Physique combinatoire
et
algèbres quadratiques

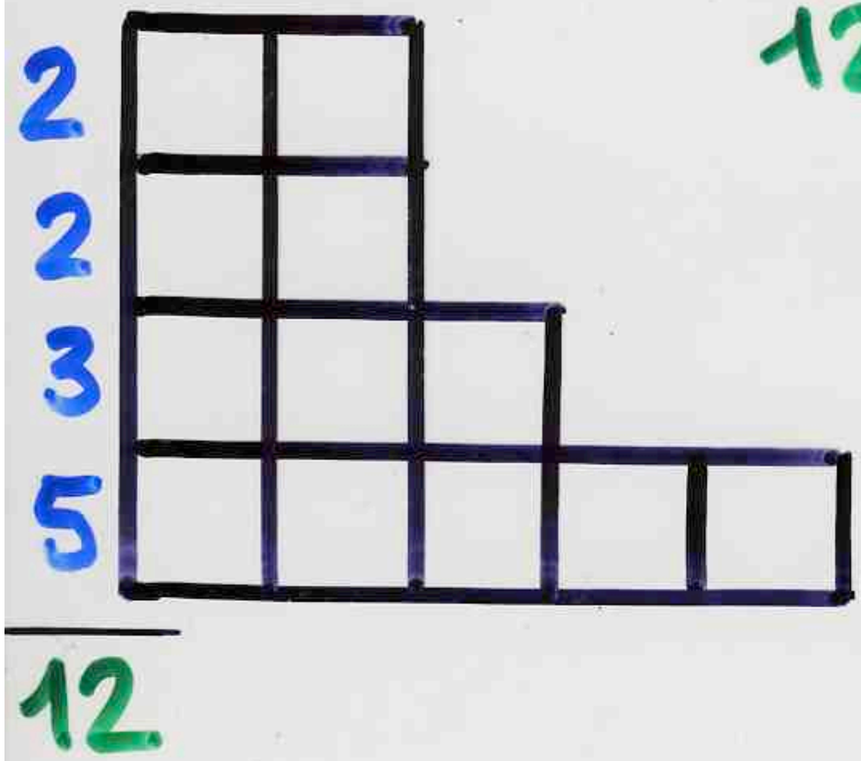
"L'Ansatz cellulaire"

7 Novembre 2011
Colloquium, Nice

Xavier Viennot
LaBRI, CNRS, Bordeaux

enumerative combinatorics:

