Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics, a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc

January-March 2017

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Chapter 4

Linear algebra revisited with heaps of pieces

(1)

IMSc, Chennai

2 February 2017

combinatorial (bijective) proofs
of classical theorem
in **linear algebra**

- MacMahon "master theorem"
  Cartier-Foata (1969)
- Matrix inversion
  Foata (1979)
- Jacobi identity (log det)
  Jackson (1977)
  Foata (1980)

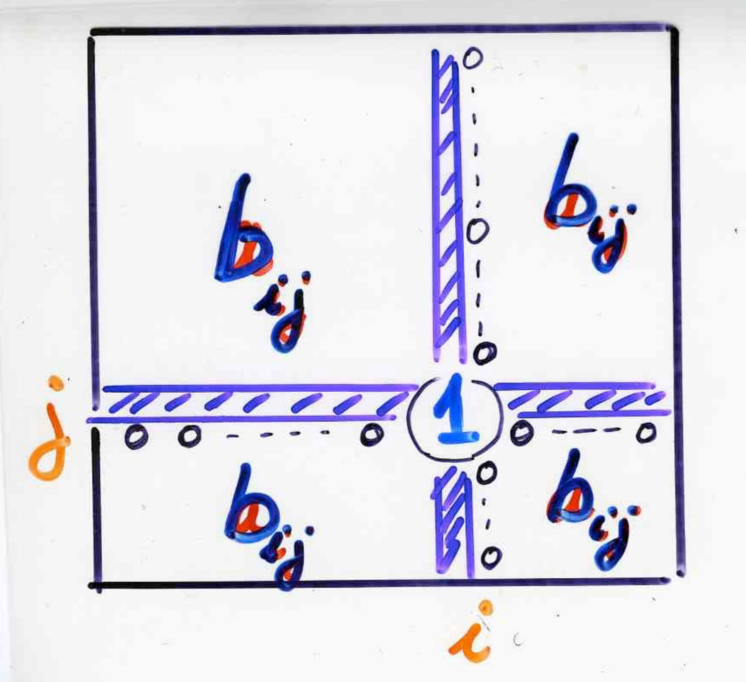
- Cayley-Hamilton theorem
  Straubing (1983)
  Zeilberger (1985)
- Jacobi identity (duality)
  Lalonde (1990, 1996)
  Fomin (2001), Talaska (2012)

Inversion of a matrix

$$\left(B^{-1}\right)_{ij} = \frac{\mathrm{cof}_{ji}(B)}{\det(B)}$$

$$\det(B) = \sum_{\substack{\sigma \\ \text{permutation} \\ \text{of } \mathfrak{S}_k}} (-1)^{\mathrm{inv}(\sigma)}\, b_{1,\sigma(1)} \cdots b_{k,\sigma(k)}$$

$$\mathrm{cof}_{ji} = (-1)^{i+j}\, \mathrm{minor}_{ji}(B)$$



$$B = I - A$$

$$A = (a_{ij})_{1 \leq i, j \leq k}$$

$$(I - A)^{-1} = I + A + \dots + A^n + \dots$$

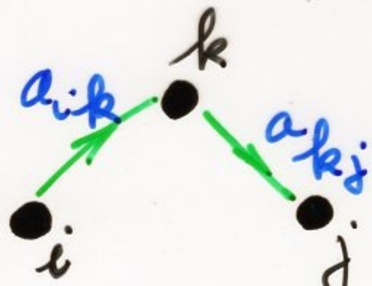
Lemma $X = \{1, 2, \dots, k\}$

$A = (a_{i,j})$ $n \times n$ matrix

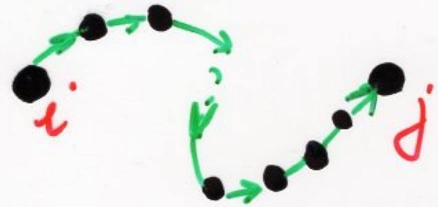
$$(I-A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

with $v(i,j) = a_{i,j}$

$(A^2)_{ij}$



$(A^m)_{ij}$



$|\omega| = m$

$$(A^m)_{ij} = \sum_{|\omega|=m} v(\omega)$$

Bijection

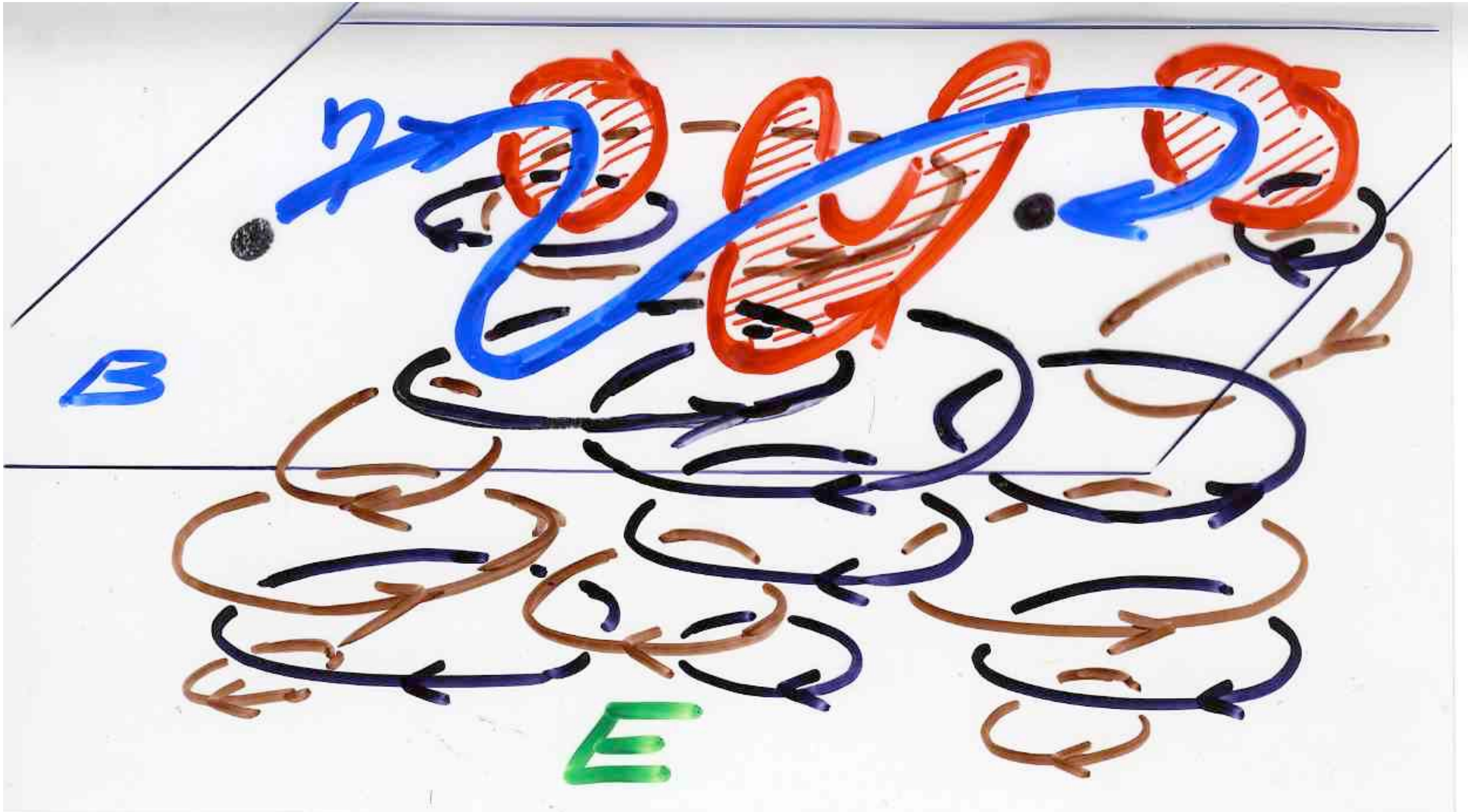
$u, v \in X$

path ω on X going from u to v $\overset{\chi}{\longleftrightarrow}$ (η, E)

- η self-avoiding path going from u to v
- E heap of cycles such that the projections $\alpha = \pi(m)$ of the maximal pieces intersect η ($\underset{\text{cycle}}{\alpha}$ and $\underset{\text{path}}{\eta}$ has a common vertex)

$$v(\omega) = v(\eta)\, v(E)$$

The bijection χ



path ω on X $\xleftrightarrow{\chi}$ (η, E)

$$\sum_{\substack{\omega \\ i \to j}} v(\omega) = \sum_{\substack{\eta \\ i \to j}} v(\eta) \sum_{E} v(E)$$

self-avoiding

heap of cycles

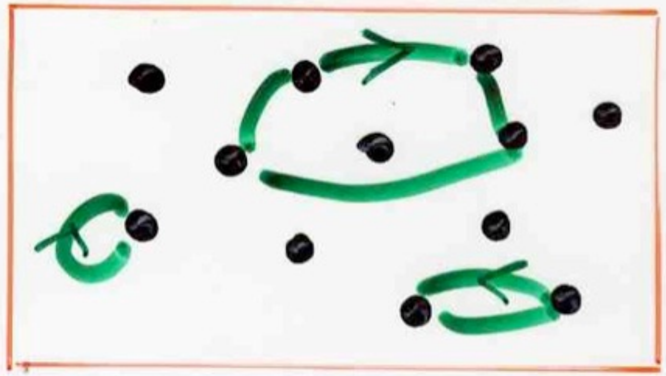
$\pi(m)$ maximal piece intersects η

$$\frac{N_\eta}{D}$$

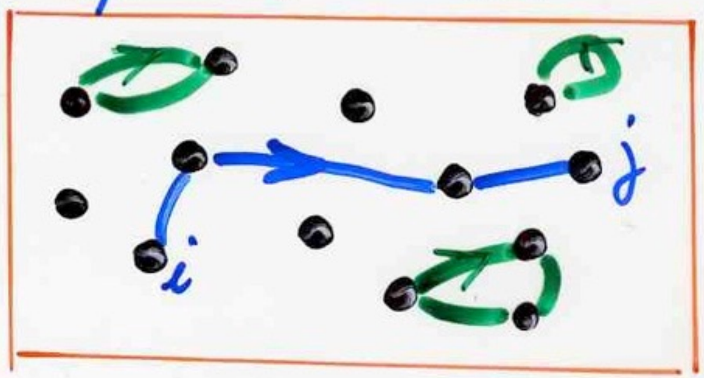
inversion lemma

$$\frac{N_\eta}{D}$$

$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint} \\ \text{cycles}}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$



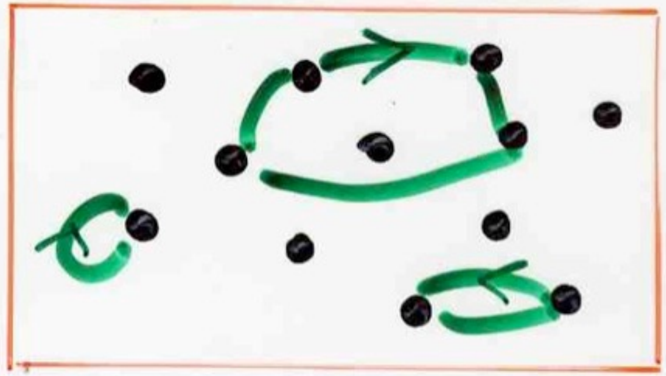
$$N_\eta = \sum_{\{\gamma_1, \dots, \gamma_r\}} (-1)^r \, v(\eta) \, v(\gamma_1) \cdots v(\gamma_r)$$



