Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

IMSc
January-March 2017

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Chapter 5

Heaps and algebraic graph theory (1)

IMSc, Chennai

16 February 2017

This class is
dedicated to
my dear friend
Jean-Pierre Muller

« en guise d'apéritif ….»

$G = (V, E) \longrightarrow$ heap monoid

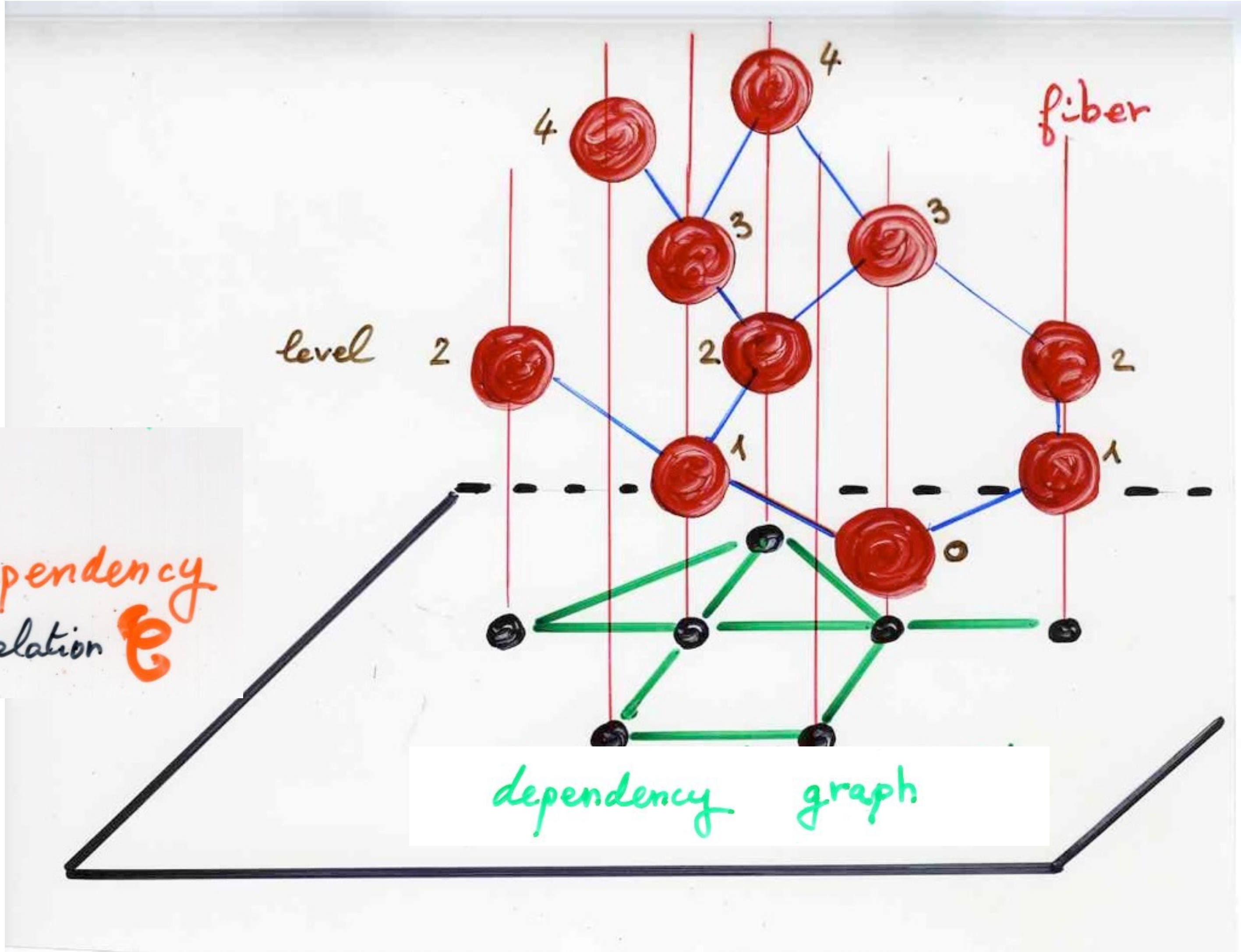
$\mathbf{H}(G) = \mathbf{H}(V, E)$

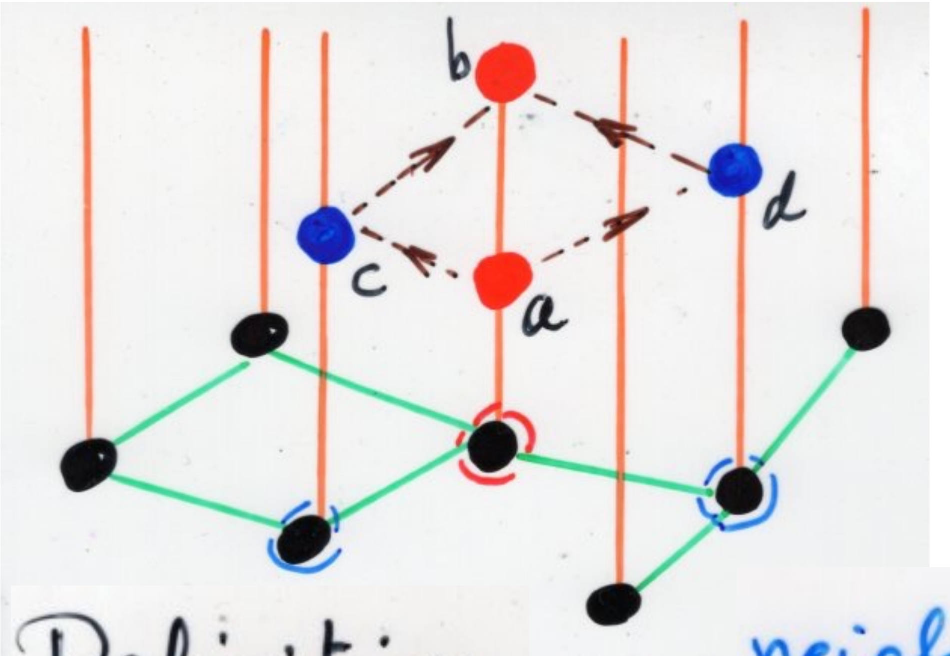
G graph

$G = (V, E)$

basic pieces

dependency relation $\mathscr{C}$





Definition

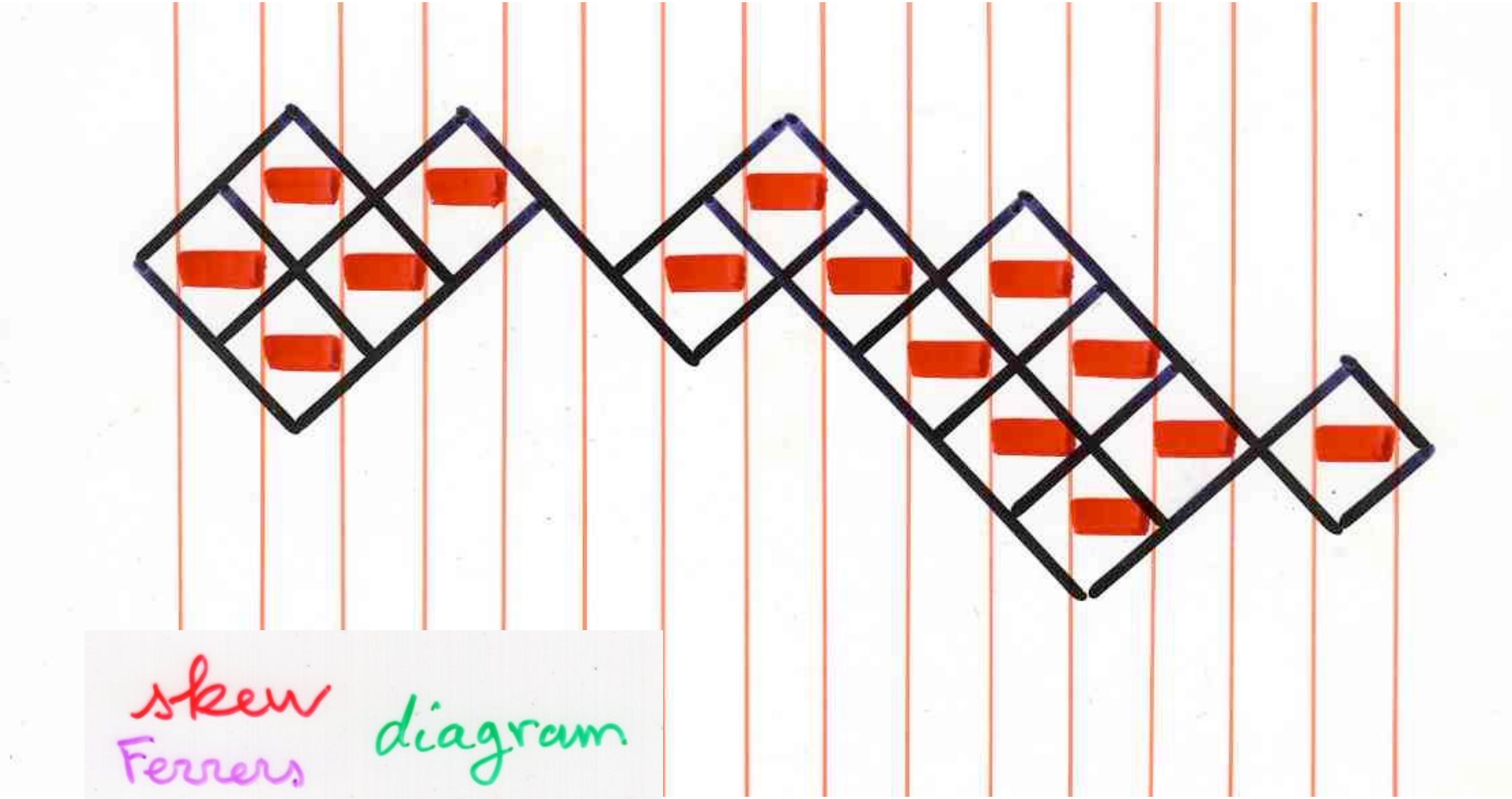
neighbourly heap

In term of graph and poset
(the underlying poset $(H, \leq)$ of the heap H)
$\pi : H \to V$ projection

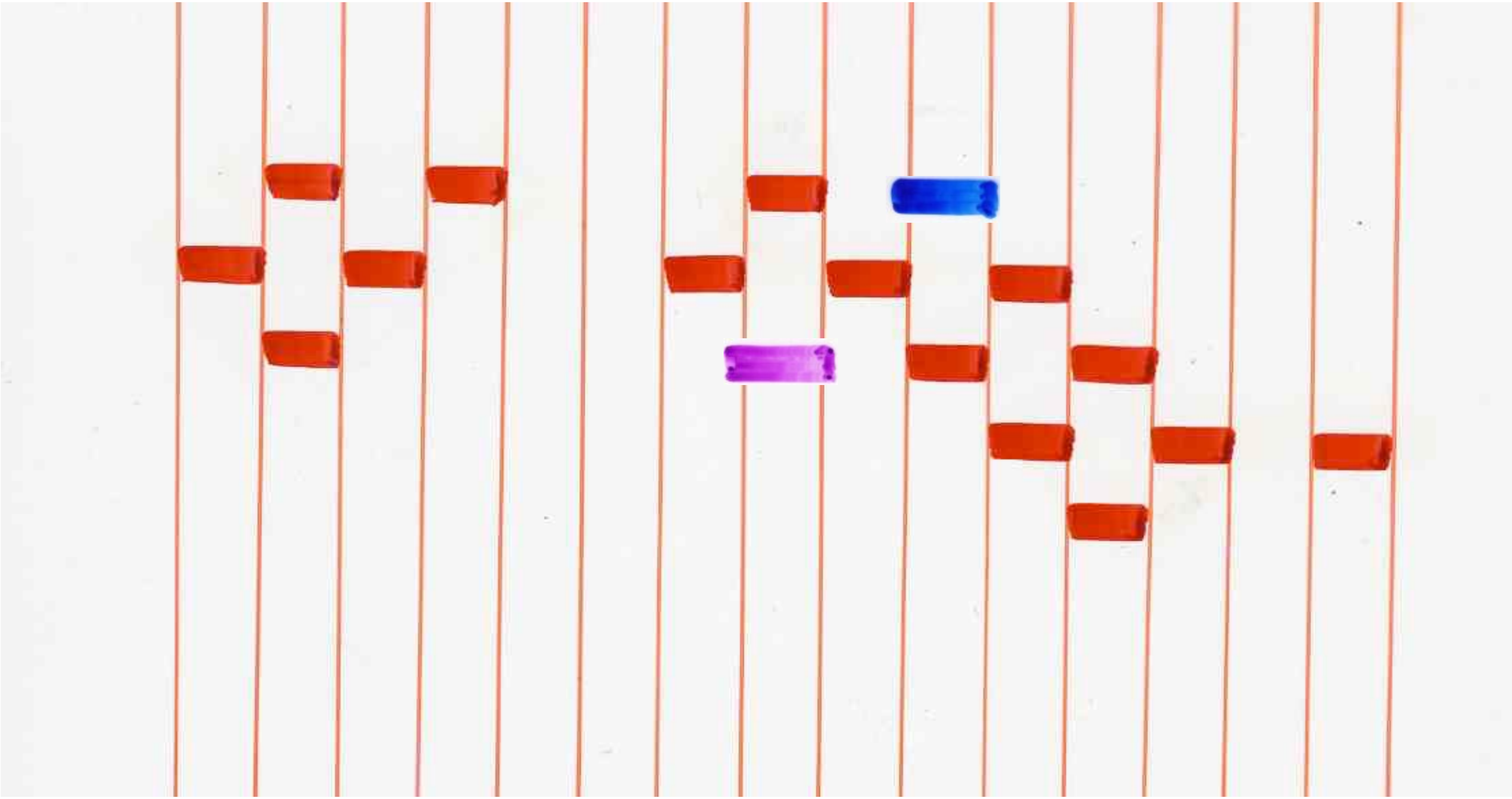
(*) for any pair $a, b \in H$ with $\pi(a) = \pi(b)$
there exist $c, d \in H$ with
$a \leq c \leq b$, $a < d < b$ and $\pi(c), \pi(d)$ are
neighbours of $\pi(a) = \pi(b)$ in G

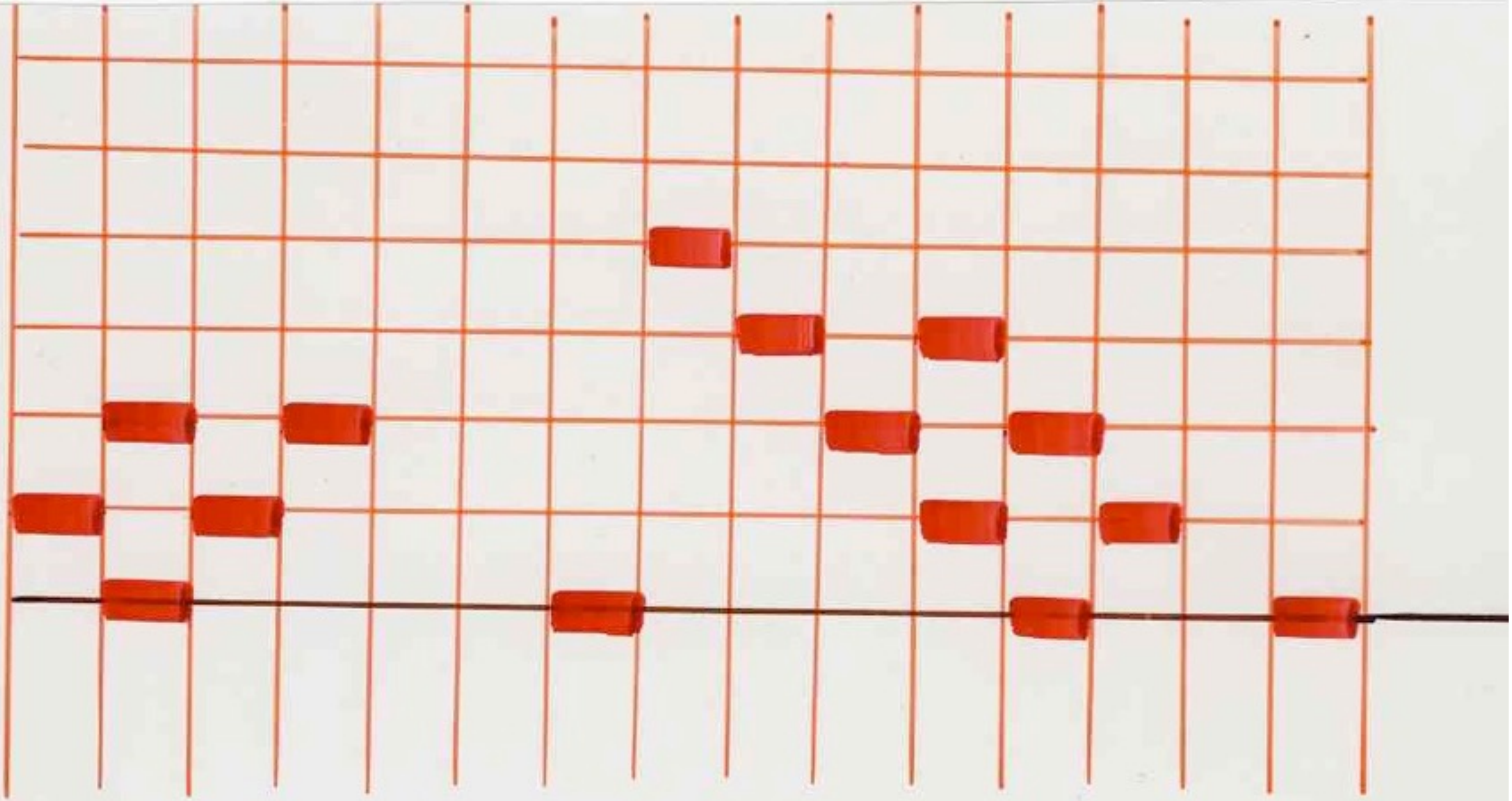
Definition

A neighbourly heap H is called maximal is H cannot be extended by the addition of a piece (in any position) to a larger neighbourly heap



skew diagram
Ferrers





Definition

A neighbourly heap H is called two-neighbourly if there are exactly two occurrences of pieces c and d in condition $(*)$.

