Course IMSc Chennai, India

January-March 2017

Enumerative and algebraic combinatorics,
a bijective approach:

commutations and heaps of pieces

(with interactions in physics, mathematics and computer science)

Monday and Thursday 14h-15h30

www.xavierviennot.org/coursIMSc2017

The Institute of Mathematical Sciences

IMSc

January-March 2017

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Chapter 5

Heaps and algebraic graph theory (2)

IMSc, Chennai

20 February 2017

from the previous lecture

Ch5a

matching
polynomial
of a graph G

$\gamma_G(\lambda)$ chromatic polynomial

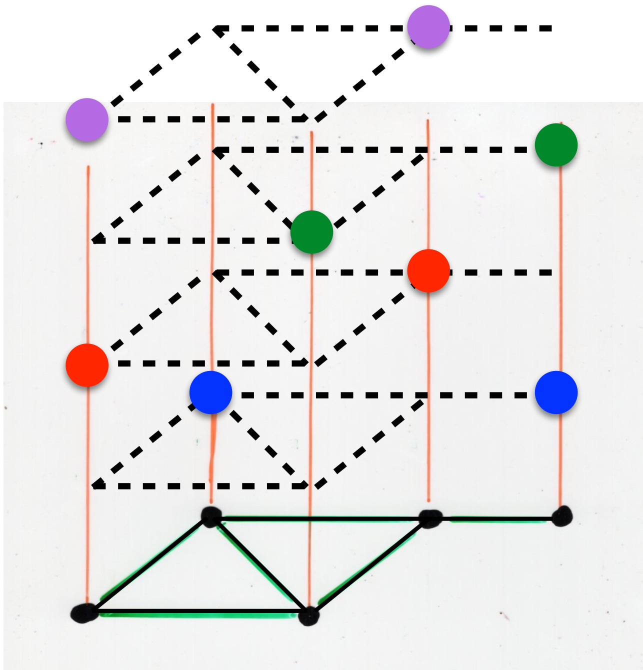
Proposition (Stanley, 1973)

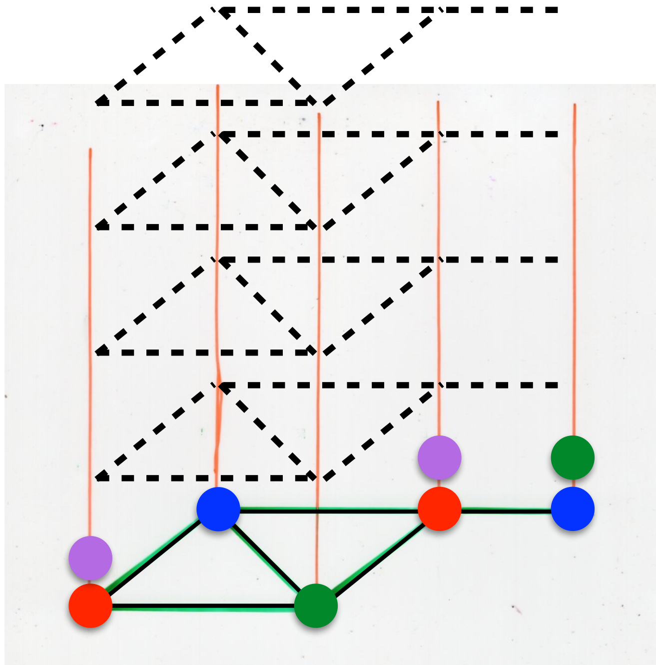
$$a(G) = (-1)^{n(G)} \gamma_G(-1)$$

$a(G)$ number of orientations of G acyclic

$n(G) = |V|$
number of vertices

$\gamma^{G}_{k}(\lambda)$ number of multicoloring associated to k with λ colors





$k = (k_1, \ldots, k_n)$

Definition Chromatic power series of the graph G (with weighted heaps)

$$\Gamma^{v}_{G}(\lambda) = \sum_{\substack{F \\ \text{heap} \\ \text{covering } G}} \sum_{k \geq 1} \beta_k(F)\, v(F)\, \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

Greene, Zaslavsky (1983)

- number of acyclic orientations with one sink $= \pm$ linear term of $\gamma_G(\lambda)$

→ proved with hyperplane arrangements

Gebhard, Sagan (2000) 3 other proofs

Ch5a, p81

Lass (2001)

algebra of "fonctions d'ensemble"
"set functions"

interpretation of all the coefficients of $\gamma_G(\lambda)$

Research? exercise

→ translate it (or work directly) with heaps over G

→ analogue for the chromatic power series ?

zeta function of a graph

Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

Ihara-Selberg zeta function of a graph

$$\zeta_G(t)$$

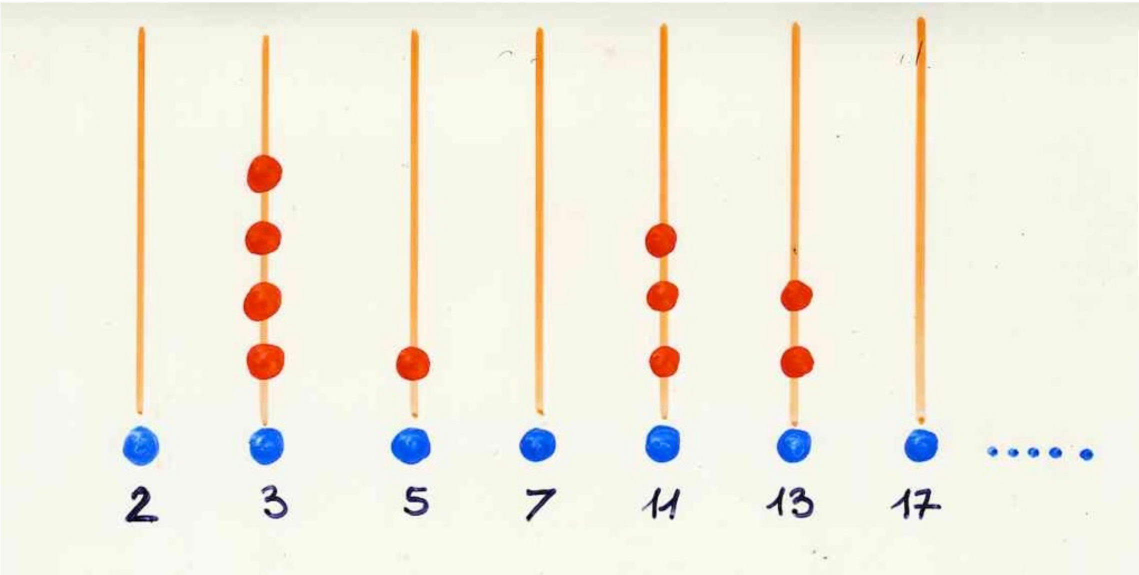
$\mathbb{N}^+ = \mathbb{N} - \{0\}$

$\mathbb{N}^+$ multiplicative monoid

$n \in \mathbb{N}^+ \longrightarrow p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

prime numbers decomposition



$H(\mathbb{N}^+, \mathscr{C})$

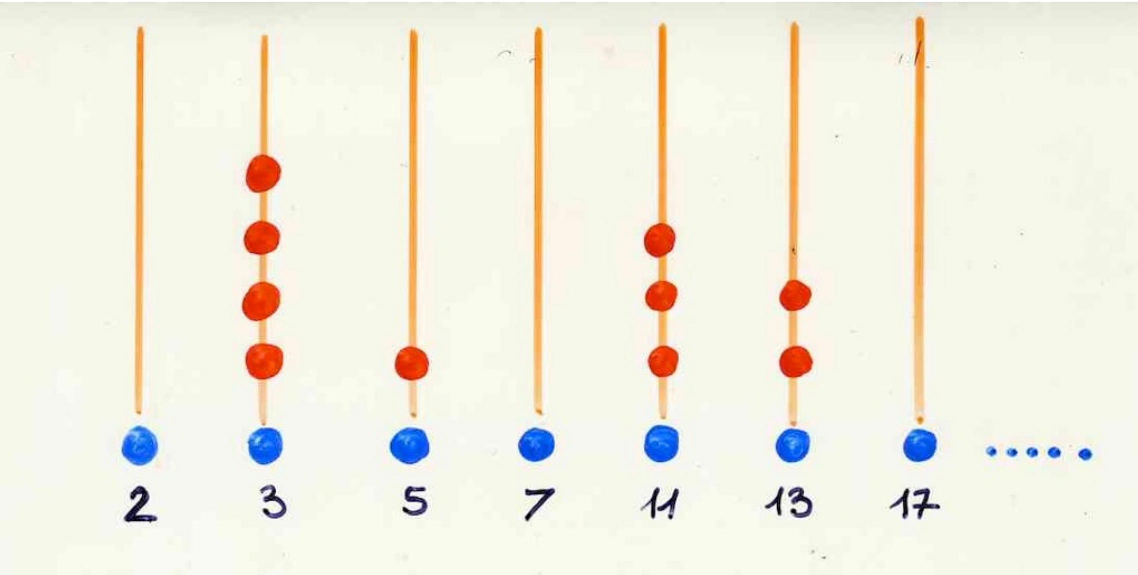
$a \not\mathscr{C} b$ for any $a, b \in \mathbb{N}^+$

except $a \mathscr{C} a$

$\zeta(s)$

$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$

$$\sum_{n\geqslant 1} n^{-s} = \left(\sum_{n\geqslant 1} \mu(n)\, n^{-s} \right)^{-1}$$



$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

Euler identity

$$\zeta(s) = \prod_{\substack{p \\ \text{prime number}}} \left(\frac{1}{1-p^{-s}} \right)$$

$$\zeta(s) = \prod_{p} \left(\frac{1}{1-p^{-s}} \right)$$

p prime number

$$Z_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

some "prime" over the graph G

Ihara-Selberg zeta function of a graph $Z_G(t)$

circuit

= path ω
$u \rightsquigarrow u$

$\omega' \cdot \omega''$
$u \rightsquigarrow s \quad s \rightsquigarrow t$

product
of two paths
circuit

prime
circuit

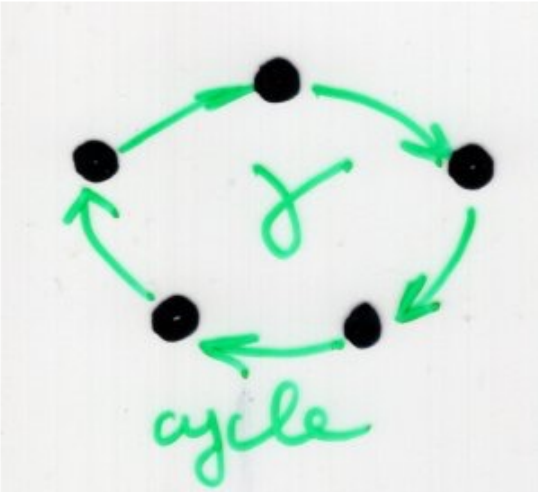
$C = \omega^p$

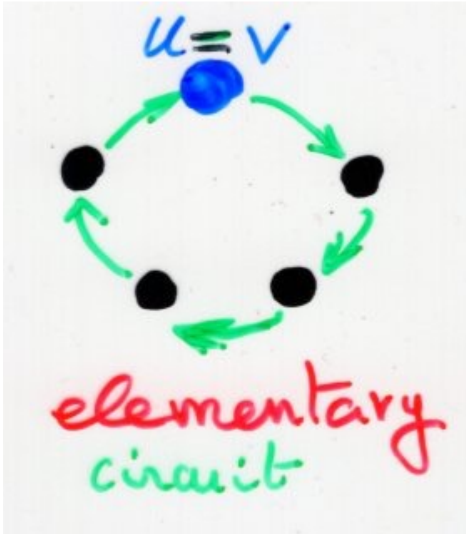
$\Rightarrow \quad p = 1$

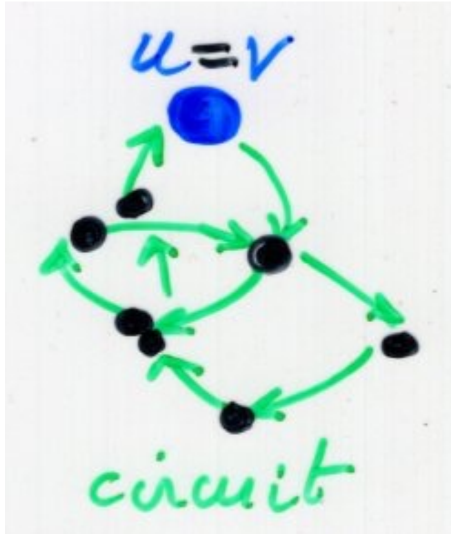
equivalence
class
of a circuit C

$[C]$

C as a word
of edges
up to a circular permutation







cycle = [C]

Ihara-Selberg zeta function of a graph

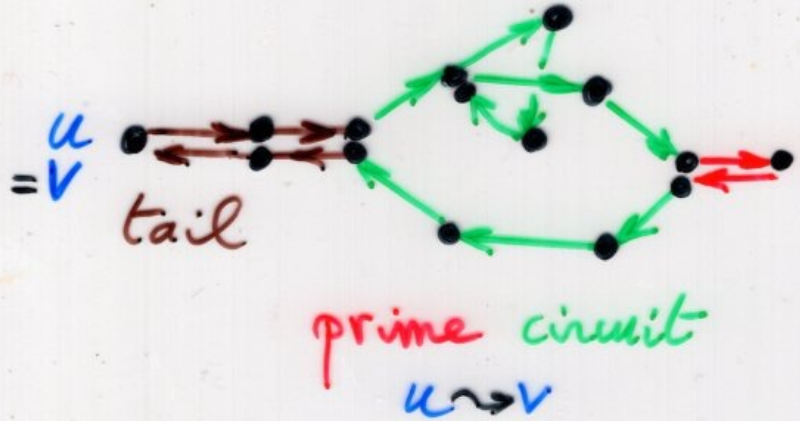
Ihara (1966)

$$(i) \quad Z_G(t) = \prod_{[C]} \frac{1}{(1 - t^{|C|})}$$

equivalence class

prime circuit

no backtracking



backtracking

(- no tail
- no backtracking

Ihara-Selberg zeta function of a graph

(i) $$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

(ii) $$\zeta_G(t) = \frac{1}{\det(1-Ht)}$$

(iii) $$\zeta_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det\left(I-tA+t^2(D-I)\right)}$$

$$t\frac{d}{dt}\log \zeta_G(t)$$

Bass formula

Bass (1992) Hashimoto (1989)
Venkov, Nikitin (1994)
Sunada (1986, 88)