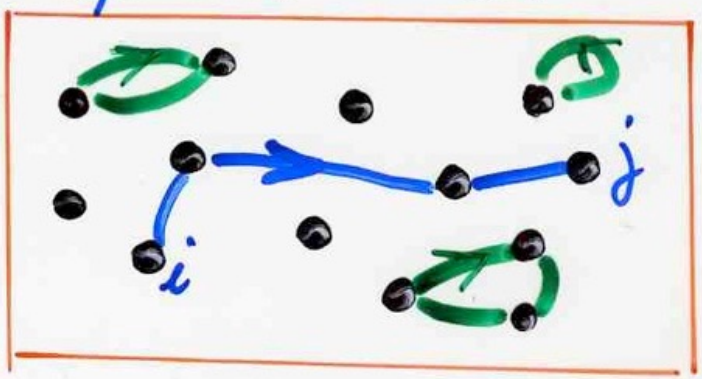
Prop. $\displaystyle\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$ $\qquad$ $\displaystyle N_{ij} = \sum_{\eta} v(\eta) N_\eta$

η self-avoiding path $i \rightsquigarrow j$

$$D = \sum_{\substack{\{\gamma_1, \ldots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$



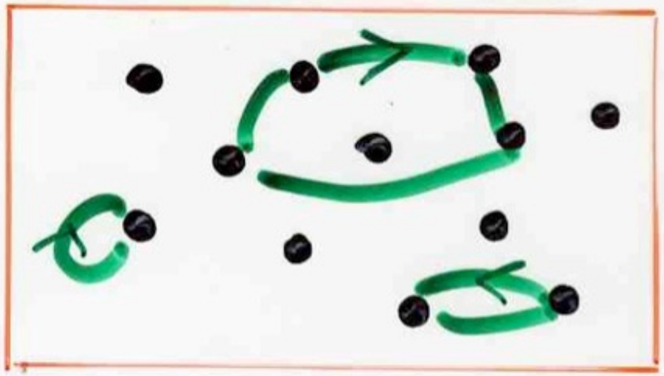
$$N_{ij} = \sum_{\{\eta; \gamma_1, \ldots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



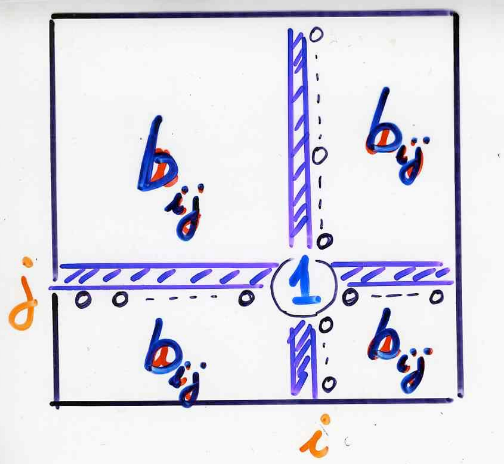
$$\det(B) = \sum_{\substack{\sigma \\ \text{permutation} \\ \text{of } \mathfrak{S}_k}} (-1)^{\mathrm{inv}(\sigma)}\, b_{1,\sigma(1)} \cdots b_{k,\sigma(k)}$$

$B = I - A$ easy exercise !

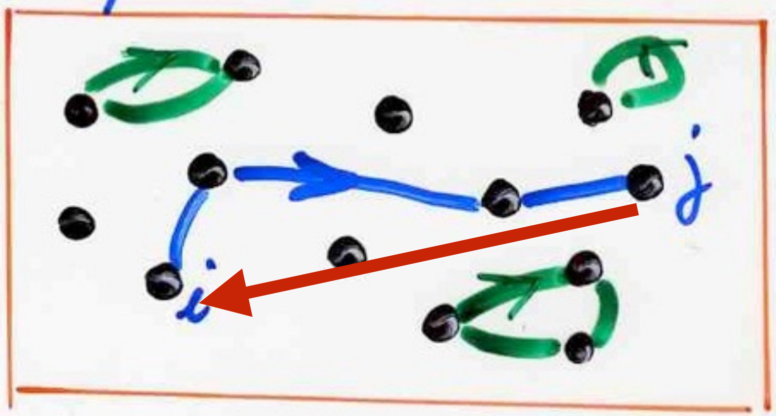
$$\det(I_k - A) = \sum_{\substack{\{\gamma_1, \ldots, \gamma_r\} \\ \text{2 by 2 disjoint} \\ \text{cycles}}} (-1)^r\, v(\gamma_1) \cdots v(\gamma_r)$$



$\mathrm{cof}_{ji}(I_k - A)$



$$N_{ij} = \sum_{\{\eta;\gamma_1, \ ,\gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



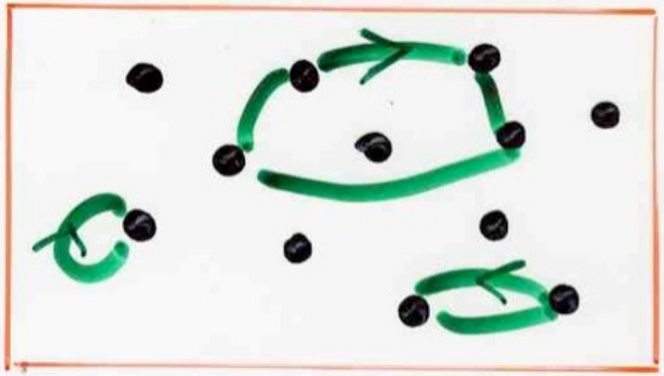
η self-avoiding path $i \rightsquigarrow j$

$\{\gamma_1, .., \gamma_r\}$
2 by 2 disjoint cycles, and disjoint from η

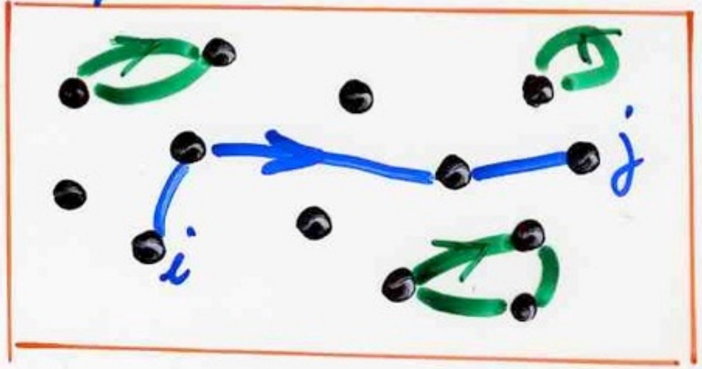
Prop. $\displaystyle\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$ $\qquad N_{ij} = \displaystyle\sum_{\eta} v(\eta) N_\eta$

η self-avoiding path $i \rightsquigarrow j$

$$D = \sum_{\substack{\{\gamma_1, \ldots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$



$$N_{ij} = \sum_{\{\eta;\gamma_1, \ldots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



Transition matrix methodology in Physics examples

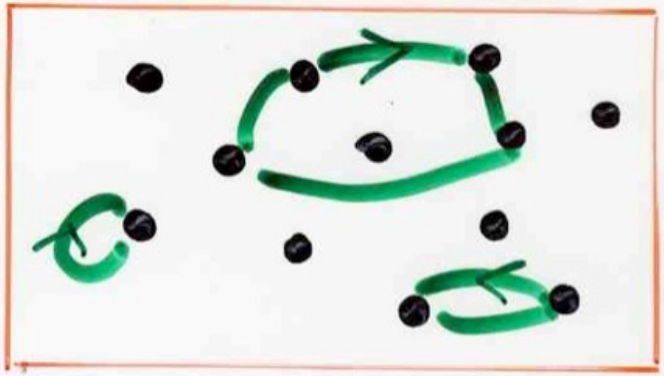
bounded Dyck paths

Prop. $$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$$

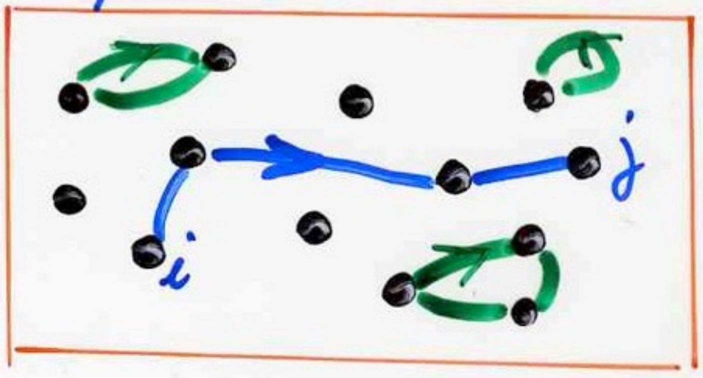
$$N_{ij} = \sum_{\eta} v(\eta) N_{\eta}$$

η self-avoiding path $i \rightsquigarrow j$

$$D = \sum_{\substack{\{\gamma_1, \ldots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$



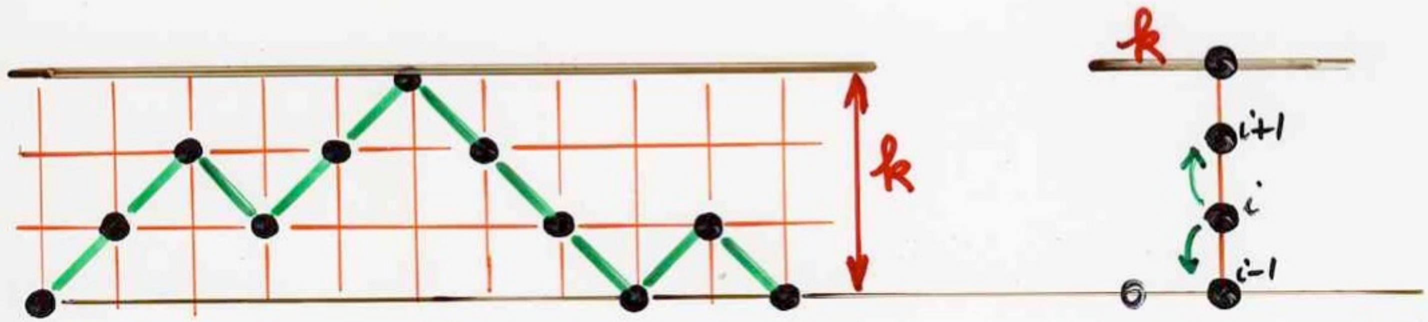
$$N_{ij} = \sum_{\{\eta; \gamma_1, \ldots, \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$