Young tableaux

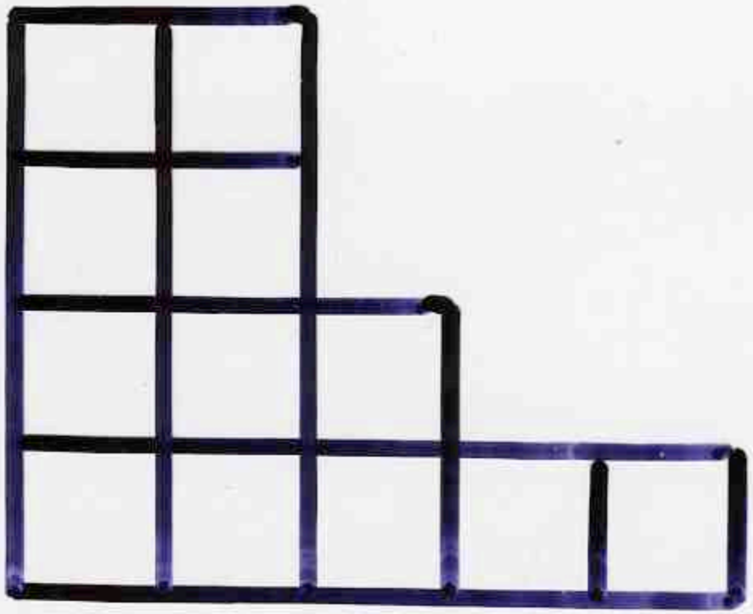


$12 = n = 5 + 3 + 2 + 2$

Ferrers

diagram

Partition of n



Young tableau

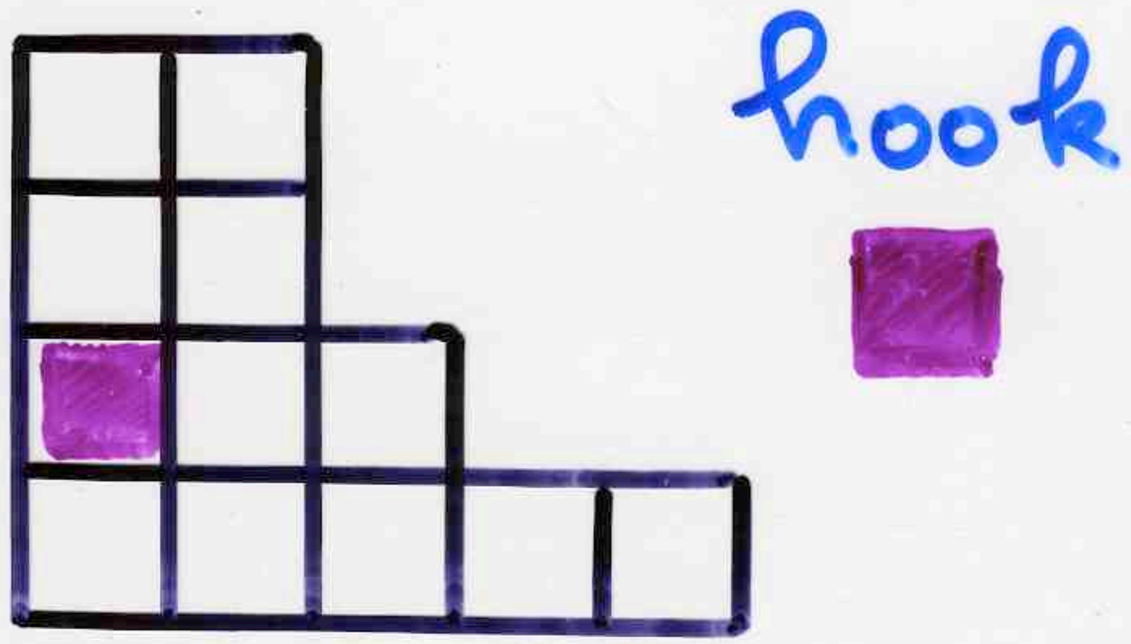
7	12			
6	10			
3	5	9		
1	2	4	8	11

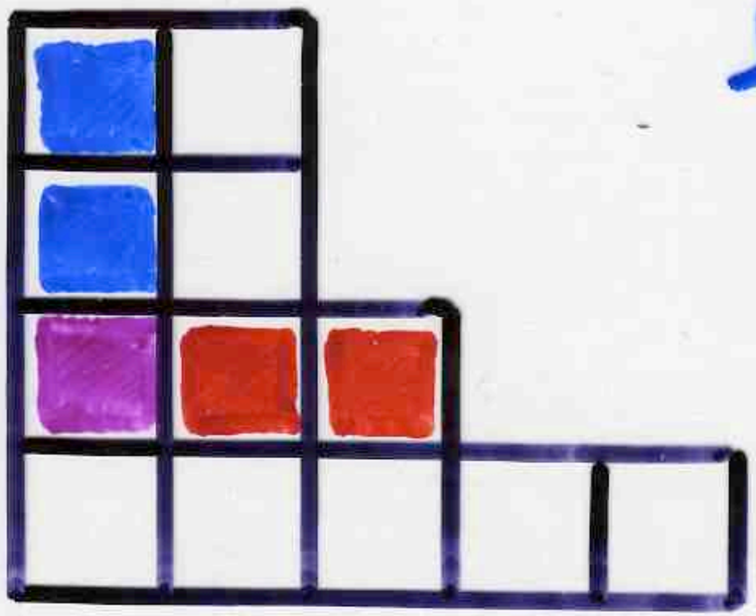