Stark, Terras (1996, 2000)
book

Northshield (1999)
Foata, Zeilberger (1999)
bijective proof

Bartholdi (1999)

Mizuno, Sato (2000, ..., 2009)

....
many others

→ quantum
walks

Giscard, Rochet (2016)
extending number theory
to paths on Graphs

(i) $\prod_{[C]} \frac{1}{(1-t^{|C|})}$ $\qquad \log Z_G(t) = \sum_{[C]} \sum_{p\geq 1} \frac{1}{p} t^{p|C|}$

$$t\frac{d}{dt}\log Z_G(t) = \sum_{[C]} \sum_{p\geq 1} |C|\, t^{p|C|}$$

equivalence class
prime
circuit

no backtracking

$$= \sum_{[C]} |C|\, t^{|\omega|}$$

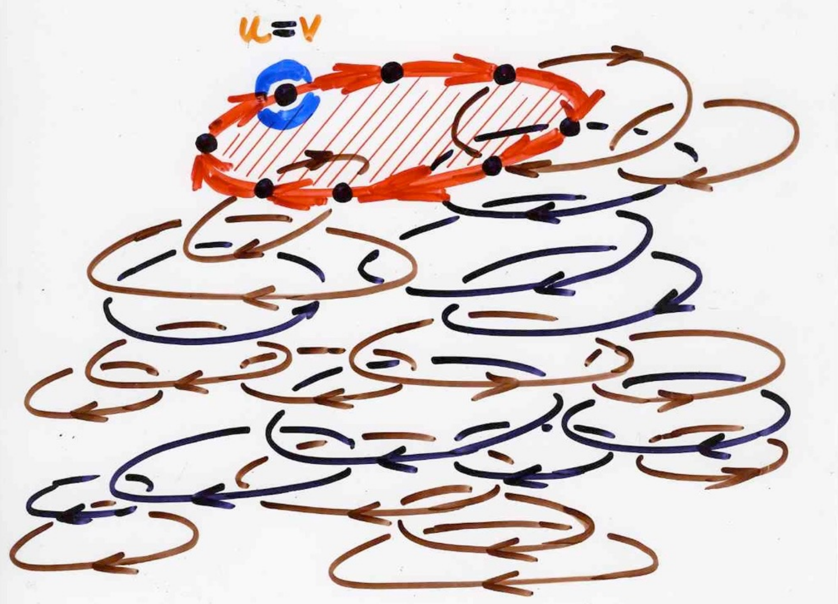
equivalence class
circuit

no backtracking

ω
circuit
$u \to u$
$= C^p$

$$= \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

(- no tail
- no back tracking



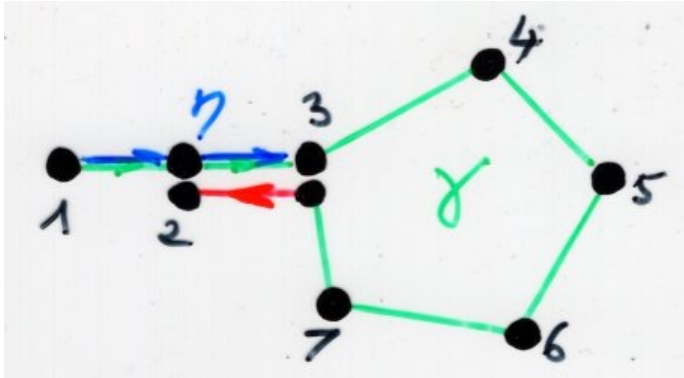
pointed cycles pyramid

$$= \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

- no tail
- no back tracking

?

no backtracking for ω $\Longleftrightarrow$ no cycle length 2 in E



$$\omega \not\longrightarrow (\eta, E)$$

$$\omega \longrightarrow (\underbrace{\bullet \to \bullet}_{\eta}, \underbrace{d \odot \gamma}_{E})$$

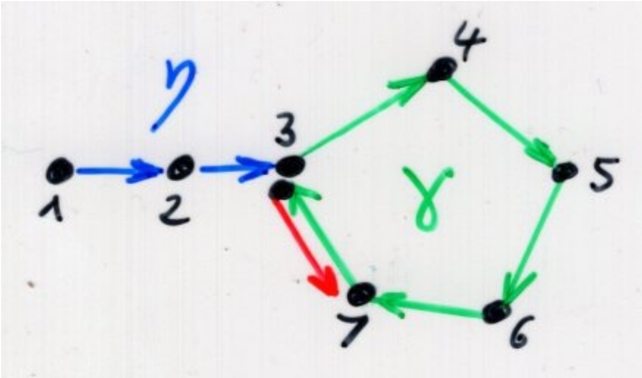
$$\omega = (1, \overbrace{2, 3}, 4, 5, 6, 7, \overbrace{3, 2})$$

$$d = [2, 3, 2] \quad \text{cycle} \quad |d| = 2$$

$$\eta = (1, 2) \qquad \gamma = [3, 4, 5, 6, 7, 3]$$

no backtracking for ω

a cycle length **2** in E



$$\omega \xrightarrow{\cancel{\gamma}} (\eta, E)$$

$$\omega = (1, 2, 3, 4, 5, 6, 7, 3, 7)$$

$$E = (\gamma)$$

$$\eta = (1, 2, 3, 7)$$

$$\gamma = [3, 4, 5, 6, 7, 3]$$

a backtracking
for ω

no cycle
length 2
in E

second bijection

$$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$$

$$\omega = (s_0, s_1, \dots, s_i, s_n)$$

ω path of G

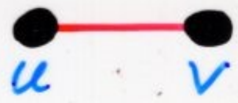
ω path on V

$$\longrightarrow \vec{L}(\omega)$$

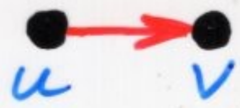
$$\big((s_0, s_1), (s_1, s_2) \cdots, (s_i, s_{i+1}), \dots, (s_{n-1}, s_n)\big)$$

path of $\vec{L}G$

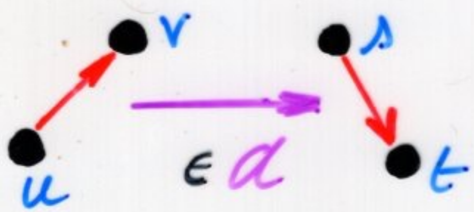
$G = (V, E)$

E 

$\overrightarrow{LG} = (\vec{E}, a)$

$\vec{E}$ 

oriented line graph



$\Longleftrightarrow v = s$

second bijection ψ

$\underset{u \rightsquigarrow v}{\omega} \xrightarrow{\psi} (\eta, F)$

$\underset{u \rightsquigarrow v}{\eta}$ trail

trail = path having all oriented edges distinct

F heap of "oriented loops"

oriented loop

equivalence class of trail

trail $\underset{u \rightsquigarrow u}{\eta}$ up to a circular permutation of its edges description of the bijection φ

ω path on V

$\omega = (s_0, \dots, s_n)$ $\quad u = s_0$, $v = s_n$

$\longrightarrow \vec{L}(\omega)$ $\quad \big((s_0, s_1), (s_1, s_2) \cdots, (s_i, s_{i+1}), \dots, (s_{n-1}, s_n)\big)$

$\vec{L}(\omega) = (e_1, \dots, e_n)$

$e_i = (s_{i-1}, s_i)$ oriented edges

at time $T=0$ - $\eta_0 = \emptyset$ $F = \emptyset$

- suppose $\omega_T = (s_0, \dots, s_T) \rightarrow (\eta_T, F_T)$

$\eta_T = (a_1, \dots, a_{i_T})$ trail going from s_0 to s_T

$a_1 = (u, s_{j_T})$

$F_T =$ heap of oriented loops $\pi(\max(\text{pieces}))$ intersect η

at time $T+1$, two cases

(i) (s_T, s_{T+1}) does not appear in η_T

then $\eta_{T+1} = \left(a_1, \dots, a_{i_T}, (s_T, s_{T+1}) \right)$

$F_{T+1} = F_T$

at time $T+1$, two cases

(ii) else $\cancel{\text{}}$ $(s_T, s_{T+1}) = a_j$ edge of η_T

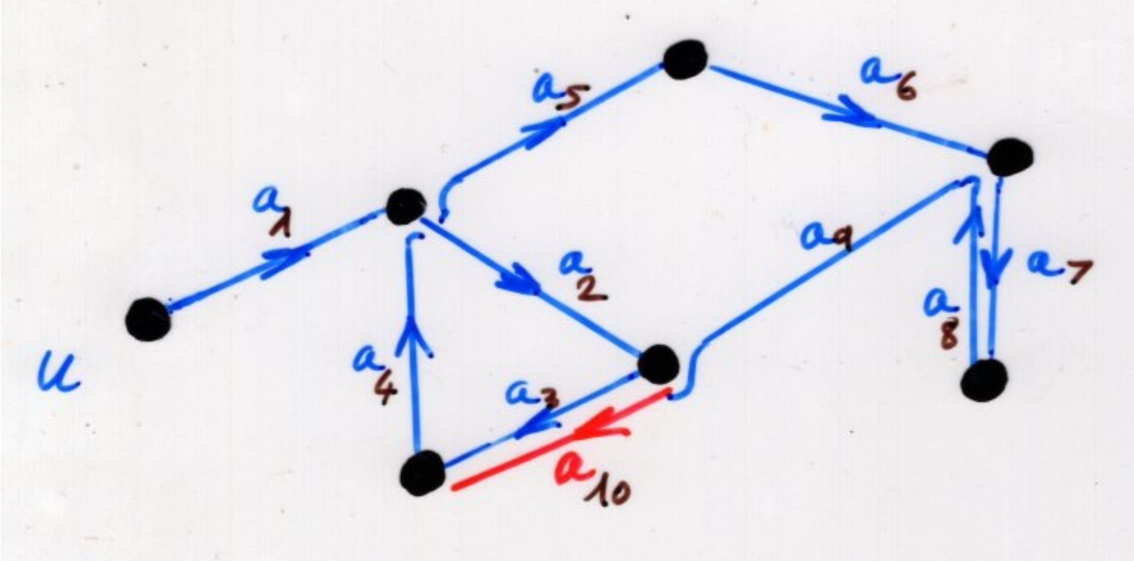
then $\eta_{T+1} = (a_1, \dots, a_j)$

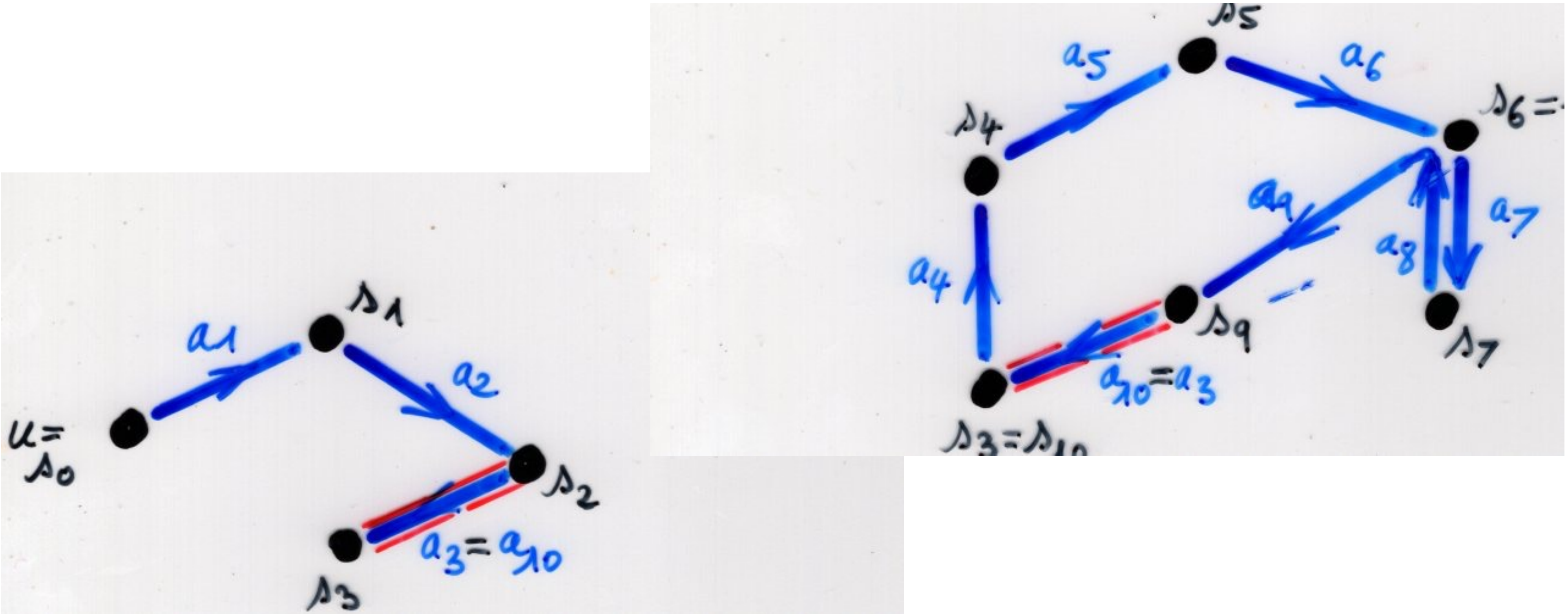
let $\Gamma_{T+1} = [a_j, \dots, a_{i_T}]$