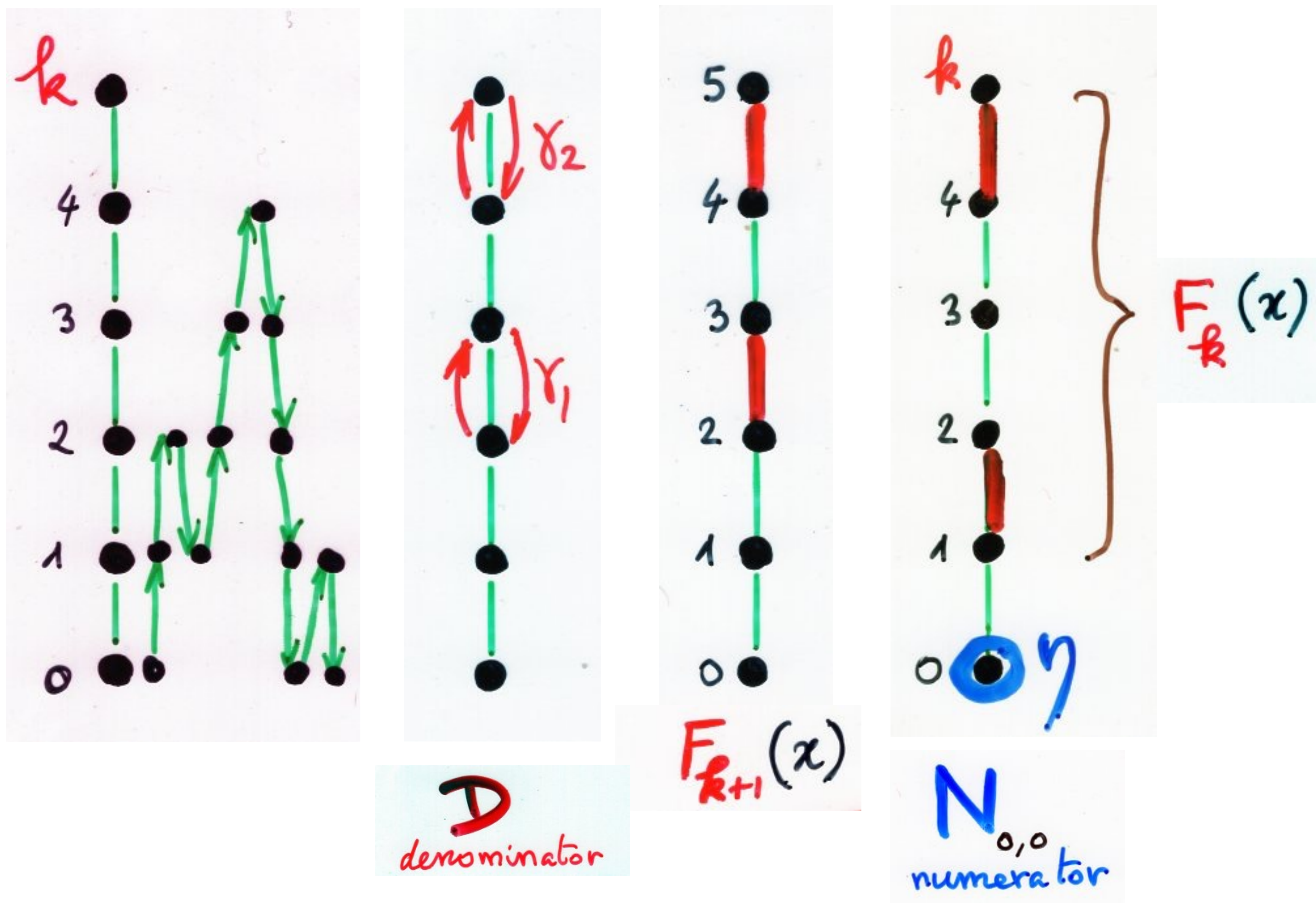


Transition matrix methodology in Physics

ex: Dyck path

bounded at height k




k
4
3
2
1
0
γ_2
γ_1
D
denominator
5
$F_{k+1}(x)$
$F_k(x)$
η
$N_{0,0}$
numerator

$$\sum_{\omega} t^{|\omega|/2} = \frac{F_k(t)}{F_{k+1}(t)}$$

Dyck paths
bounded k

$$A = (a_{i,j}) = \begin{pmatrix} 0 & t & & \cdots & 0 \\ t & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & t \\ 0 & \cdots & & t & 0 \end{pmatrix}$$

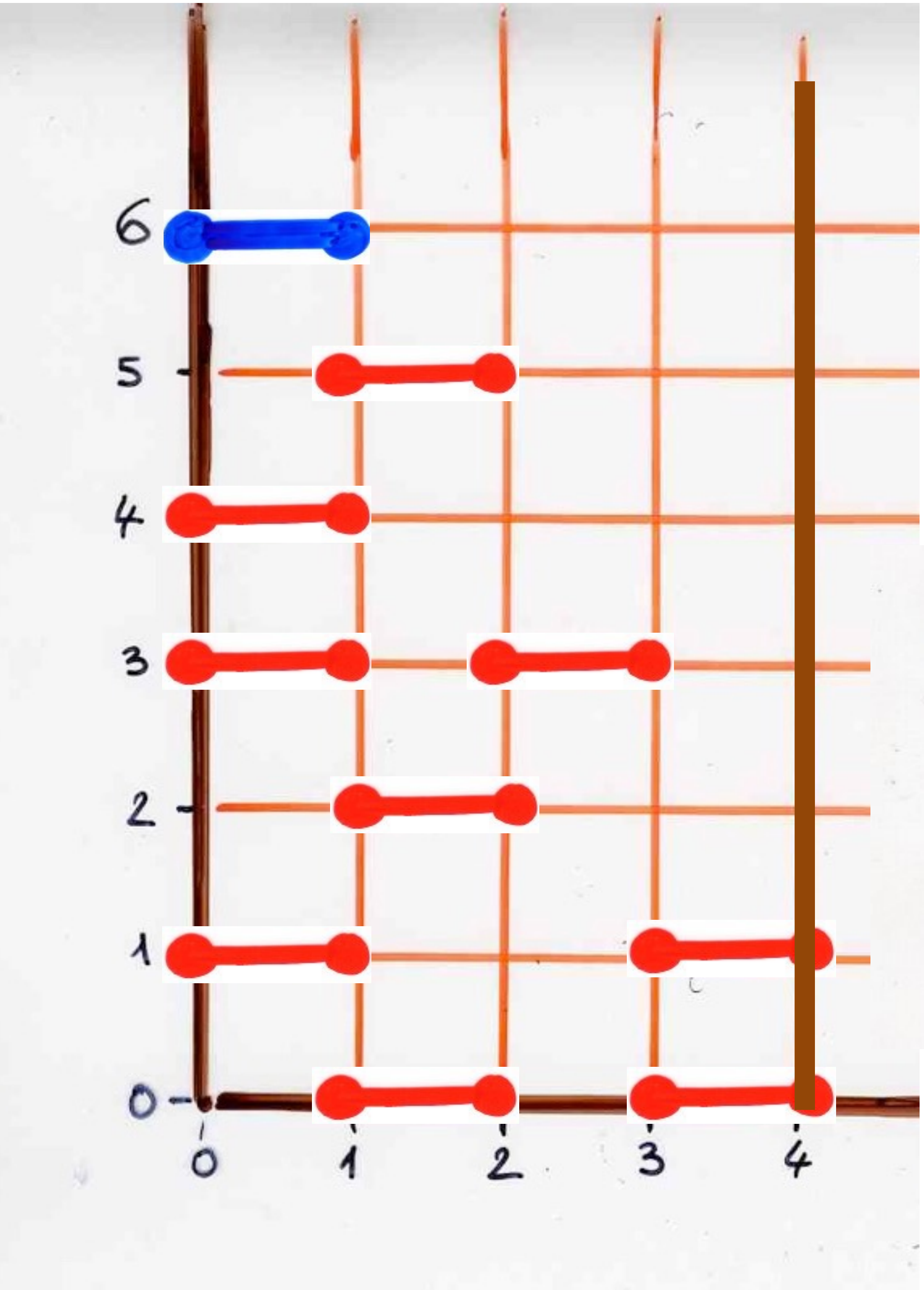
Fibonacci polynomials

$$F_n(x) = \sum_{k \geq 0} (-1)^k a_{n,k} x^k$$

$$= \sum_{\substack{M \\ \text{matchings} \\ \text{of } \{1,\dots,n\}}} (-x)^{|M|}$$

$= n$

$a_{n,k}$ = number of matchings of $\{1,2,\dots,n\}$ with k dimers



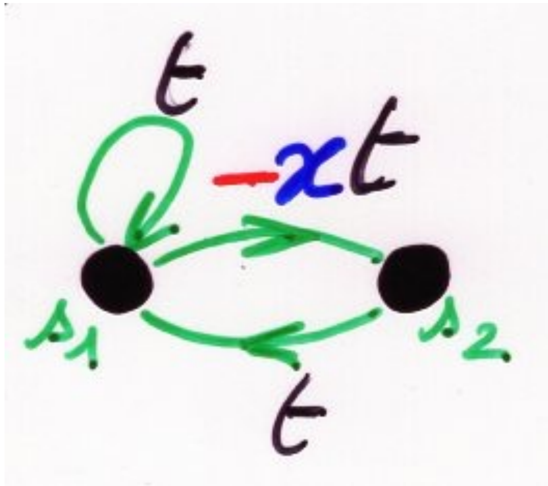
$$\frac{F_k(t)}{F_{k+1}(t)}$$

generating function of semi-pyramids of dimers on the segment $[0, k]$ (enumerated by the number of dimers)

$= n$

bijection

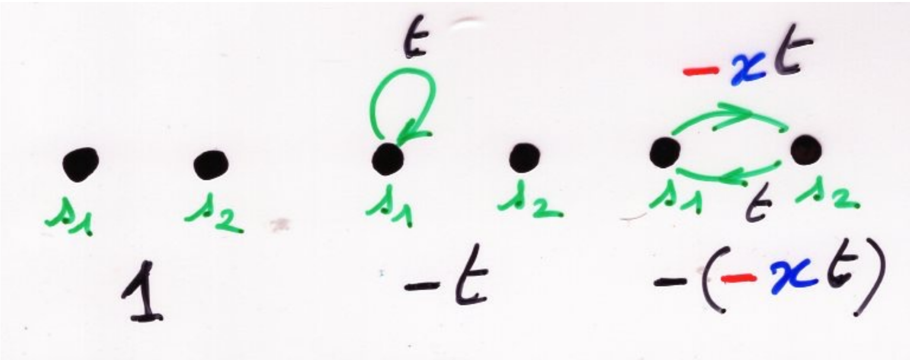
matchings of $[1, n]$ $\longleftrightarrow$ paths ω length n going from s_1 to s_1

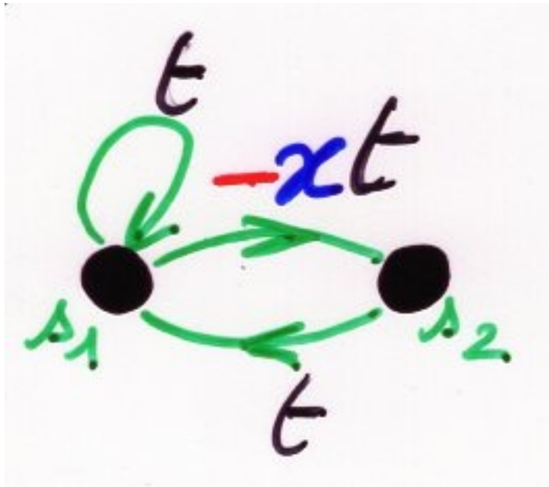


such that
$$v(\omega) = (-x)^k t^n$$

k = number of dimers of the matching

$= n$

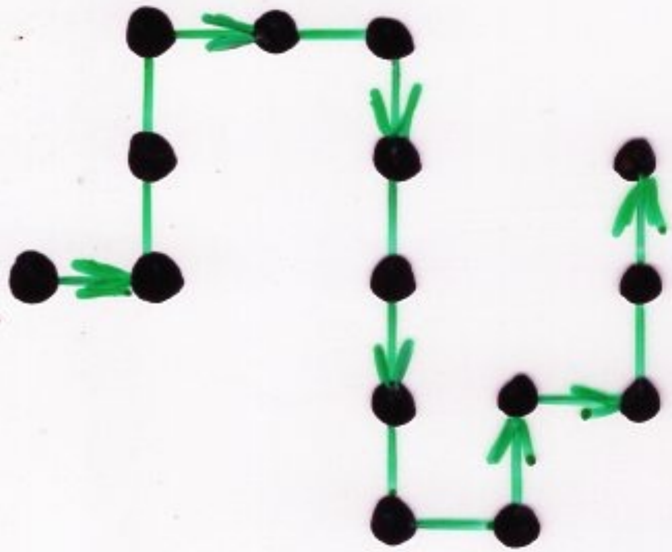




$$\sum_{n \geq 0} F_n(x)\, t^n = \frac{1}{1 - t + x t^2}$$

exercise

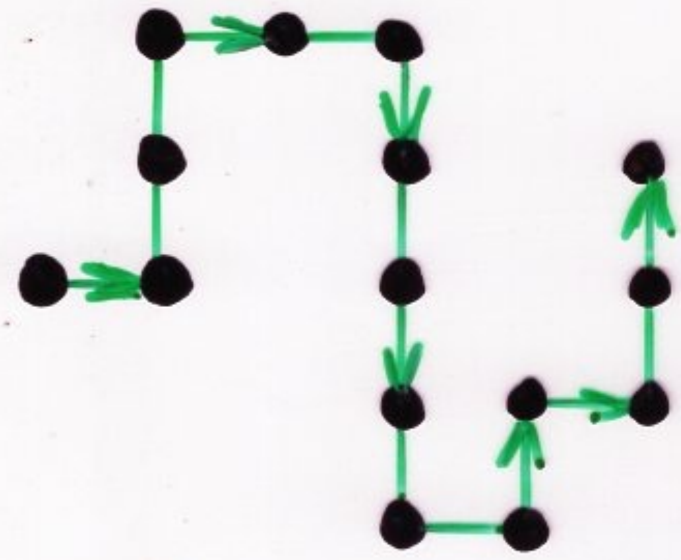
directed paths
(on a square lattice)



