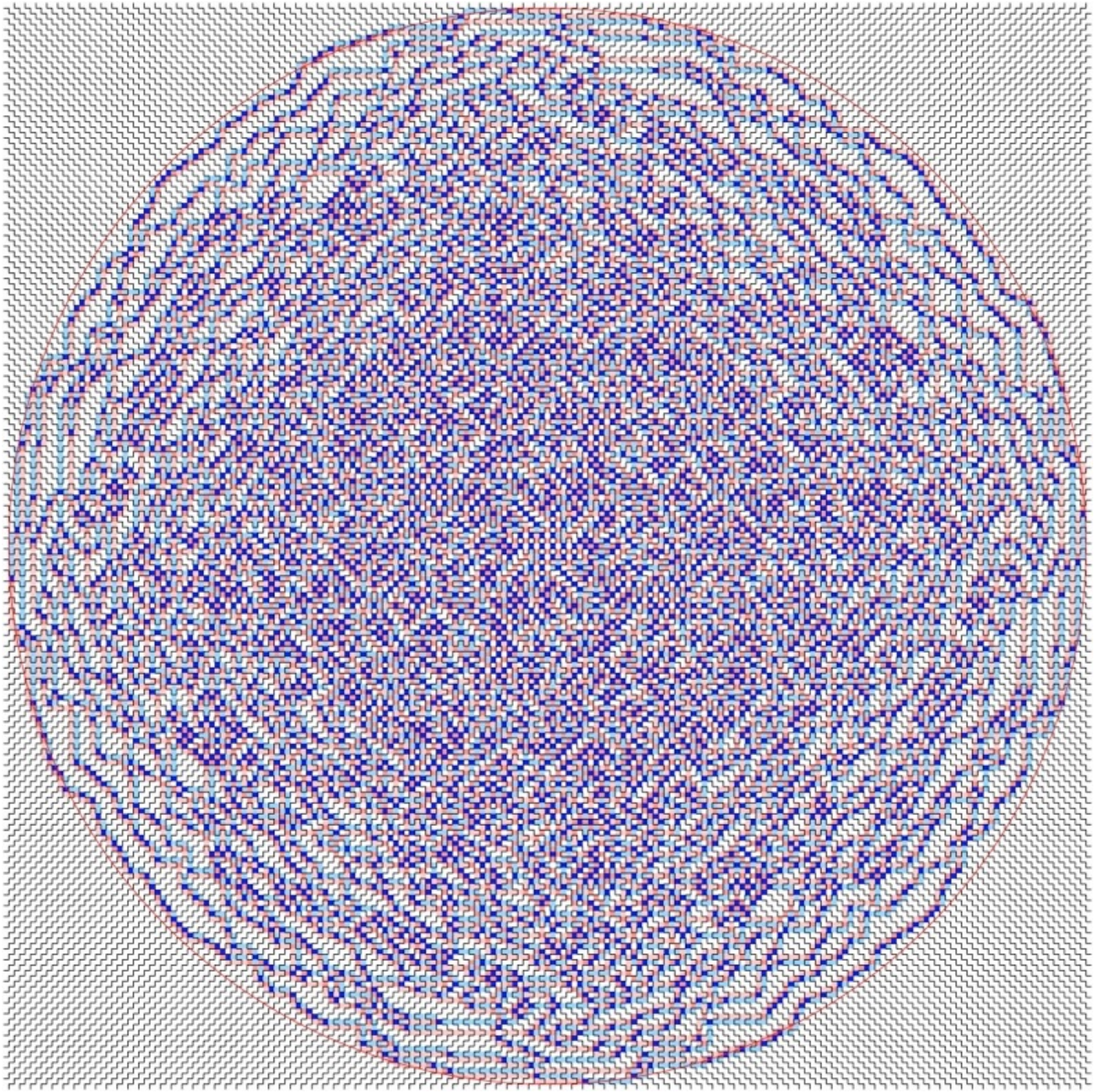
FPL

(P.Duchon)