$F_{T+1} = \Gamma_{T+1} \odot F_T$

$$\psi(\omega) = (\eta_n, F_n)$$

$T = n$


u
a1
a2
a3
a4
a5
a6
a7
a8
a9
a10


u = s0
s1
s2
s3
a1
a2
a3 = a10
s4
s5
s6 =
s7
s9
a4
a5
a6
a7
a8
a9
a10 = a3

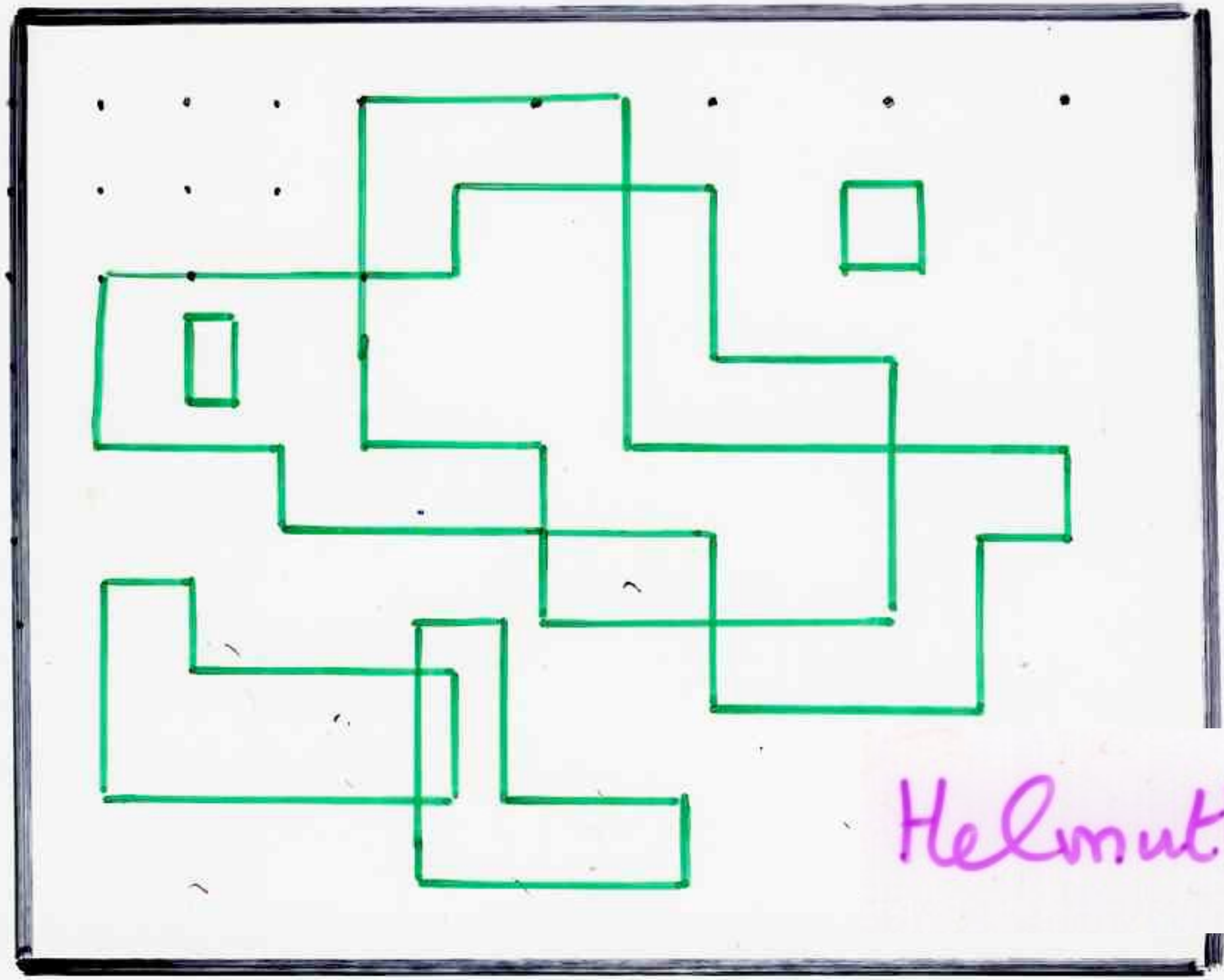
Proposition

$$\underset{\text{path}}{\omega} \xrightarrow{\psi} (\eta, F)$$

ω (no tail / non backtracking

$\Longleftrightarrow$

$\left\{ \begin{array}{l} \bullet\ \eta \text{ is non backtracking, no tail} \\ \text{and} \\ \bullet\ \text{each oriented loops of } F \text{ is non backtracking} \end{array} \right.$



Helmuth (2012)

"closed" graph

Ising model

second definition for zeta

(ii) $\zeta_G(t) = \dfrac{1}{\det(1 - Ht)}$

$H = T - B$

T = adjacency matrix of the oriented line graph $\vec{LG} = (\vec{E}, a)$

$T = \left(t_{(i,j),(k,\ell)} \right)$

$t_{(i,j),(k,\ell)} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$

B submatrix of T

$B = \left(b_{(i,j)(k,\ell)} \right)$

$b_{(i,j),(j,i)} = \begin{cases} 1 & \\ 0 & \text{else} \end{cases}$

backtracking

$$t \frac{d}{dt} \log \frac{1}{\det(1-\mathbf{H}t)}$$

= generating function (by number of edges)

pointed pyramids of non backtracking oriented loops

pointed = one of the edge of the maximal piece is pointed

$$= \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

- no tail
- no back tracking

$$t\frac{d}{dt}\log\frac{1}{\det(1-\mathbf{H}t)} = \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

$$t\frac{d}{dt}\log Z_G(t)$$

$\Big($ - no tail
  - no backtracking

third definition for zeta

(iii)

$$Z_G(t) = \frac{1}{(1-t^2)^{m-n}} \frac{1}{\det\left(I - tA + t^2(D-I)\right)}$$

$G = (V, E)$

$m = |E|$ number of edges

$n = |V|$ number of vertices

$A = (a_{ij})$ incidence matrix of G

D diagonal matrix

$D = (d_{ii})$

$d_{ii} = \deg v_i$

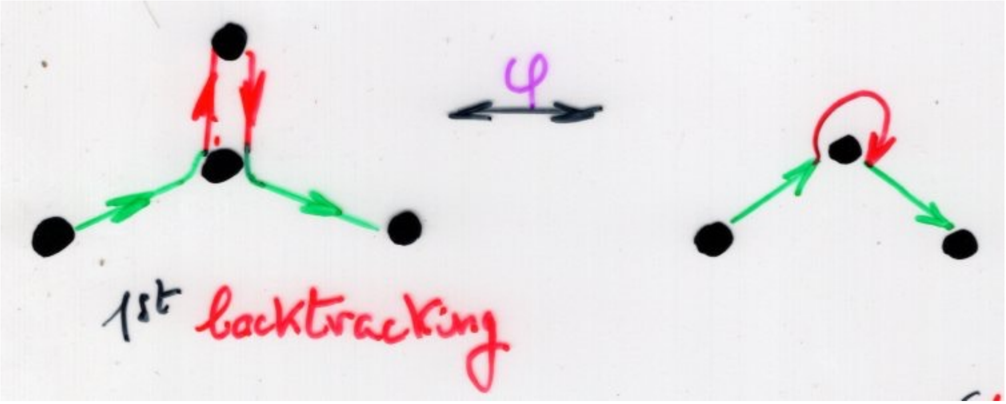
$V = \{v_1, \ldots, v_n\}$

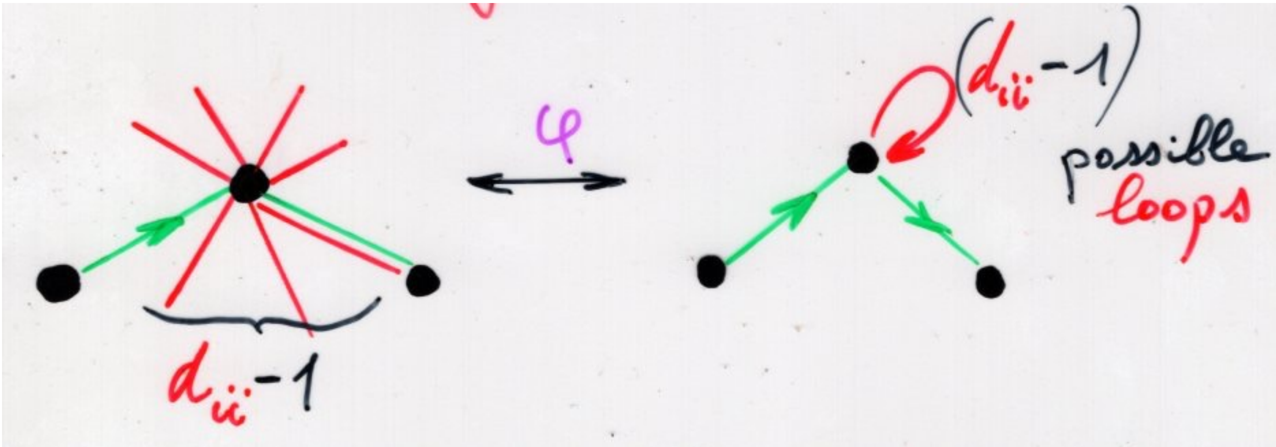
$$t \frac{d}{dt} \log \frac{1}{\det\left(I - tA + t^2(D - I)\right)}$$

$$= \sum_{\substack{\omega \\ \text{circuit}}} v(\omega)$$

$$V = \{1, \dots, n\}$$

$$\begin{cases} v(i,j) = t \\ v(i,i) = -t^2\left((\deg i) - 1\right) \end{cases}$$


φ
1st backtracking


φ
$(d_{ii}-1)$
possible loops
$d_{ii}-1$

ω

circuit

$u \rightsquigarrow u$

(with possible loops)

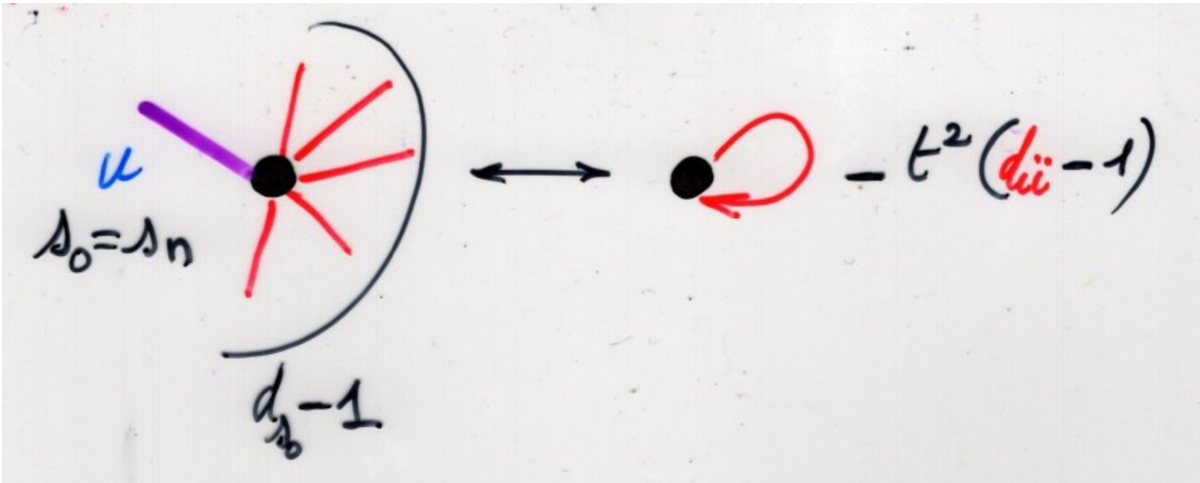
following ω after $u = s_0$

take the first following event

$\left\{ \begin{array}{l} \text{– there exist a backtracking} \\ \text{– there exist a loop} \end{array} \right.$

φ "exchanges" $\left(\begin{array}{l} \text{backtracking } (i,j,i) \\ \text{loop pointed on } (i,j) \end{array} \right.$


special
edge

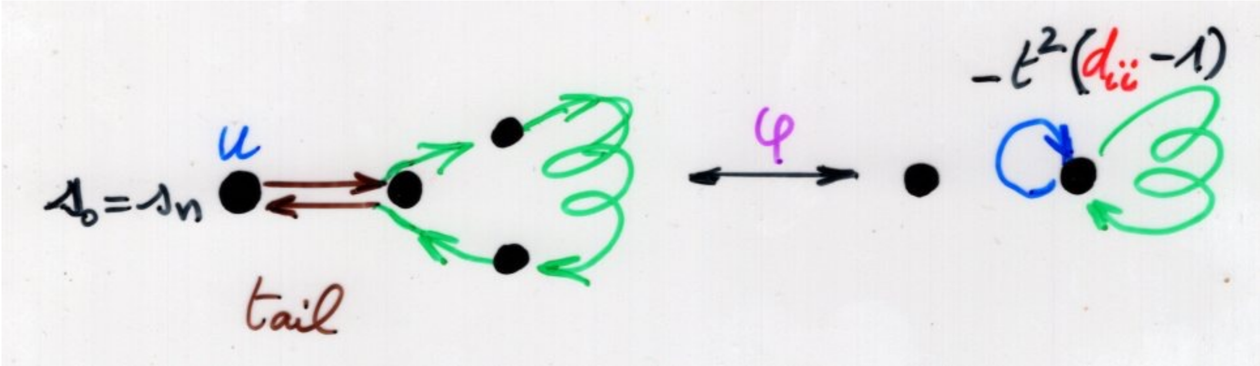

u
$s_0 = s_n$
$d_{s_0} - 1$
$- t^2 (d_{ii} - 1)$

pointed pyramids of cycles $= \displaystyle\sum_{\substack{\omega \\ \text{circuit}}} v(\omega)$