Wildberger (2001)

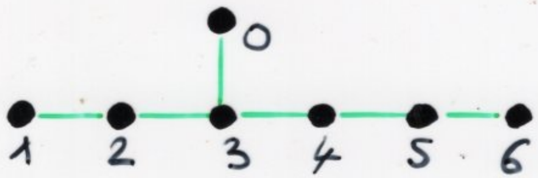
Proposition Let G be a graph for which there exists a maximal neighbourly heap H which is two-neighbourly. Then G is one of the following graphs:

number of such heaps

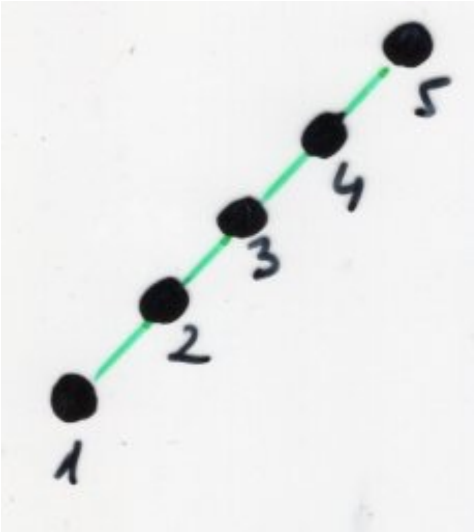
A_n — n

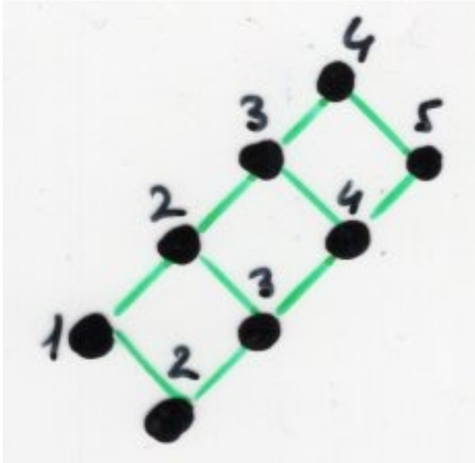
D_n — 3

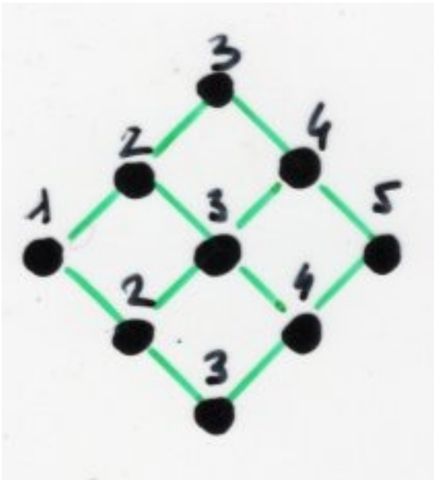
E_6 — 2

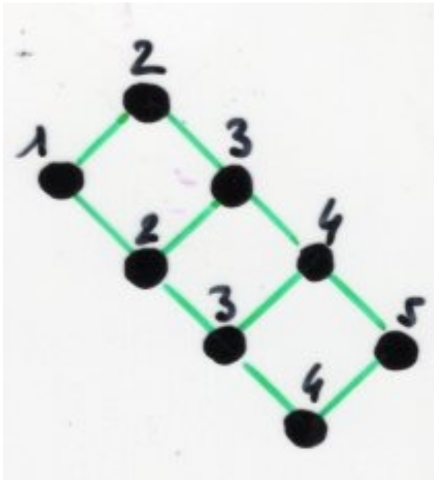


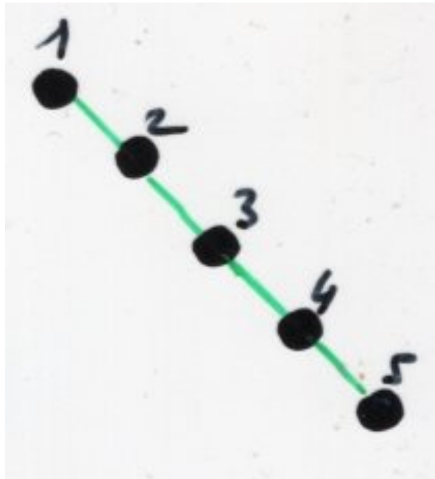
E_7 — 1

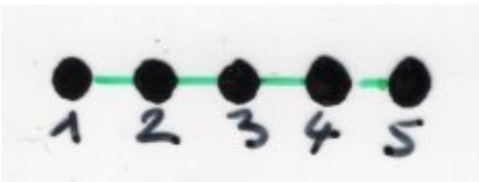




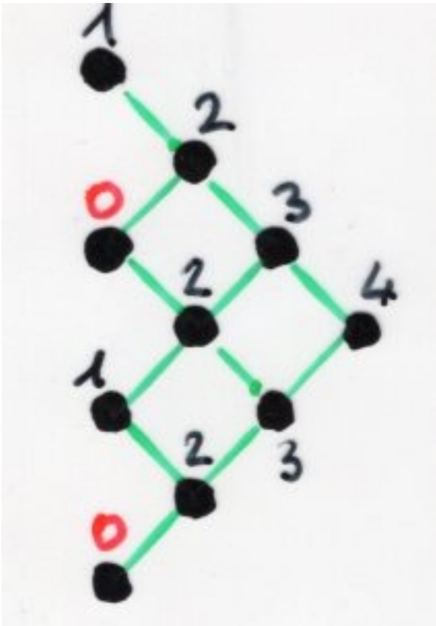


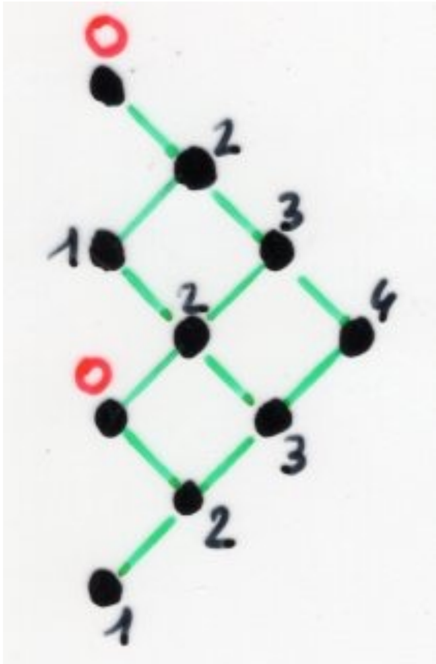


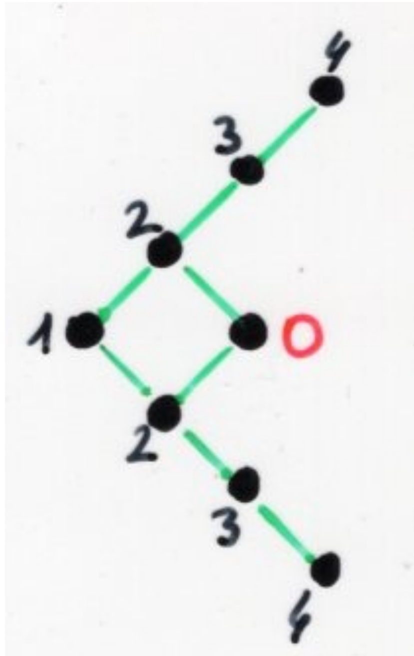


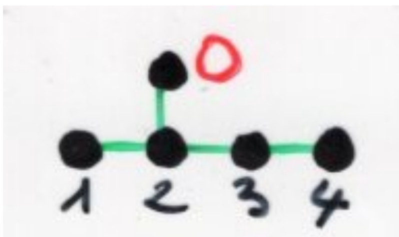


A_5

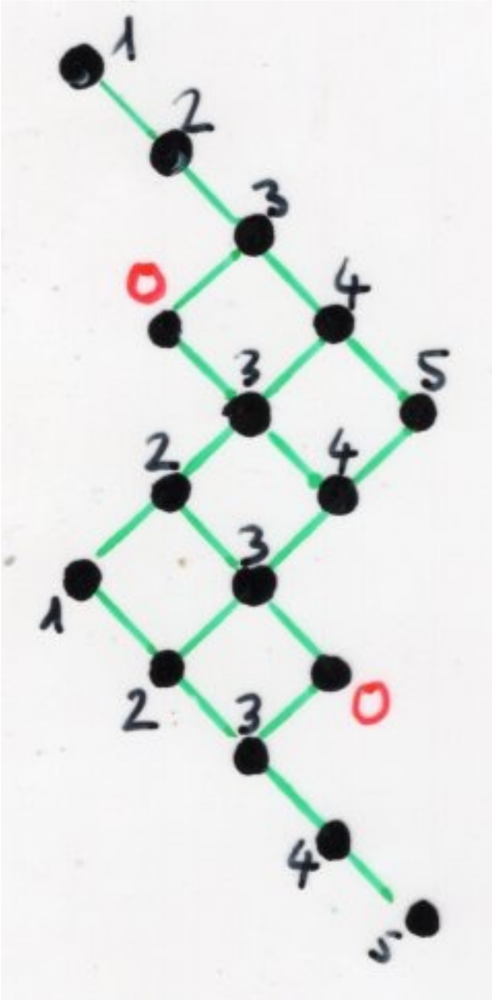




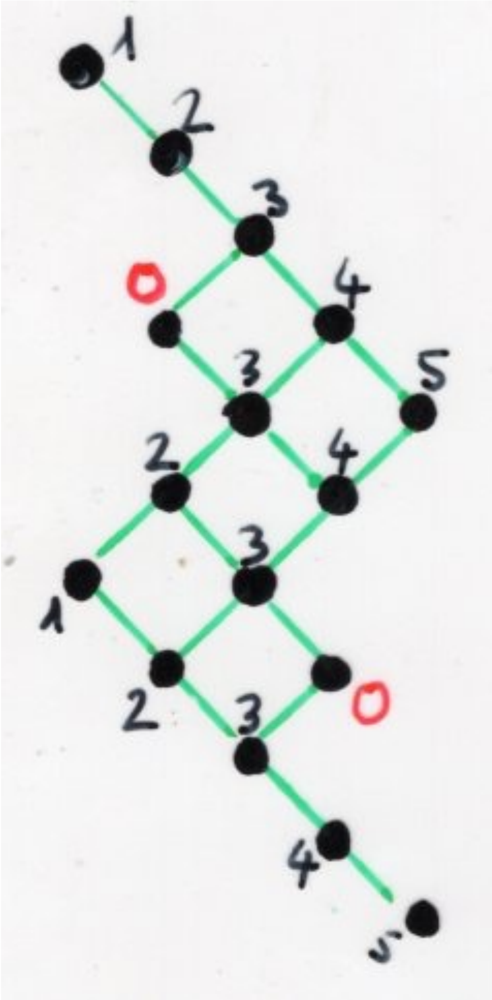


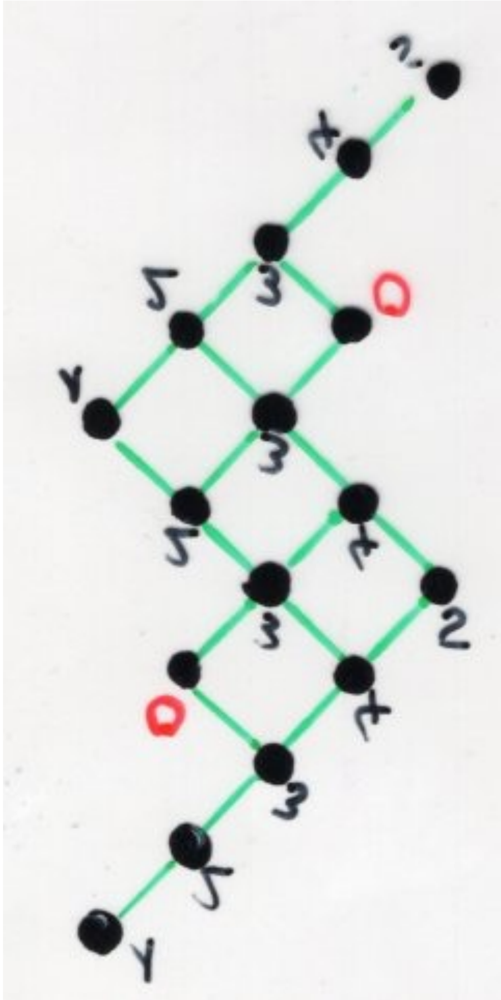


D_5

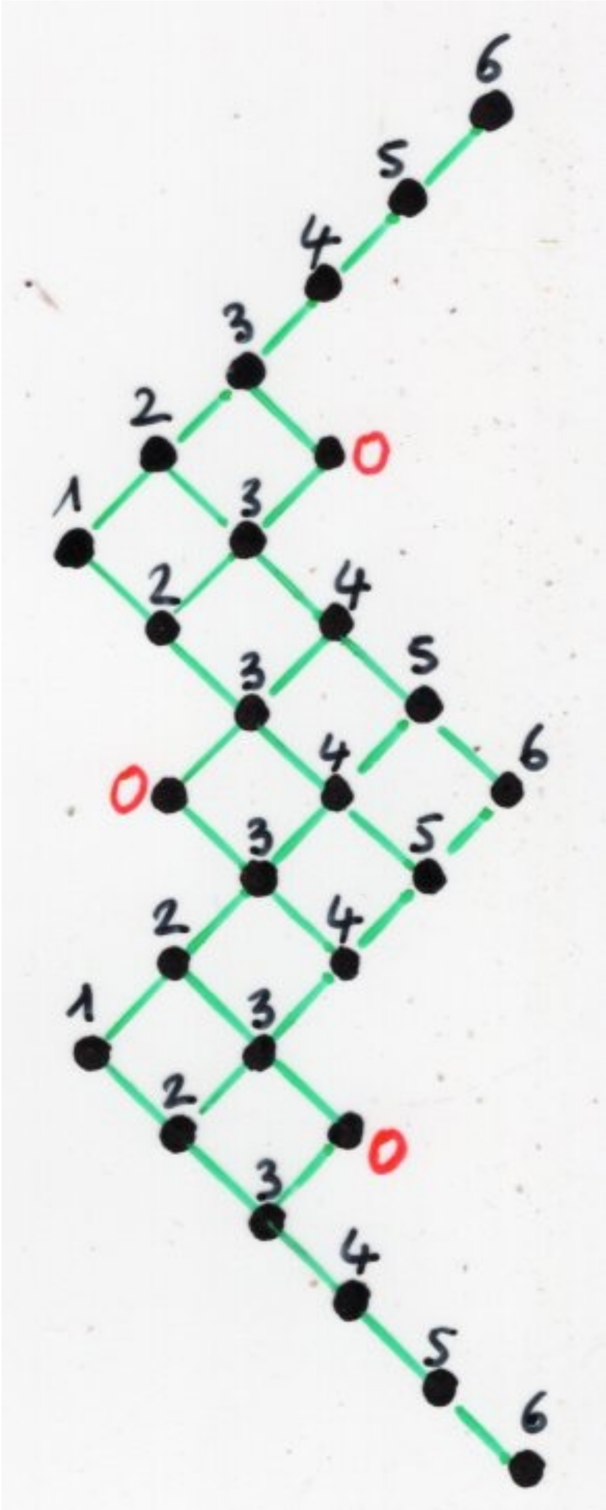


E_6

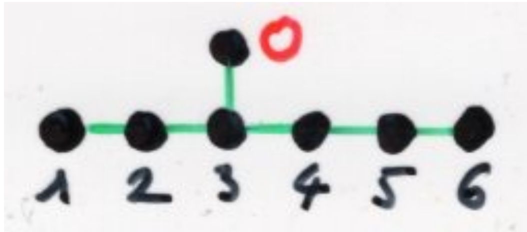




E_6



swallow



E_7

complements

Proposition. If the graph G has a maximal neighbourly heap H, then G is a tree and the poset $(H, \preccurlyeq)$ is a lattice

Wildberger (2001)

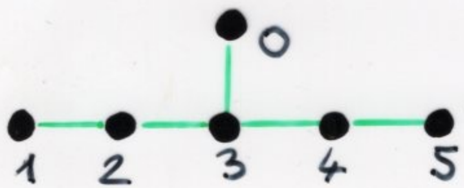
exercise (easy)

prove the first part (G is a tree)

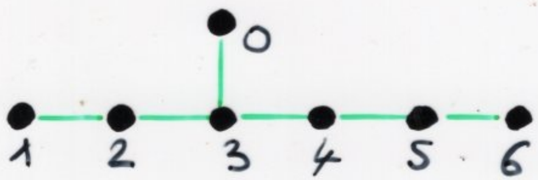
Dynkin diagrams

A_n

D_n



E_6



E_7

irreducible
"minuscule" posets

Proctor (1984)

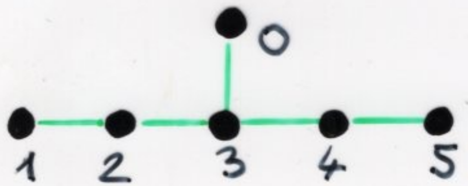
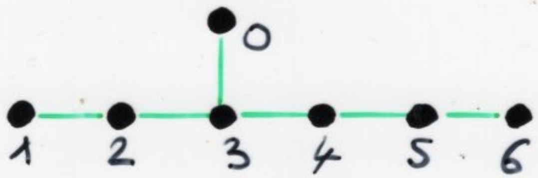
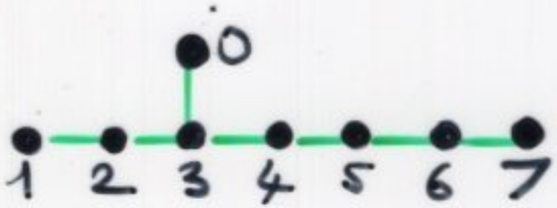
minuscule representation