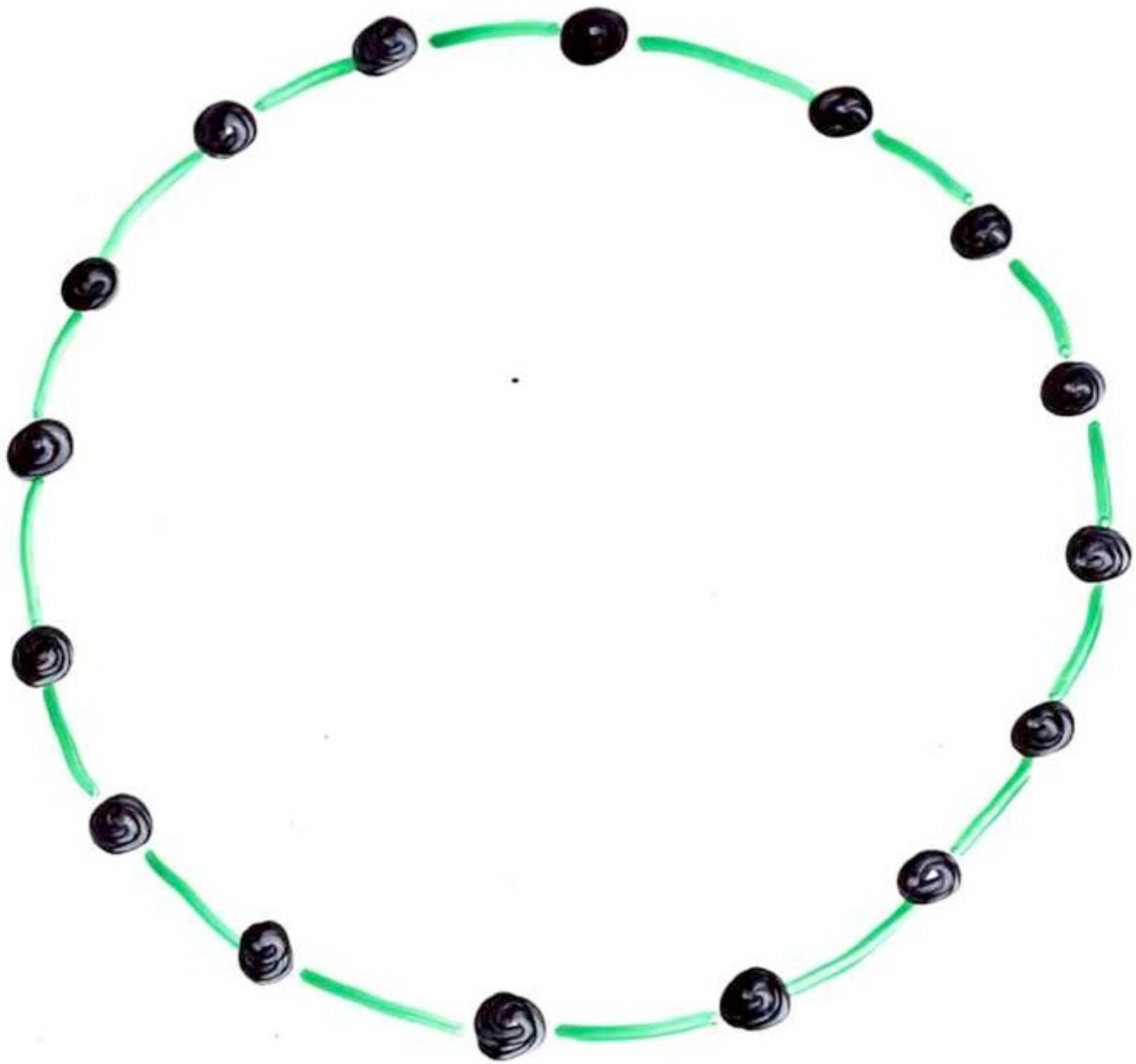


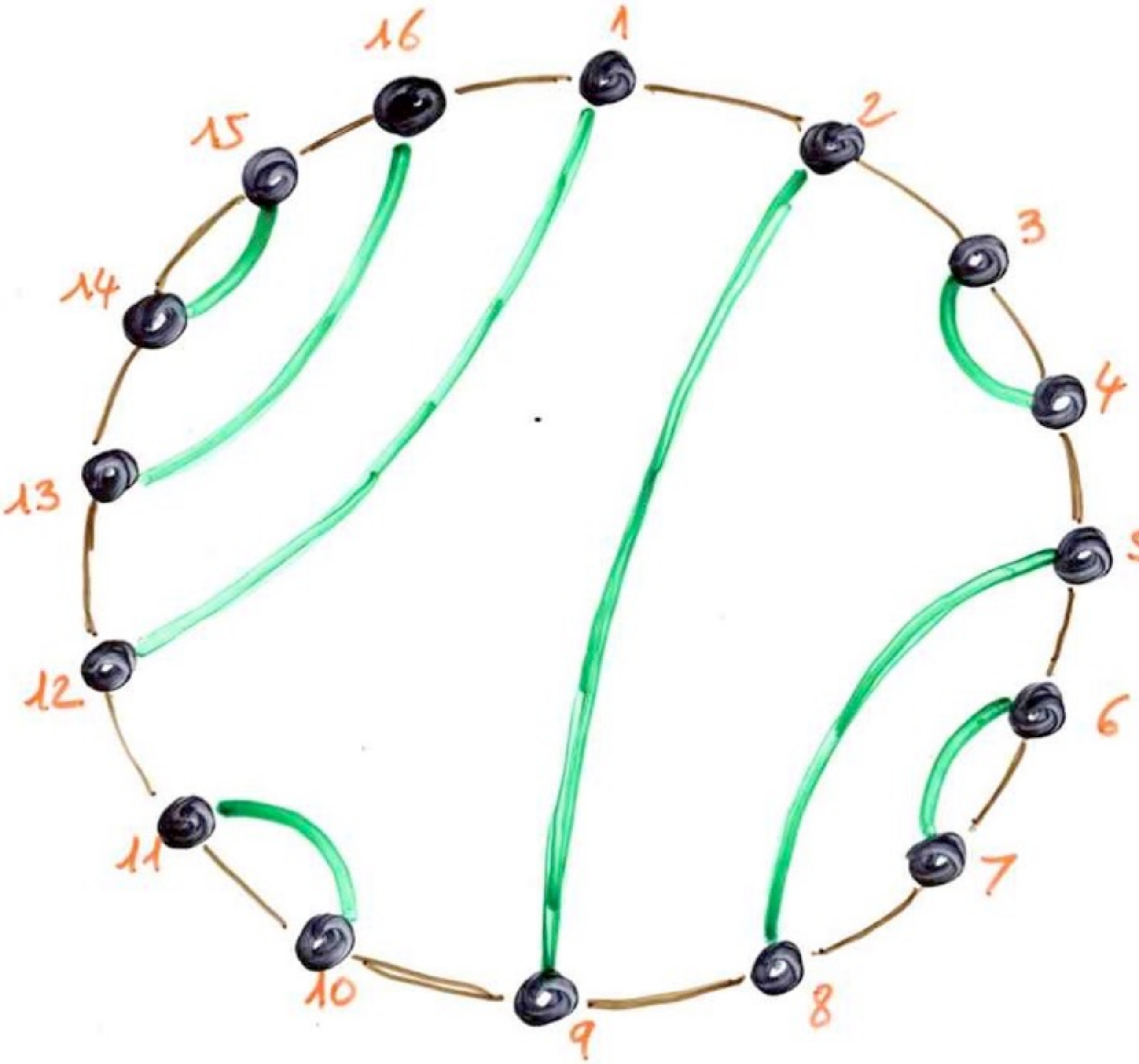
random
FPL

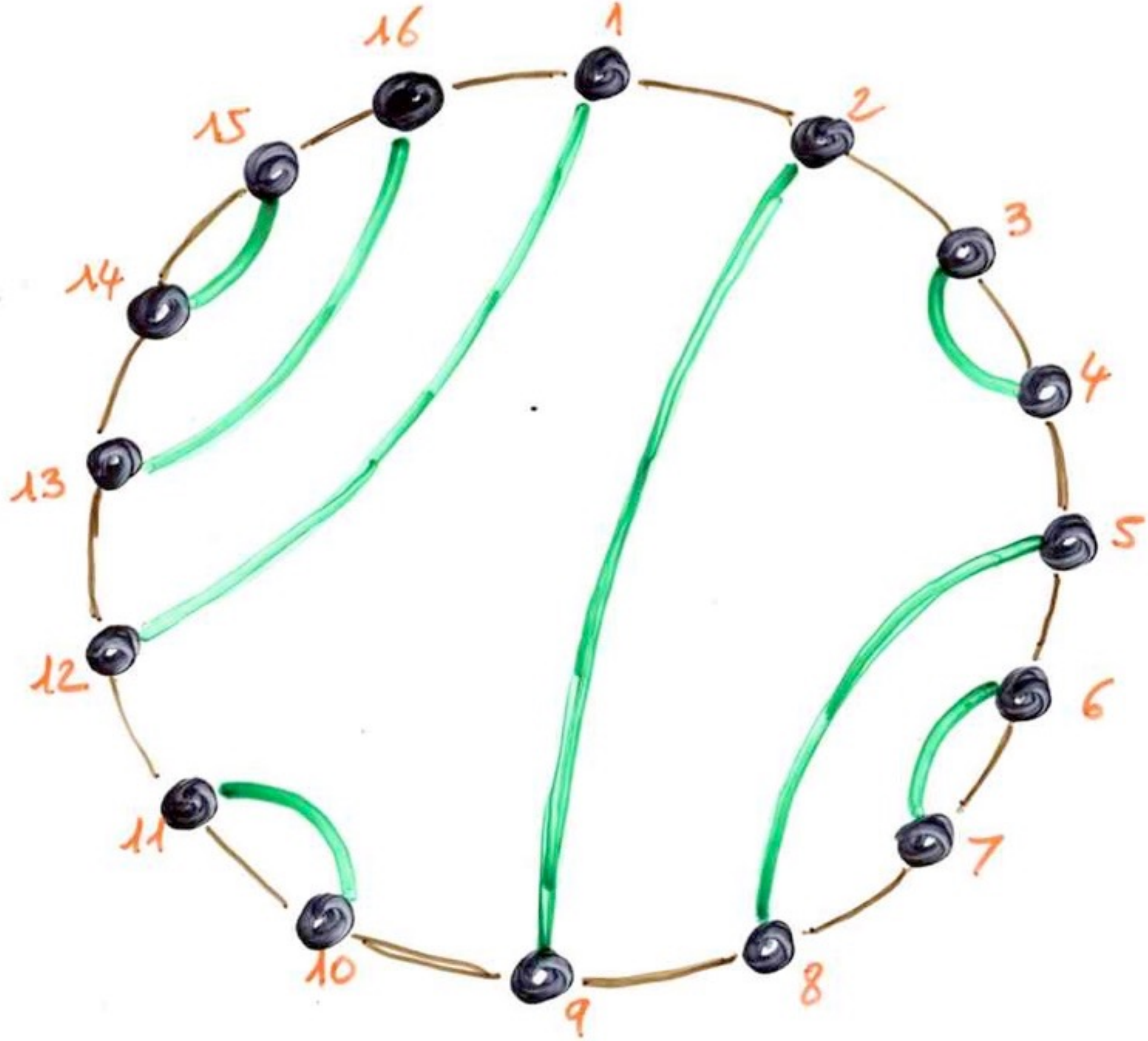
(P.Duchon)