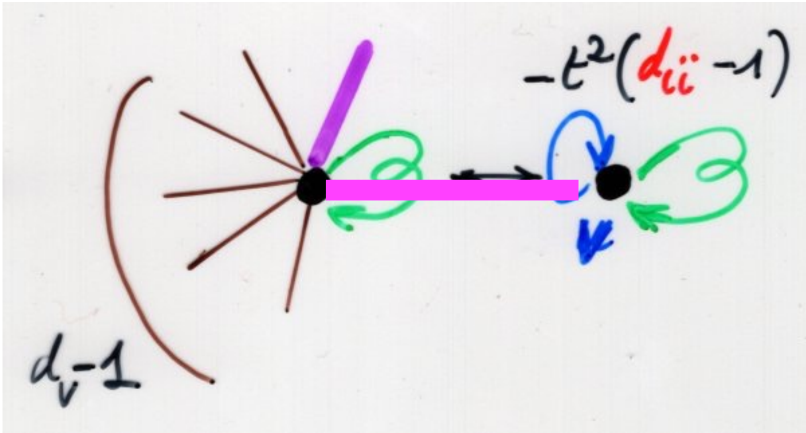
one edge of the max piece is pointed

if the max piece is  $\rightarrow$ **2**  $-t^2(d_{ii}-1)$

 $-t^2(d_{ii}-1)$  $-t^2(d_{ii}-1)$



$s_0 = s_n$
u
tail
φ
$-t^2(d_{ii}-1)$



$-t^2(d_{ii}-1)$
$d_v - 1$

special
edge

$$\sum_{\substack{\omega \\ \text{circuit}}} v(\omega)$$

remain ω

(*) circuit

$u \to u$

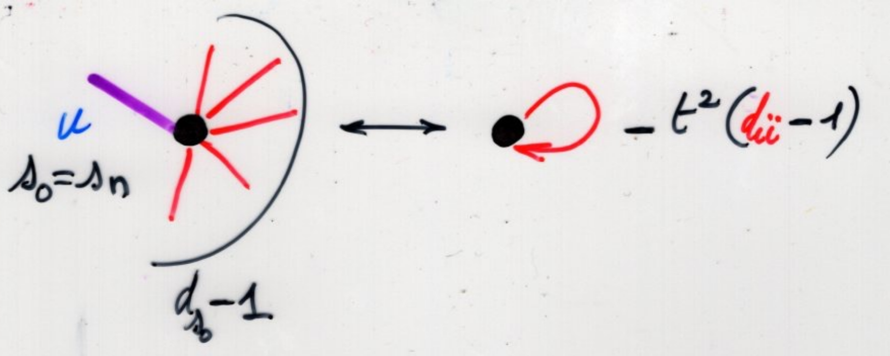
with no loops

no backtracking

except may be at the origin u

with backtracking

on special edge associated to u



$$t\frac{d}{dt}\log \frac{1}{(1-t^2)^{m-n}} \cdot \frac{1}{\det\left(I - tA + t^2(D-I)\right)} =$$

$$t\frac{d}{dt}\log \left(\frac{1}{(1-t^2)^{m-n}}\right)$$

$$(m-n)\sum_{n\geq 1} \frac{1}{n} t^{2n}$$

$$2(m-n)\sum_{n\geq 1} t^{2n} = \frac{2(m-n)t^2}{(1-t^2)}$$

$$+$$

$$t\frac{d}{dt}\log \frac{1}{\det\left(I - tA + t^2(D-I)\right)}$$

$$\sum_{\substack{\omega \\ \text{circuit}}} v(\omega) \quad (*)$$

$$t\frac{d}{dt}\log Z_G(t) = \sum_{\substack{\omega \\ \text{circuit}}} t^{|\omega|}$$

(- no tail
- no back tracking)

?

$$= \frac{2(m-n)t^2}{(1-t^2)} + \sum_{\substack{\omega \\ \text{circuit}}} v(\omega)$$

(*)

$$\zeta(s) = \prod_{\substack{p \\ \text{prime} \\ \text{number}}} \left(\frac{1}{1-p^{-s}} \right)$$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

some "prime" over the graph G

$$\zeta(s) = \prod_{p} \left(\frac{1}{1-p^{-s}} \right)$$

p prime number

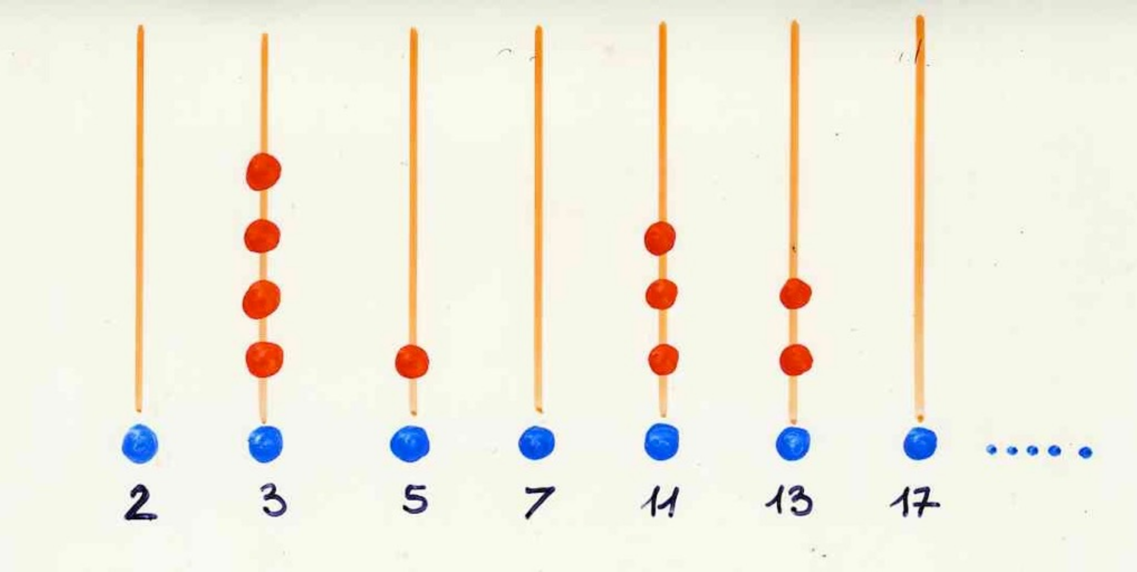
$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

equivalence class

prime circuit

no backtracking

$$\sum_{n\geq 1} n^{-s} = \left(\sum_{n\geq 1} \mu(n)\, n^{-s} \right)^{-1}$$

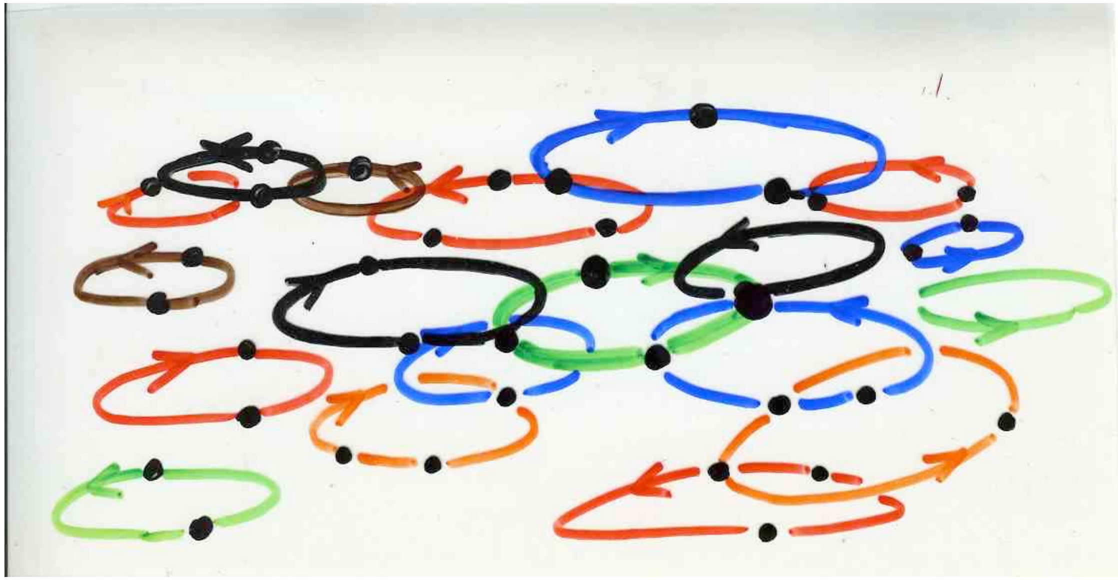


$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

Euler identity

$$\zeta(s) = \prod_{\substack{p \\ \text{prime number}}} \left(\frac{1}{1-p^{-s}} \right)$$

$$\sum_{n\geq 1} n^{-s} = \left(\sum_{n\geq 1} \mu(n)\, n^{-s} \right)^{-1}$$



$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

$$\zeta_G(t) = \prod_{[C]} \frac{1}{\left(1 - t^{|C|}\right)}$$

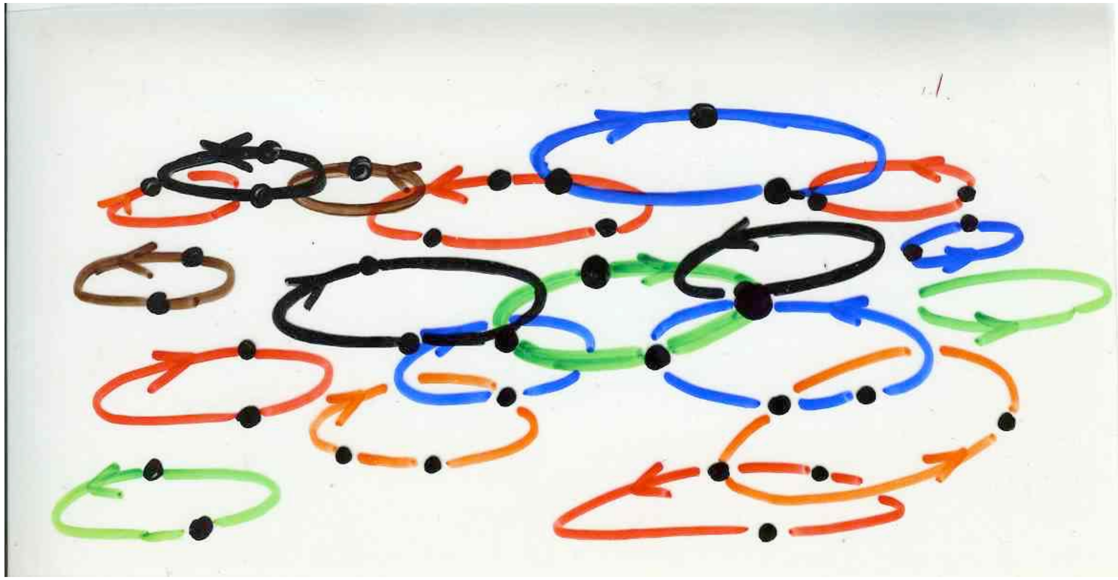
equivalence class

prime

circuit

$$\zeta_G(t) = \frac{1}{\det(I-A)}$$

Giscard, Rochet (2016)
extending number theory to paths on Graphs



$$n^{-s} = p_1^{-s\alpha_1} \cdots p_k^{-s\alpha_k}$$

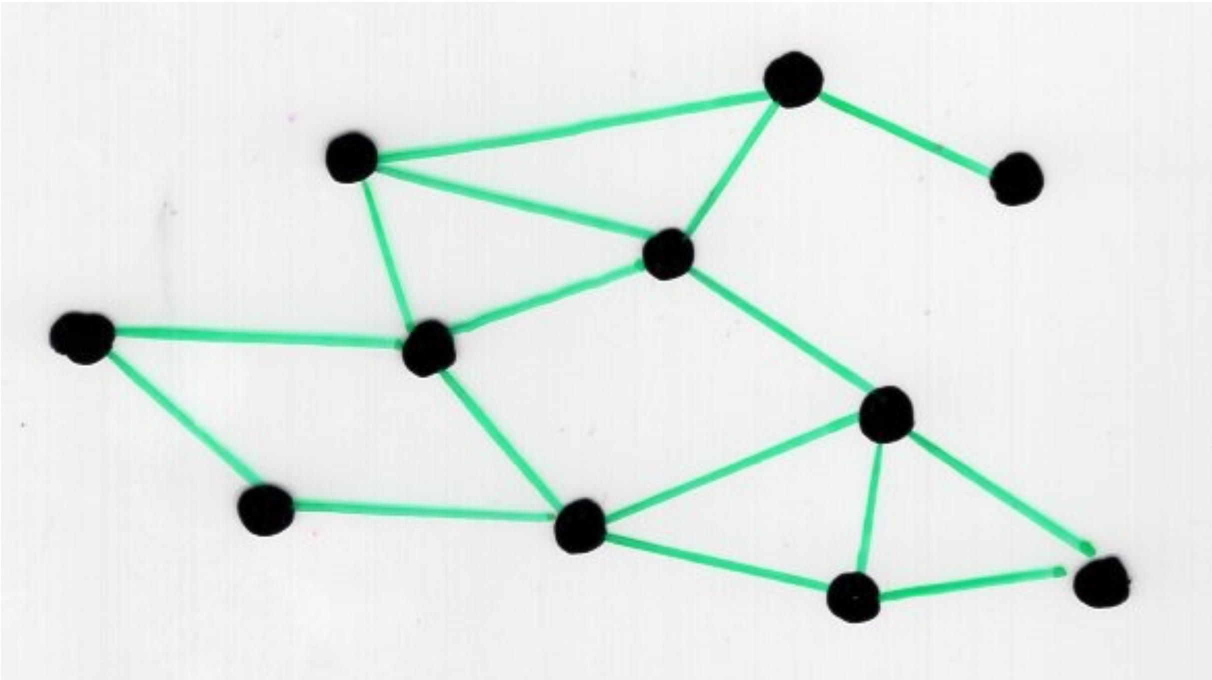
$$\zeta_G(t) = \prod_{[C]} \frac{1}{(1-t^{|C|})}$$