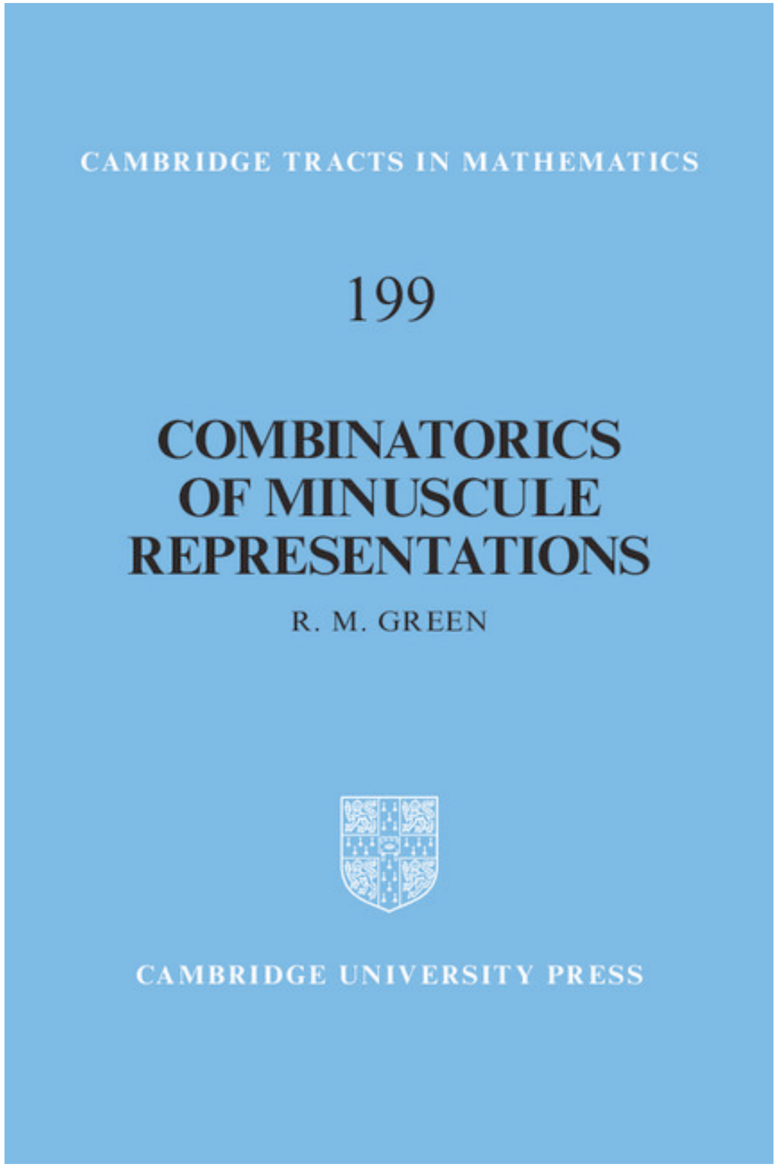
raising
lowering } operators

on spaces of ideals
of heaps

Wildberger (2000)

all the simply-laced
simple Lie algebras
have minuscule representations
with the sole exception of E_8

Dynkin diagrams		number of such heaps
	A_n	n
	D_n	3
	E_6	2
	E_7	1
	E_8	0

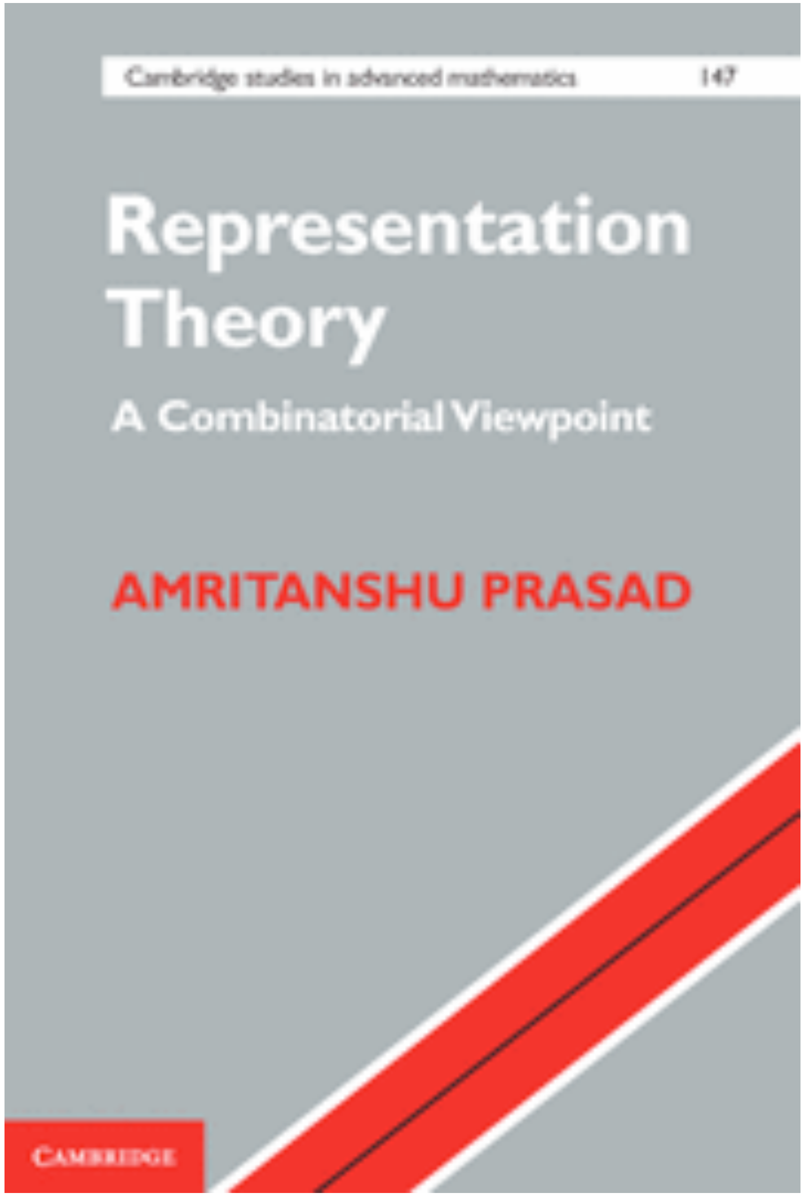


R. Green (2013)

Introduction;
1. Classical Lie algebras and Weyl groups;
2. . Heaps over graphs;
3. 3. Weyl group actions;
4. 4. Lie theory;
5. 5. Minuscule representations;
6. 6. Full heaps over affine Dynkin diagrams;
7. 7. Chevalley bases;
8. 8. Combinatorics of Weyl groups;
9. 9. The 28 bitangents;
10. 10. Exceptional structures; 1
11. 1. Further topics;
12. Appendix A. Posets graphs and categories;
13. Appendix B. Lie theoretic data; References;
14. Index.

Team R. Green: Hugh Denoncourt, Brent Pohlmann, Dana Ernst, Richard Green, Jacob Harper, Strider McGregor-Dorsey, Tyson Gern

Combinatorics Of Coxeter Groups
AMS Special Session, 2011 Spring Eastern Sectional Meeting
College of the Holy Cross, Worcester, MA, April 9-10, 2011


Cambridge studies in advanced mathematics 147
Representation Theory
A Combinatorial Viewpoint
AMRITANSHU PRASAD
CAMBRIDGE

algebraic graph theory

$G = (V, E)$

graph $\begin{cases} V & \text{vertices} \\ E & \text{(non-oriented) edges } \{u, v\} \end{cases}$

combinatorial properties of graphs $\longleftrightarrow$ algebraic objects

- polynomials
- vector spaces
- power series
- ....

N. Biggs "algebraic graph theory" (1974)

connection with
- Statistical physics
- Knots theory
- Lie algebra
- Heaps theory

some polynomials or numbers associated to a graph

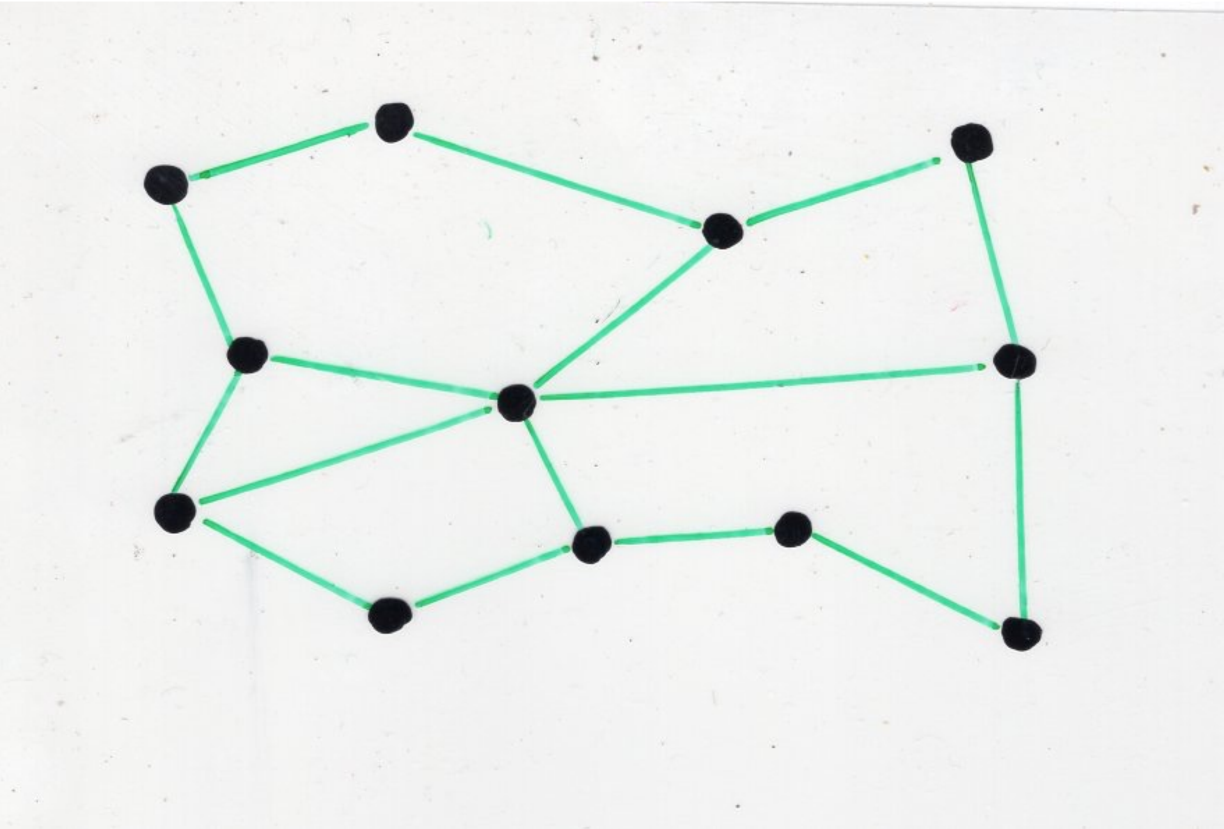
characteristic polynomial of a graph G

$$A = (a_{ij})_{1 \leq i, j \leq n}$$

adjacency matrix

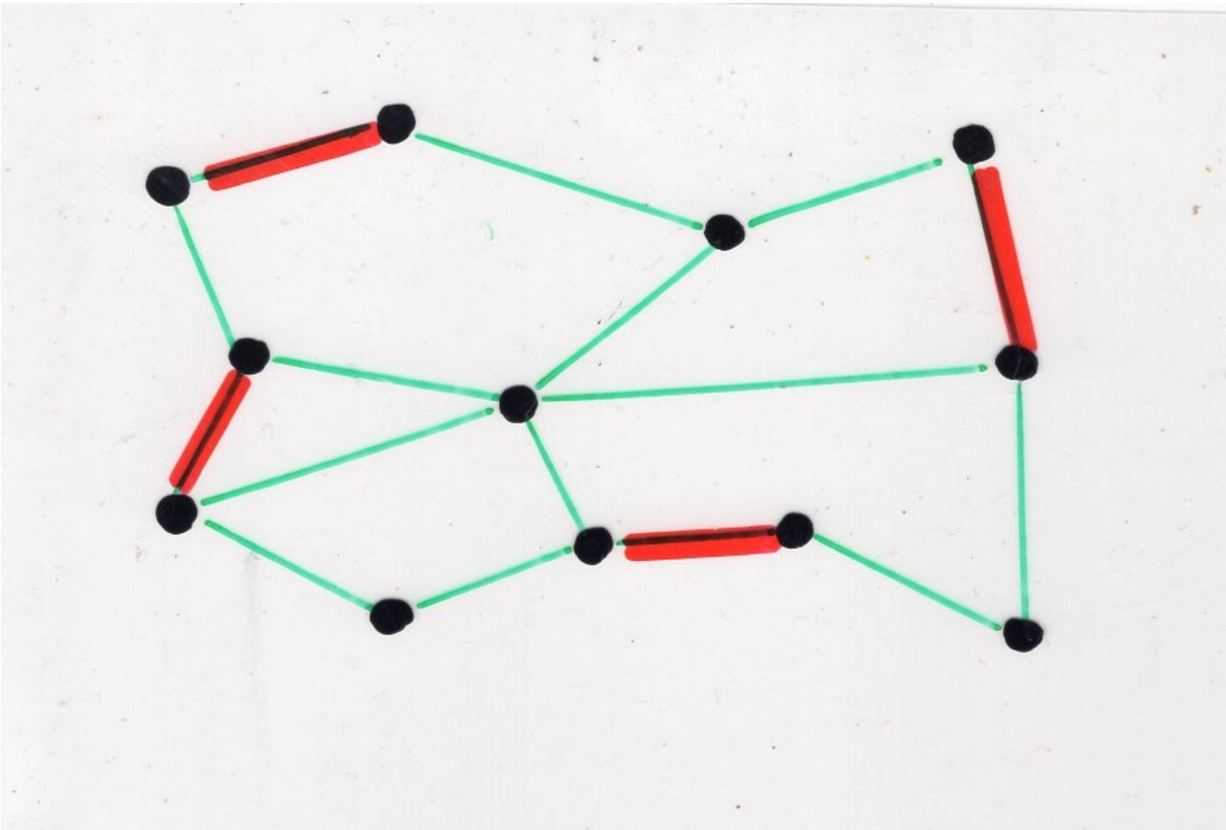
$$a_{ij} = \begin{cases} 1 & i \bullet\!\!-\!\!\bullet j \\ 0 & \text{no edge} \end{cases}$$

$$\chi(x) = \det(Ix - A)$$

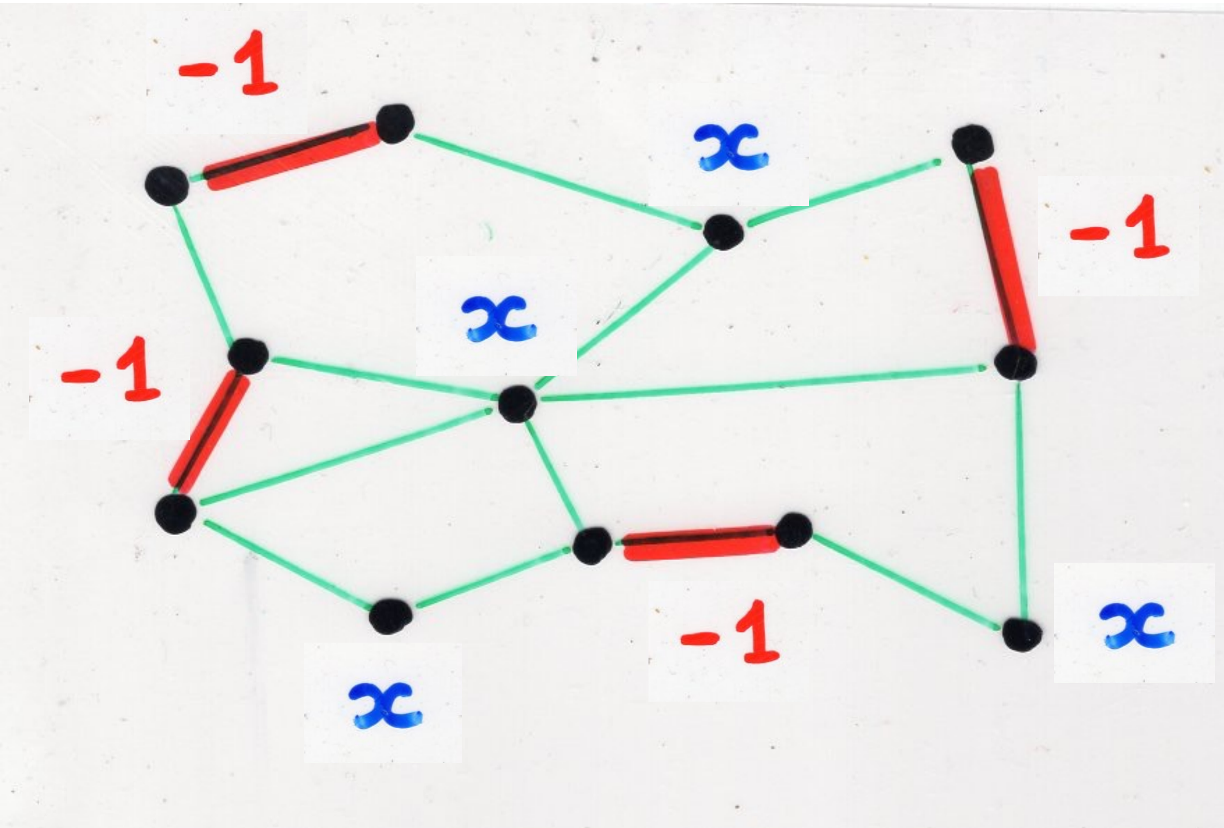


→ Ch 2c

matching polynomial of a graph G



matching
of a graph G = set of 2 by 2
disjoint edges



→ Ch 2c

matching polynomial of a graph G

- number of perfect matchings

  constant term
  of the matching polynomial

→ Pfaffian, determinant ...
(for planar graph).