- elementary steps E, N, S
- self-avoiding

prove that the generating function (paths enumerated by the length) is:

$$\frac{1+\quad t}{1-2t-t^2}$$



hint: find a bijection with paths on a graph

with 3 vertices

MacMahon Master theorem

$$\frac{\mathrm{cof}_{ji}(I-A)}{\det(I-A)}$$

Lemma $X = \{1, 2, \dots, k\}$

$A = (a_{i,j})$ $n \times n$ matrix

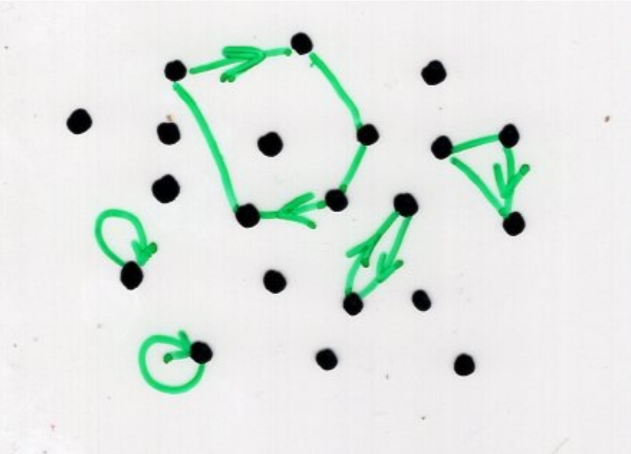
$$(I-A)^{-1}_{i,j} = \sum_{\substack{\omega \\ \text{path on } S \\ i \rightsquigarrow j}} v(\omega)$$

with $v(i,j) = a_{i,j}$

$$\frac{1}{\det(I-A)} \quad ?$$

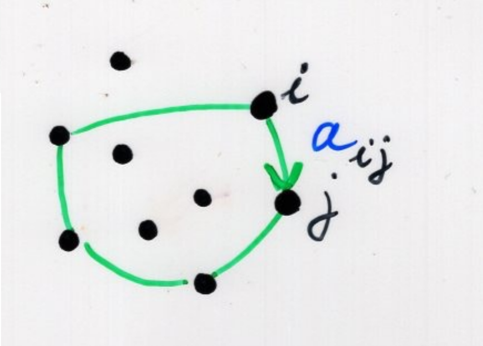
$$A = (a_{ij})_{1 \leq i,j \leq k}$$

$$\det(I - A) = \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \text{permutation}}} (-1)^{\mathrm{inv}(\sigma)} a_{1\sigma(1)} \cdots a_{k\sigma(k)}$$



$$\sum_{\substack{\gamma_1, \ldots, \gamma_r \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$