f_λ = nb of Young tableaux shape λ

Hook length formula

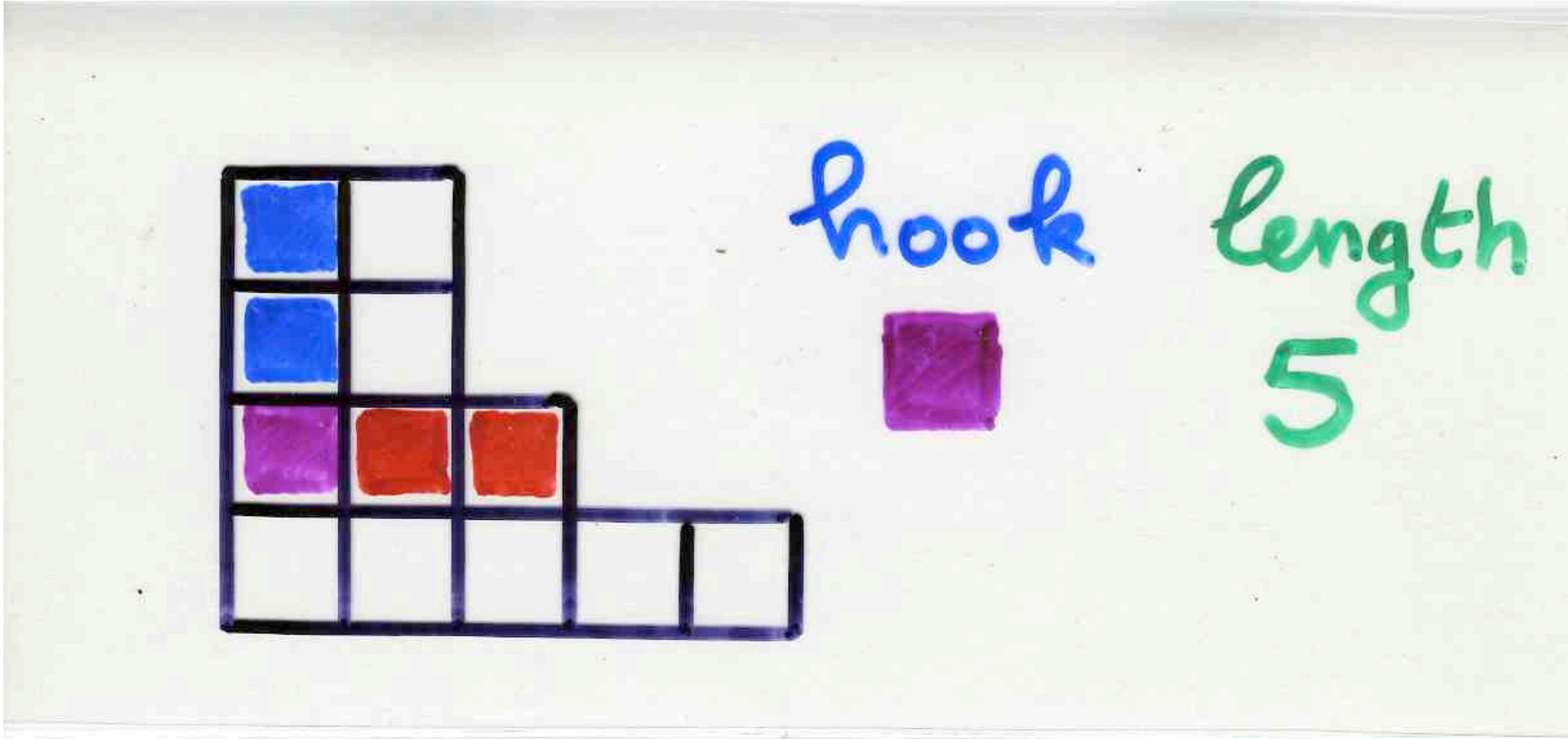
J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954

..... Franzblau-Zeilberger, Remmel, Greene-Wilf, Krattenthaler,
Novelli- Pak-Stoyanovski, ...