→ statistical mechanics
Ising model for magnetism

$\gamma_G(\lambda)$ chromatic polynomial

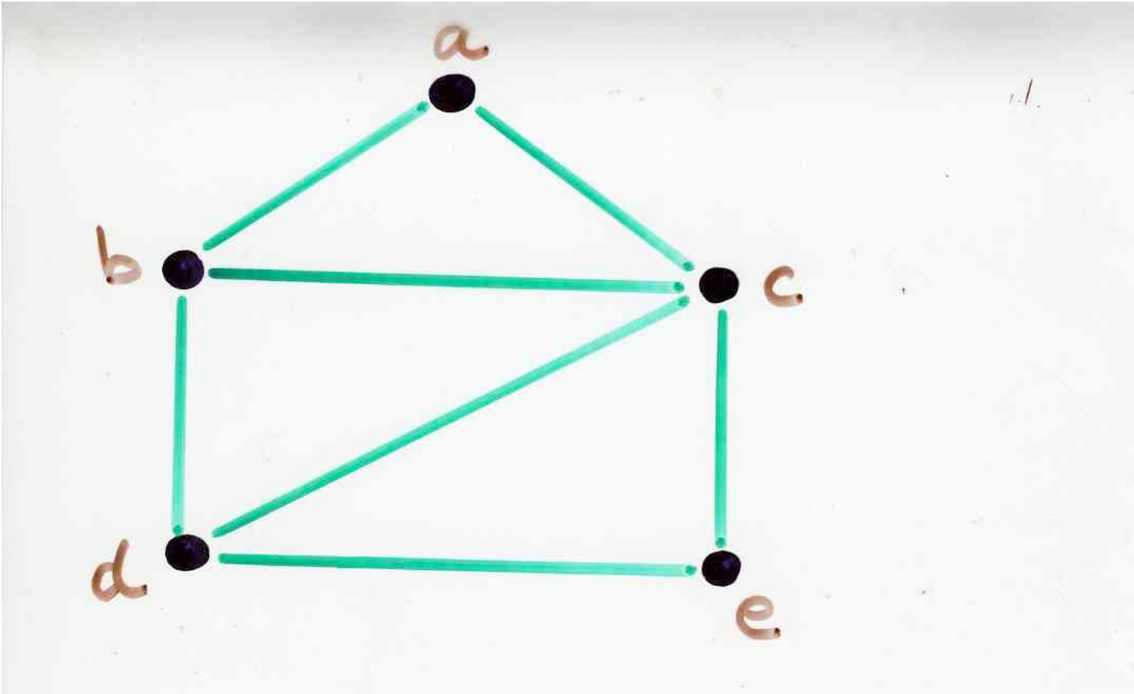
chromatic number $\gamma(G)$

= smallest number γ
such that $\gamma_G(\gamma) \neq 0$

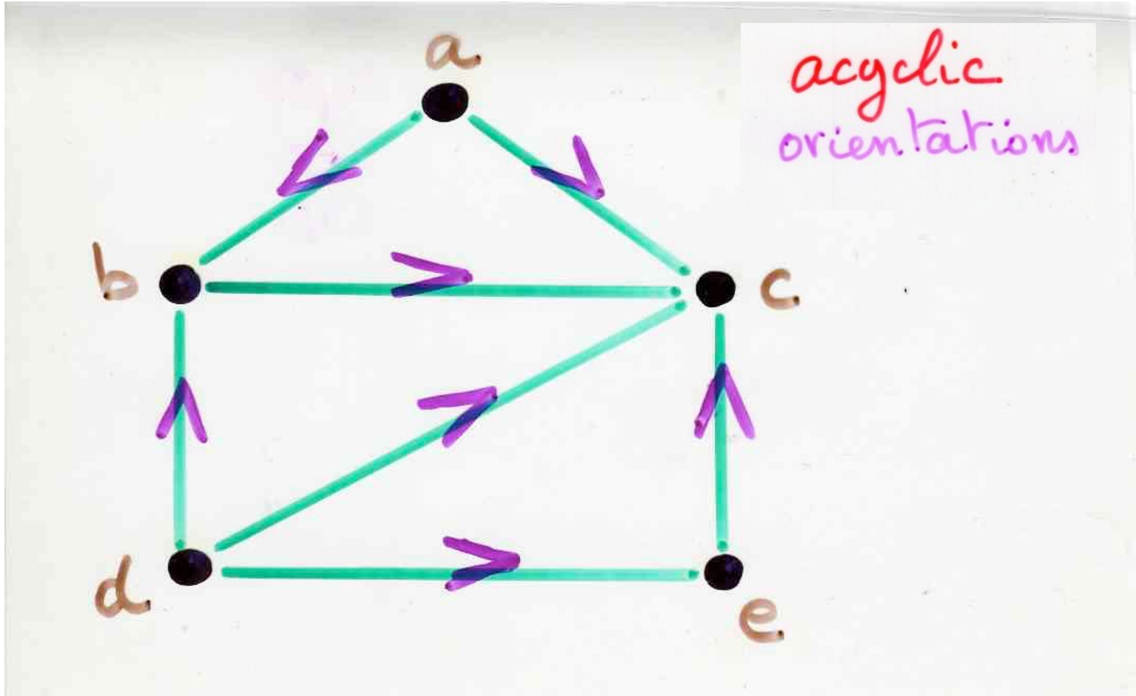
$\to$ zeros of $\gamma_G(\lambda)$

The 4 colors theorem is
"almost" false ....

- number of acyclic orientations of a graph

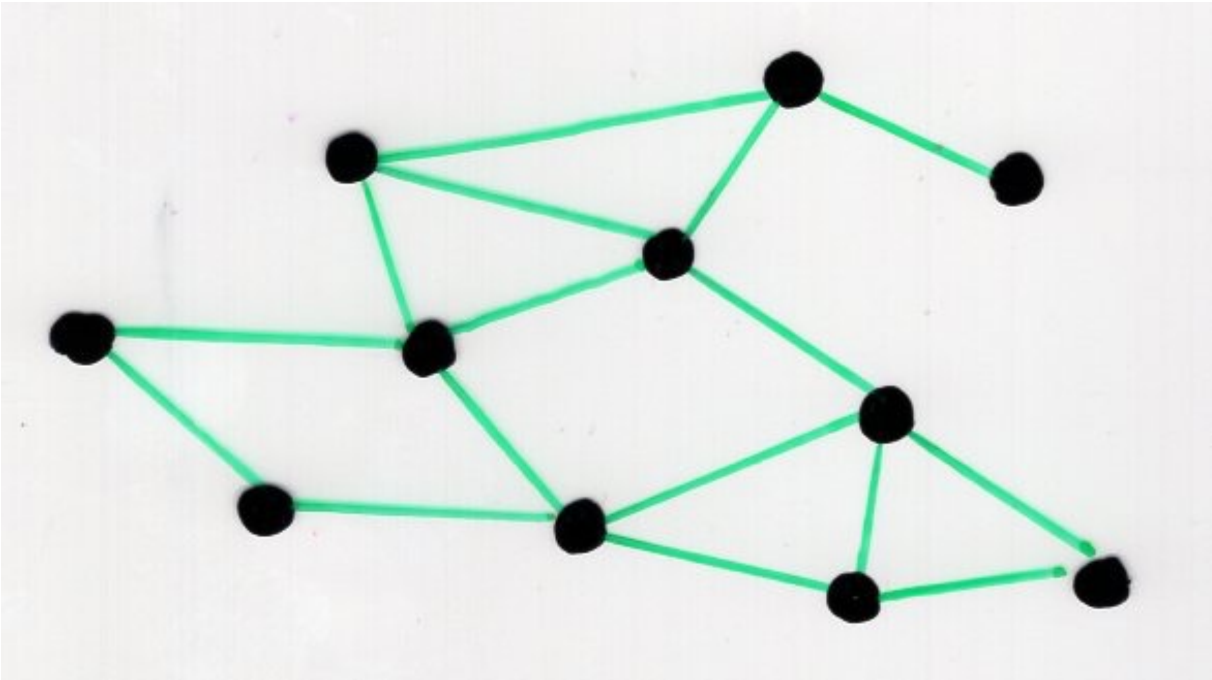


• number of acyclic orientations of a graph

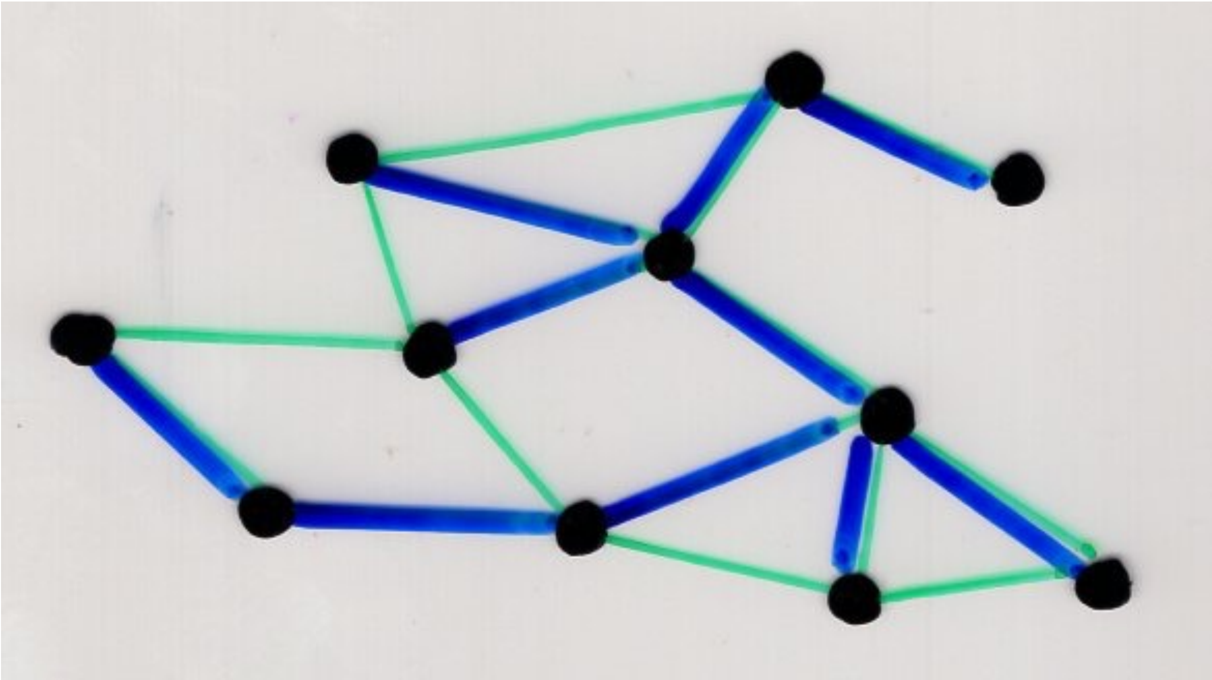


spanning tree

of a graph $G = (V, E)$



spanning tree
of a graph $G = (V, E)$



• number of spanning tree

Tutte polynomial

$$T(x,y) \quad \sum_{T} x^{i(T)} y^{e(T)}$$

spanning trees

→ Potts model

$T(1,1)$ = number of spanning trees

$T(2,0)$ = chromatic number

Ihara-Selberg zeta function of a graph

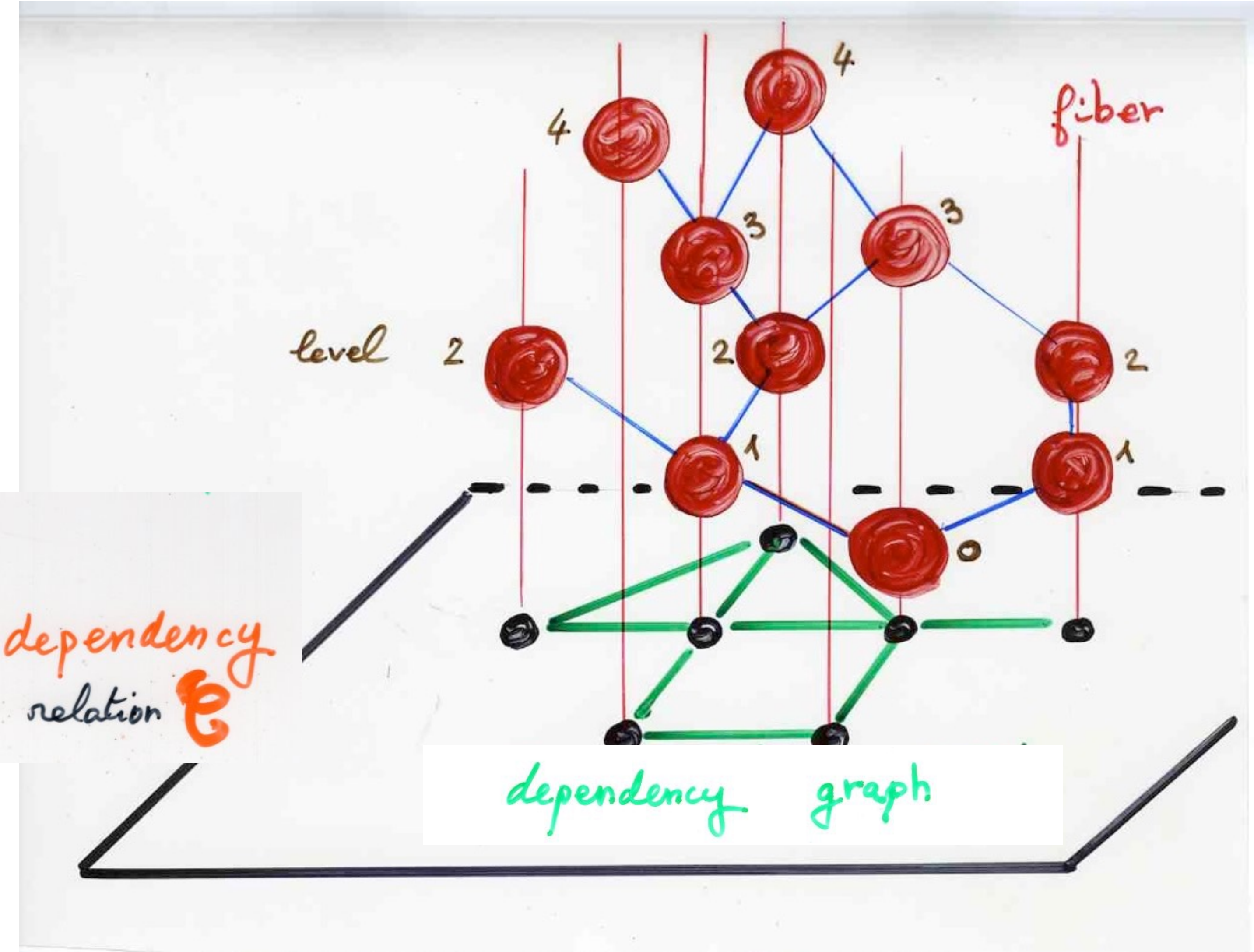
→ Ch5b

extension of Riemann zeta function

$$\sum_{n\geq 1} n^{-s}$$

$G = (V, E) \longrightarrow$ heap monoid

$$\mathbf{H}(G) = \mathbf{H}(V, E)$$



$G = (V, E)$

basic pieces

dependency relation $\mathcal{E}$

chromatic polynomial
and
acyclic orientations of a graph

graph $G = (V, E)$

$\gamma_G(\lambda)$ chromatic polynomial

number of (proper) coloring of the graph G with λ colors

$\alpha(G)$ number of acyclic orientations of G

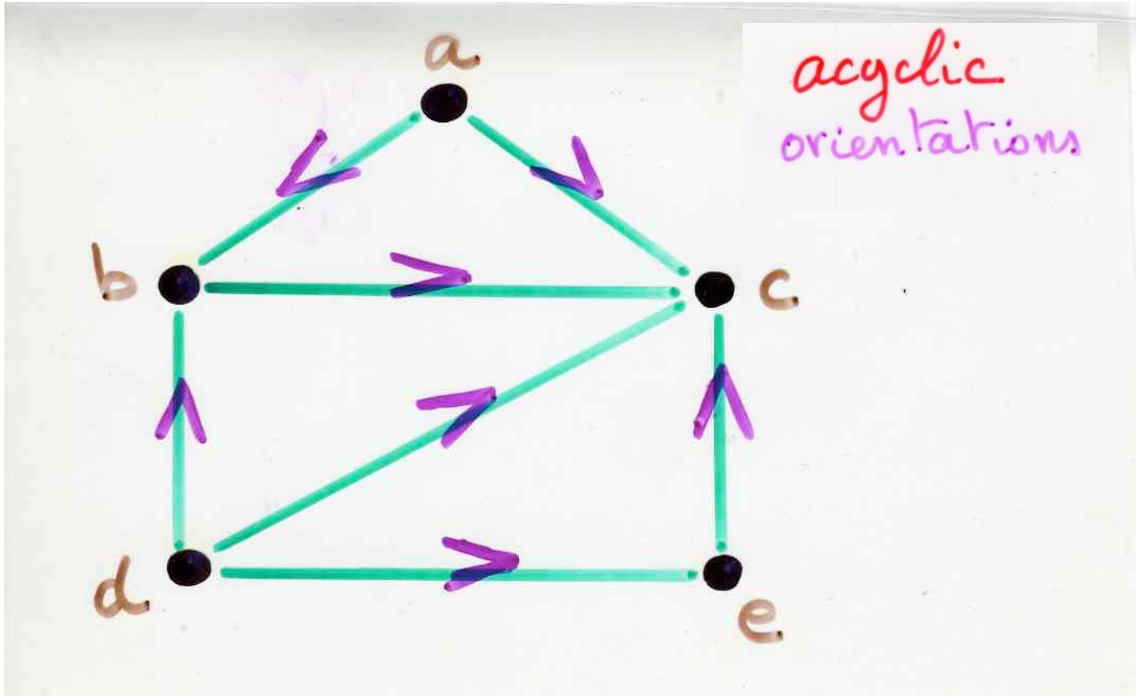
$n(G) = |V|$ number of vertices

Proposition (Stanley, 1973)

$$\alpha(G) = (-1)^{n(G)} \gamma_G(-1)$$

Proposition (Stanley, 1973)

$$a(G) = (-1)^{n(G)} \gamma_G(-1)$$



proof using commutation (Cartier-Foata) monoid

from Gessel (1985)?