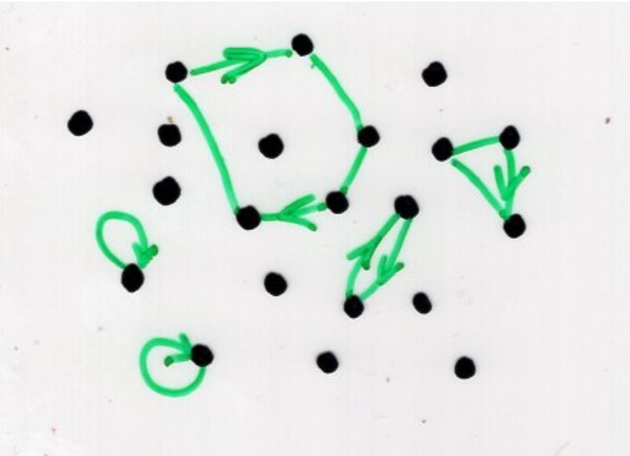
$$X = [1, k]$$

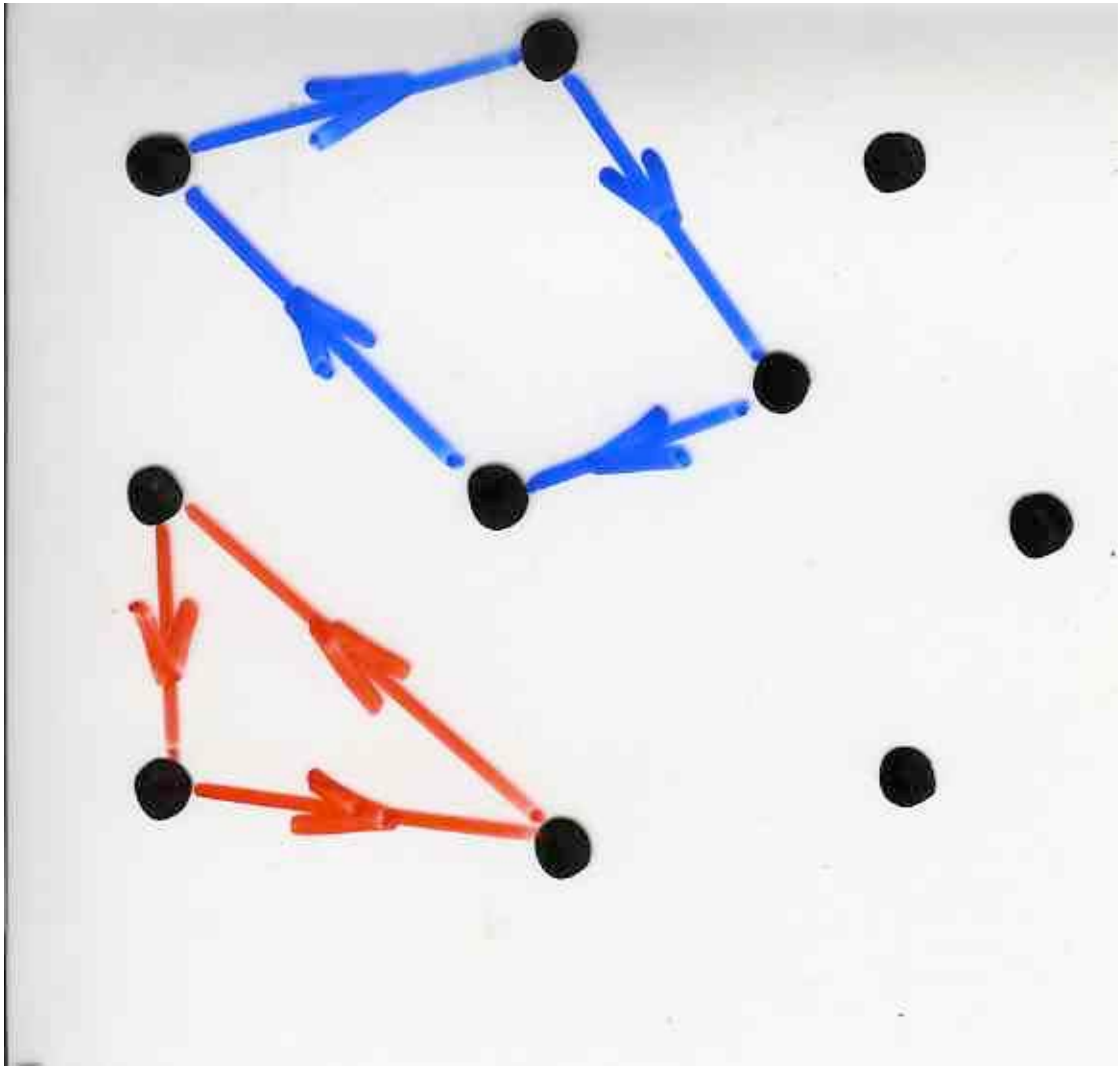


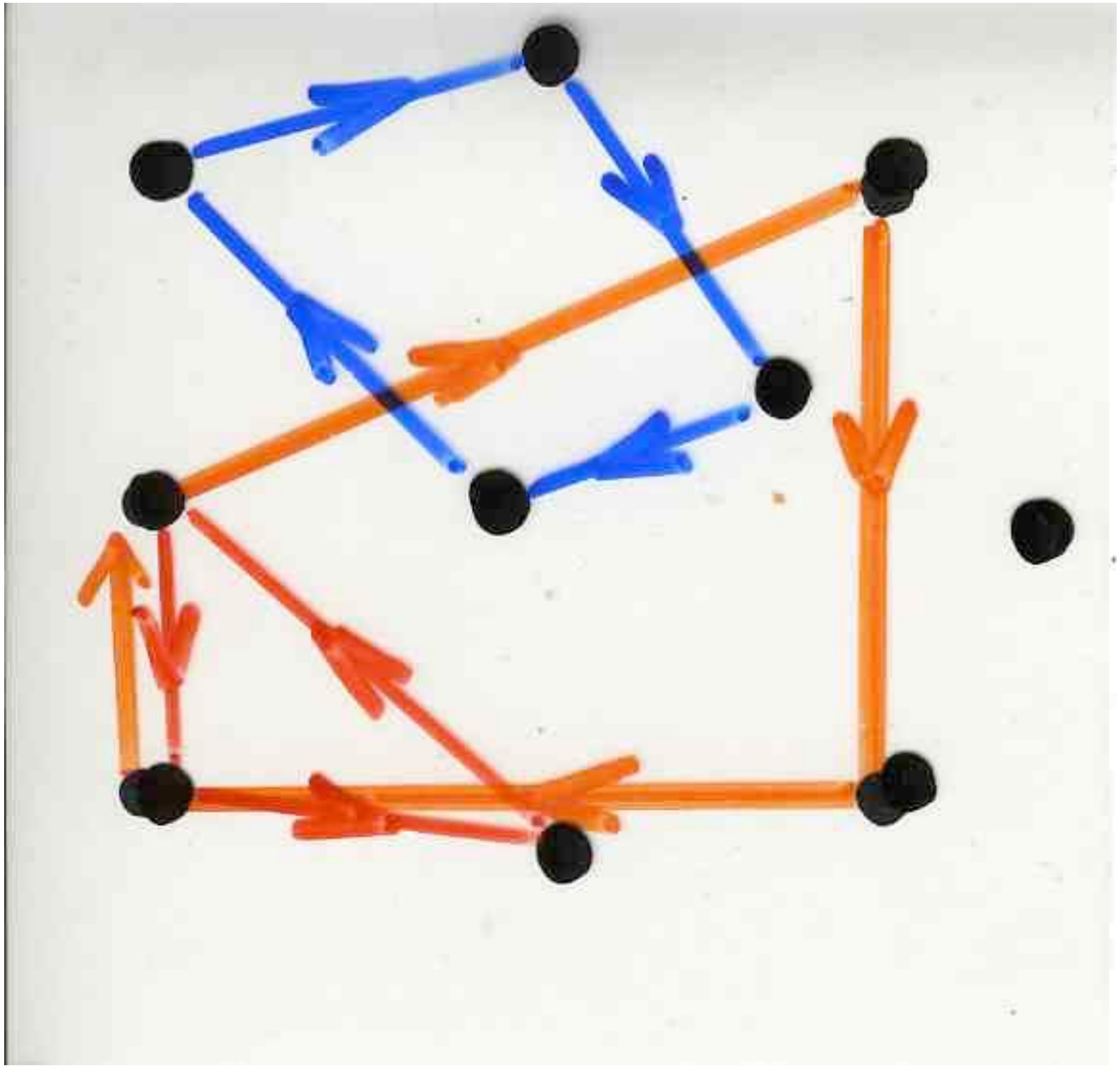
inversion lemma

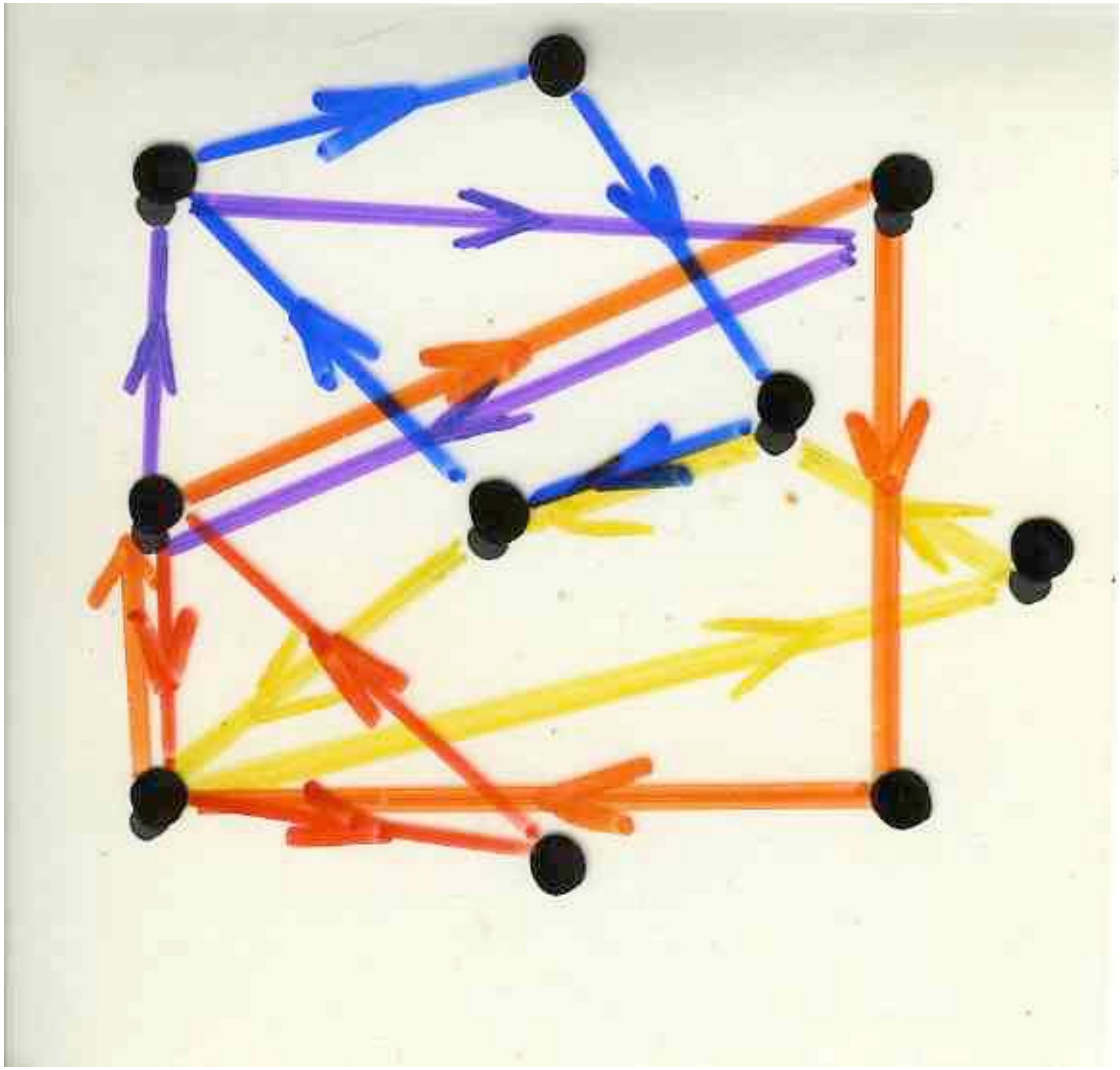
$$\frac{1}{\det(I-A)} = \sum_{E} v(E)$$

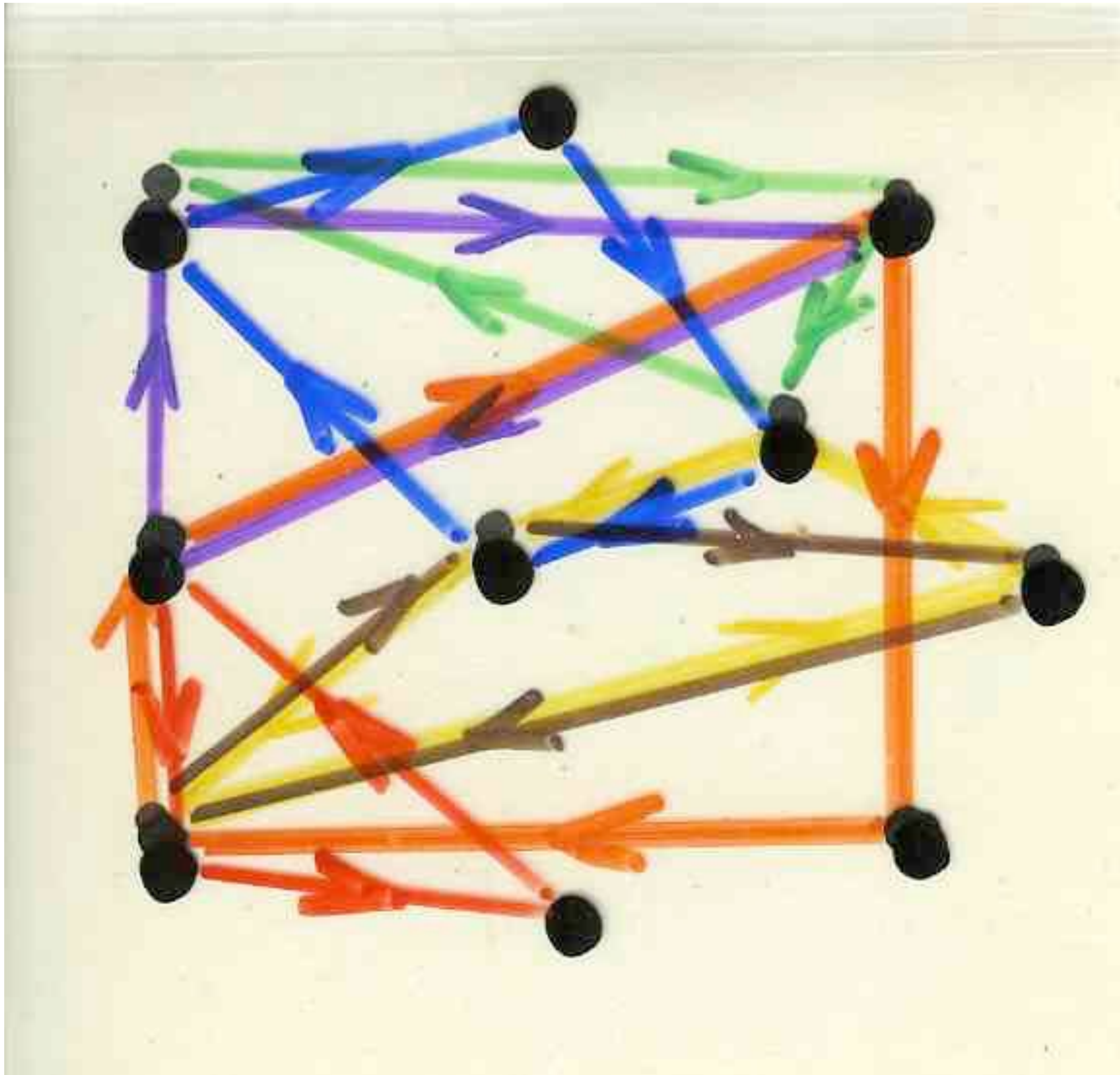
E heap of cycles on $[1,k]$

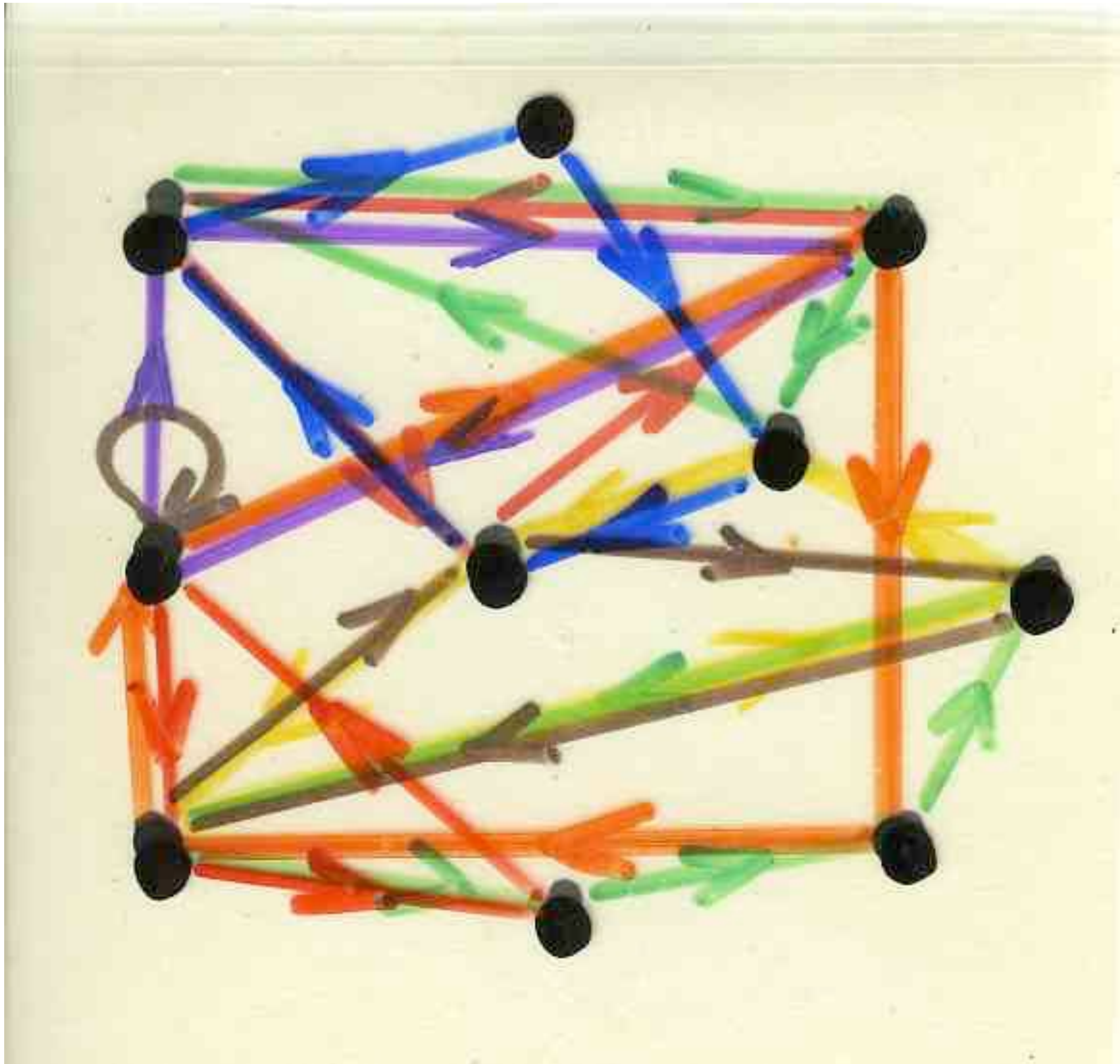




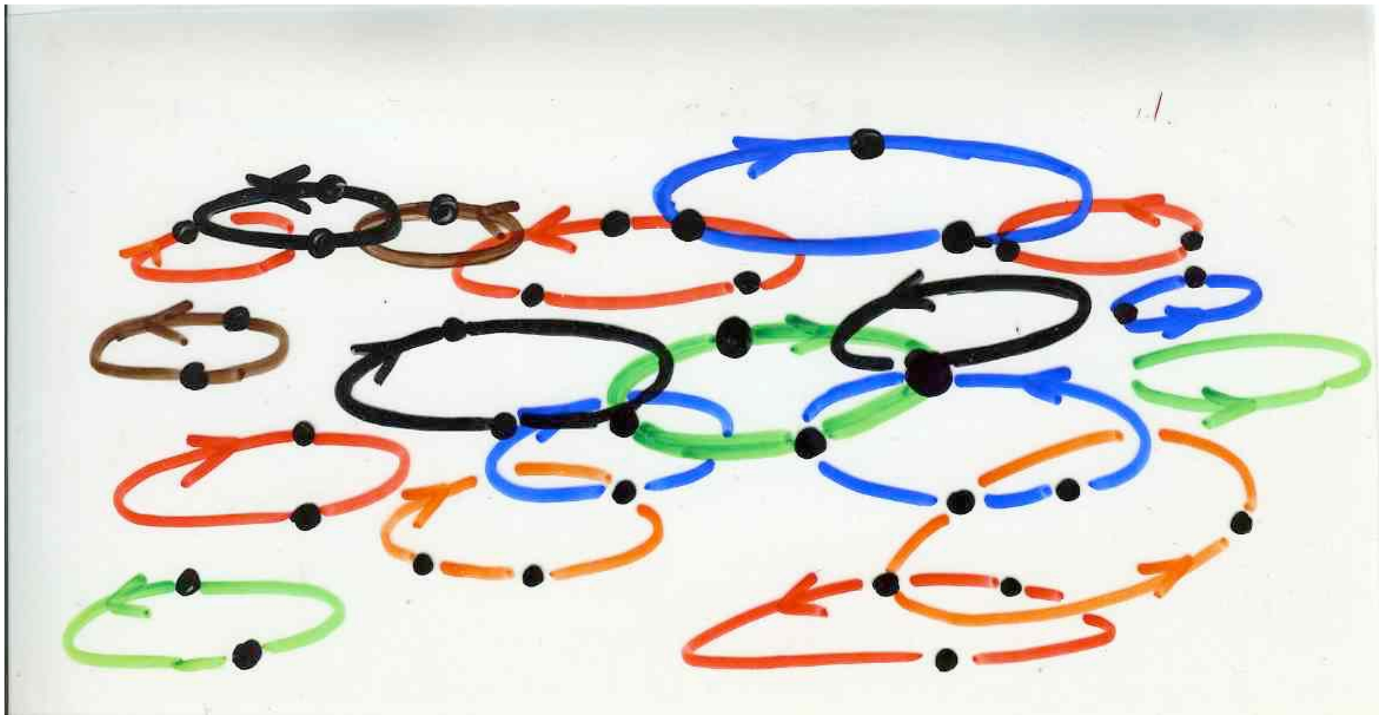






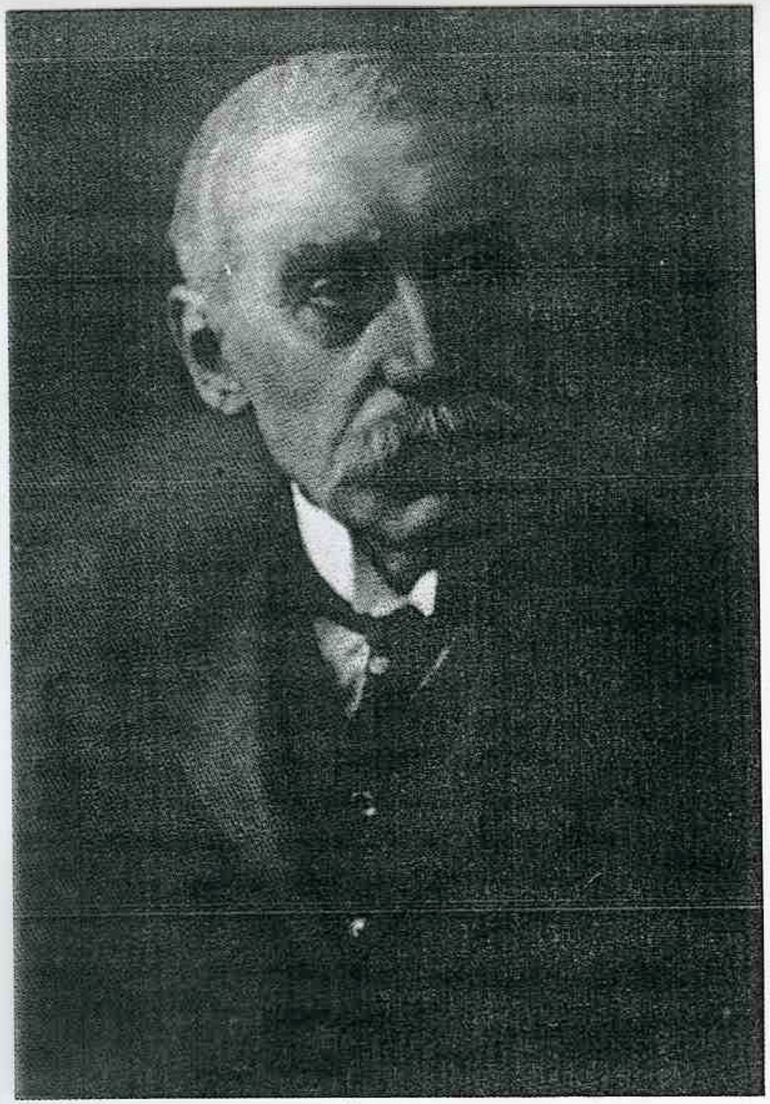


$$\frac{1}{\det(I-A)}$$



$$\frac{1}{\det(I - A)} = \sum_{\substack{E \\ \text{heap of cycles} \\ \text{on } [1,k]}} v(E)$$

$$= \sum_{\substack{\Phi \\ \text{rearrangements} \\ \text{on } [1,k]}} v(\Phi)$$



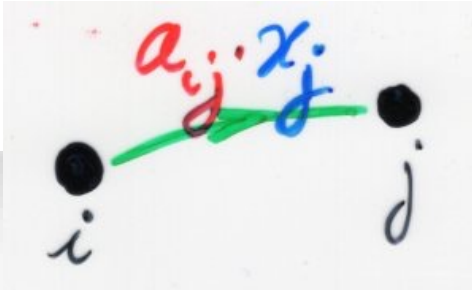
Percy Alexander MacMahon

$$\frac{1}{\det(I-A)} = \sum_{\Phi} v(\Phi)$$

rearrangements on $[1,k]$

Where is my MASTER THEOREM ?

MacMahon master theorem



$$A = (a_{ij})_{n\times n}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$AX = (a_{ij}\, x_j)_{n\times n}$$

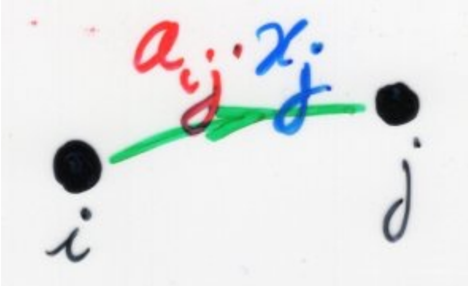
$$Y_i = \sum_{j=1}^{n} a_{ij}\, x_j$$

$$\frac{1}{\det(I - AX)}$$

coeff. de $x_1^{\alpha_1} \cdots\cdots x_n^{\alpha_n}$

$$Y_1^{\alpha_1} \cdots\cdots Y_n^{\alpha_n}$$

MacMahon master theorem



The coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ in $\dfrac{1}{\det(I - AX)}$

is the same as

the coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$

in $Y_1^{\alpha_1} \dots Y_n^{\alpha_n}$

string theory

"Quivers, Words and Fundamentals"

Paolo Mattioli, Sanjaye Ramgoolam (2014)

arXiV:1412.5991

ABSTRACT