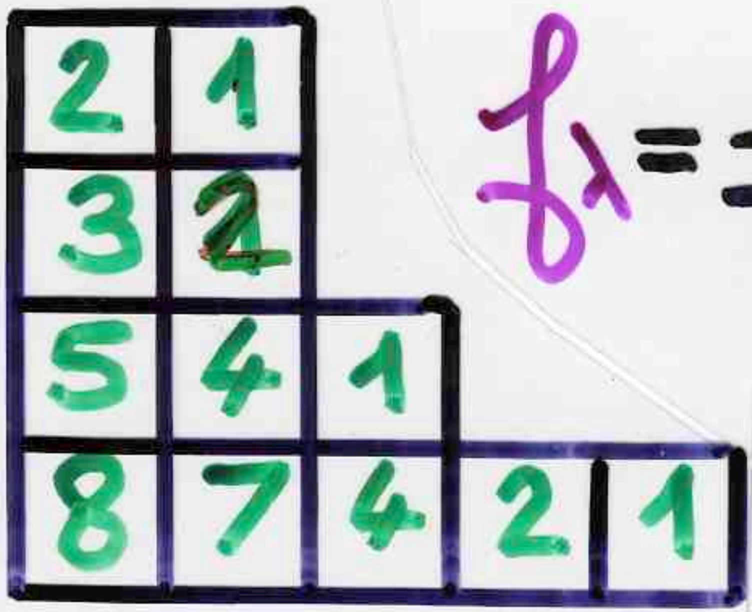

hook



hook

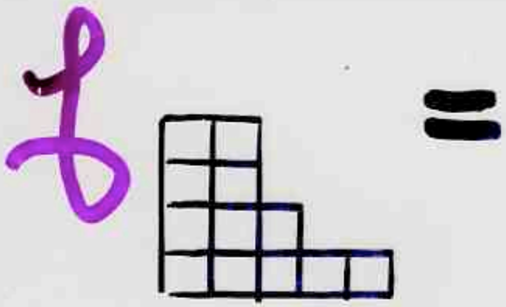

hook length
5

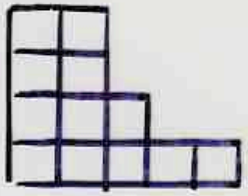
2	1			
3	2			
5	4	1		
8	7	4	2	1

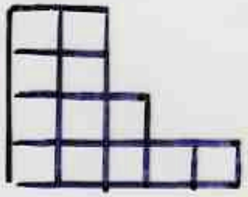


$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula



f  $= \dfrac{1 \cdot 2 \times 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \times 2^3 \times 3 \times 4^2 \cdot 5 \cdot 7 \cdot 8}$

f 

$$= \frac{1 \cdot 2 \times 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \times 2^3 \times 3 \times 4^2 \cdot 5 \cdot 7 \cdot 8}$$

$$= 3^4 \times 5 \times 11 = 4455$$

algebraic combinatorics:

Young tableaux
and
representation of the symmetric group

G fini

$$|G| = \sum_{\varphi} \deg^2(\varphi)$$

φ représentation irréductible

$$n! = \sum_{\lambda} f_{\lambda}^{2}$$

$n!$: ordre groupe fini $\mathfrak{S}_n$

λ : représentations irréductibles

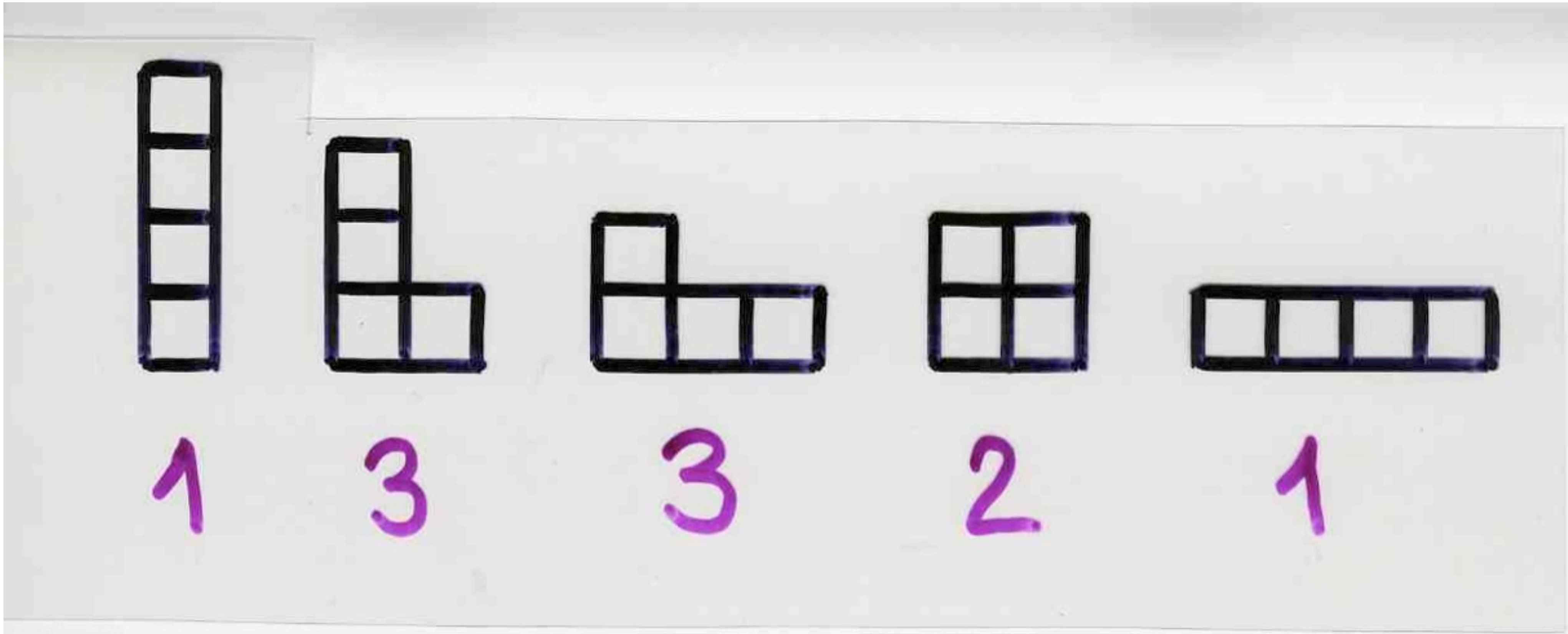
f_λ : degré

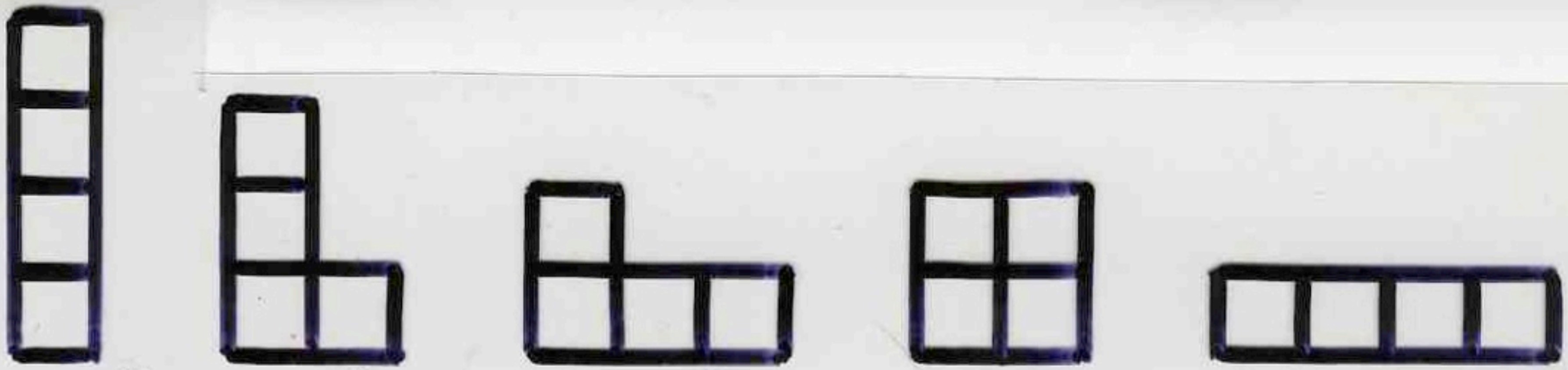
nombre de permutations

$$n! = \sum_{\lambda} \left(f_{\lambda} \right)^2$$

λ : forme, n cases

f_{λ} : nombre de tableaux de Young de forme λ


1 3 3 2 1

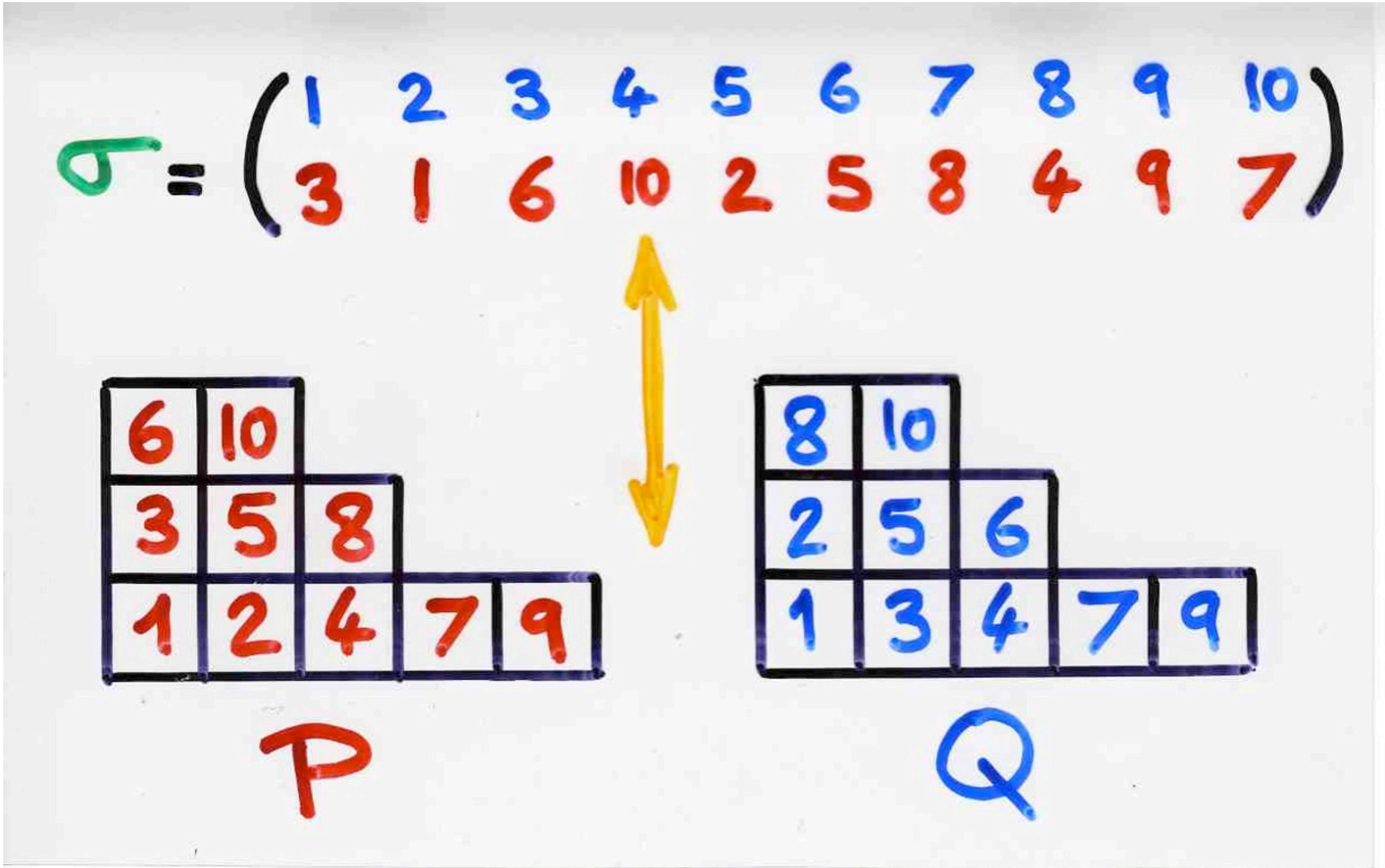


$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

bijective combinatorics:



The Robinson-Schensted correspondence (RSK) between permutations and pair of (standard) Young tableaux with the same shape combinatorial physics:

quantum mechanics:
spin chain model

Spin chains and combinatorics

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Institute for High Energy Physics
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(J. R. R. Tolkien)

The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$N = 3: \quad \psi_{001} = 1;$$
$$N = 5: \quad \psi_{00011} = 1, \quad \psi_{00101} = 2;$$
$$N = 7: \quad \psi_{0000111} = 1, \quad \psi_{0001101} = \psi_{0001011} = 3, \quad \psi_{0010011} = 4 \quad \psi_{0010101} = 7.$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101} = \psi_{000010111} = 4.$$

1, 2, 7, 42, 429, ...

?

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)\,a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

$$\text{E.g.f.:}\ \ (1-x)^{-3}\ e^{-x}.$$

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of ln $(1 + \ln(1 + x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)\,a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,.\,.\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 6453983693872074973935 6
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n) = n(n-1)a(n-1)/2 + a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 10339081666350180289849
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 5695183504479116640376509, 16217557574922386301420514191523784895639577710480
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 5695183504489239067484387, 16217557574922386301420531277071365103168734284282
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727

Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500

Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,.\,.\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356

Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \,..\, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 6453983693872074973956
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