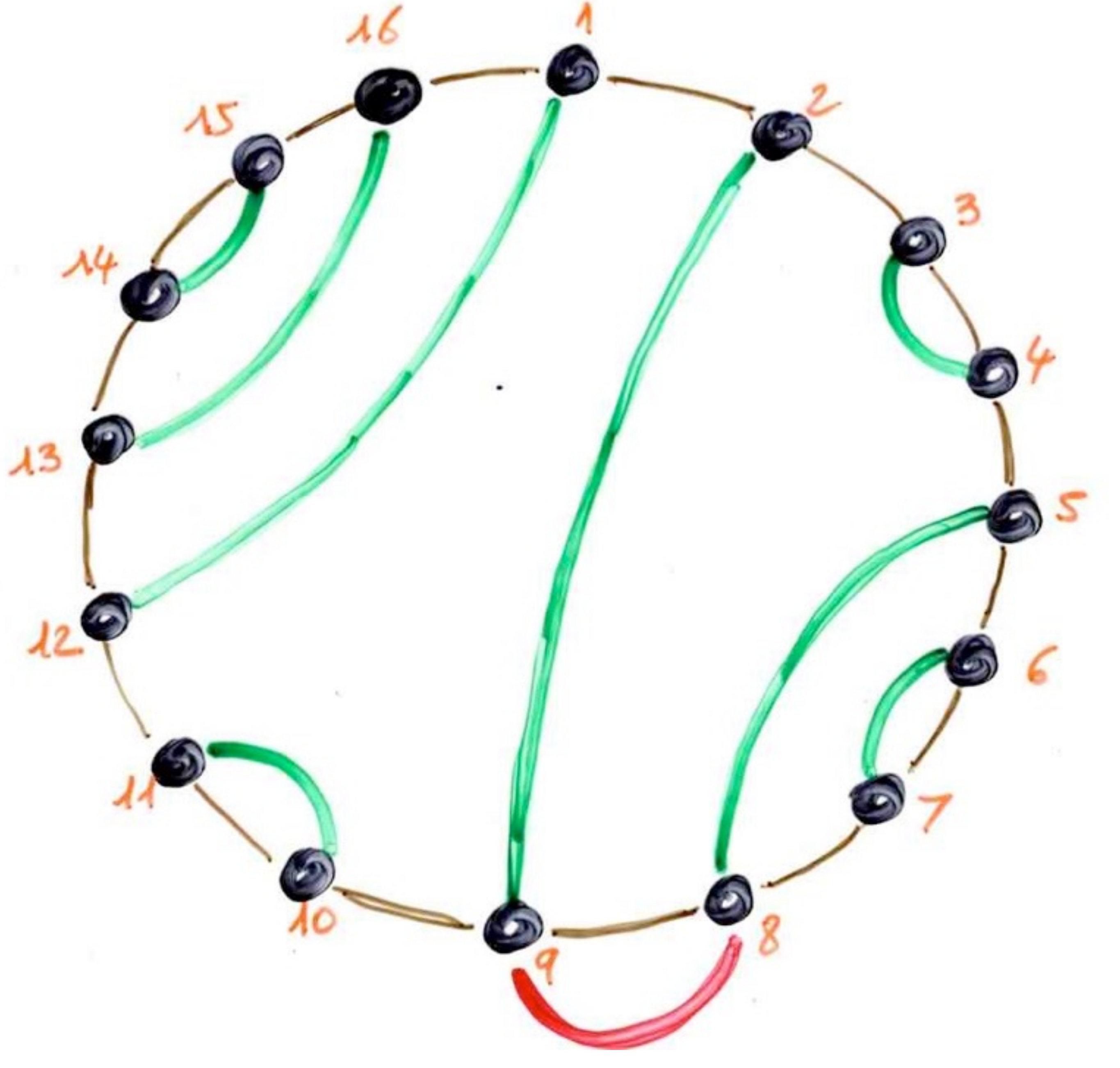


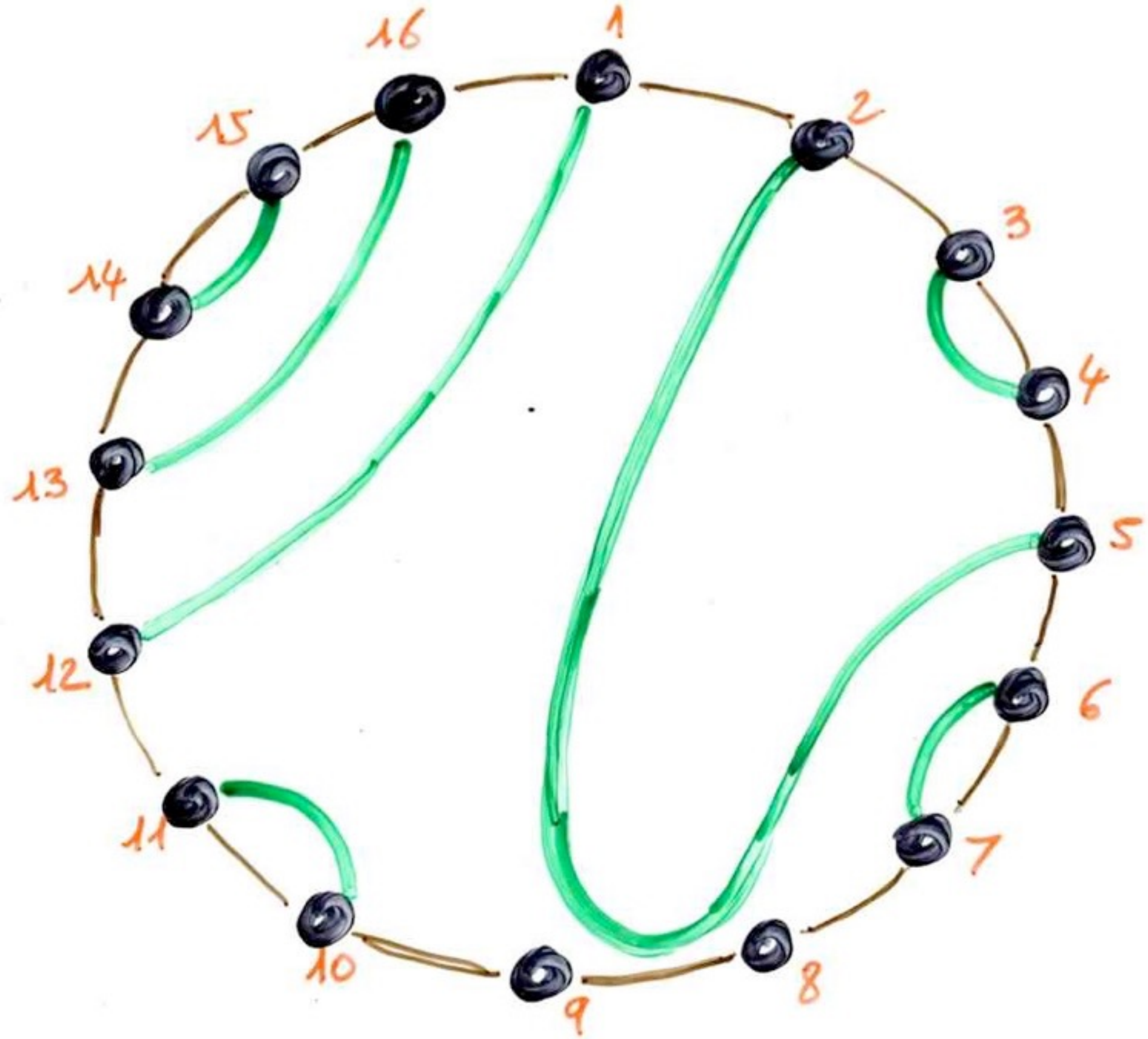
Razumov - Stroganov
(ex) - conjecture

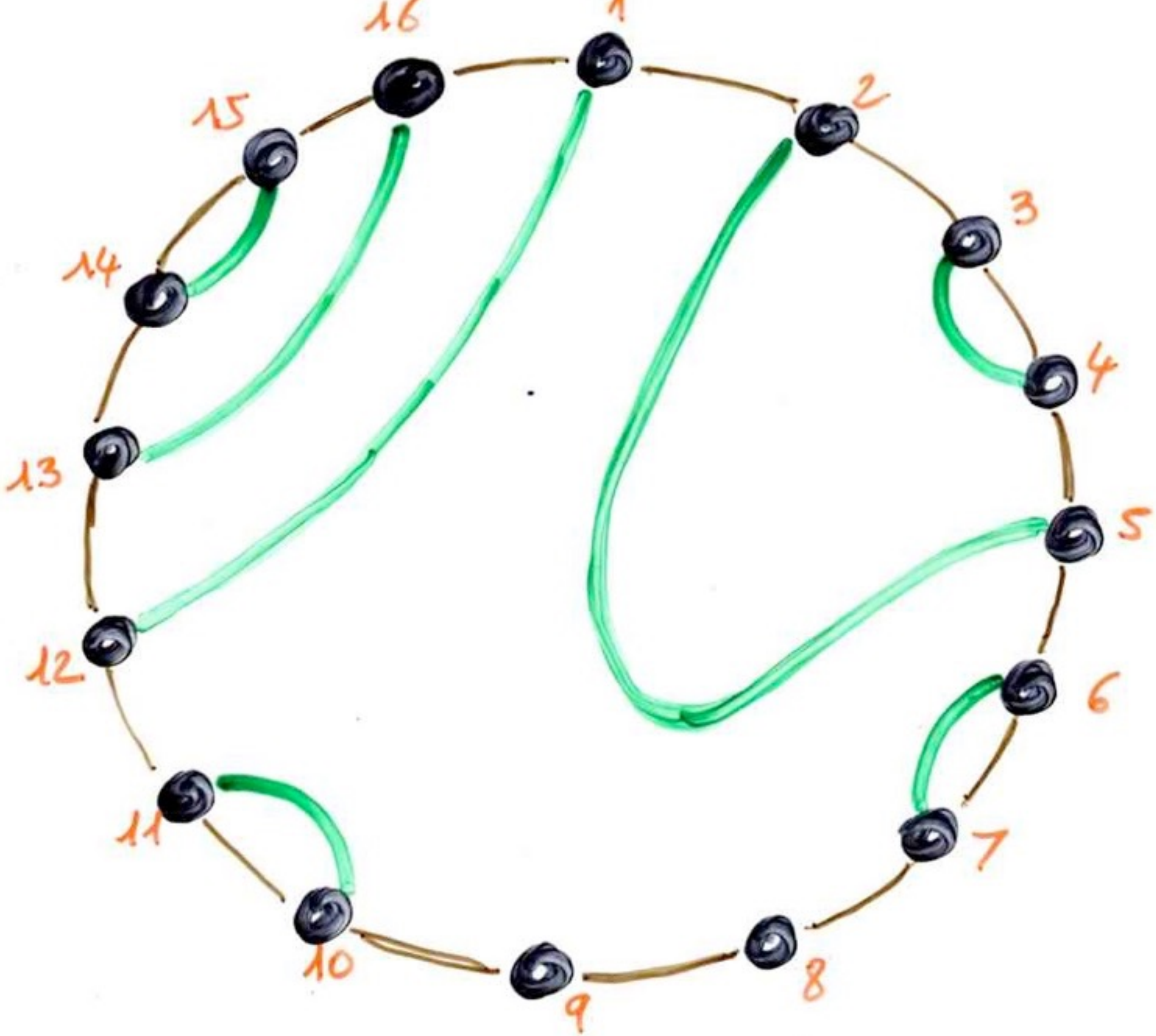



1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

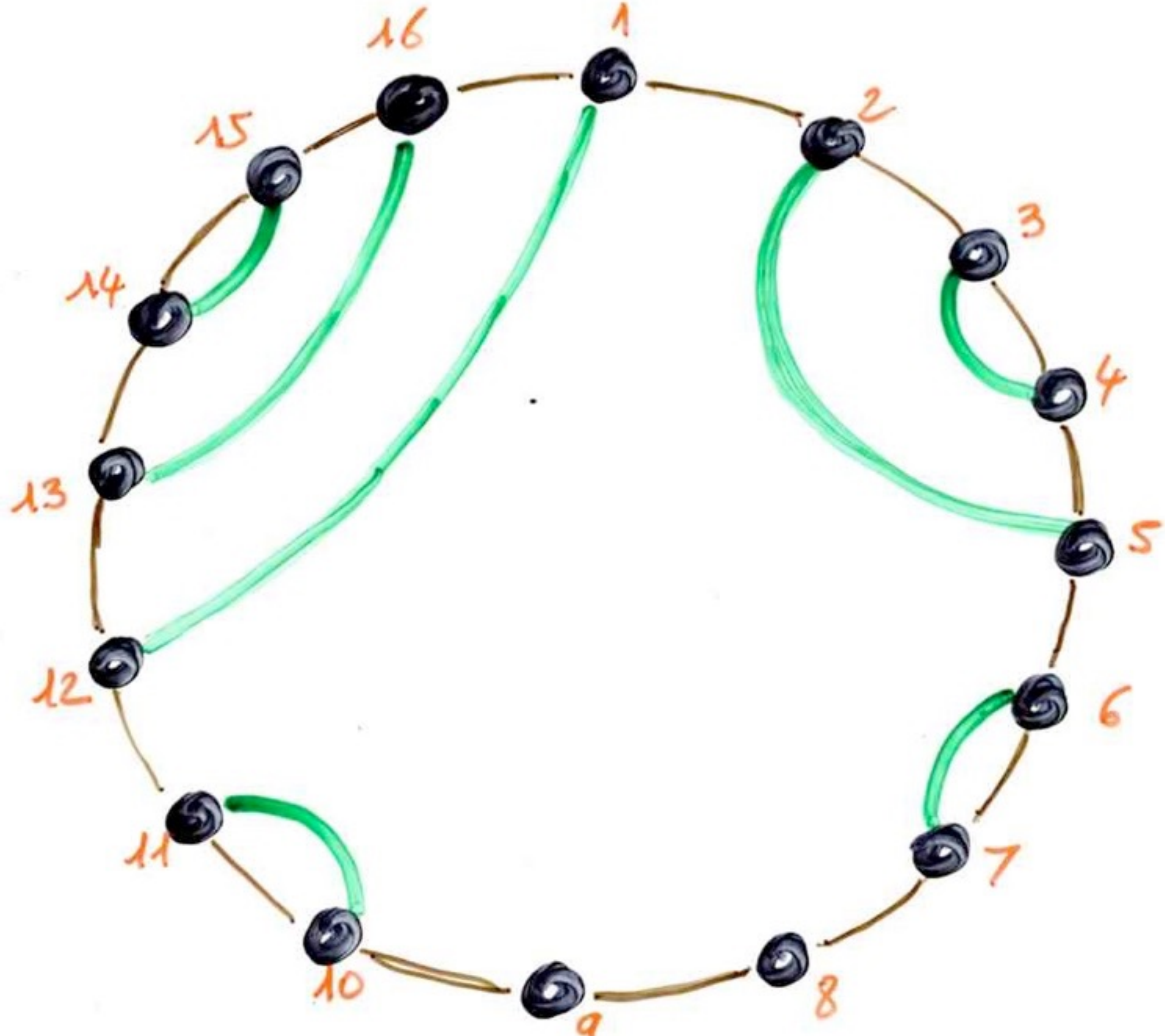

16
1
2
15
3
14
4
13
5
12
6
11
7
10
8
9


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

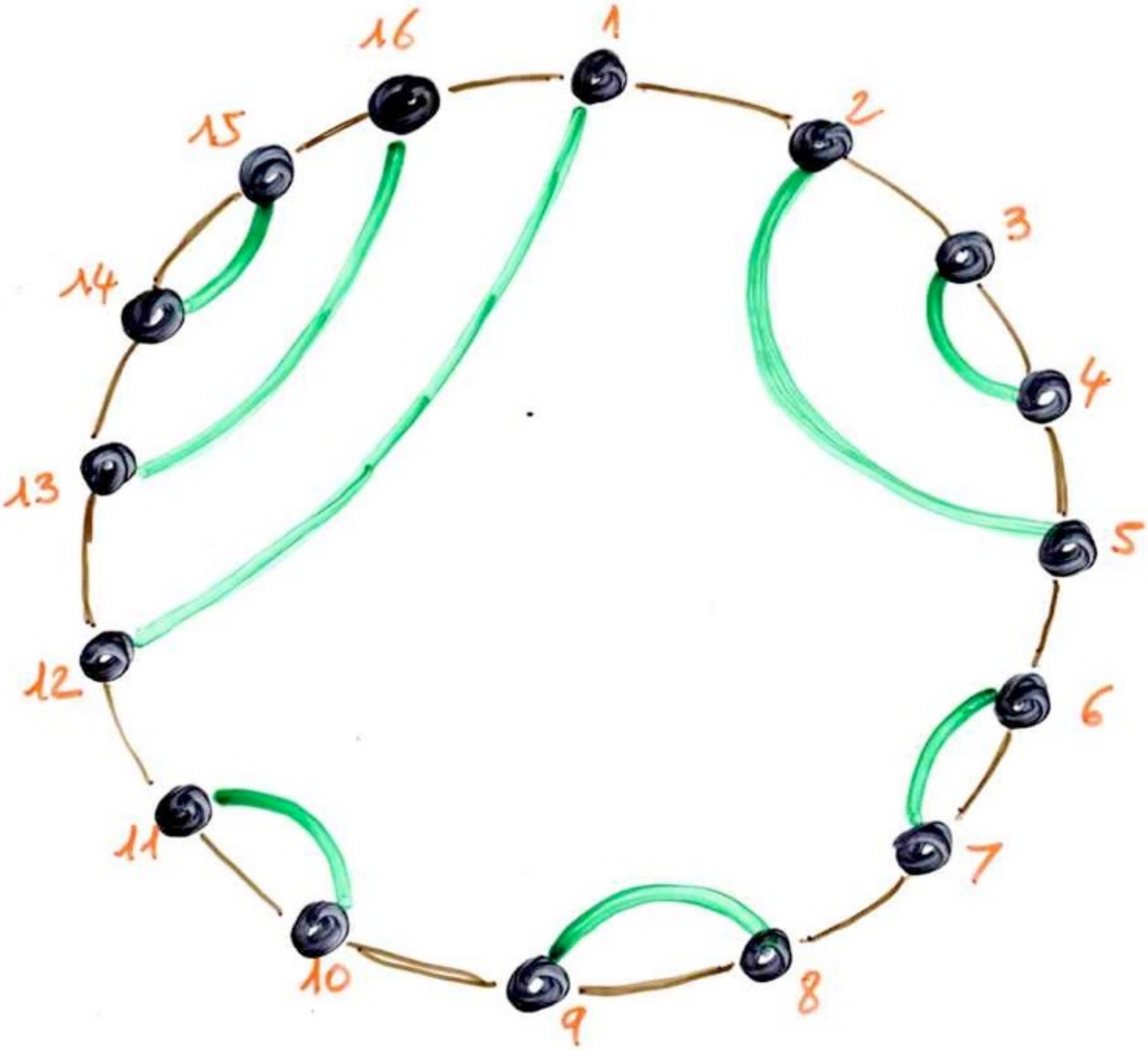