equivalence class
prime circuit

spanning tree

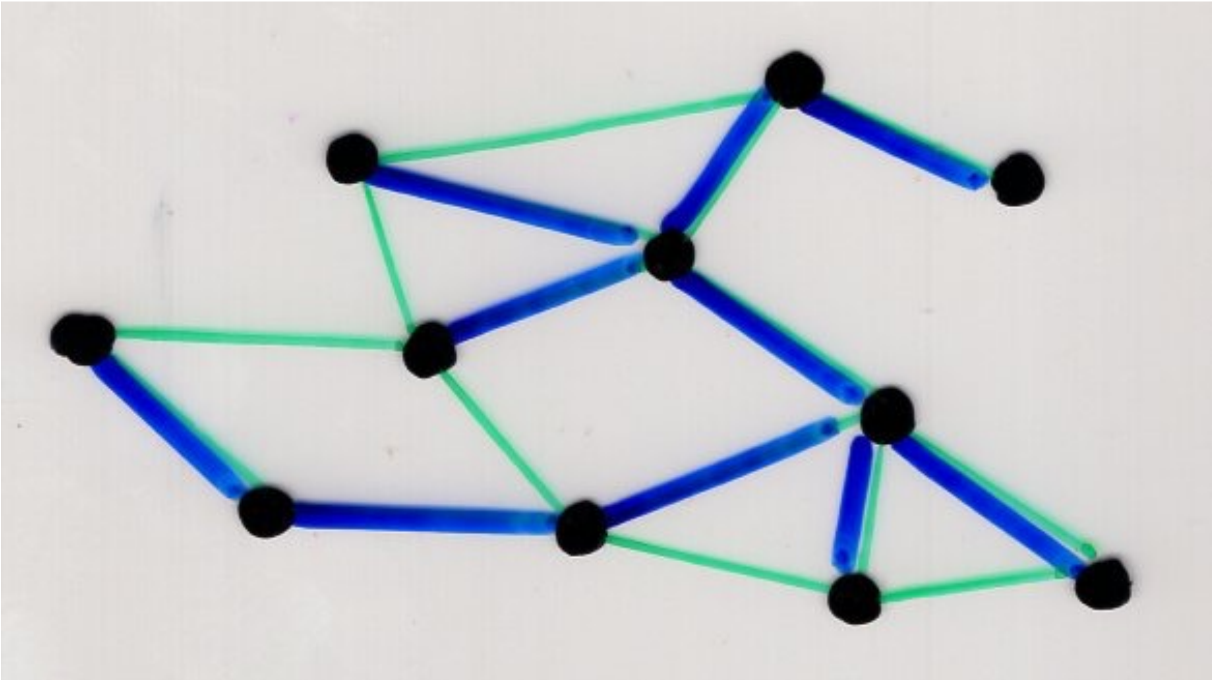
spanning tree

of a graph $G = (V, E)$



spanning tree

of a graph $G = (V, E)$



• number of spanning tree

G graph

$G = (V, E)$

Laplacian matrix

$L(G)$

$L = D - A$

$D = (d_{ii})$

$A = (a_{ij})$

diagonal matrix

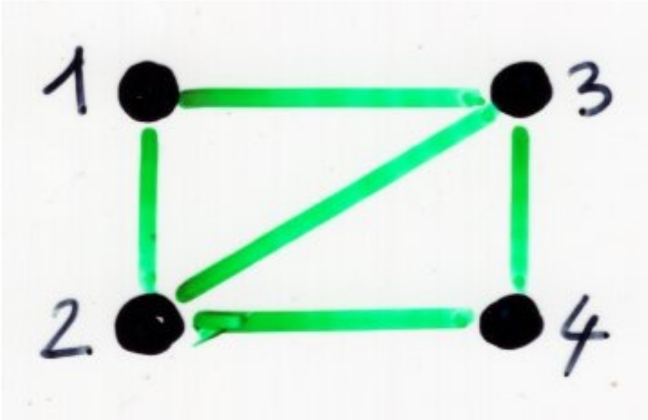
incidence matrix

A

matrix-tree theorem

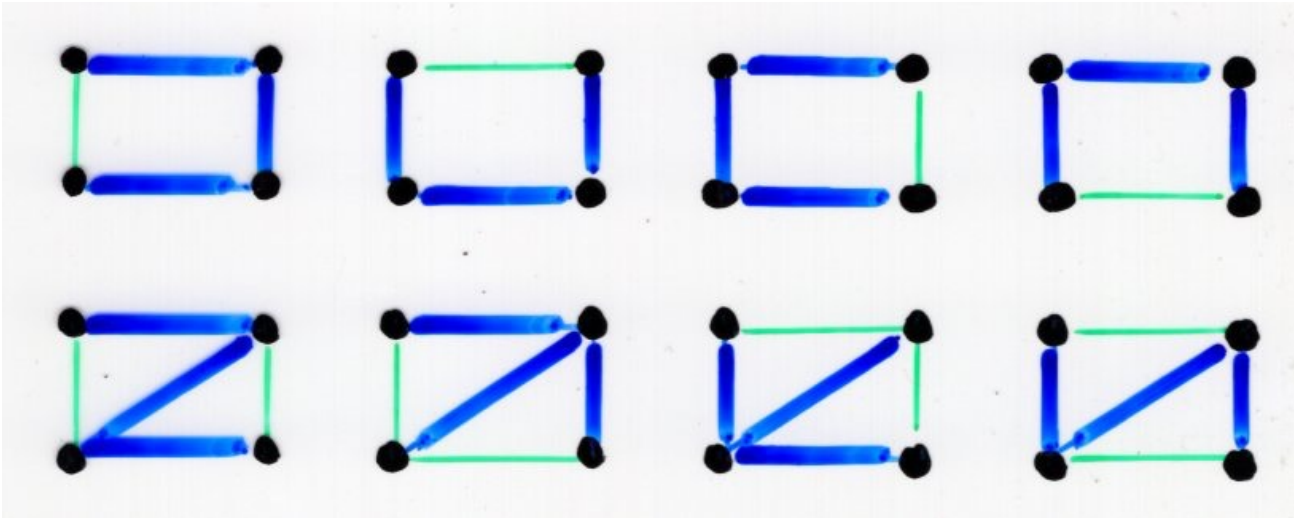
number of spanning trees of the graph G $=$ $\text{minor}_{i,i}(L(G))$

$$L = D - A$$



$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$\det(L_{11}(G)) = 8$$

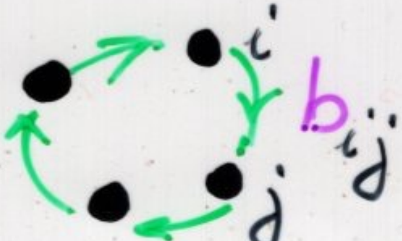


$$B = (b_{ij})_{1 \le i,j \le k}$$

$$\det(B) = \sum_{\sigma = \{\gamma_1, \dots, \gamma_r\}} (-1)^{\mathrm{Inv}(\sigma)} v(\gamma_1) \cdots v(\gamma_r)$$

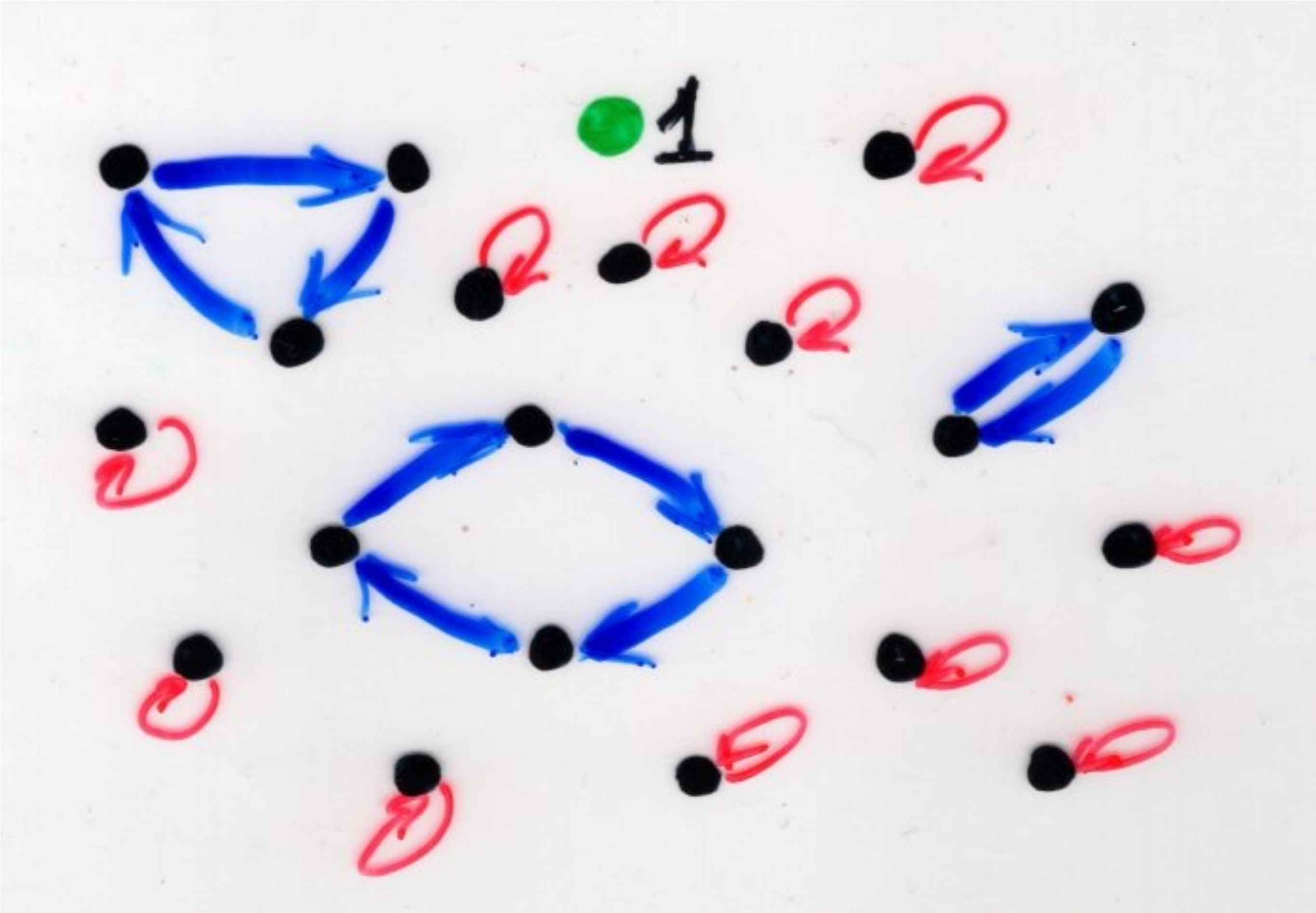
pairwise disjoint
covering $\{1, 2, \dots, k\}$

$$v(\gamma) =$$



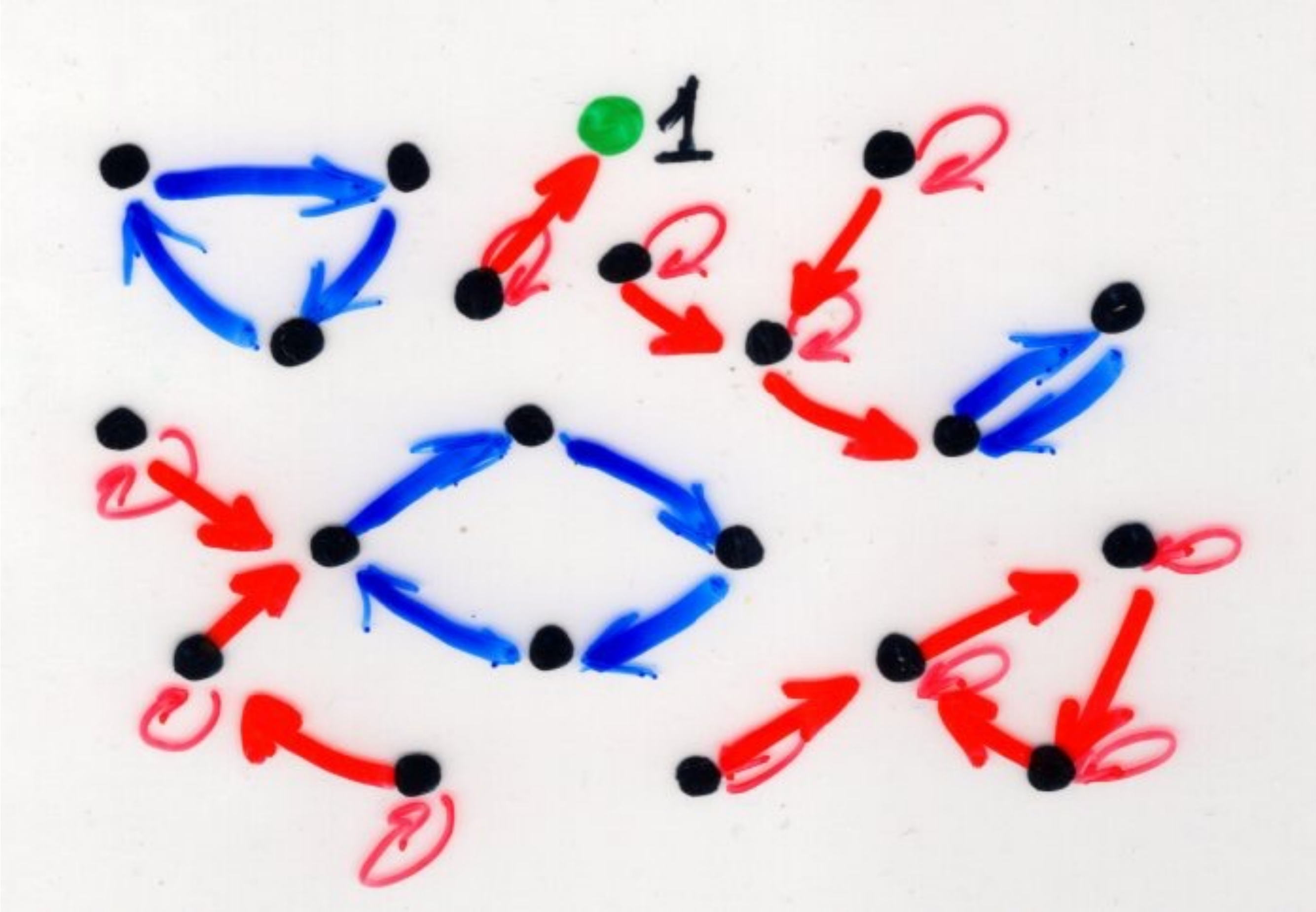
$$\mathrm{Inv}(\sigma) = \sum_{\gamma_i} \mathrm{Inv}(\gamma_i)$$