4 ideas

- (proper) coloring gives a partition of the vertices V of the graph G into trivial heaps
(called in graph theory indepent sets)

sequence of trivial heaps
$\longrightarrow$ a heap on the graph G

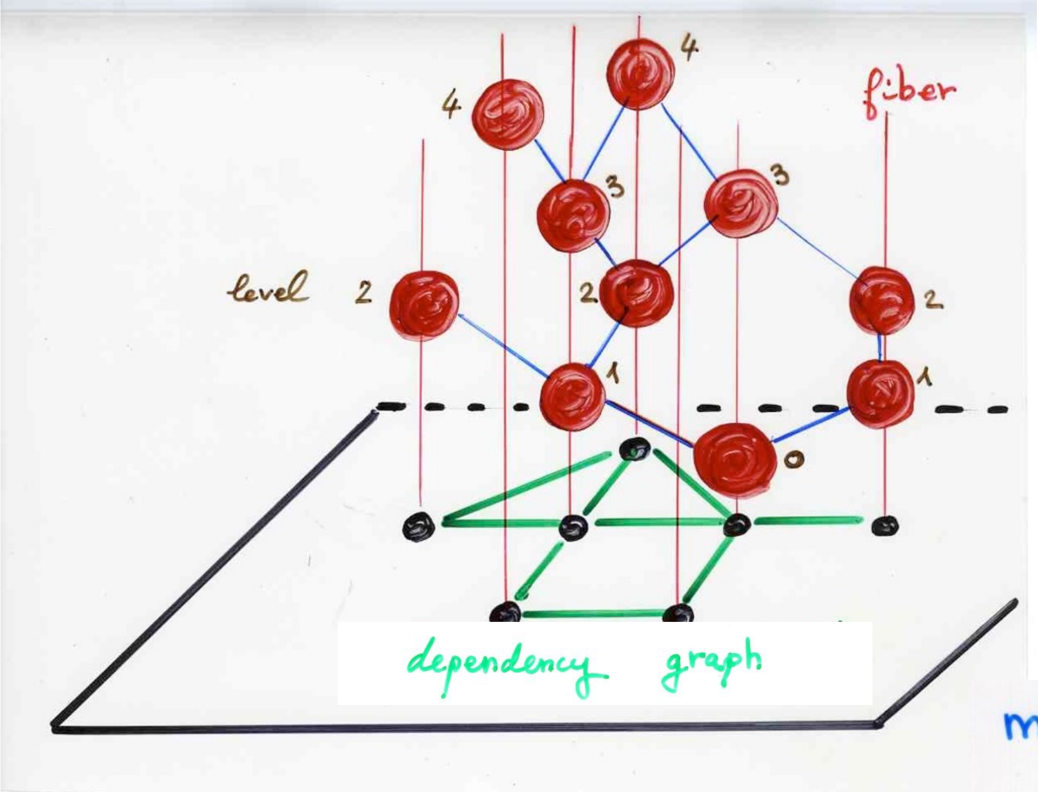
- if f is the generating function of combinatorial objects

  $\dfrac{1}{1-f}$ g.f. of sequences of such objects

- Inversion Lemma for heaps (or commutation) monoids

- multilinear heaps

Definition A heap F is multilinear iff in each fiber $\pi^{-1}(v)$, $v \in V$ there is one and only one piece of F

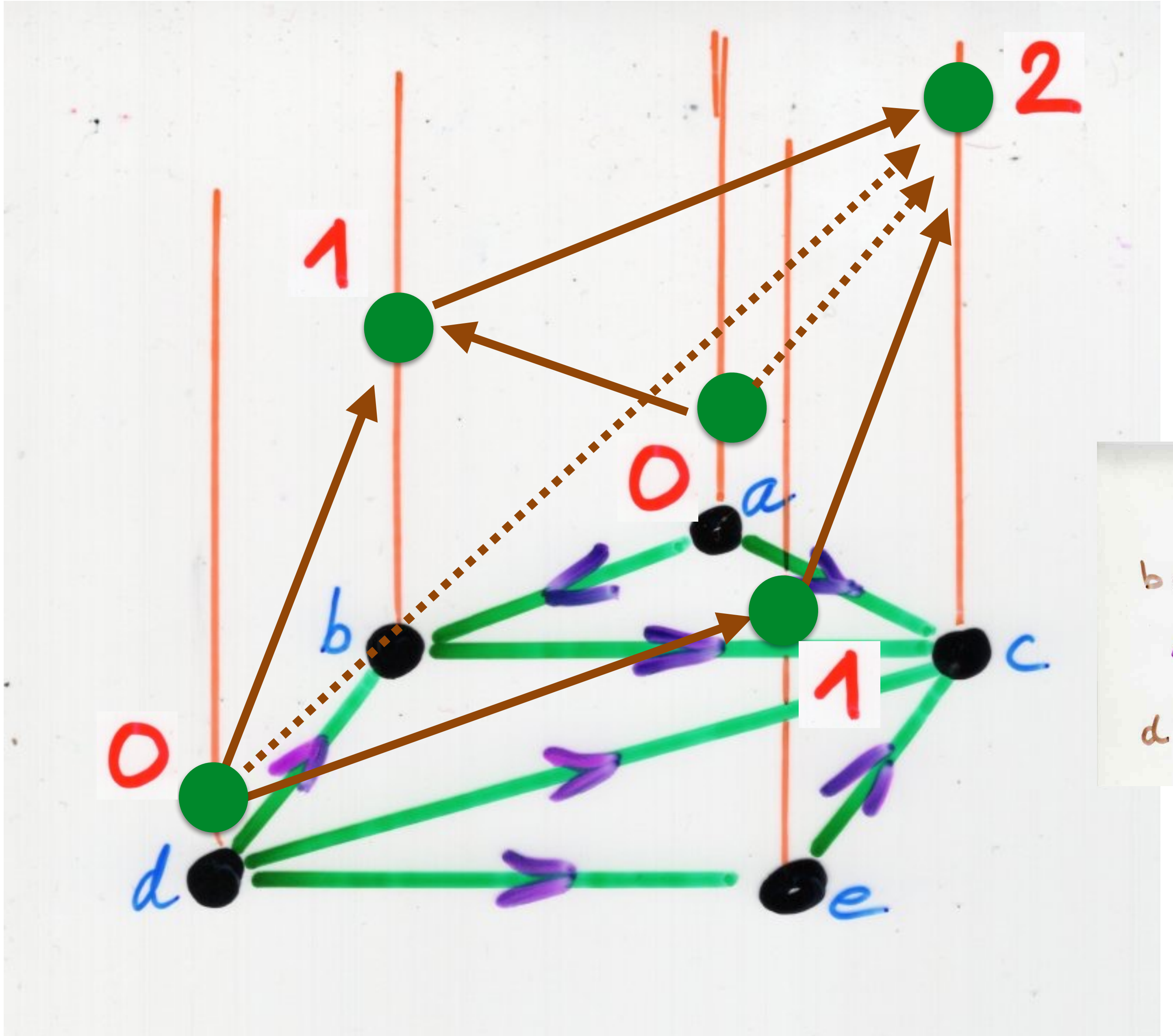


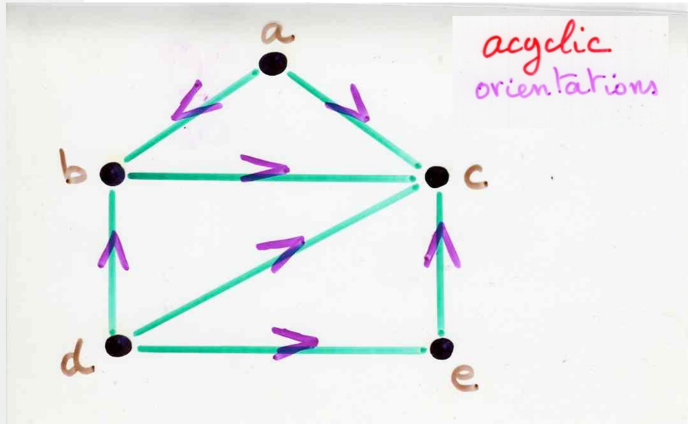
Bijection

multilinear heaps on G $\longleftrightarrow$ acyclic orientations of G

Bijection

multilinear heaps on G $\longleftrightarrow$ acyclic orientations of G





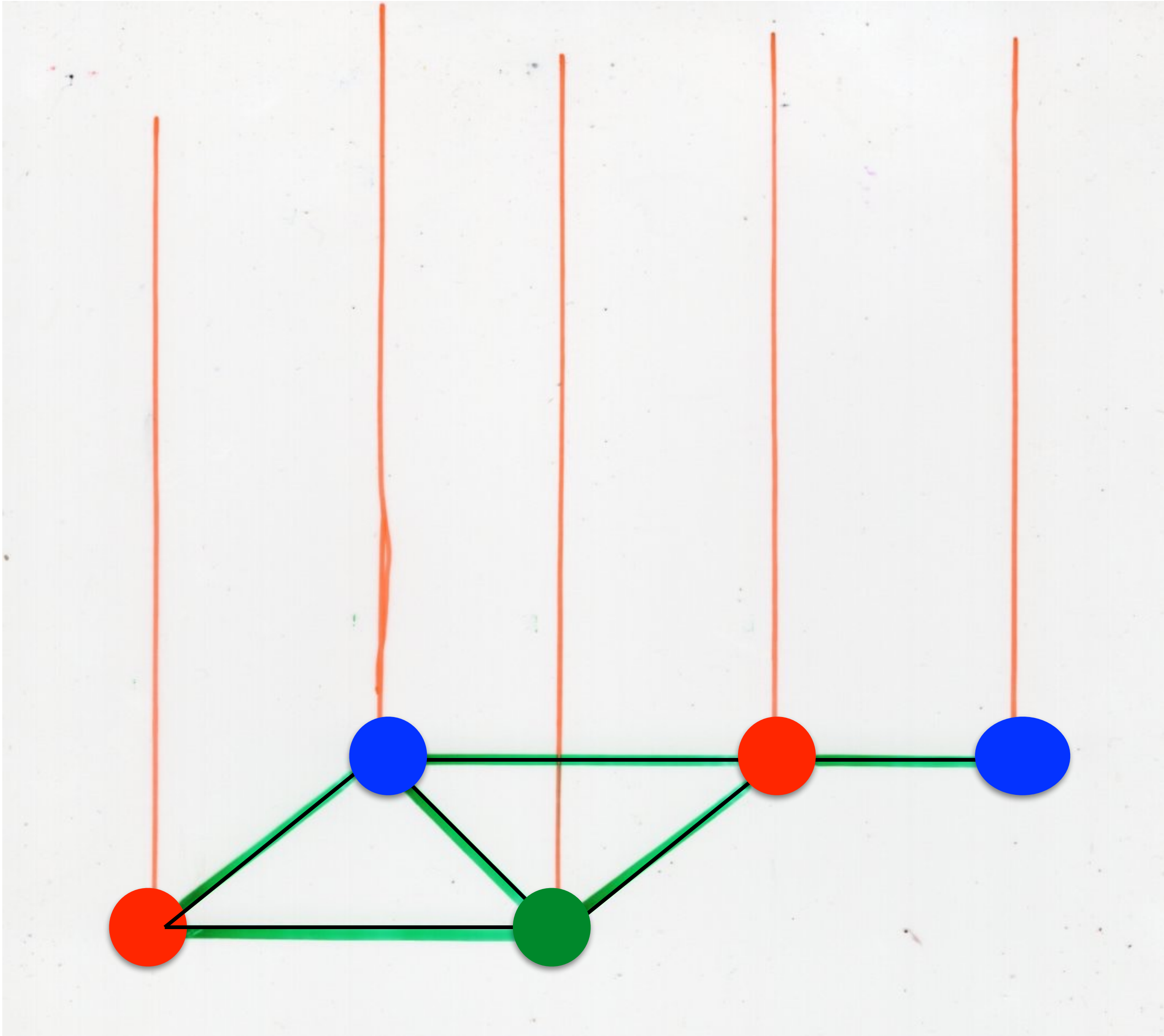
λ possible colors k are used

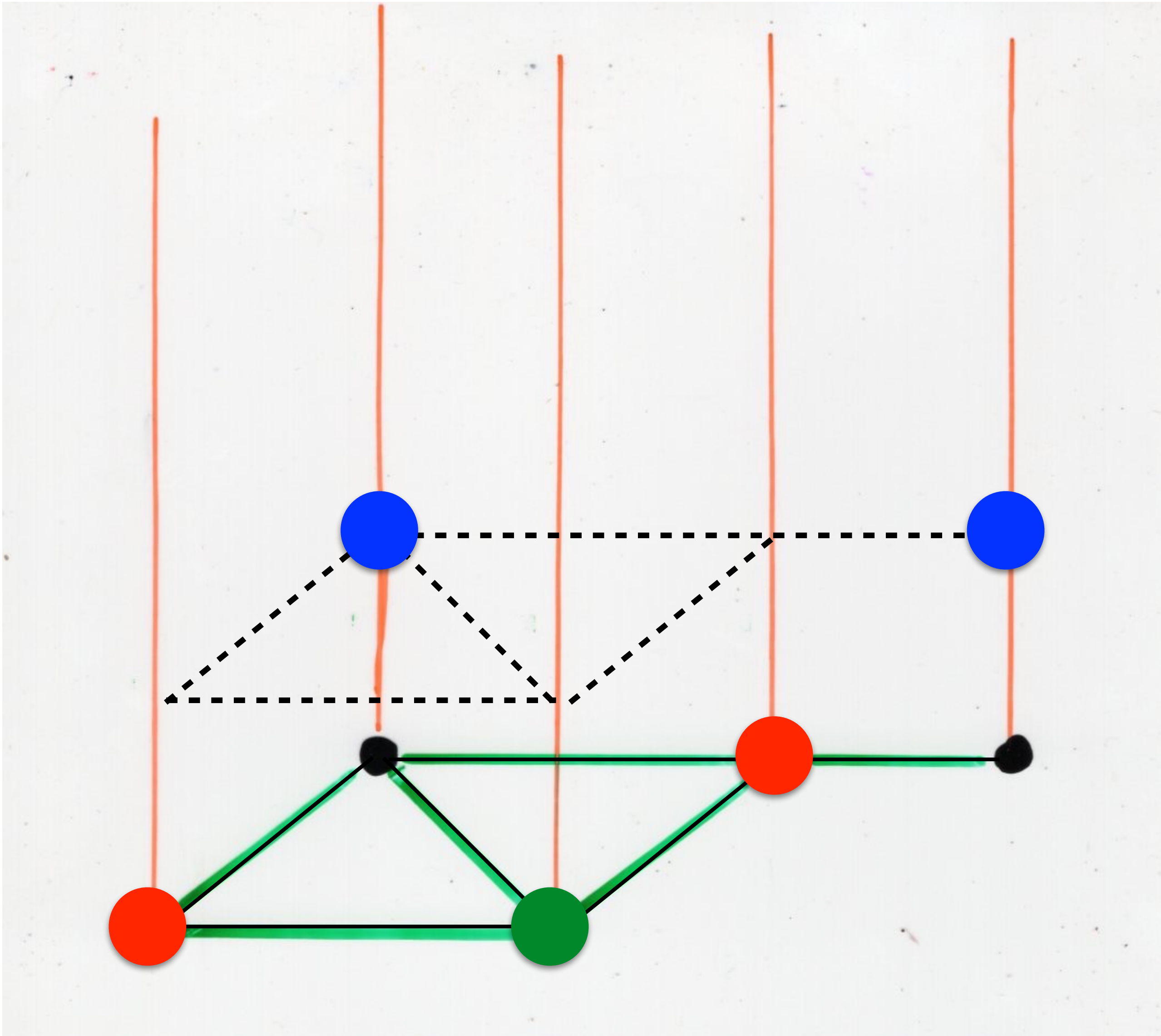
define a total order on the colors
$c_1, \dots, c_k$

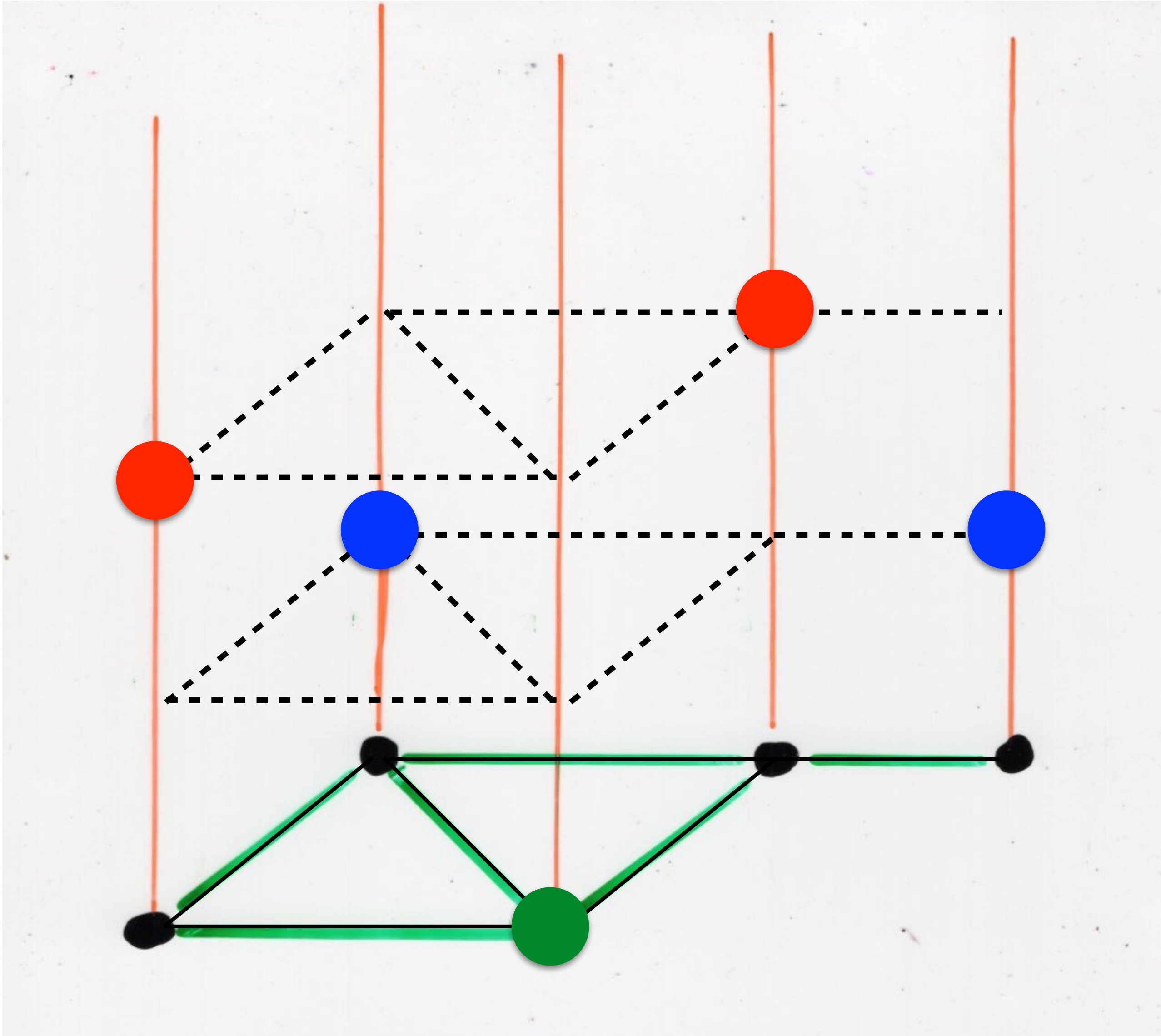
$\longrightarrow$ $(T_1, \dots, T_k)$

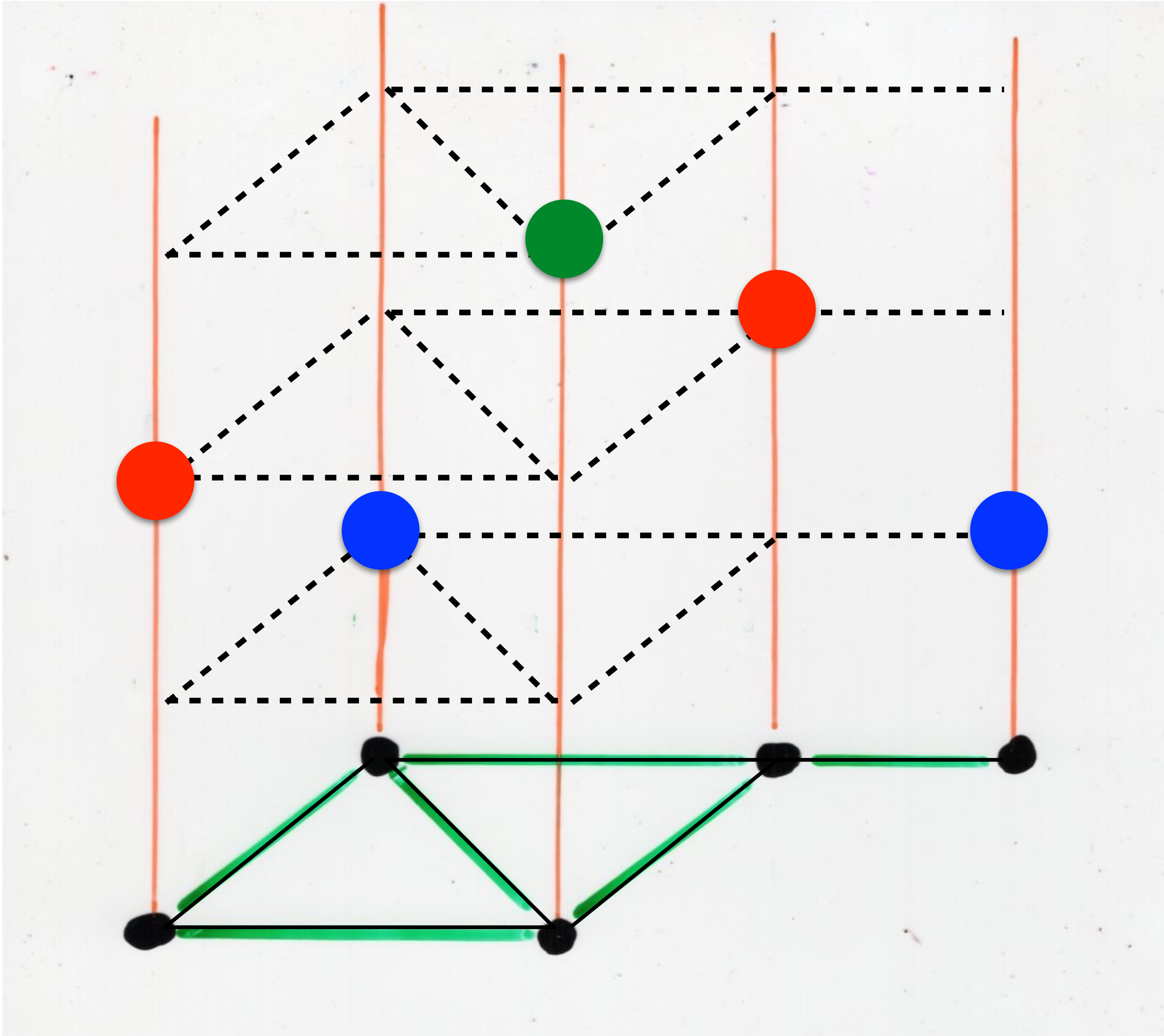
sequence of trivial heaps

$F = T_1 \odot \cdots \odot T_k$
is a multilinear heap









Stanley (1973)