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16



1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

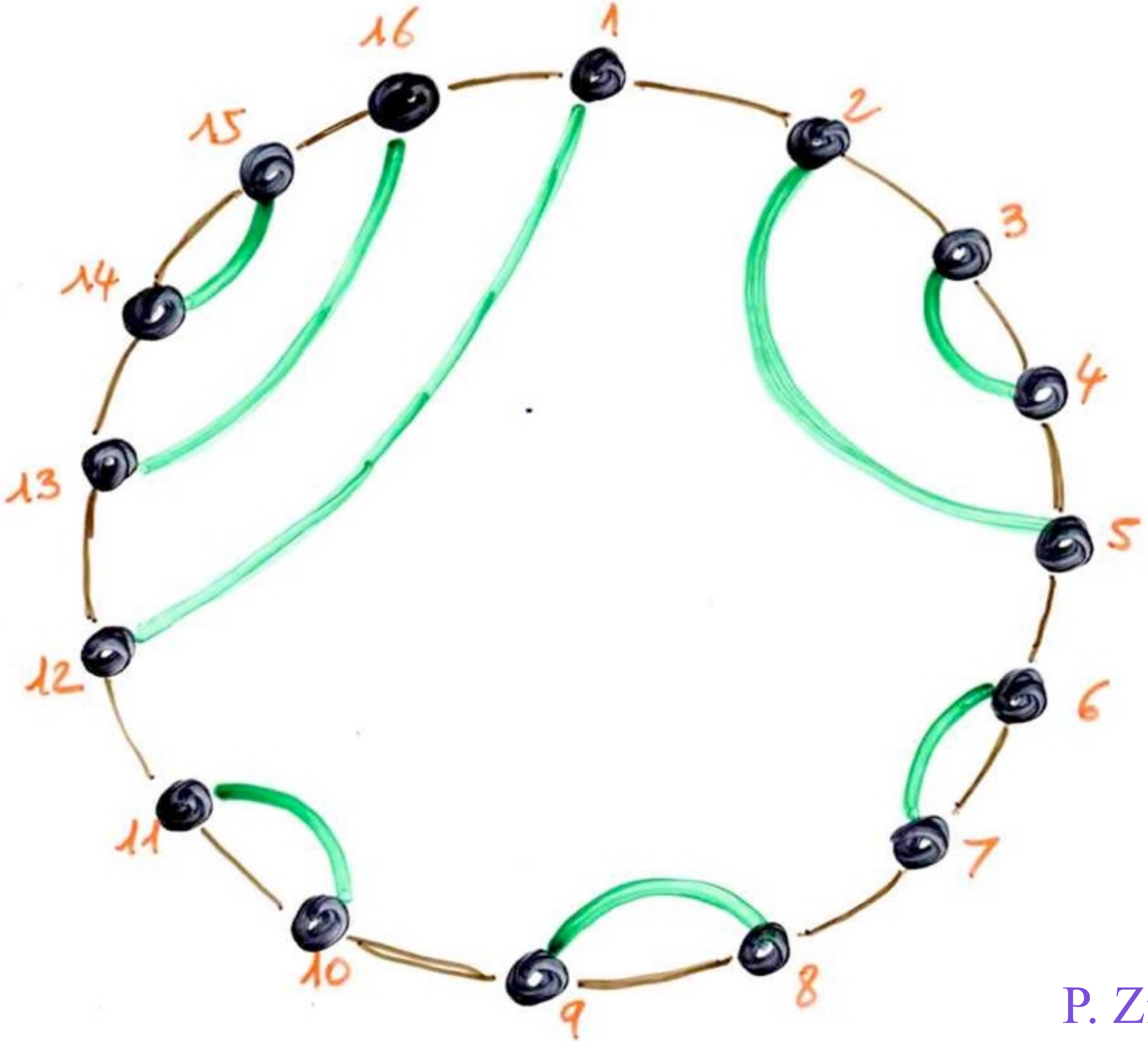


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16



1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16

Razumov-Stroganov conjecture



stationary probabilities

Di Francesco,
P. Zinn-Justin (2005)