$$\sum_{\gamma_i} (-1)^{|\gamma_i| - 1}$$



$\mathrm{minor}_{1,1}(D - A)$

$L = D - A$



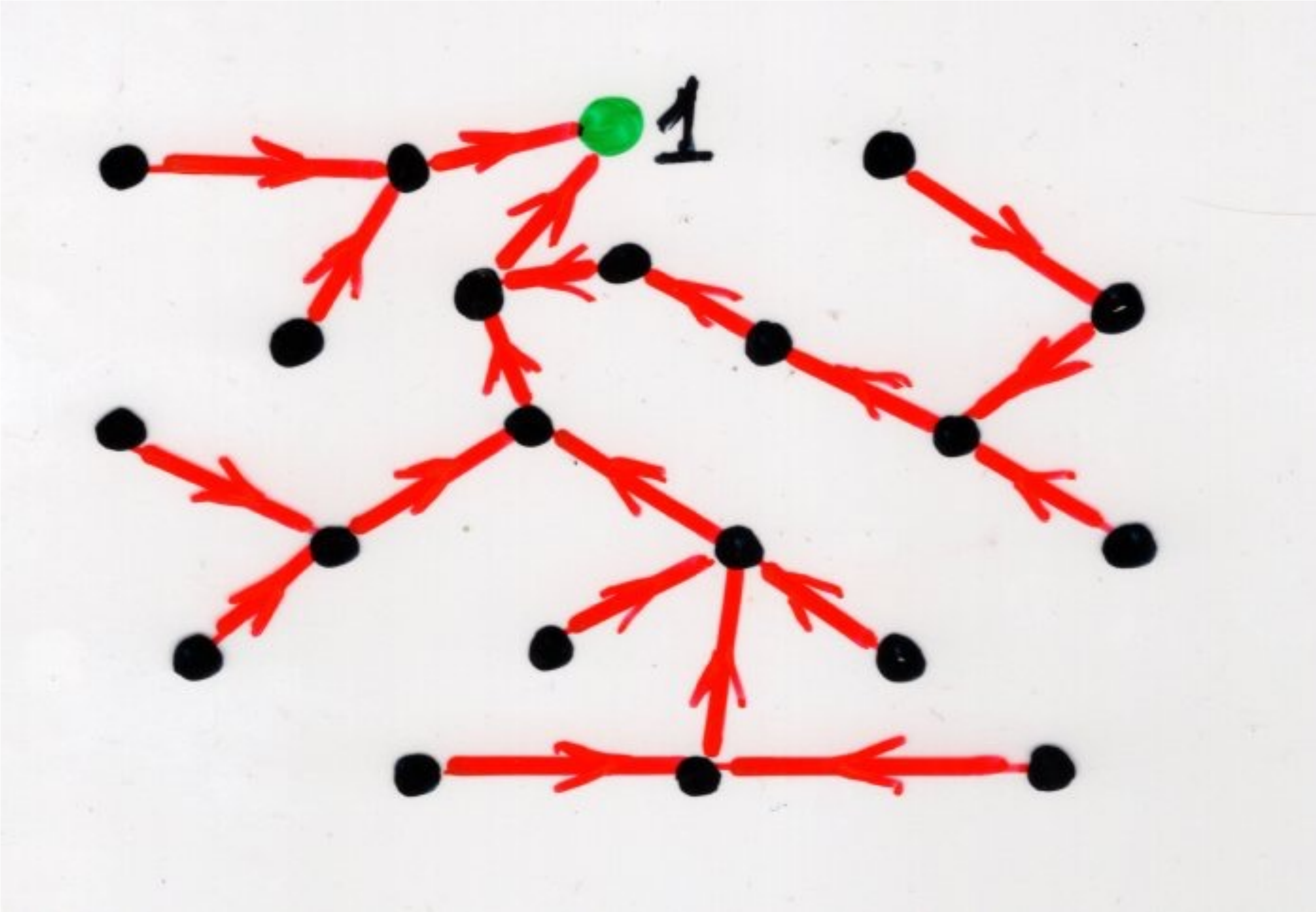
Involution φ

red cycle γ

$+1$

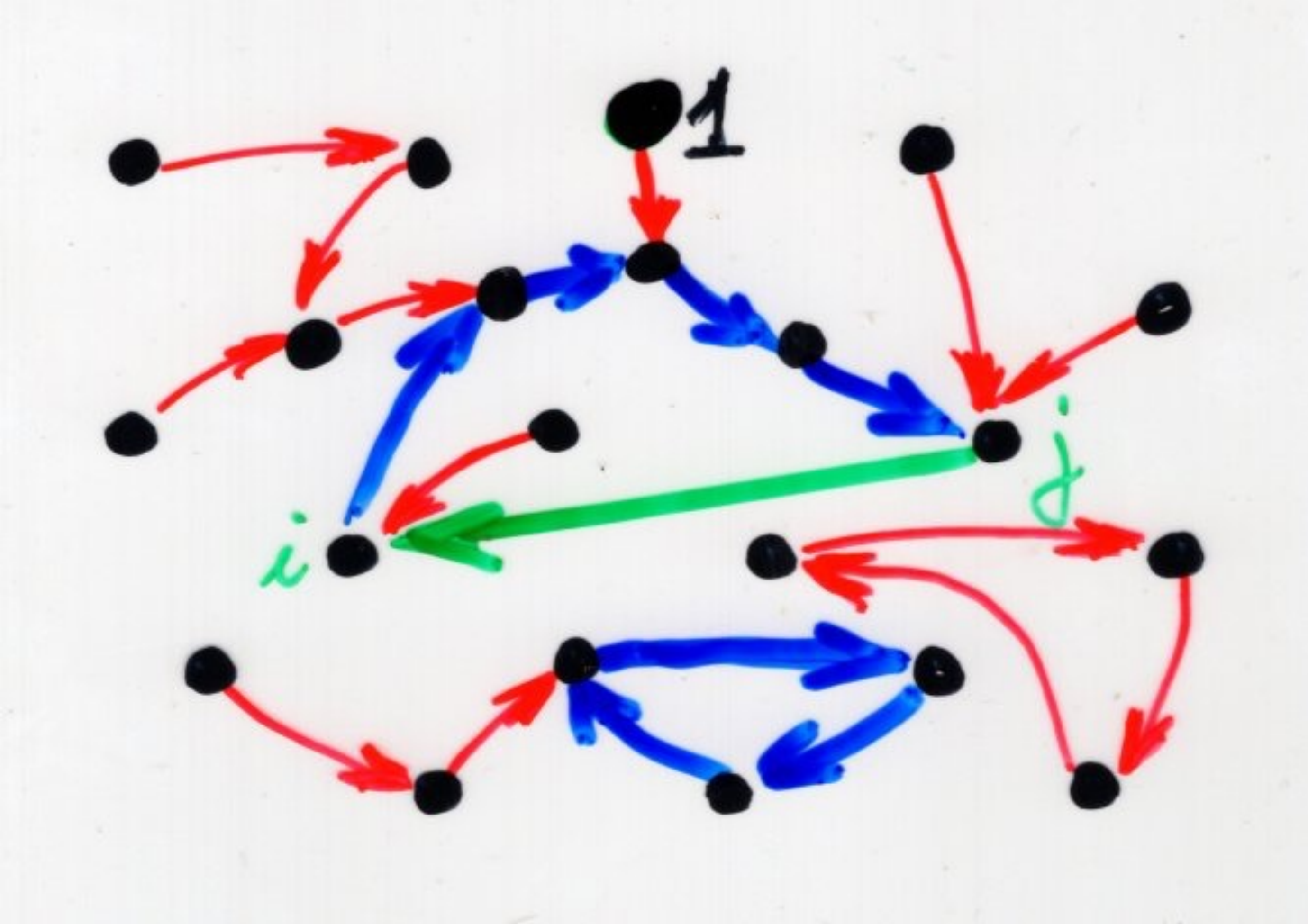
blue cycle γ

$$(-1)^{|\gamma|}(-1)^{\mathrm{Inv}(\gamma)} = -1$$



after the action
of the involution φ ...

a spanning tree



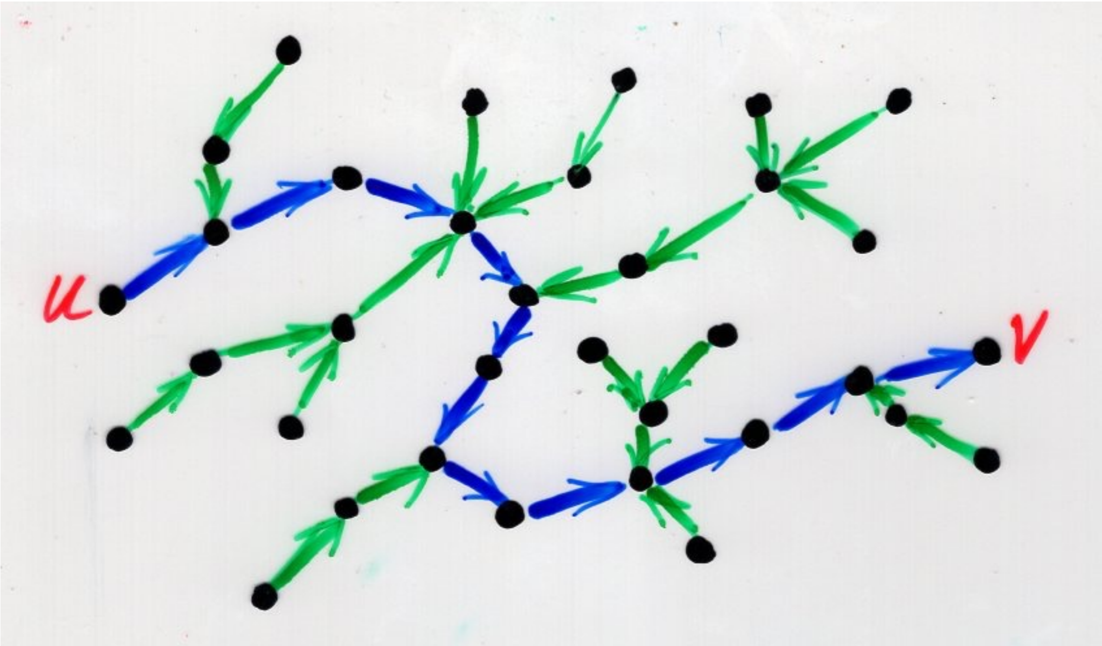
the case of
a general
cofactor (j, i)

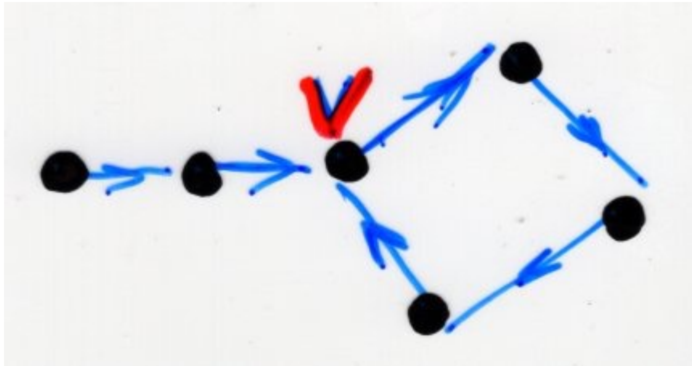
for path $\omega \to f(\omega) \in F(X)$

- what do you "see" above $f(\omega)$
- " " " below $f(\omega)$

$\omega \longleftrightarrow (\eta, E) \xrightarrow{\cancel{\chi}}$

$\longrightarrow T(\omega)$ spanning tree on $\mathrm{supp}(\omega)$





for path $\omega \longrightarrow f(\omega) \in F(X)$

- what do you "see" above $f(\omega)$
- " " " below $f(\omega)$

$\cancel{\omega \longleftrightarrow (\eta, E)}$ $\longrightarrow T(\omega)$ spanning tree on $\mathrm{supp}(\omega)$