Proposition The chromatic polynomial $\gamma_G(\lambda)$ is the number of pairs (σ, α), $\sigma : V \to \{1, 2, .., \lambda\}$ and α is an orientation of the edges of G such that:

(i) α is acyclic

(ii) if $u \to v$ is in the orientation α $u, v \in V$, then $\sigma(u) < \sigma(v)$

Definition

F heap of $H(V, E)$

a **layer factorization** of F is a sequence $(T_1, \dots, T_k)$ of trivial heaps such that $F = T_1 \odot \cdots \odot T_k$

(product of heaps)

$(F; (T_1, \dots, T_k))$ is called a layered heap

$\beta_k(F)$ number of layer factorizations of F

<u>Definition</u> <u>colored layered heap</u> is a layered heap $(F; (T_1, \ldots, T_k))$ where each layer T_i is colored
(i.e. all the pieces of T_i have the same color)
with the condition that all layers have distinct colors

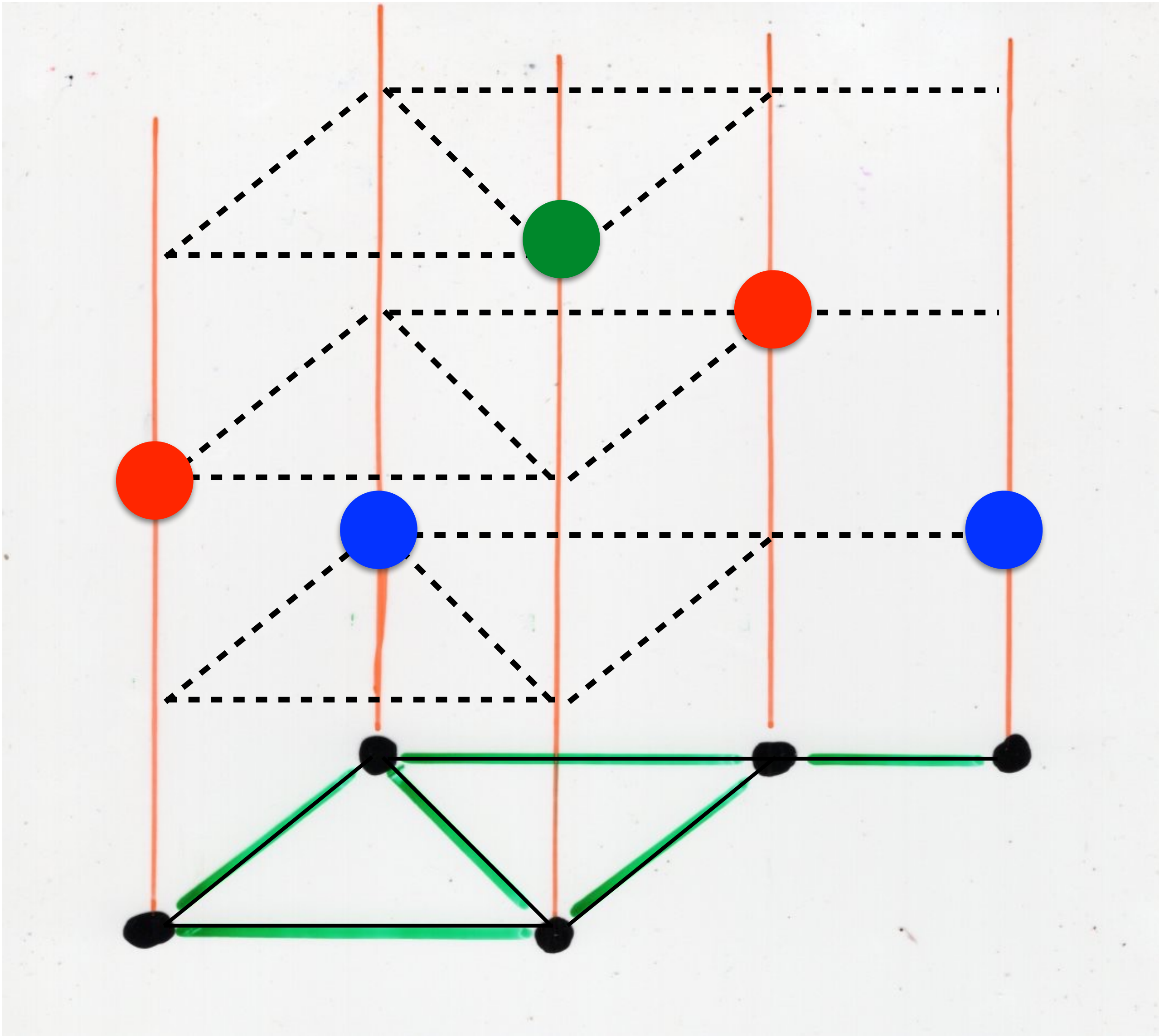
If λ is the number of possible colors, the number of colored layer associated to the heap F is:

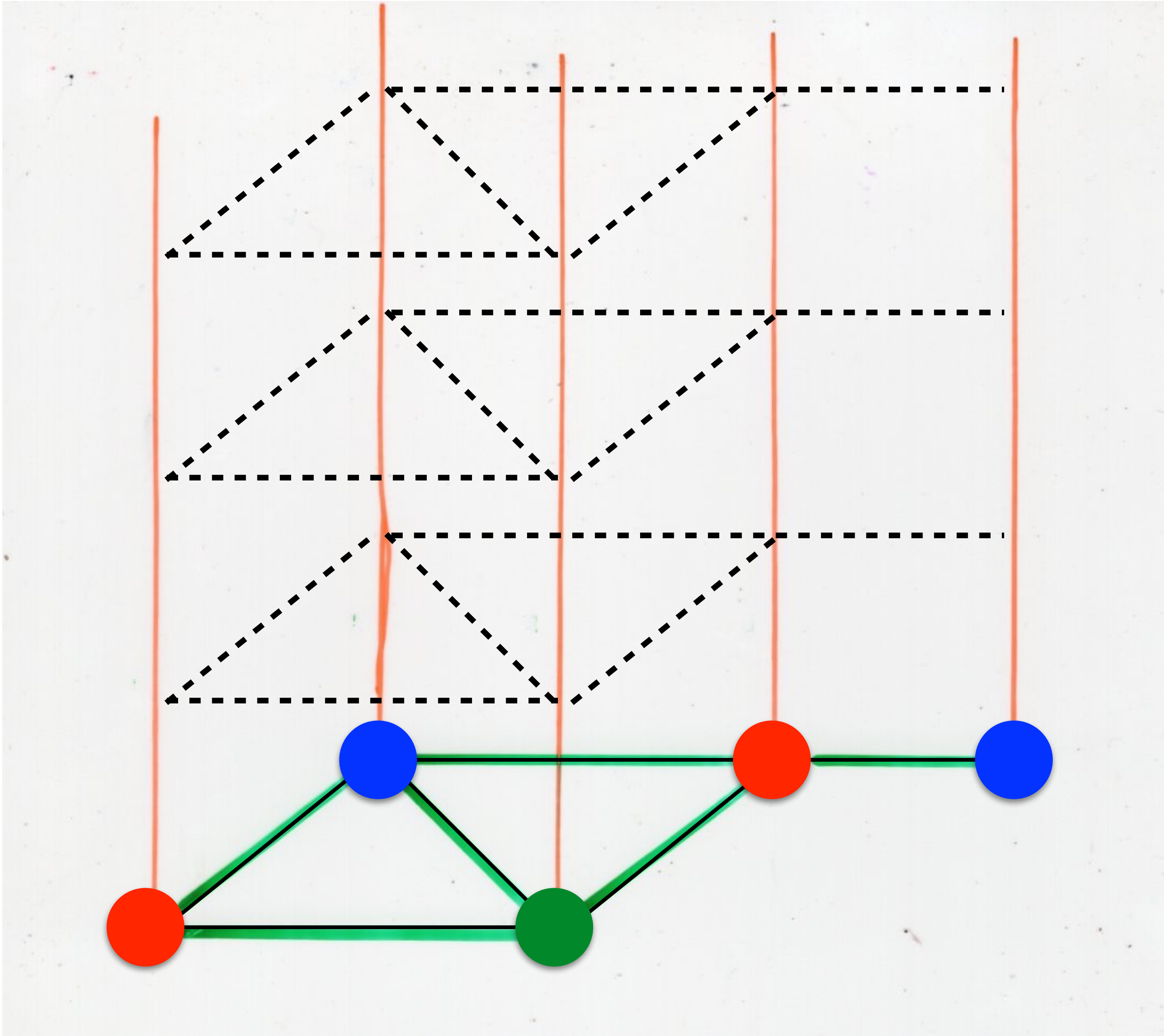
$$\beta_k(F)\, \lambda(\lambda-1)\cdots(\lambda-k+1)$$

<u>Definition</u> A heap F is covering the graph G iff for any vertex $v \in V$ of G the fiber above v is not empty
(the fiber is the chain of pieces of F with projection on v)

(the fiber above v is the chain $\pi^{-1}(v)$)

multilinear $\leftrightarrow$ ordered coloring
colored layered
heap





Proposition

$$\gamma_G(\lambda) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap over } G}} \sum_{k \geq 1} \beta_k(F) \frac{\lambda(\lambda-1)\ldots(\lambda-k+1)}{k!}$$

$$\binom{\lambda}{k}$$

$$\gamma_G(-1) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap over } G}} \sum_{k \geq 1} \beta_k(F) (-1)^k$$

Definition multicoloring of the graph G
associated to $\mathbf{k} = (k_1, \ldots, k_n)$

$|V| = n \quad V = (1, 2, \ldots, n)$

is an assignement of colors to the vertices of G in vertex $i \in V$ receives k_i colors, such that adjacent vertices receive only disjoint colors.

$\gamma^{G}_{\mathbf{k}}(\lambda)$

number of multicoloring associated to $\mathbf{k}$ with λ colors

Bijection

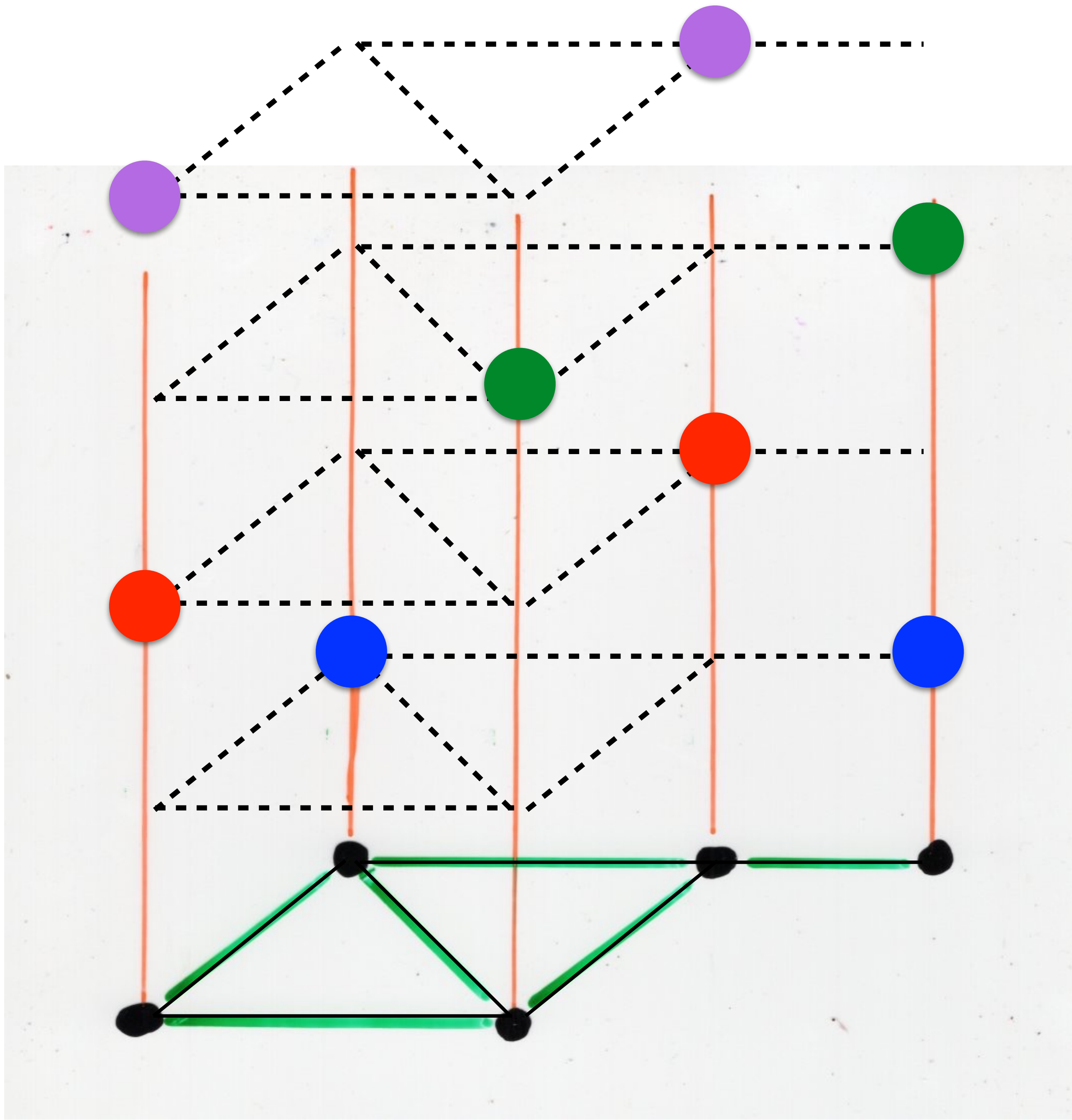
colored layered heap covering G
(having k layers)

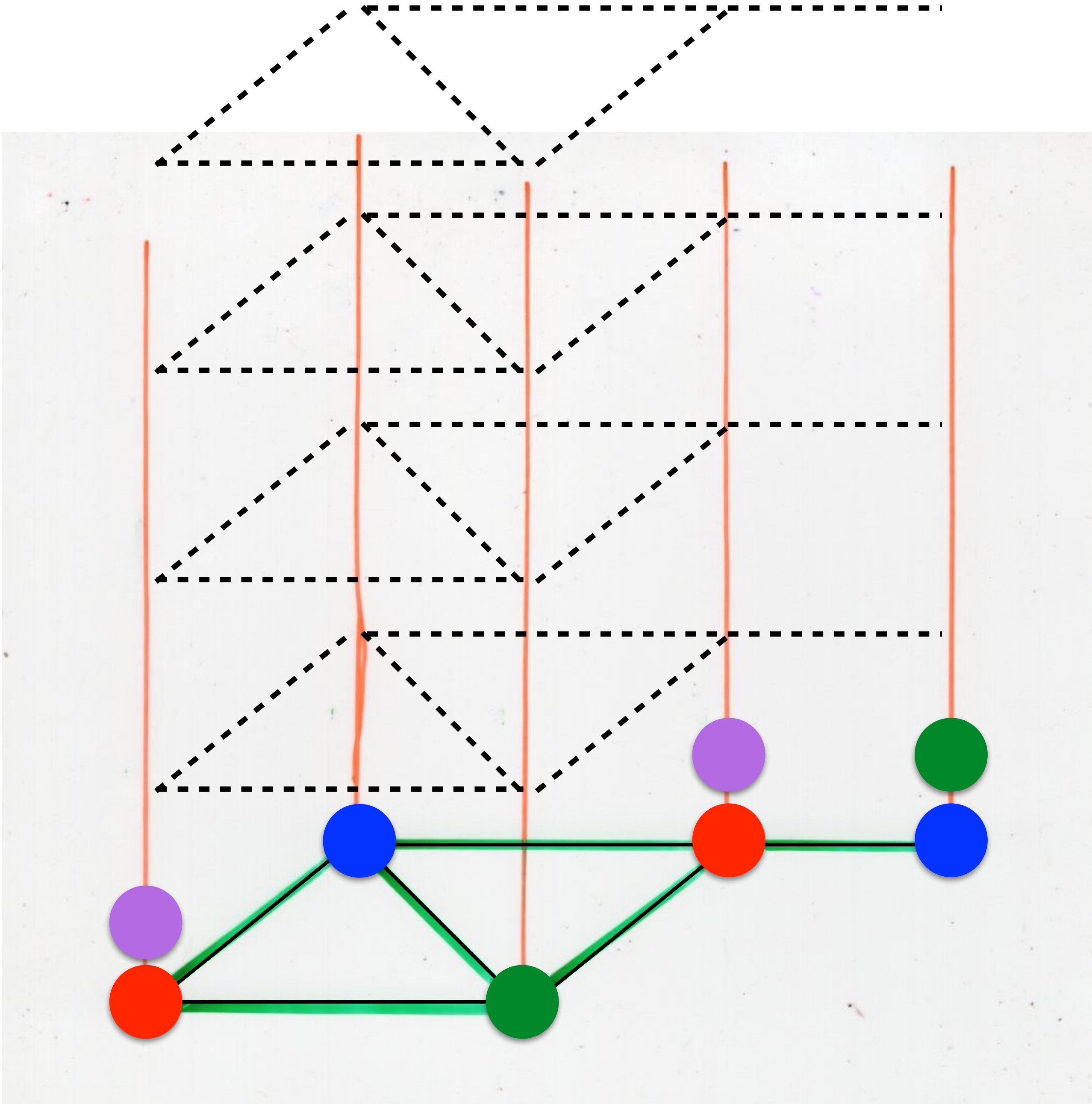
$\updownarrow$

ordered multicoloring
(i.e. the k colors used in the multicoloring are totaly ordered)

multilinear colored layered heap $\leftrightarrow$ ordered coloring

$k = (1, 1, \ldots, 1)$





« Behind the walls »
Jean-Pierre Muller 2013

« Behind the walls »
Jean-Pierre Muller 2013

« Behind the walls »
Jean-Pierre Muller   2013

« Behind the walls »
Jean-Pierre Muller 2013

- multi-chromatic polynomial related to root multiplicities for Borcherds-Kac-Moody algebras