Razumov - Stroganov (ex) - conjecture

proof by :
L. Cantini and A.Sportiello (March 2010)
arXiv: 1003.3376 [math.CO]
based on «Wieland rotation»
completely combinatorial proof

Philippe Di Francesco, Paul Zinn-Justin (2005 - 2009)

Knizhnik - Zamolodchikov
equation

qKZ

ASM

Around the Razumov-Stroganov conjecture

Philippe Di Francesco, Paul Zinn-Justin (2005 - 2009)

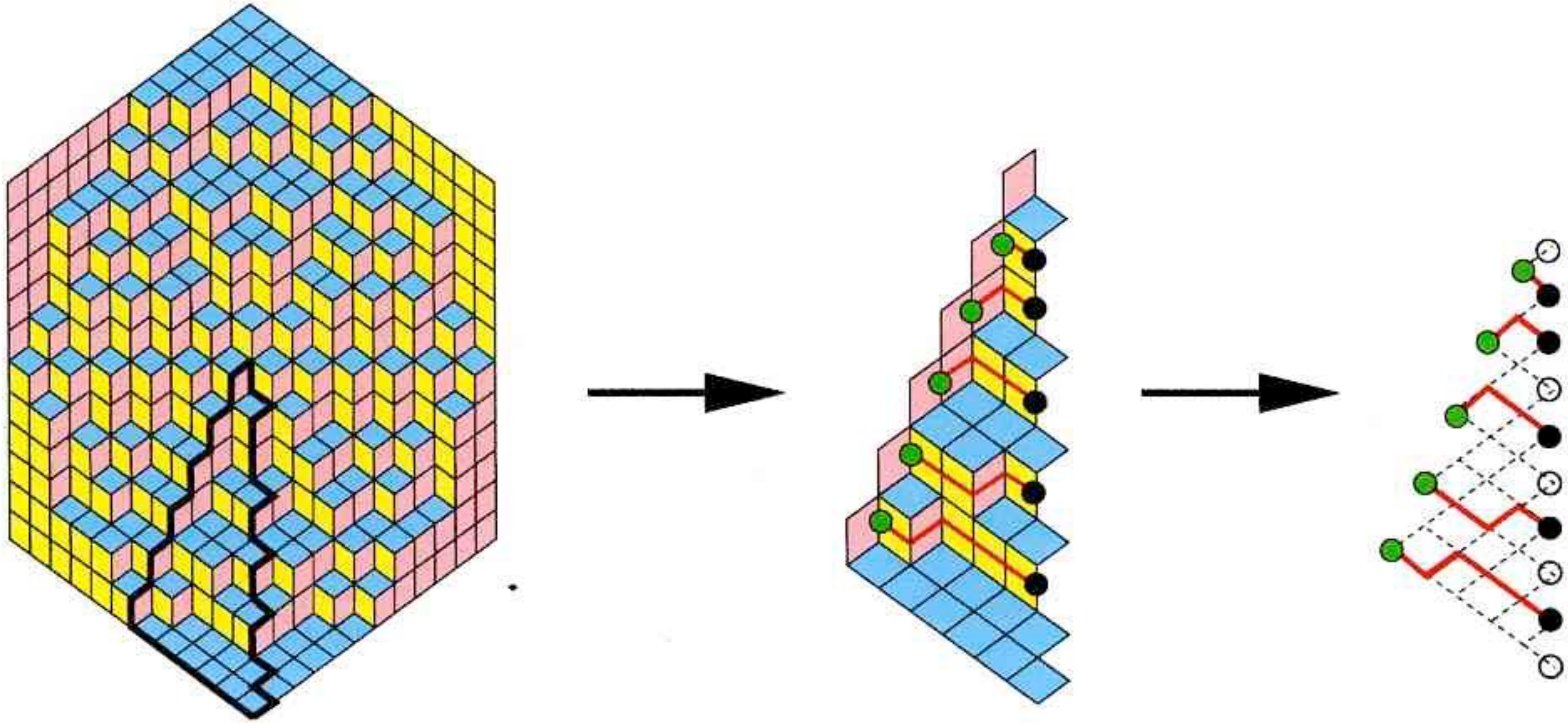
De Gier, Pyatov (2007)

Knizhnik - Zamolodchikov
equation

qKZ

TSSCPP

ASM



S. Dulucq (1985)
Di Francesco (2006)

ASM

1-, 2-, 3- enumeration $A_n(x)$

Colomo, Pronco, (2004)

Hankel determinants

(continuous) Hahn, Meixner-Pollaczek,
(continuous) dual Hahn orthogonal polynomials

Ismail, Lin, Roan (2004)
XXZ spin chains and Askey-Wilson operator

Schubert and Grothendick polynomials
Lascoux, Schützenberger

correlations functions in XXZ spin chains

LPENSL-TH-05

Exact results for the σ^z two-point function of the XXZ chain at $\Delta = 1/2$

N. Kitanine[1], **J. M. Maillet**[2], **N. A. Slavnov**[3], **V. Terras**[4]

Abstract

We propose a new multiple integral representation for the correlation function $\langle\sigma_1^z\sigma_{m+1}^z\rangle$ of the XXZ spin-$\frac{1}{2}$ Heisenberg chain in the disordered regime. We show that for $\Delta = 1/2$ the integrals can be separated and computed exactly. As an example we give the explicit results up to the lattice distance $m = 8$. It turns out that the answer is given as integer numbers divided by $2^{(m+1)^2}$.

[1]LPTM, UMR 8089 du CNRS, Université de Cergy-Pontoise, France, kitanine@ptm.u-cergy.fr
[2]Laboratoire de Physique, UMR 5672 du CNRS, ENS Lyon, France, maillet@ens-lyon.fr
[3]Steklov Mathematical Institute, Moscow, Russia, nslavnov@mi.ras.ru
[4]LPTA, UMR 5207 du CNRS, Montpellier, France, terras@lpta.univ-montp2.fr

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

intergers ?

positivity ?

$$
\begin{aligned}
&P_1(\kappa) = 1 + \kappa,\\
&P_2(\kappa) = 2 + 12\kappa + 2\kappa^2,\\
&P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,\\
&P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4,\\
&P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,\\
&P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4\\
&+96289380\kappa^5 + 7436\kappa^6,\\
&P_7(\kappa) = 218348 + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3\\
&+265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + 218348\kappa^7,\\
&P_8(\kappa) = 10850216 + 8485108350684\kappa + 39461894378292782\kappa^2\\
&+3224112384882251896\kappa^3 + 11919578544950060460\kappa^4 + 3224112384882251896\kappa^5\\
&+39461894378292782\kappa^6 + 8485108350684\kappa^7 + 10850216\kappa^8\\
&P_9(\kappa) = 911835460 + 5649499685353257\kappa + 177662495637443158524\kappa^2\\
&+77990624578576910368767\kappa^3 + 1130757526890914223990168\kappa^4
\end{aligned}
\qquad (12)
$$

e^{2z_j}, it reduces to the derivatives of order $m-1$ with respect to each x_j at $x_1 = \cdots = x_n = e^{\frac{i\pi}{3}}$ and $x_{n+1} = \cdots = x_m = e^{-\frac{i\pi}{3}}$. If the lattice distance m is not too large, the representations (9), (11) can be successfully used to compute $\langle Q_\kappa(m) \rangle$ explicitely. As an example we give below the list of results for $P_m(\kappa) = 2^{m^2} \langle Q_\kappa(m) \rangle$ up to $m = 9$:

po

intergers ?

positivity ?