A systematic study of holomorphic gauge invariant operators in general N = 1 quiver gauge theories, with unitary gauge groups and bifundamental matter fields, was recently presented in [1]. For large ranks a simple counting formula in terms of an infinite product was given. We extend this study to quiver gauge theories with fundamental matter fields, deriving an infinite product form for the refined counting in these cases. The infinite products are found to be obtained from substitutions in a simple building block expressed in terms of the weighted adjacency matrix of the quiver. In the case without fundamentals, it is a determinant which itself is found to have a counting interpretation in terms of words formed from partially commuting letters associated with simple closed loops in the quiver. This is a new relation between counting problems in gauge theory and the Cartier-Foata monoid. For finite ranks of the unitary gauge groups, the refined counting is given in terms of expressions involving Littlewood-Richardson coefficients.

complements

An identity of Bauer
for loop-erased random walks

weighted path

$s, t \in X$ $w(s,t) \in \mathbb{K}[Z]$

can be $w(s_i, s_j) = a_{ij}$

or path on a graph

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\chi} \underset{u \rightsquigarrow v}{(\eta, E)}$$

$\eta = LE(\omega)$

loop-erased

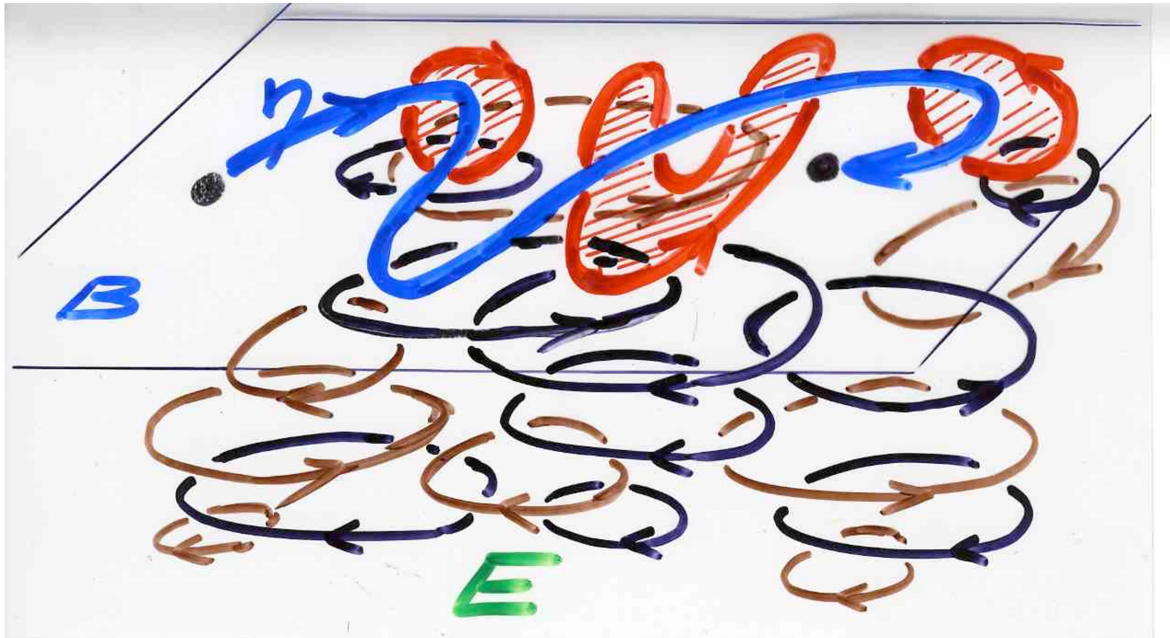
probability law on η

$\rightarrow$ LERW

Loop-erased random walks

Ch 3b, p 66–79

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\chi} (\underset{u \rightsquigarrow v}{\eta}, E)$$



$\eta = LE(\omega)$

loop-erased

define

$$v(\eta) = \sum_{\substack{\omega \\ u \rightsquigarrow v \\ \eta = LE(\omega)}} w(\omega)$$

Bauer identity

$$V(\eta) = \sum_{\substack{\omega \\ u \rightsquigarrow v \\ \eta = LE(\omega)}} w(\omega)$$