ASM

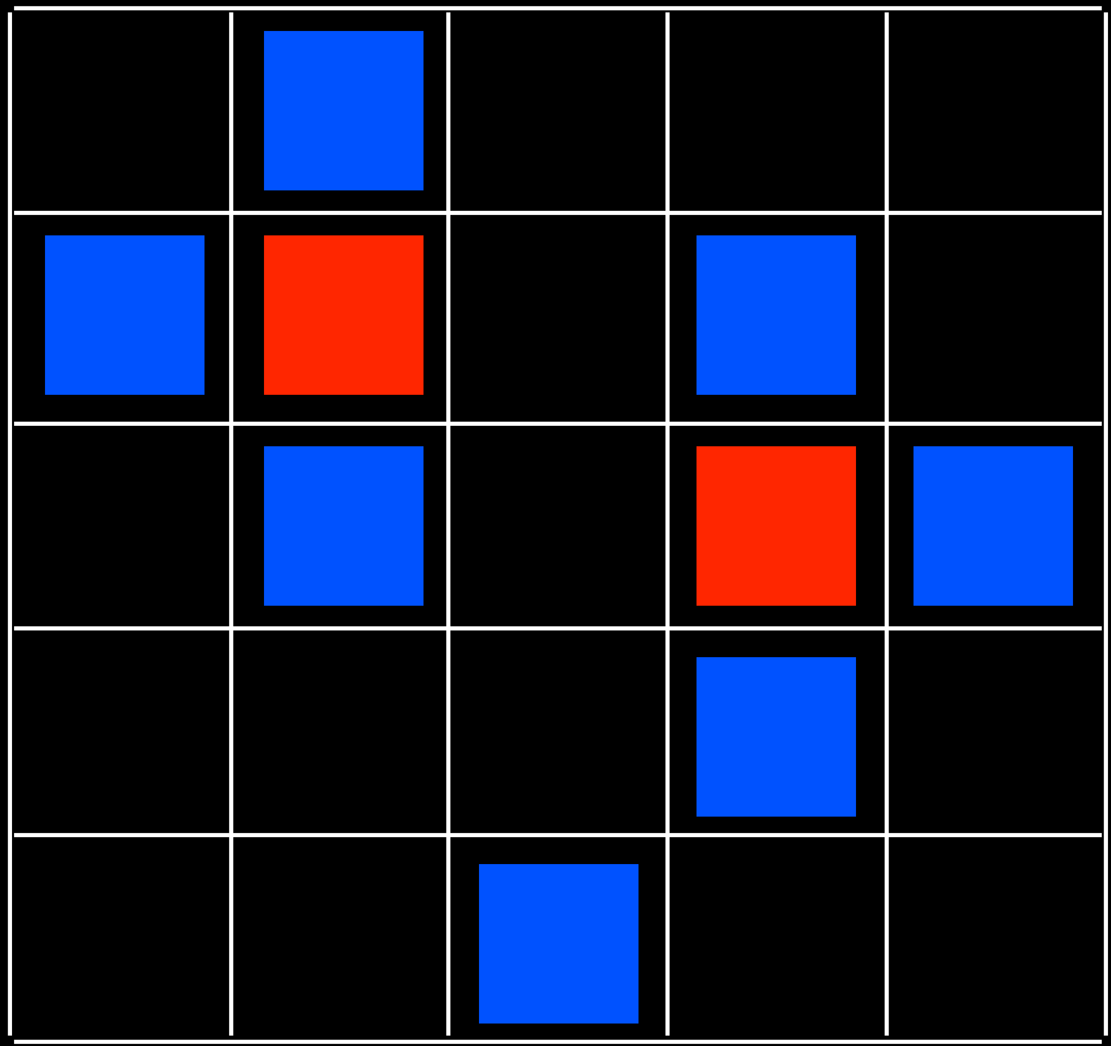
Alternating sign matrices

Alternating sign matrices

- entries: 0, 1, -1
- sum in rows and columns = 1
- non 0 entries alternate in sign in each row and column

ex:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \; 6$$

permutations

1, 2, 7, 42, 429, ...

"What else have you got in your pocket?" he went on, turning to A"

"Only a thimble,"