ω stops the first time it arrives in v

$\rightarrow$ maximal edges form a tree

E heaps of cycles not containing v

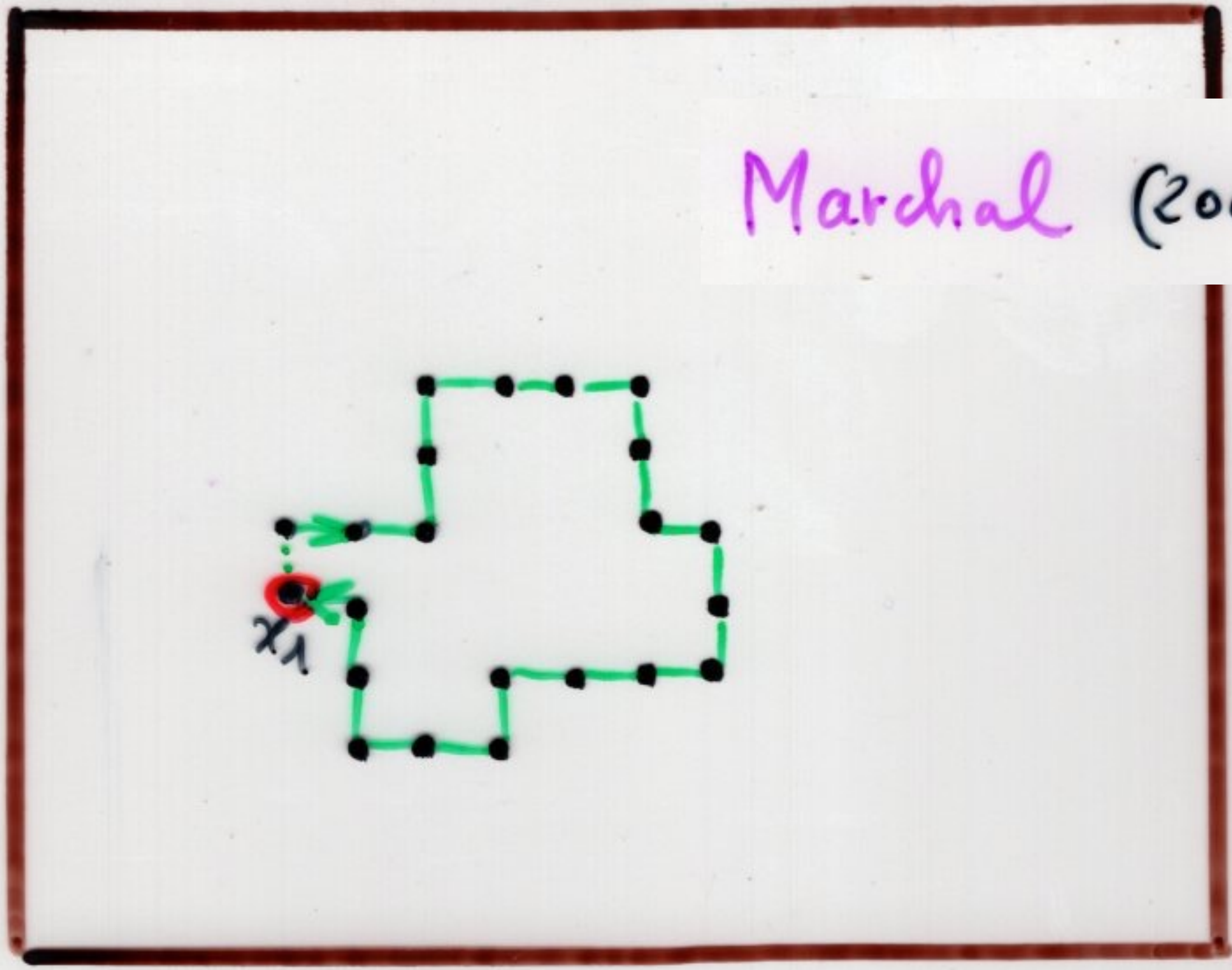
complements

Wilson's algorithm
for
uniform random spanning tree

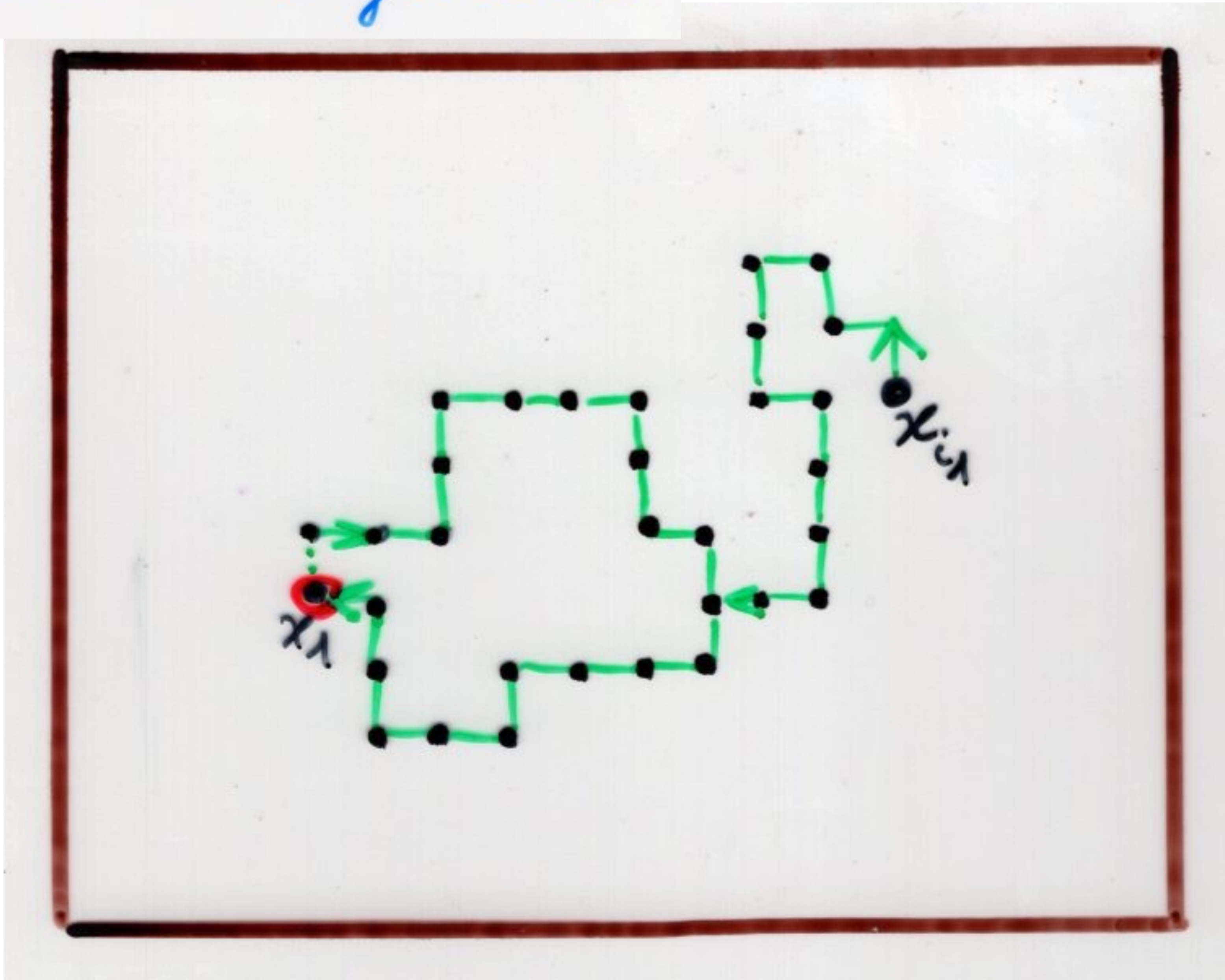
Ch3b, p80

Wilson's algorithm

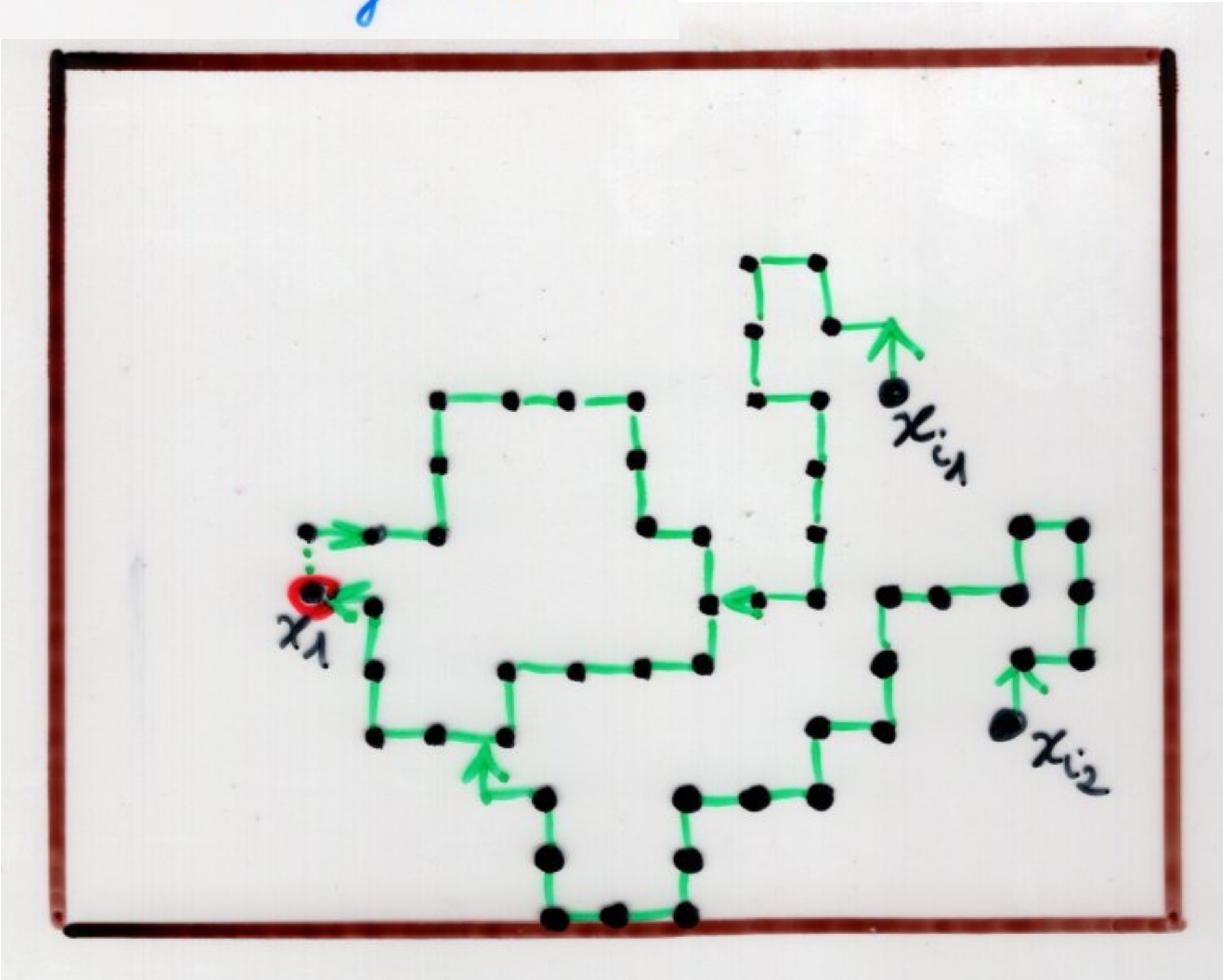
Marchal (2001)