Prop M. Bauer (2007) $\eta = (s_0 = u, s_1, \dots, s_k = v)$

$$V(\eta) = \frac{w(\eta)}{\det(I - K_{ij})_{0 \le i,j \le k}}$$

$$\underset{0 \le i,j \le k}{K_{ij}} = \sum_{\substack{\omega \\ s_i \rightsquigarrow s_j}} w(\omega)$$

ω avoiding η (except for the first and last vertex)

$$\frac{1}{\det(I - K_{ij})} = \sum_{\substack{E \\ \text{heaps of cycles } \gamma \text{ on } S}} v_K(E)$$

$\eta = (s_0, \dots, s_k)$

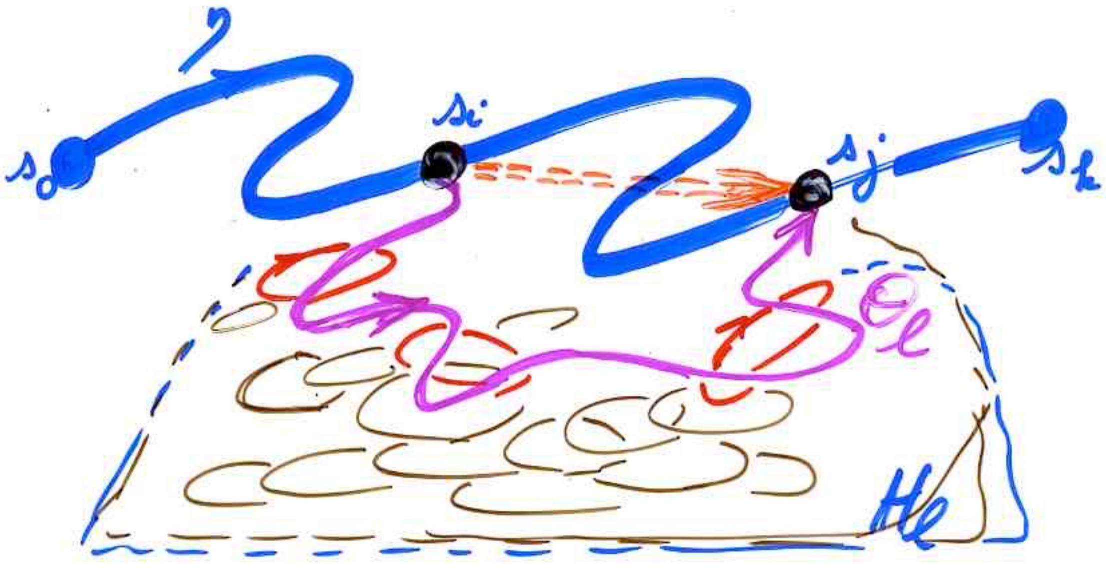
$S = \{s_0, \dots, s_k\}$

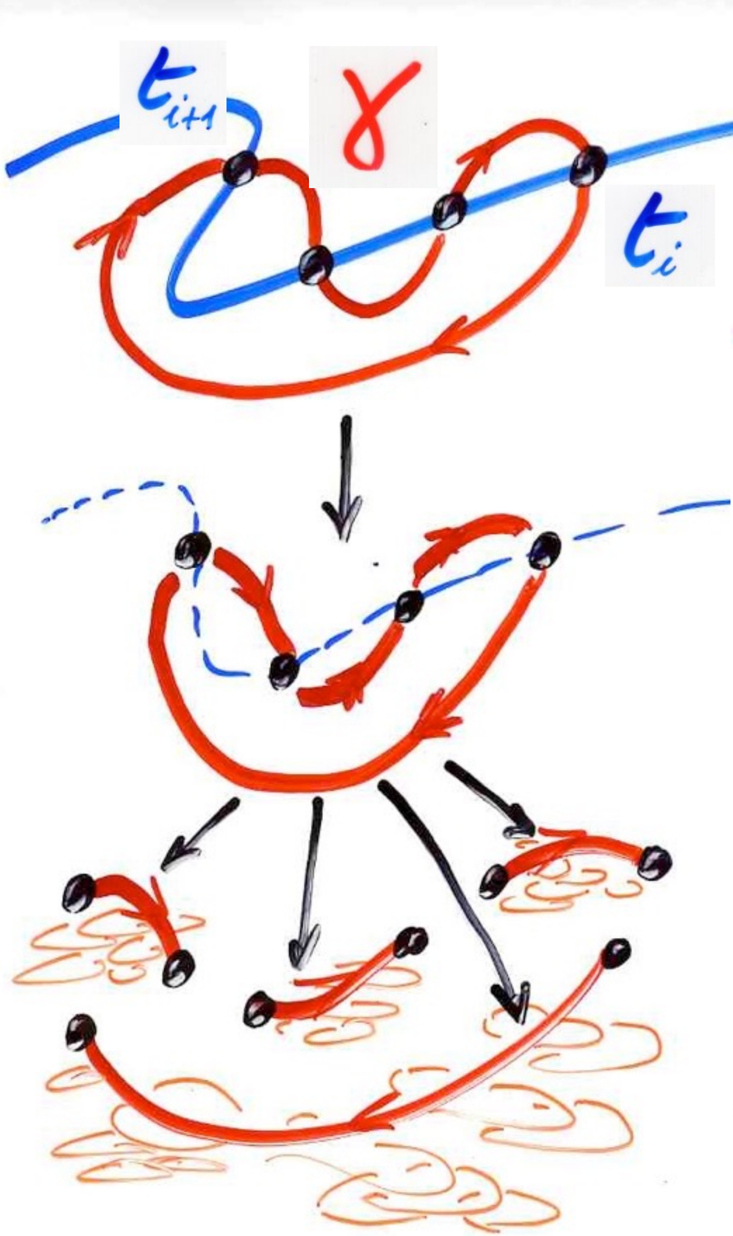
$$\underbrace{v_K(s_i, s_j)}_{K_{ij}} = \sum_{\substack{\omega \\ s_i \rightsquigarrow s_j}} w(\omega)$$

ω walk on 

avoid η

(except the first and last vertex)





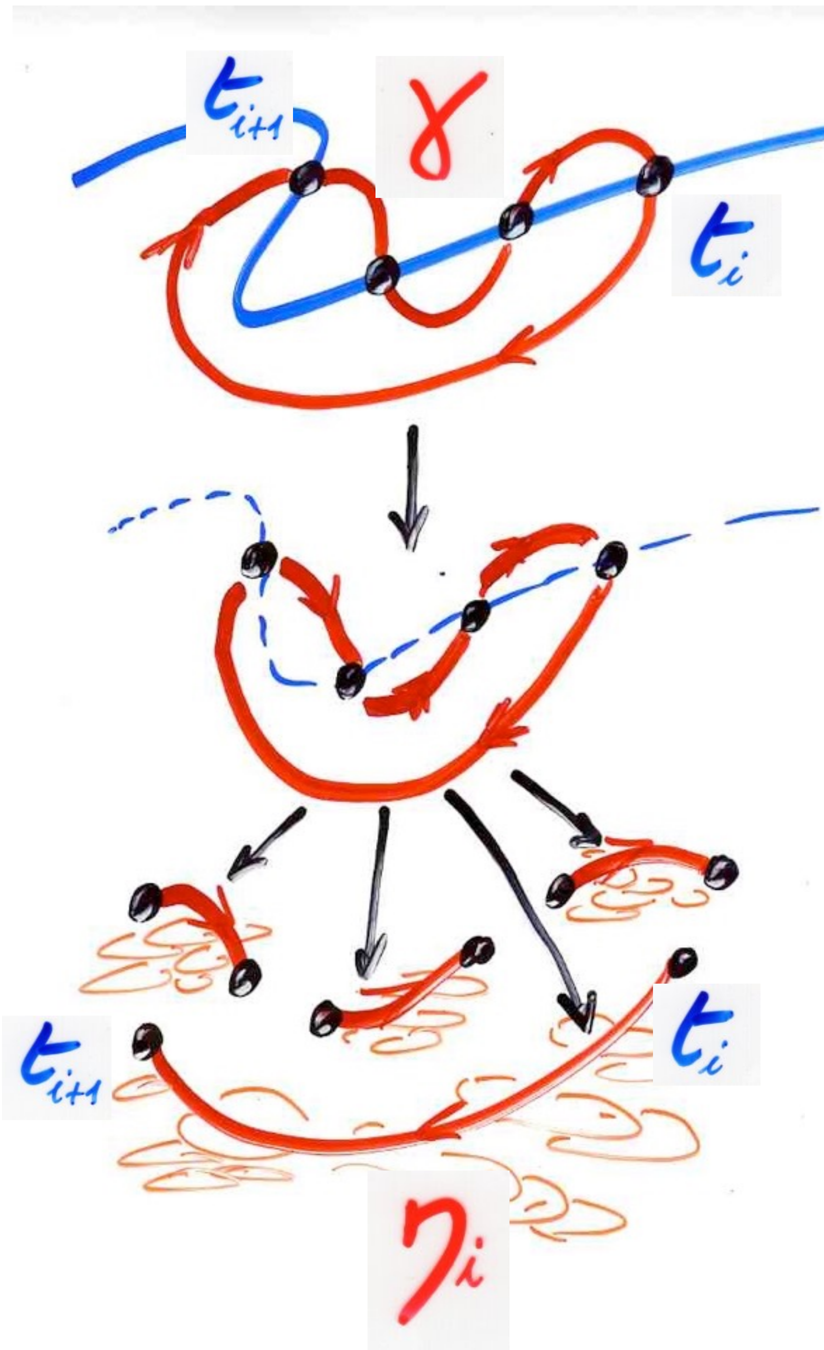
$$v_K(\gamma) = v_K(t_1, t_2) \cdots v_K(t_i, t_{i+1}) \cdots v_K(t_r, t_{r+1} = t_1)$$

$$\gamma = (t_1, \dots, t_r)$$

$$t_i \in \eta$$

$$\sum_{\substack{\omega_i \\ t_i \rightsquigarrow t_{i+1} \\ \text{avoid } \eta \\ (\text{except for } t_i,\, t_{i+1})}} w(\omega_i)$$

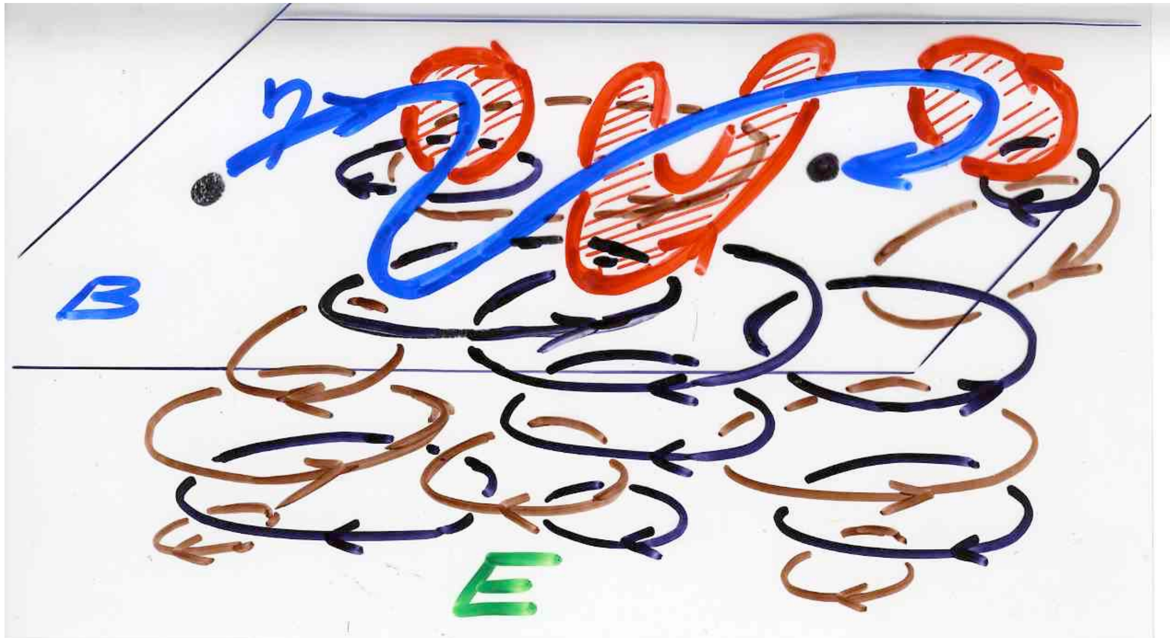
$$v_K(\gamma) = \sum_{\substack{(\omega_1, \dots, \omega_r) \\ \omega_i \;\; t_i \rightsquigarrow t_{i+1} \\ \text{avoid } \eta \\ (\text{'' \quad ''})}} \prod_i w(\omega_i)$$



$$\omega_i \xleftrightarrow{\chi} (\eta_i, E_i)$$

$$v_K(\gamma) = \sum_{\substack{(\omega_1, \ldots, \omega_r) \\ \omega_i \ t_i \rightsquigarrow t_{i+1} \\ \text{avoid } \eta \\ (\text{'' \quad ''})}} \prod_i w(\omega_i)$$