Arunkumar, Kus, Venkatesh (2016)

- chromatic polynomials from Kac-Moody algebras

Venkatesh, Viswanath (2015)

<u>Definition</u> Chromatic power series of the graph G (with weighted heaps)

$$\Gamma_G^{v}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F)\, \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

$$\gamma_G(\lambda)$$

multilinear

$\longrightarrow$ interpretation for $\gamma_{\mathbf{k}}^{G}(\lambda)$ $\quad \mathbf{k} = (k_1, \ldots, k_n)$

sequence of trivial heaps

$(T_1, \dots, T_k)$

$$f = \sum_{T} v(T)$$

generating function of trivial heaps

$$\frac{1}{1-f}$$

g.f. of sequence of trivial heaps

add a variable t for taking account of the parameter k

$$\frac{1}{1-t\left(\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} v(T)\right)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, t^k$$

$t = -1$ $\qquad \bar{v}(\alpha) = -v(\alpha)$

α basic piece
= vertex of G

$$\frac{1}{1+\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} (-1)^{|T|}\, \bar{v}(T)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \bar{v}(F)$$

$$\frac{1}{1+\left(\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} v(T)\right)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k$$

$$t = -1 \qquad \overline{v}(\alpha) = -v(\alpha)$$

α basic piece
= vertex of G

$$=$$

$$\frac{1}{1+\sum_{\substack{T \\ \text{trivial} \\ \text{heap}}} (-1)^{|T|}(T)} = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

bijective proof
with involution

Definition Chromatic power series of the graph G (with weighted heaps)

$$\Gamma_G^{v}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F)\, \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G \\ \text{covering } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G \\ \text{covering } G}} (-1)^{|F|}\, v(F)$$

$\lambda = -1$

<u>Definition</u> Chromatic power series of the graph G (with weighted heaps)

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covering G

covering G

$$\Gamma_G^{v}(-1)$$

$$\Gamma_G^{v}(-1) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} (-1)^{|F|}\, v(F)$$

$$\sum_{\substack{F \\ \text{heap} \\ \text{on } G}} \beta_k(F)\, v(F)\, (-1)^k = \sum_{\substack{F \\ \text{heap} \\ \text{on } G}} (-1)^{|F|}\, v(F)$$

covering G

multilinear

covering G

multilinear

$$\gamma_G^v(-1)$$

$$\gamma_G^v(-1) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap on } G}} (-1)^{n(G)}\, v(F)$$

$$\gamma_G^{\,v}(-1) = \sum_{\substack{F \\ \text{multilinear} \\ \text{heap on } G}} (-1)^{n(G)} \, v(F)$$

$\downarrow$

number of acyclic orientations of G

$$v(\alpha) = 1 \qquad \alpha \in V$$

Bijection

multilinear heaps on G $\longleftrightarrow$ acyclic orientations of G

$\square$

end of proof

<u>exercise</u> prove with heaps the following theorem known as

Gallai - Hasse - Roy - Vitaver
(1968) (1965) (1967) (1962) (conjecture Berge (1958))

$\Updownarrow$

- G has an acyclic orientation in which the longest path has at most k vertices
- G can be colored with at most k colors

Stanley (1973)

~~Proposition~~ The chromatic polynomial $\gamma_G(\lambda)$ is the number of pairs (σ, α), $\sigma : V \to \{1, 2, .., \lambda\}$ and α is an orientation of the edges of G such that:

(i) α is acyclic

(ii) if $u \to v$ is in the orientation α, $u, v \in V$, then $\sigma(u) \leq \sigma(v)$

Research? exercise

$$\overline{\gamma}_G(\lambda) = (-1)^{n(G)}\gamma(-\lambda)$$

prove using heaps "philosophy"

analogous to $\binom{n+k-1}{k} = (-1)^k \binom{-n}{k}$

$\binom{n}{k}$ combinations without repetition

combinations with repetition

Research ? exercise

Greene, Zaslavsky (1983)

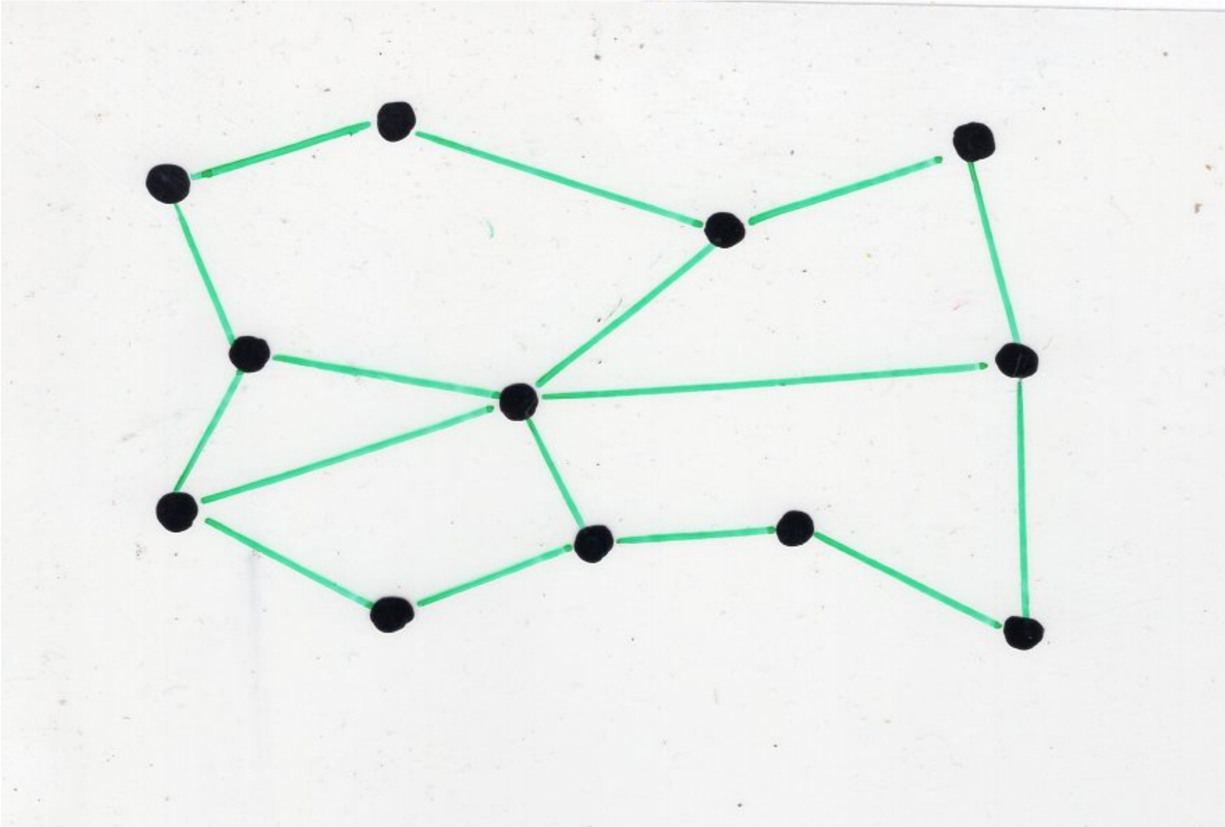
- number of acyclic orientations with one sink $= \pm$ linear term of $\gamma_G(\lambda)$

→ proved with hyperplane arrangements

Gebhard, Sagan (2000) 3 other proofs

→ Lass (2001)
proof with heaps

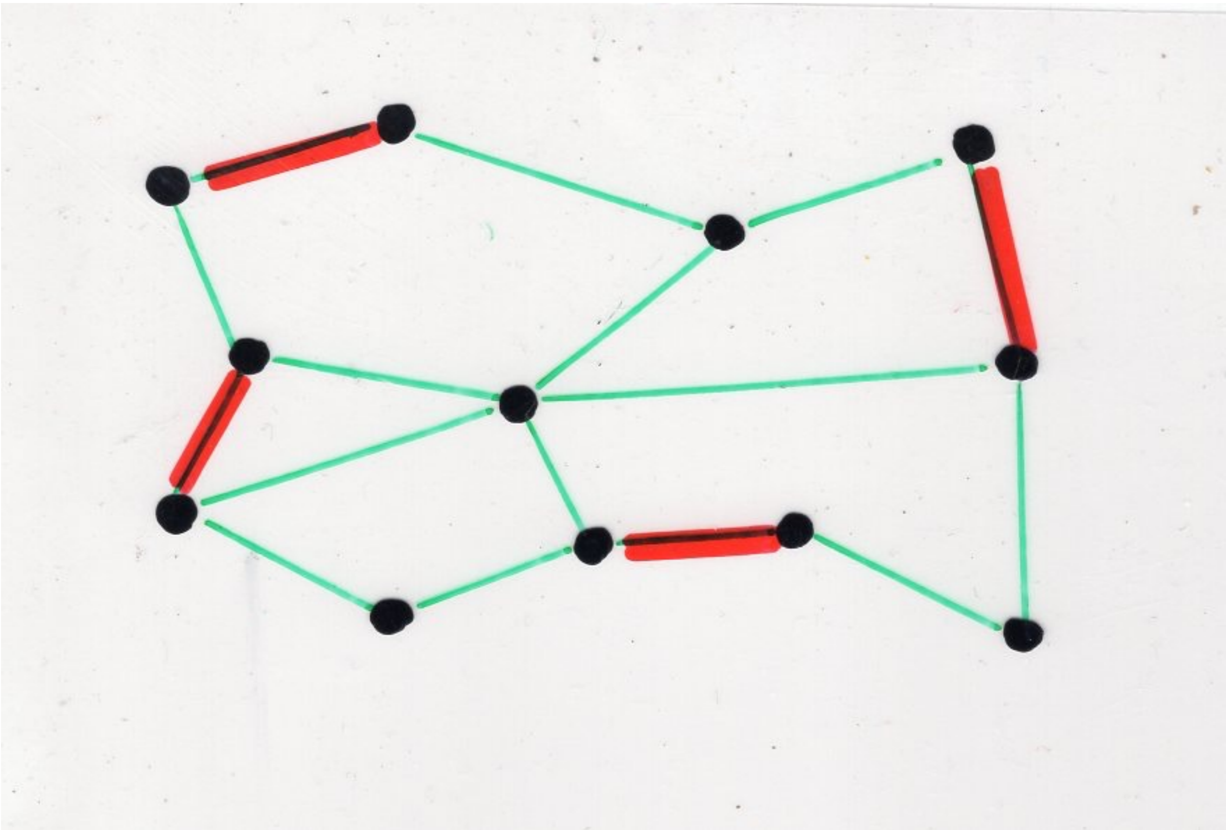
matching polynomial



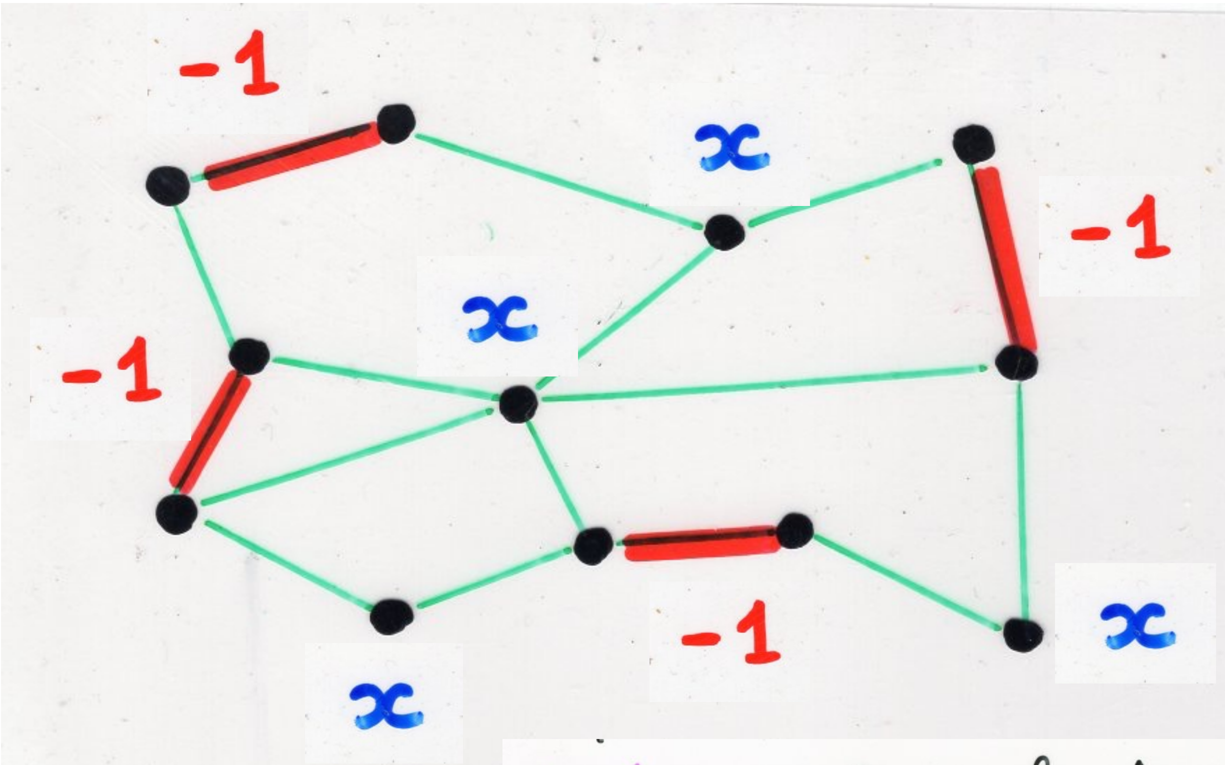
matching
polynomial
of a graph G

→ Ch 2c

Tchebycheff 1st, 2nd kind
Fibonacci, Lucas polynomials



matching of a graph G = set of 2 by 2 disjoint edges



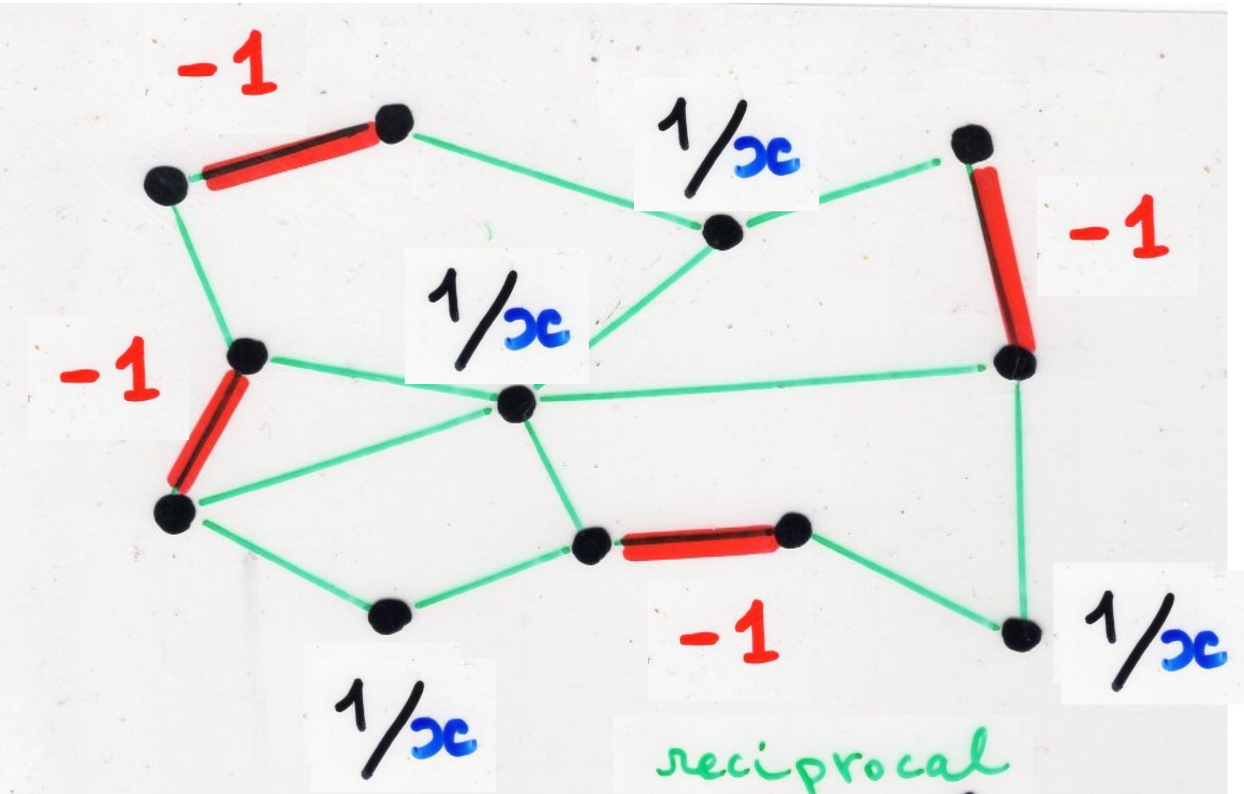
Matching polynomial of a graph G

$$M_G(x) = \sum_{\substack{\text{matchings } M \\ \text{of } G}} (-1)^{|M|} \, x^{ip(M)}$$