FPL

ASM

combinatorial interpretation

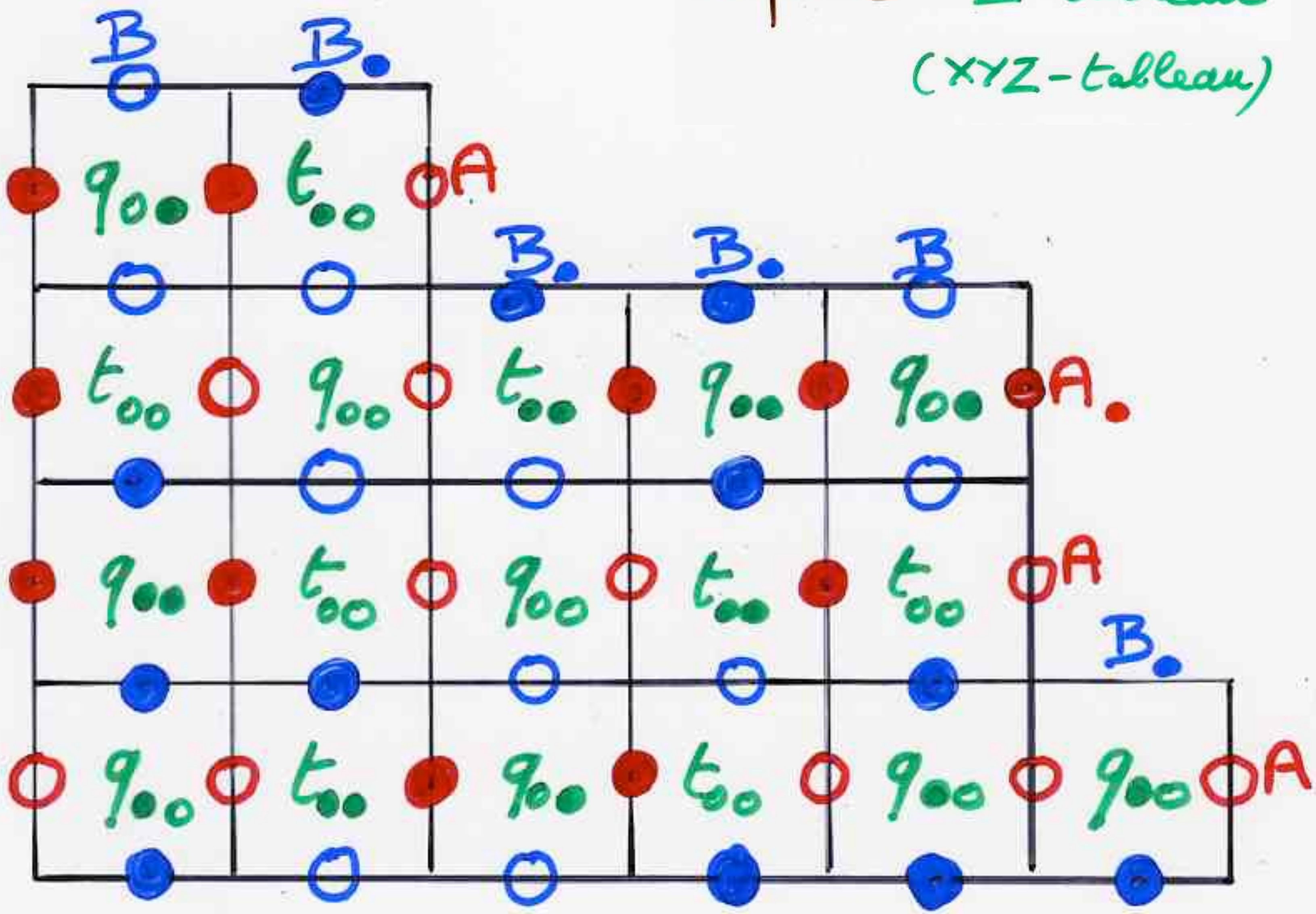
?

$$
\begin{aligned}
P_1(\kappa) &= 1 + \kappa, \\
P_2(\kappa) &= 2 + 12\kappa + 2\kappa^2, \\
P_3(\kappa) &= 7 + 249\kappa + 249\kappa^2 + 7\kappa^3, \\
P_4(\kappa) &= 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4, \\
P_5(\kappa) &= 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5, \\
P_6(\kappa) &= 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 \\
&+ 96289380\kappa^5 + 7436\kappa^6, \\
P_7(\kappa) &= 218348 + 21798199390\kappa + 15663567546585\kappa^2 + 265789610746333\kappa^3 \\
&+ 265789610746333\kappa^4 + 15663567546585\kappa^5 + 21798199390\kappa^6 + 218348\kappa^7, \\
P_8(\kappa) &= 10850216 + 8485108350684\kappa + 39461894378292782\kappa^2 \\
&+ 3224112384882251896\kappa^3 + 11919578544950060460\kappa^4 + 3224112384882251896\kappa^5 \\
&+ 39461894378292782\kappa^6 + 8485108350684\kappa^7 + 10850216\kappa^8 \\
P_9(\kappa) &= 911835460 + 5649499685353257\kappa + 177662495637443158524\kappa^2 \\
&+ 77990624578576910368767\kappa^3 + 1130757526890914223990168\kappa^4
\end{aligned}
\tag{12}
$$

(XYZ)-tableaux
and
B.A.BA configurations
(or XYZ- configurations)

complete Z-tableau

(XYZ-tableau)



The quadratic algebra $\mathbb{Z}$

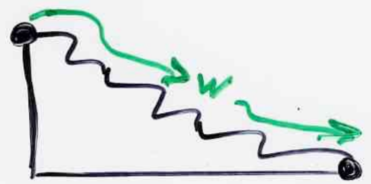
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

$$
\begin{cases}
B A = \square \, A B + \blacksquare \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = \square \, A_\bullet B_\bullet + \blacksquare \, A B \\
B_\bullet A = \square \, A B_\bullet + \blacksquare \, A_\bullet B \\
B A_\bullet = \square \, A_\bullet B + \blacksquare \, A B_\bullet
\end{cases}
$$

Configurations B.A.BA on a Ferrers diagram F

word $w \in \{B_\bullet, A_\bullet, B, A\}^*$ $\longrightarrow$ diagram $F(w)$



$w = w_1 \dots w_m$

i^{th} step of

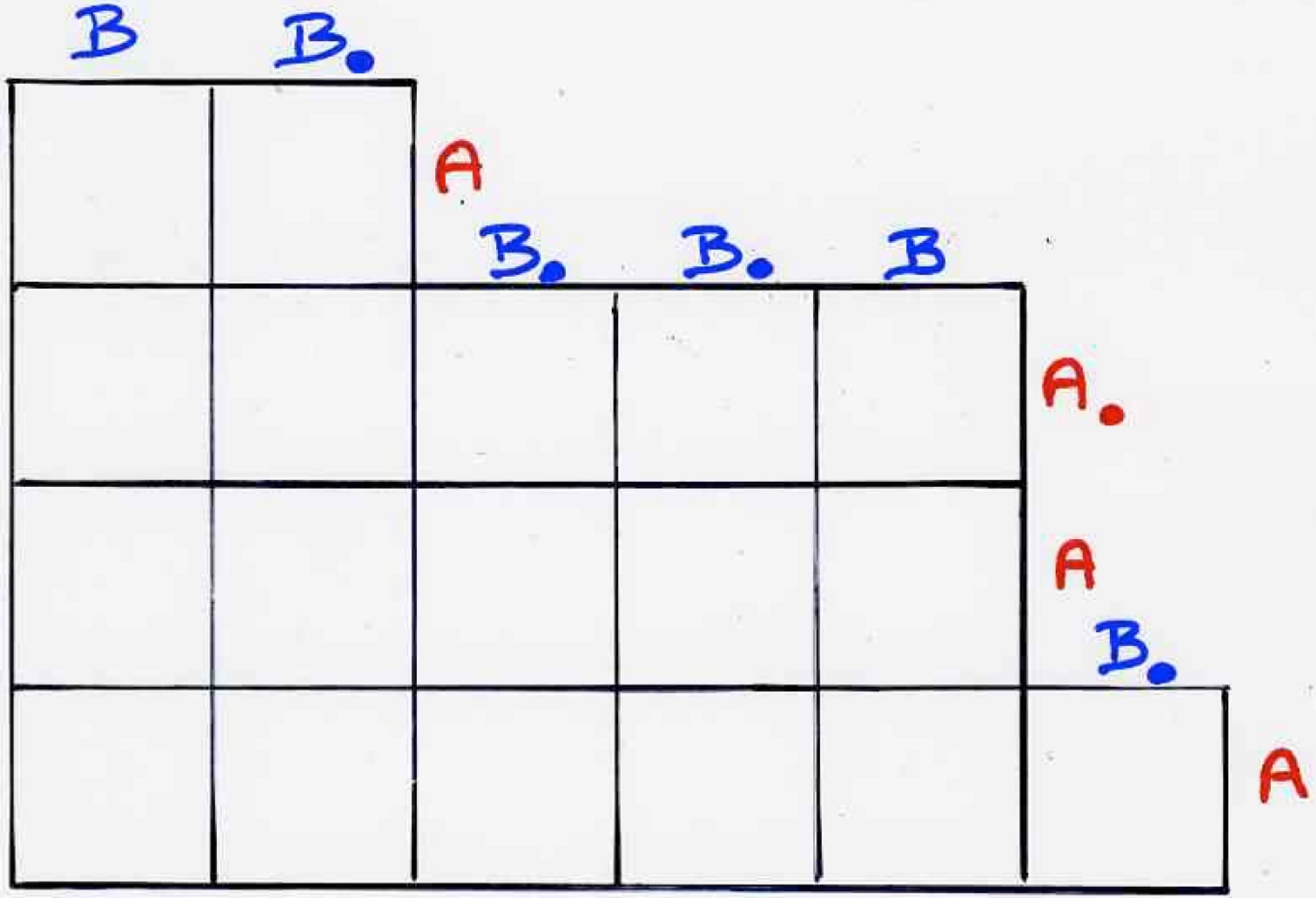
is $=$ if $w_i = B_\bullet, B$

$\|$ if $w_i = A_\bullet, A$

Bijection(s)