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\ \chi\ } \underset{u \rightsquigarrow v}{(\eta, E)}$$

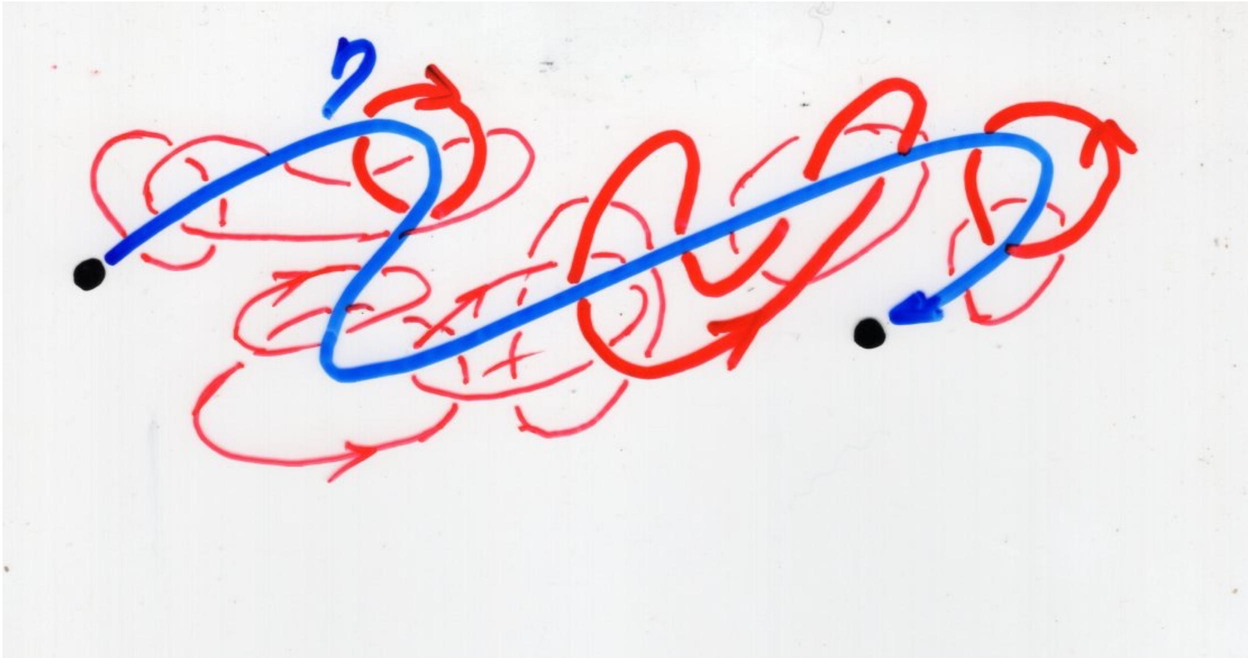


$\eta = LE(\omega)$
loop-erased

$\eta = (s_0, \dots, s_k)$

extract from the heap of cycles E
all the cycles which intersect η
$\longrightarrow$ "sub-heap" of cycles
on $S = \{s_0, \dots, s_k\}$

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\ \chi\ } (\underset{u \rightsquigarrow v}{\eta}, E)$$



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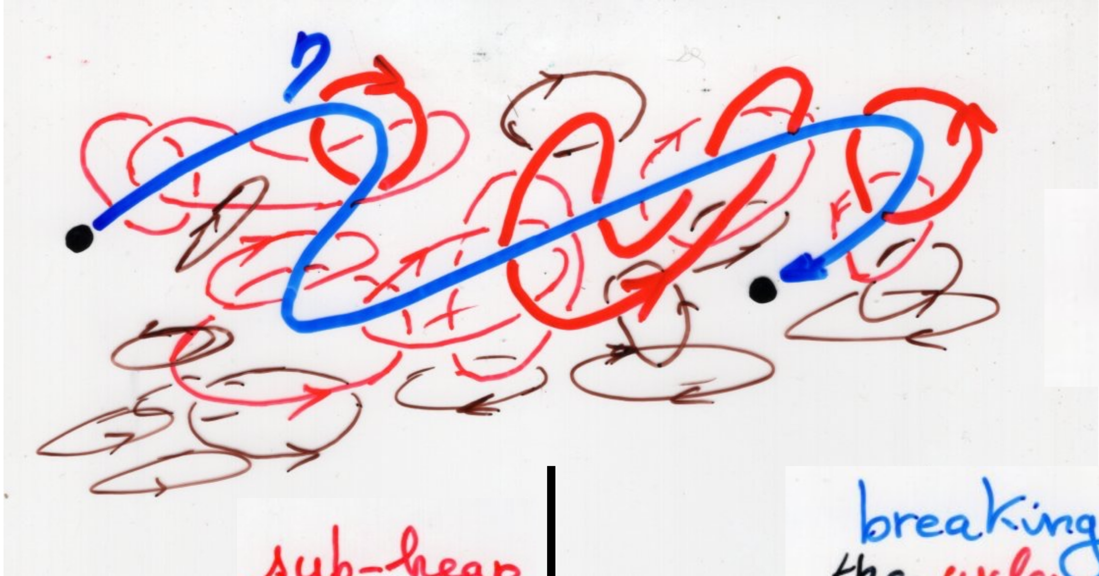
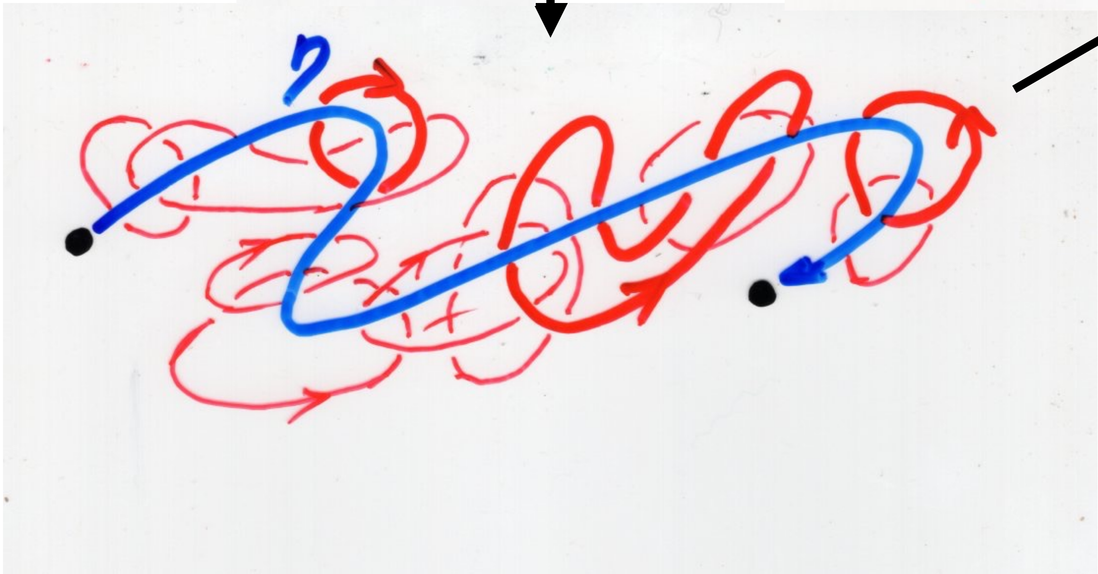
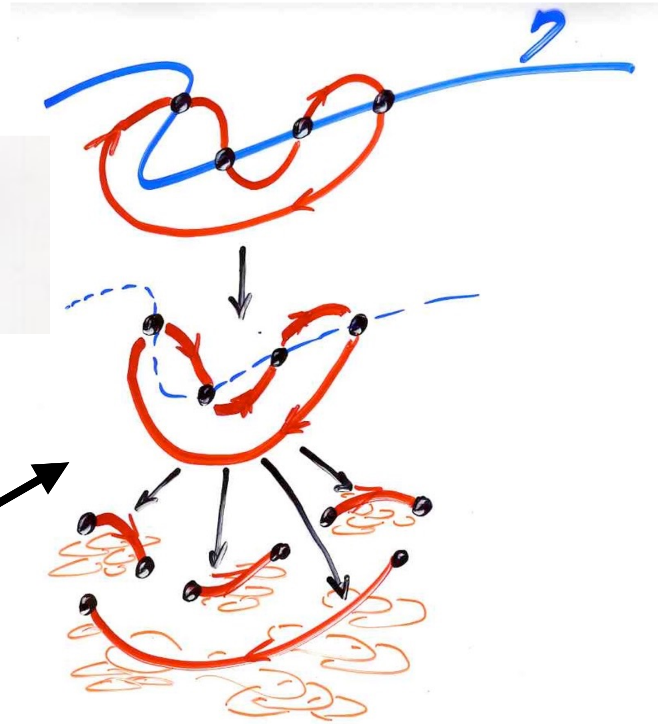


sub-heap

breaking the cycles



$=$



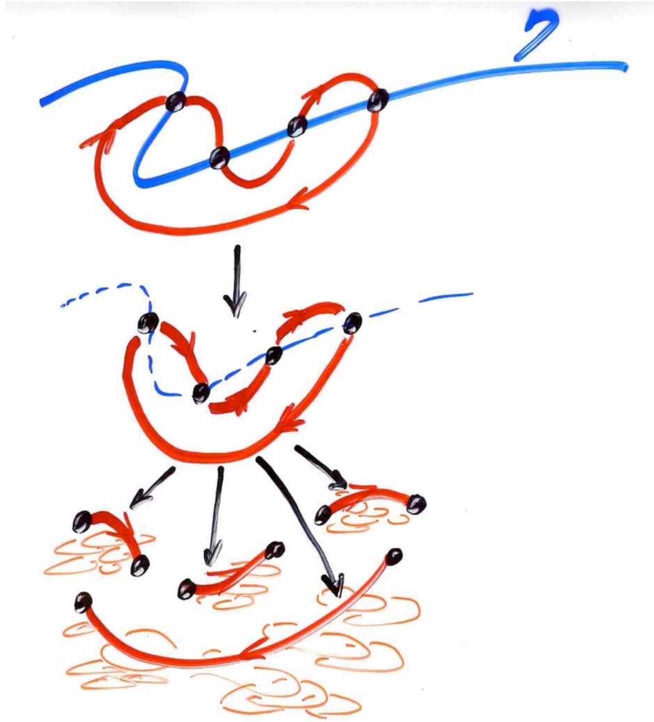
"substitution"

$$\omega_i \xleftrightarrow{\chi} (\eta_i, E_i)$$

Research problem 3

Give a bijective proof of Bauer identity using the theory of heaps

define the notion of substitution in heaps



$\mathbf{H}(P, \mathcal{C})$

$\alpha \in P \longrightarrow E_\alpha$

heap of $\mathbf{H}(Q, \mathcal{D})$