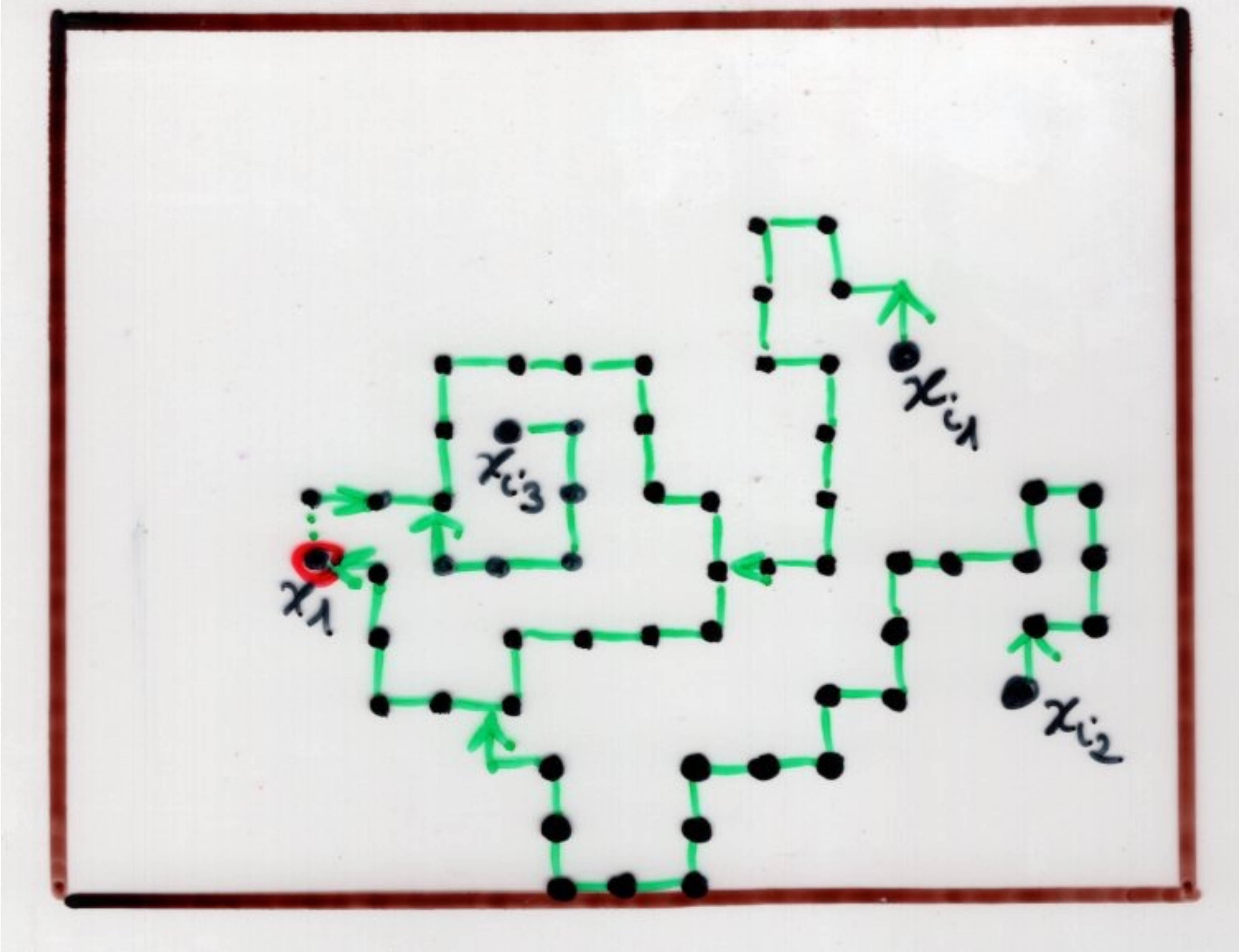
Wilson's algorithm



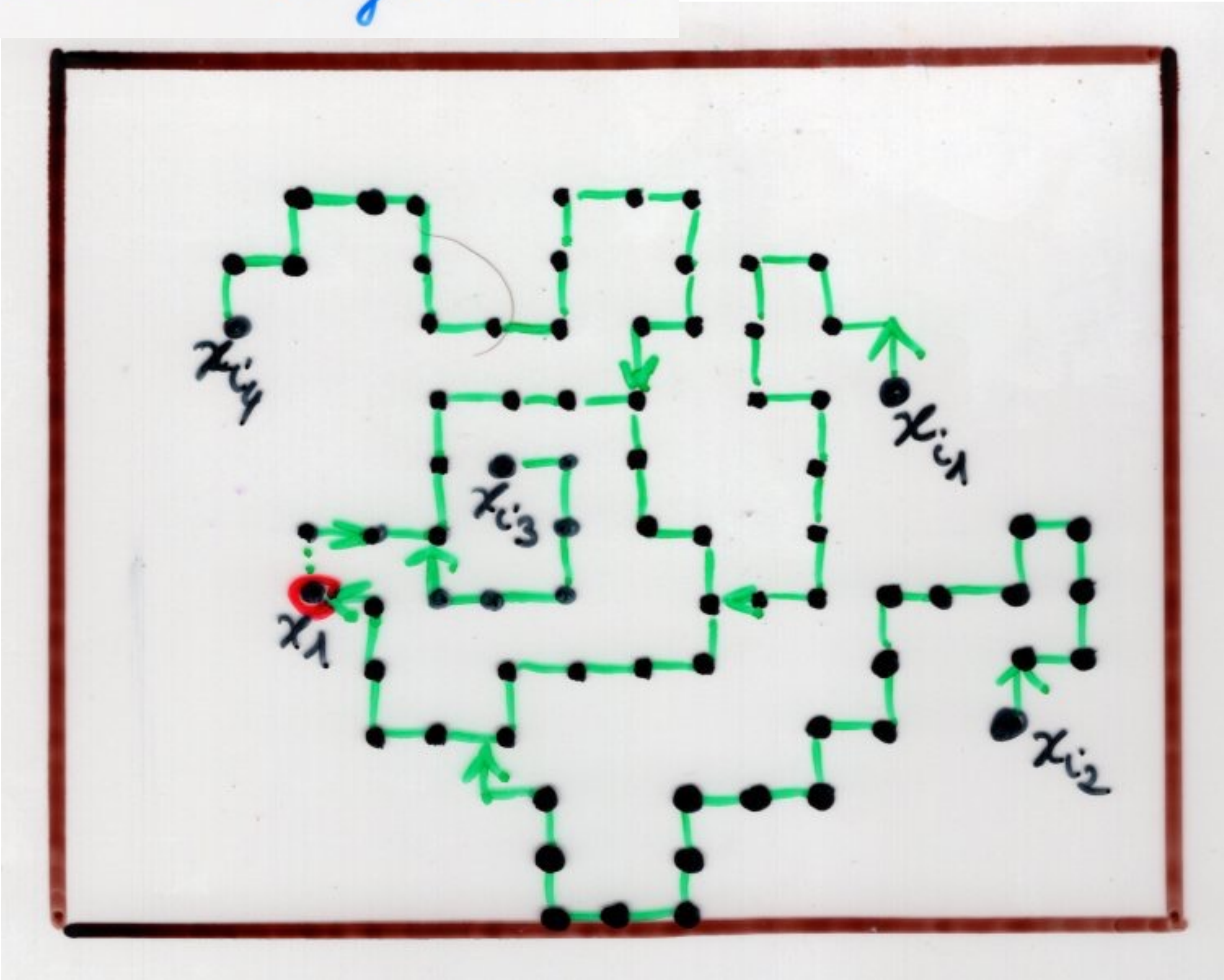
Wilson's algorithm



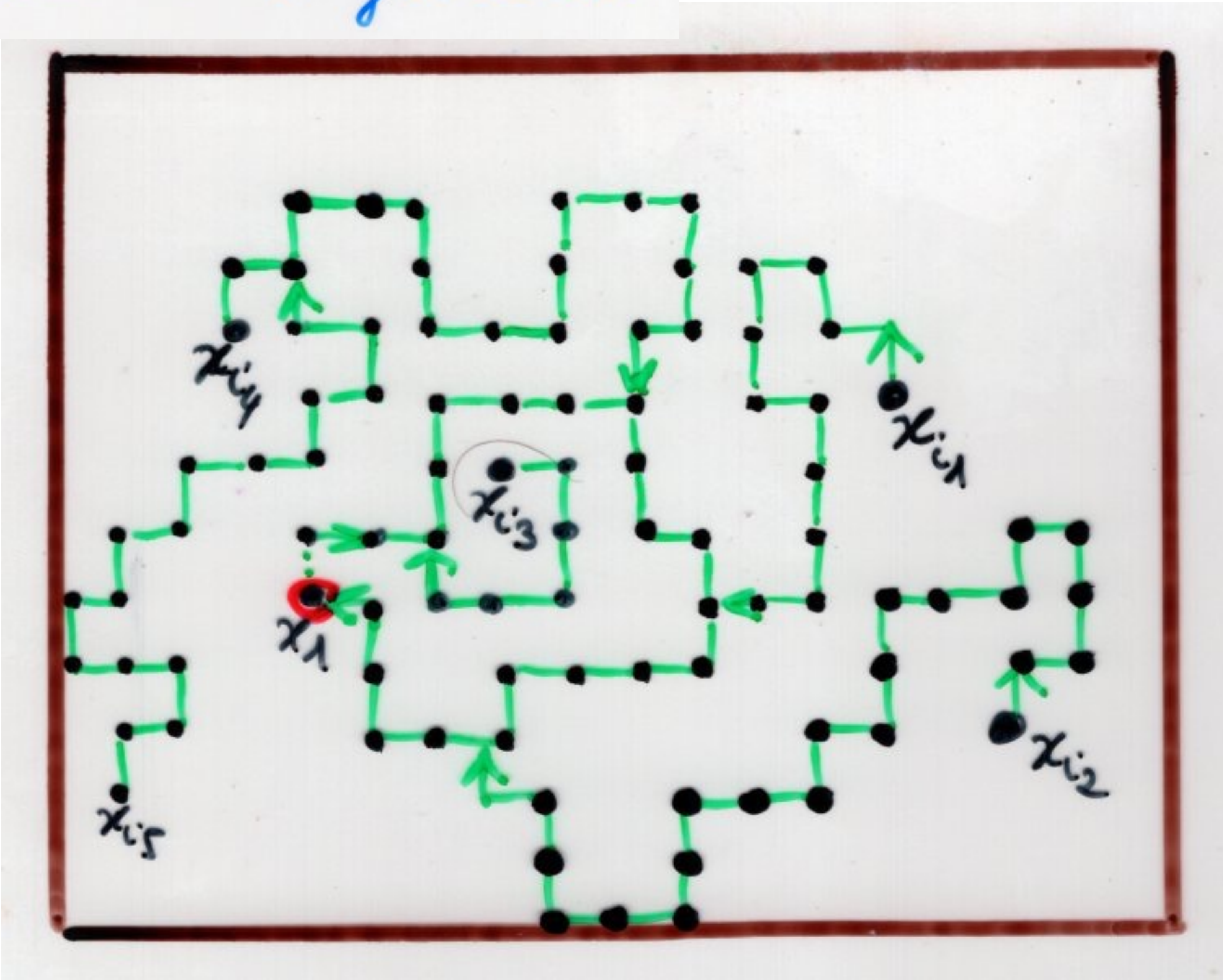
Wilson's algorithm



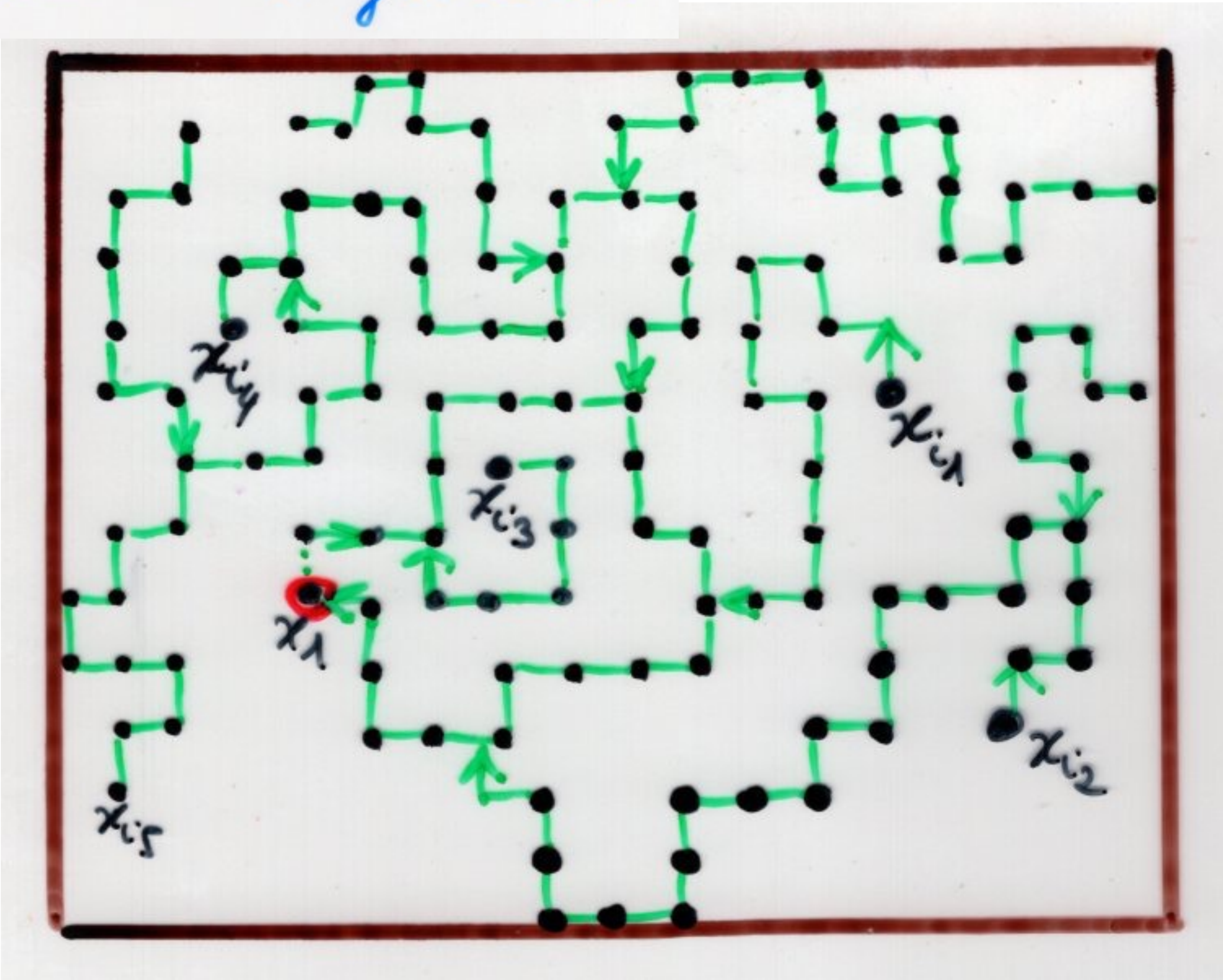
Wilson's algorithm



Wilson's algorithm



Wilson's algorithm



Proposition Propp-Wilson

The joint law of T and of the occupation measure at the stopping time of the algorithm does not depend on the ordering chosen on $V - x_1$

Proof Marchal (2001)

$$\text{Propp-Wilson algorithm} \longrightarrow (\underset{\text{spanning tree}}{T}, \underset{\text{heap of cycles}}{E})$$

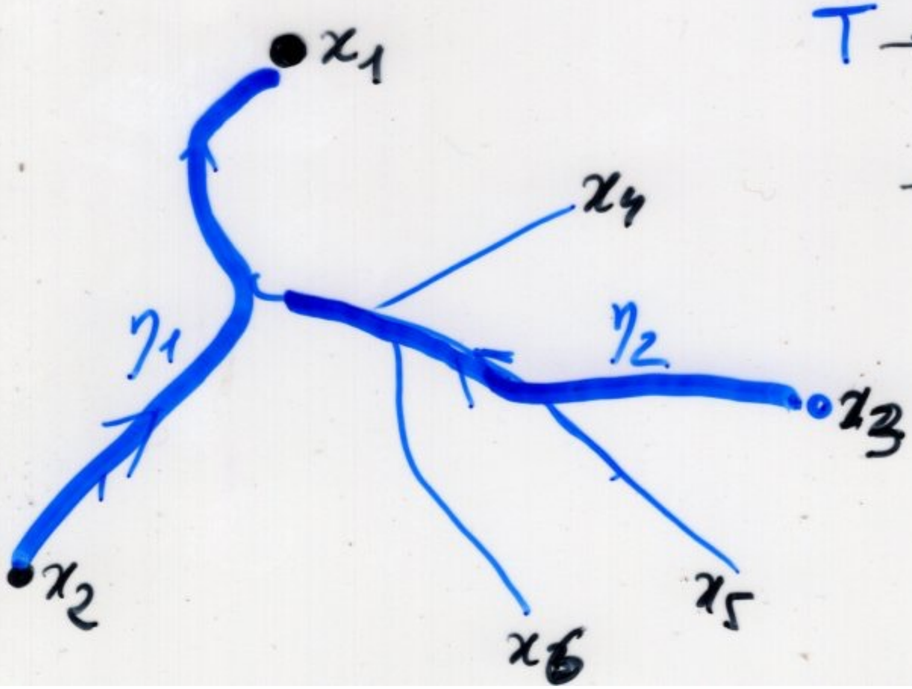
The sequence of operations of the Propp-Wilson algorithm are encoded in the pair (T, E)

(T, E)

spanning tree

heap of cycles

$\pi(\max(E))$ intersect T



$T \to x_2, \eta_1, E_1$

$\to x_3, \eta_2, E_2$

$- - - - - -$

— x_2 smallest vertex in T

η_1 path from x_1 to x_2

"push" $\eta_1 \rightarrow E = E_1 \odot F_1$

the cycles of F_1 do not intersect η_1

$\pi(\max(E_1))$ intersect η_1

— x_3 smallest vertex in $T - \eta_1$

$\rightarrow \eta_2$ path from x_3 to T

"push" $\eta_2 \rightarrow F_1 = E_2 \odot F_2$

etc ....

$\pi(\max(E_2))$ intersect η_2

$$E = E_1 \odot E_2 \cdots \odot E_k$$

probability $= V(T)\,V(E)$

does not depend of the total order of the points of $V - x_1$

$\square$

end of (summary of) the proof

Wilson's algorithm

animation: see the video

by Mike Rostock

https://bl.ocks.org/mbostock/11357811

next lecture:

fully commutative elements
in Coxeter groups