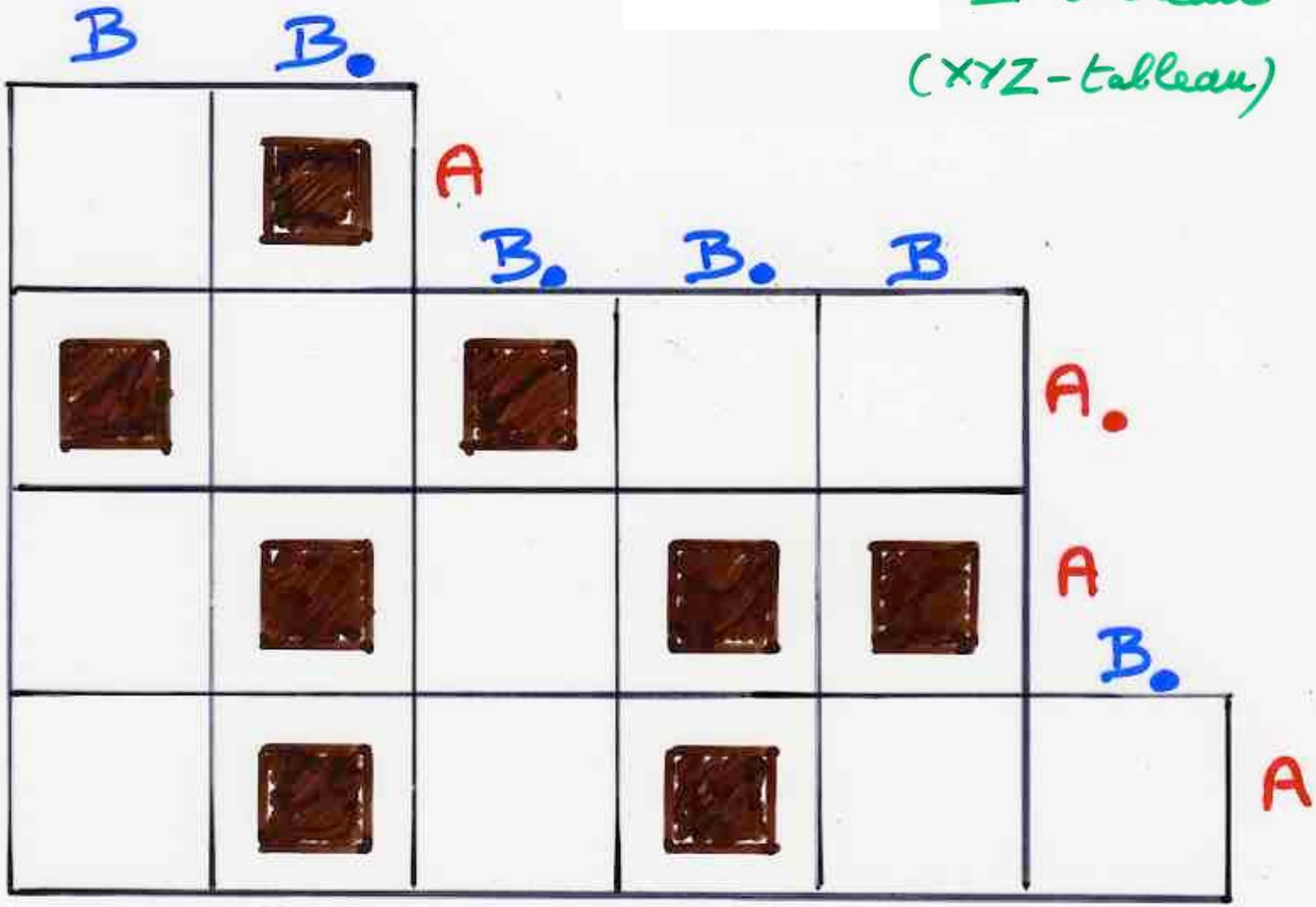
(word w, C) $\longleftrightarrow$ T complete Z-tableau (with diagram $F(w)$)

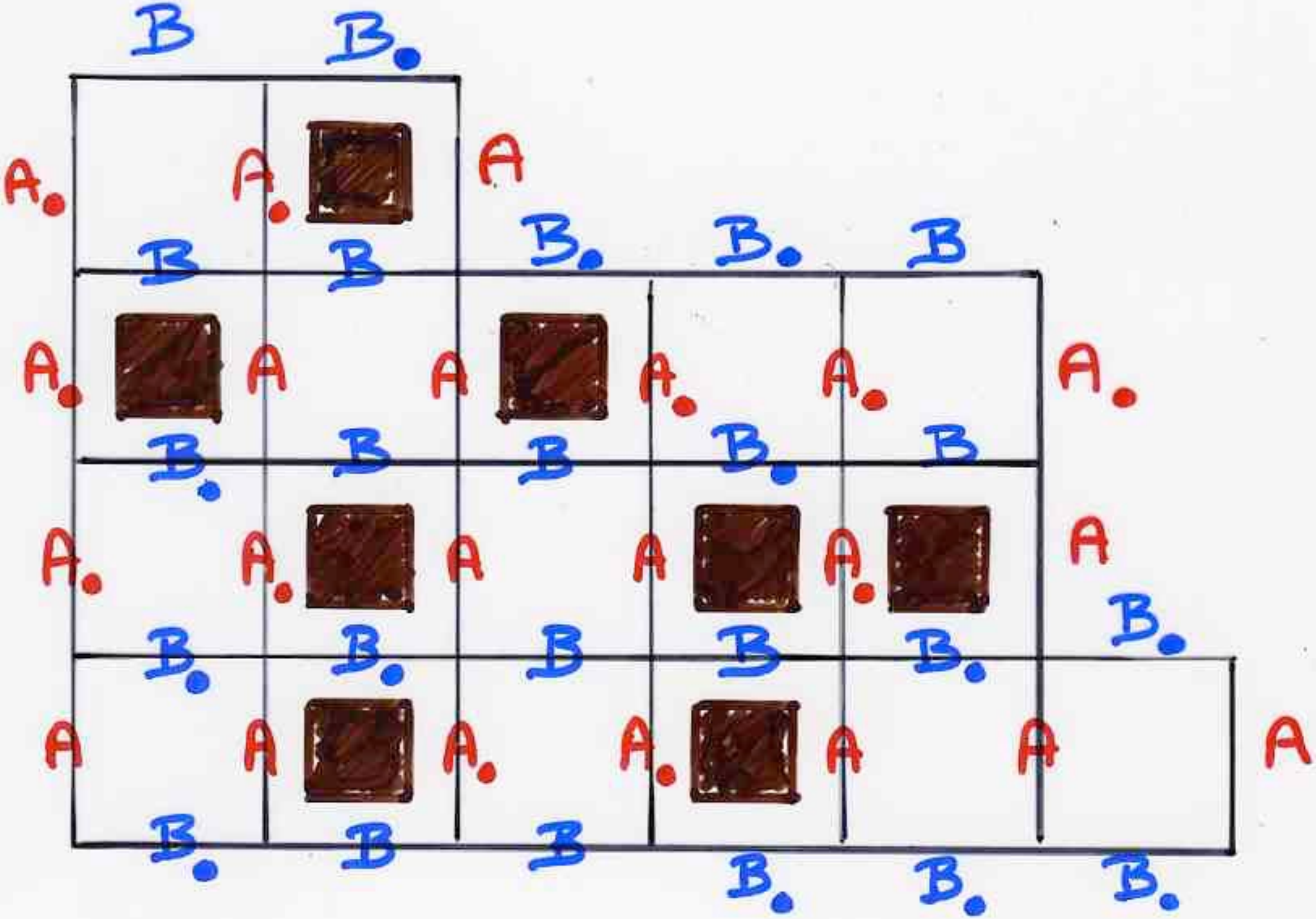
C: B.A.BA configuration on the diagram $F(w)$