$ip(M)$ = number of isolated vertices of G

$$= \sum_{M} (-1)^{|M|} \, x^{n-2|M|}$$

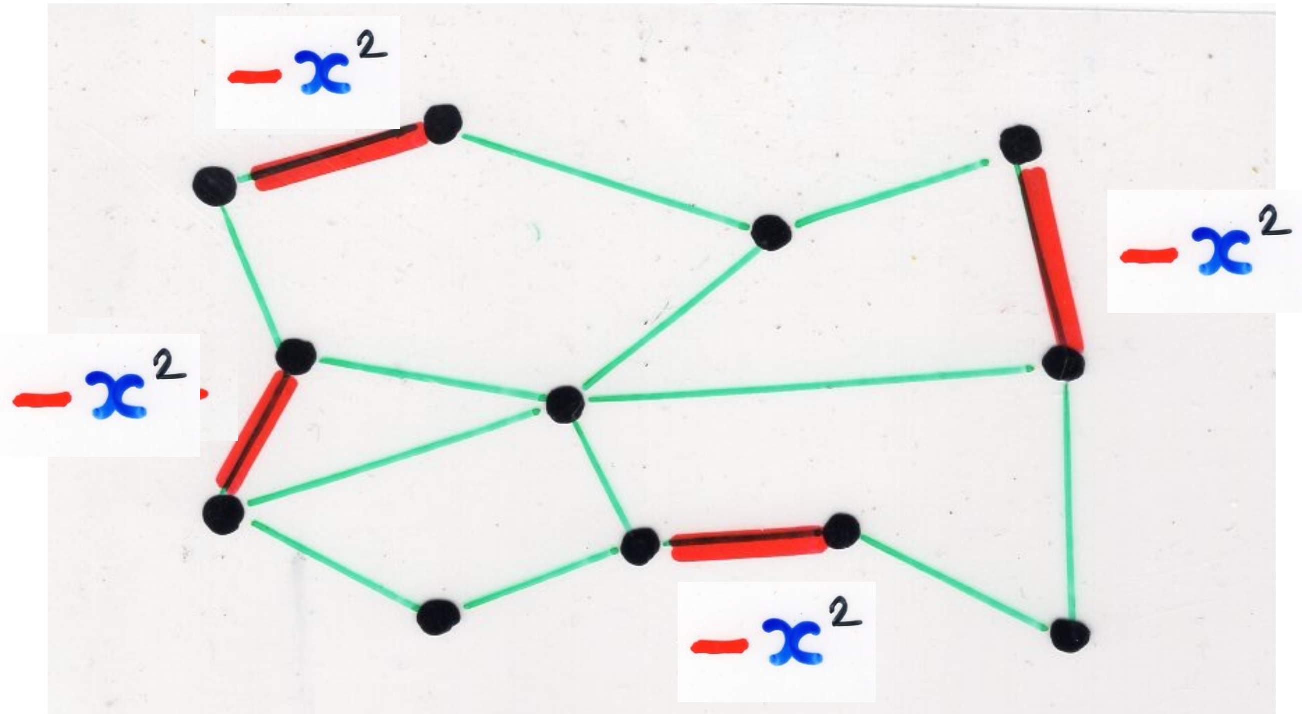
n = nb of vertices of G



reciprocal polynomial

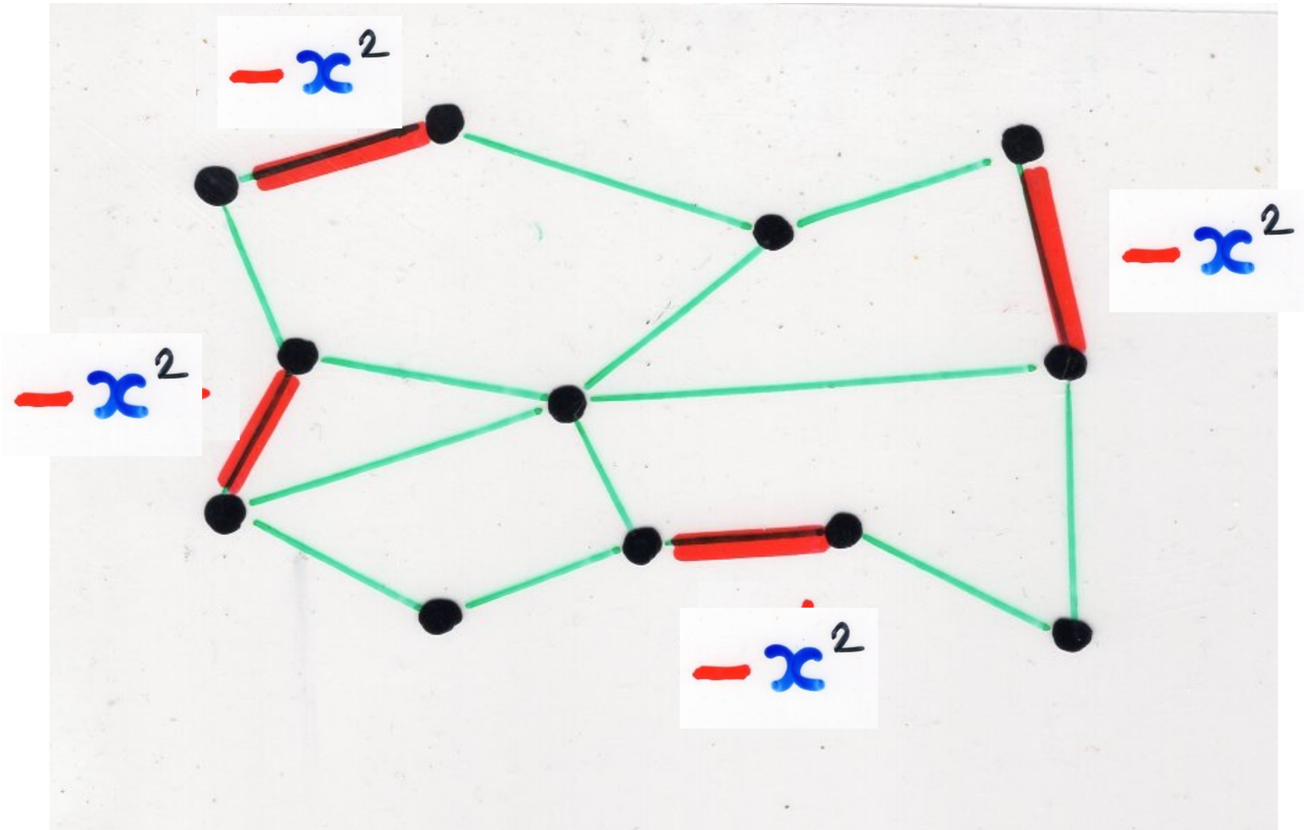
$$M^*_G(x) = x^n M_G(1/x)$$

$n = \deg(M_G)$
= number of vertices of G



reciprocal polynomial

$$M^{*}_{G}(x) = x^{n}\, M_{G}(1/x) = \sum_{\substack{M \\ \text{matchings} \\ \text{of } G}} (-x^{2})^{|M|}$$



generating function for heaps of edges on a graph G

$$\frac{1}{M^{*}_{G}(t)}$$

$$t = x^2$$

(enumerated by number of edges)

Proposition For every graph G, the zeros of the matching polynomial $M_G(x)$ are real numbers

If G is a tree, then

$$M_G(x) = \chi(x) \quad \text{the characteristic polynomial } \det(xI - A)$$

definition G graph, χ

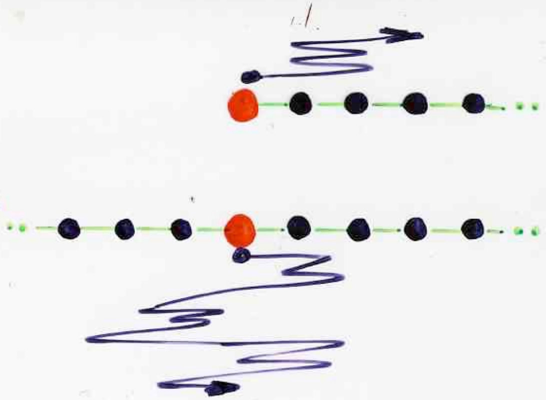
ω path on G with $\omega \longrightarrow (\eta, E)$.

ω is tree-like iff the heap E contains only cycles of length 2.

Godsil (1981)

Particular cases.

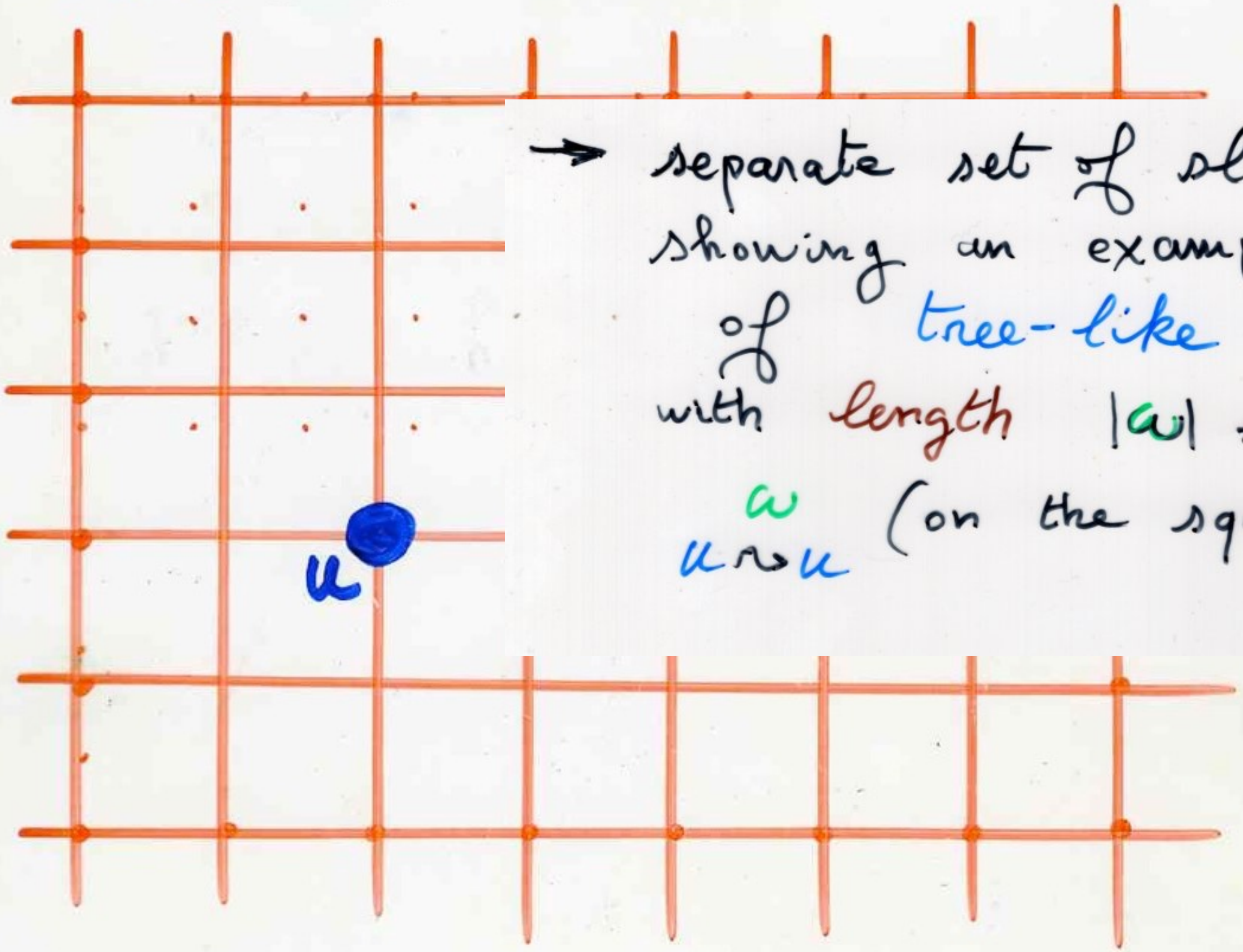
- Dyck path
- bilateral Dyck paths



paths on a tree

Tree-like paths on a graph G

Godsil (1981)



→ separate set of slides showing an example of tree-like paths with length $|\omega| = 20$

$u \overset{\omega}{\rightsquigarrow} u$ (on the square lattice)

$$\left\{\begin{array}{l} \bullet\ \text{Inversion Lemma} \\ \bullet\ \omega \xrightarrow{\not{}} (s, F) \end{array}\right. \qquad \frac{N}{D}$$

ω path on the graph $G=(V,E)$

ω

$s \rightsquigarrow s$ for $s \in V$ vertex of G

$$\sum_{\substack{\omega \\ \text{tree-like path on } G \\ s \rightsquigarrow s}} t^{|\omega|} = \frac{M^*_{G-s}(t)}{M^*_G(t)}$$

Ch 3b, p65

<u>exercise</u> **3** G graph, s vertex of G

Construct a tree T such that the tree-like paths on G starting at s are in bijection (preserving the length) with the paths on T starting at the root of T

G graph $s \in V$

$T_s(G)$ tree

vertices: the self-avoiding paths η starting from s

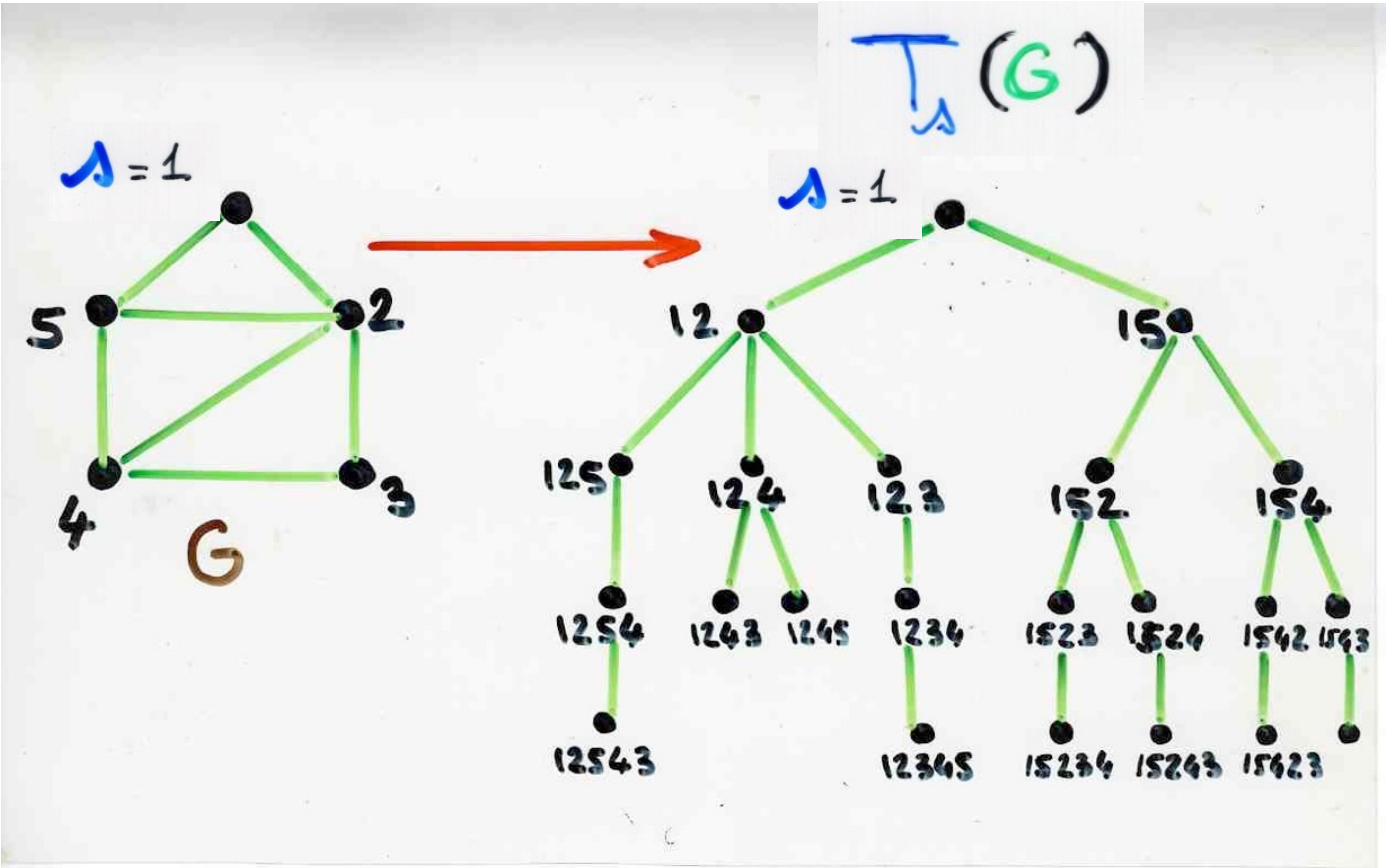
the root is s

edge $\eta \rightarrow \eta'$ iff

$\eta = (s_0 = s, s_1, \dots, s_k)$

$\eta' = (s_0 = s, \dots, s_k, s_{k+1})$

thus $\{s_k, s_{k+1}\}$ is an edge of G and $s_{k+1} \notin \eta$



s = 1
5
2
4
3
G
T
(G)
s = 1
12
15
125
124
123
152
154
1254
1243
1245
1234
1523
1524
1542
1543
12543
12345
15234
15243
15423

$$T_s(G) = T$$

<u>Lemma</u>

$$\sum_{\substack{\omega \\ \text{tree-like} \\ \text{path on } G \\ s \rightsquigarrow s}} t^{|\omega|} = \sum_{\substack{\omega \\ \text{paths on } T \\ s \rightsquigarrow s}} t^{|\omega|}$$

$$\frac{M^*_{G-s}(t)}{M^*_G(t)} \qquad\qquad \frac{M^*_{T-s}(t)}{M^*_T(t)}$$

$$M^*_T(t) = \chi^*_T(t)$$

characteristic polynomial of the tree T

$$\chi(x) = \det(Ix - A)$$

<u>Proposition</u> The zeros of the characteristic polynomial of a graph G are real numbers

the zeros are the eigenvalues of the adjacency matrix A

$$M^{*}_{T}(t) = \chi^{*}_{T}(t)$$

real zeros

→ By recurrence on the number of vertices of G

zeros → level of energy
in quantum chemistry