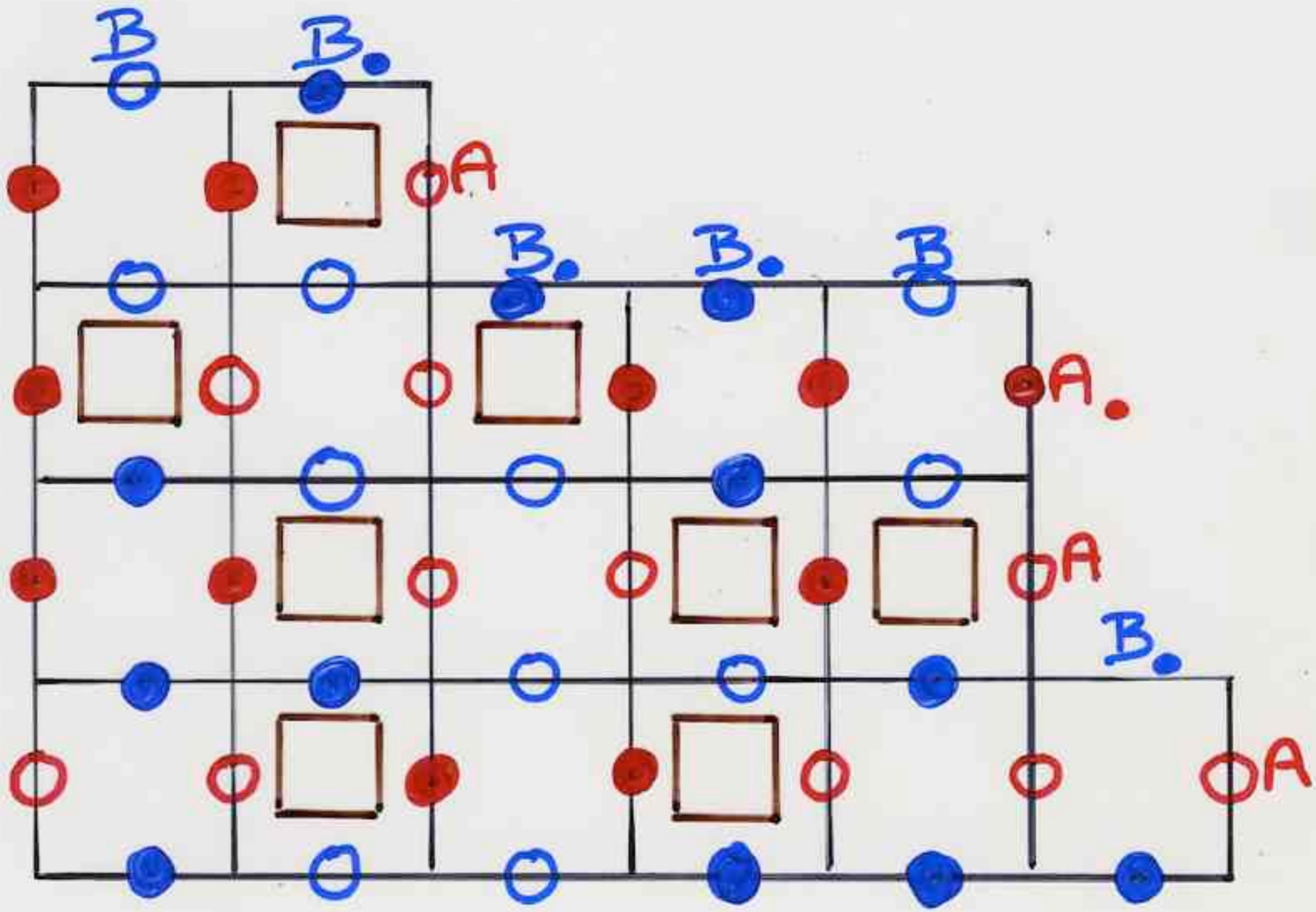

B
B
A
B
B
B
A
A
B
A

Z-tableau

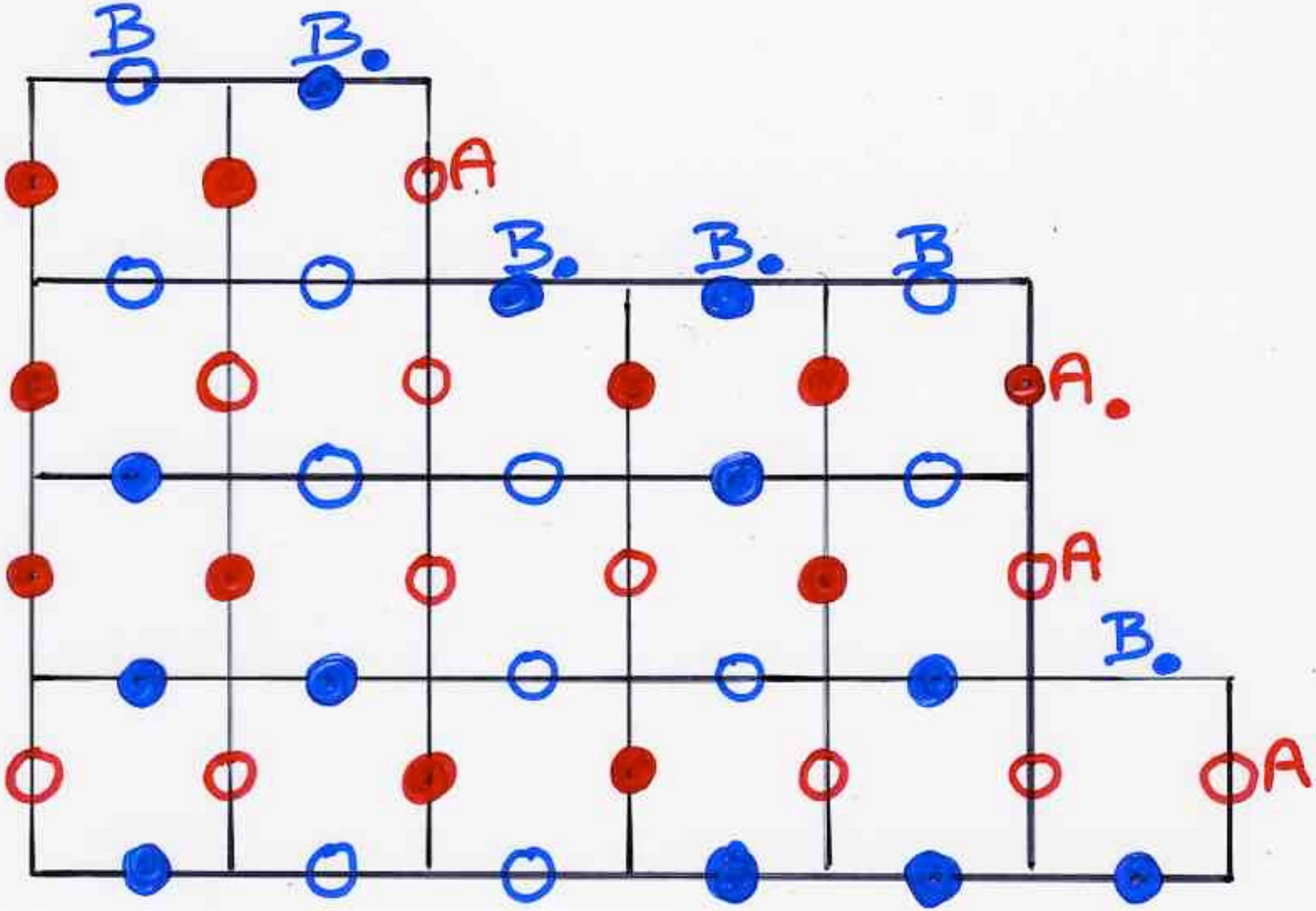
(XYZ-tableau)

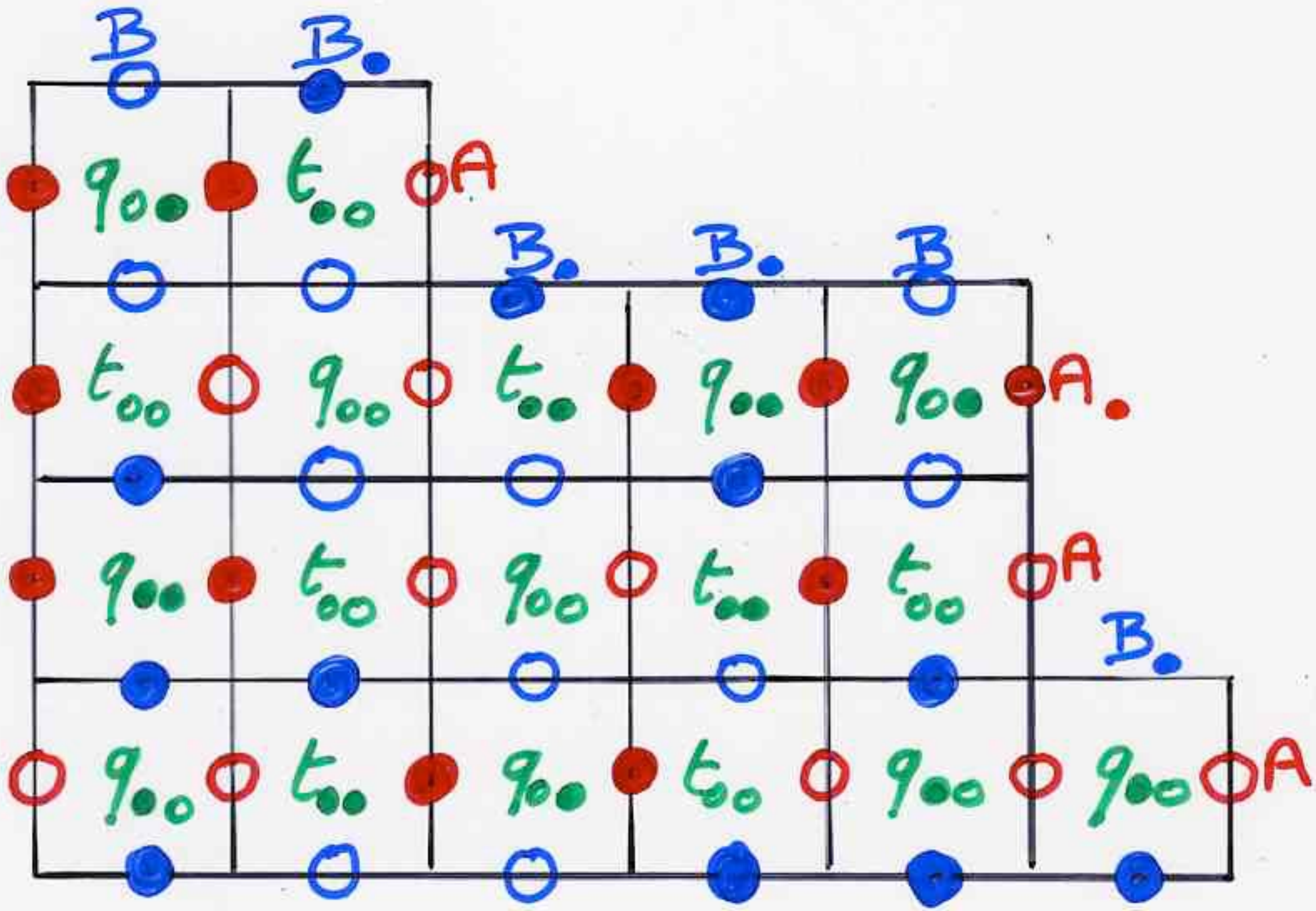



A
B



B
B
A
B
B
B
A
A
B
A


B
B
A
B
B
B
A
A
B
A


B
B
A
B
B
B
A
A
B
A

8 - vertex model

XYZ- spin chains model

analog of

Razumov - Stroganov conjecture ?

"The cellular Ansatz"

Physics

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

DE = qED + E + D

PASEP

dynamical systems in physics
stationary probabilities

quadratic algebra Q

commutations
rewriting rules

planarization

combinatorial
objects
on a 2d lattice

rooks placements

permutations

alternative tableaux

tree-like tableaux

reverse Q-tableaux

Q-tableaux

the XYZ algebra

ASM, (alternating sign matrices)

FPL (Fully packed loops)

tilings, non-crossing paths

planar
automata

representation
by operators

data structures
"histories"

orthogonal
polynomials

bijections

RSK

permutations ⟷ pairs of Tableaux Young

alternative tableaux ⟷ permutations

Laguerre histories

demultiplication
of equations
in algebra Q

RSK automata

reverse planar
automata

bijection
BABA - pair (P,Q)

the very end of the course ...

thank you very much !