"Hand it over her

Then they all crow

while the Dodo solem

Lewis Carroll

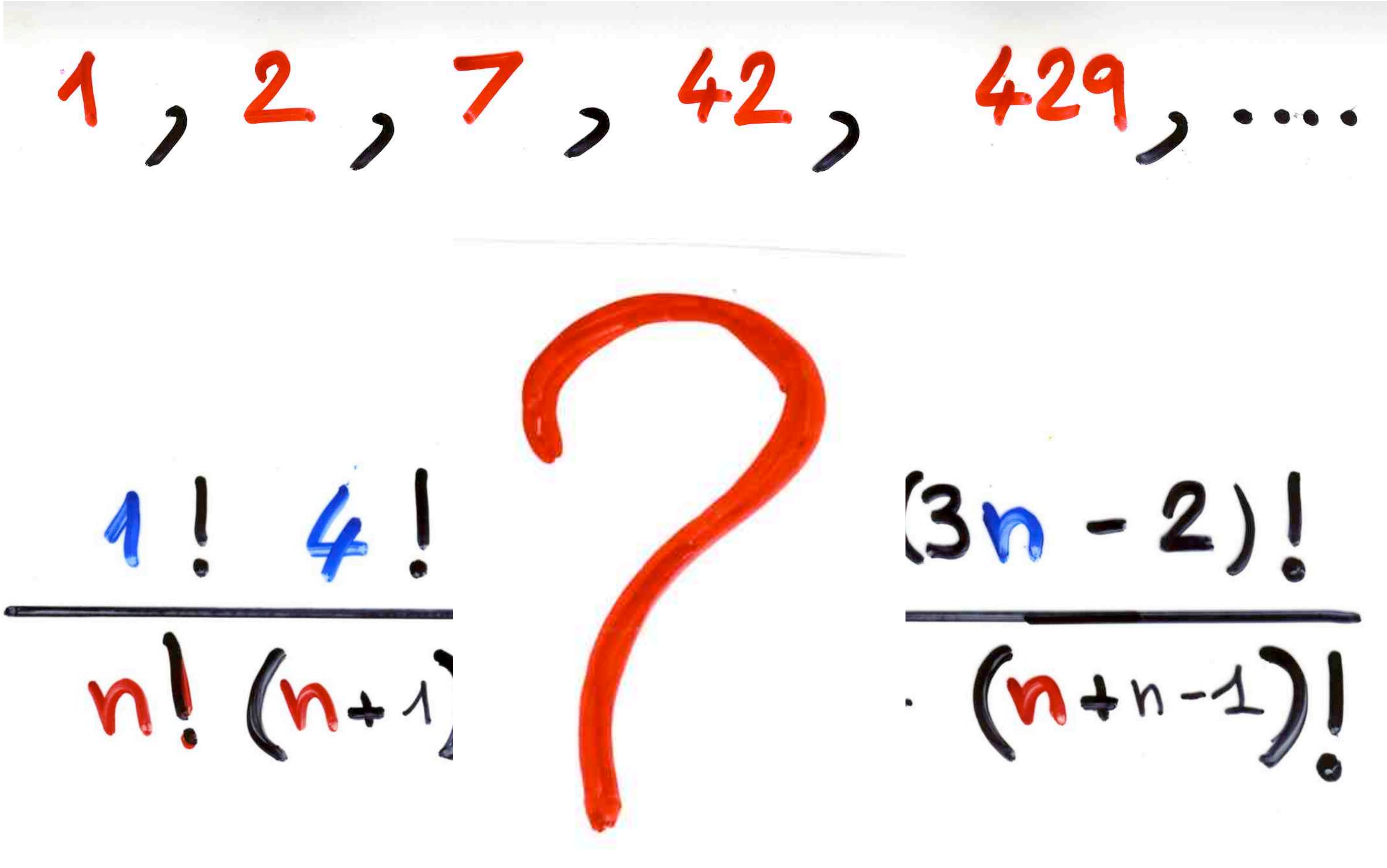
"Alice aux pays des merveilles"

C. I. Dodgson (1866)

Condensation of determinants

$$\det(M) = \frac{M_{NO} M_{SE} - M_{NE} M_{SO}}{M_C}$$

enumeration of ASM



alternating sign matrices conjecture

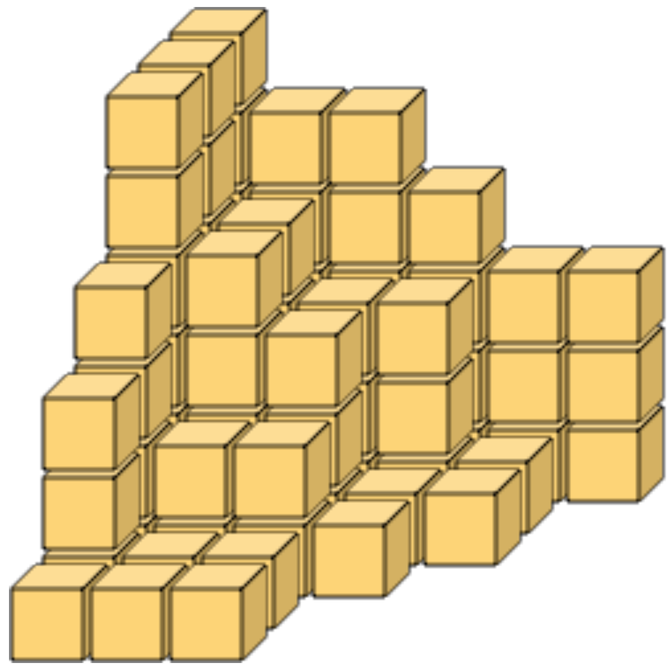
Mills, Robbins, Rumsey (1982)

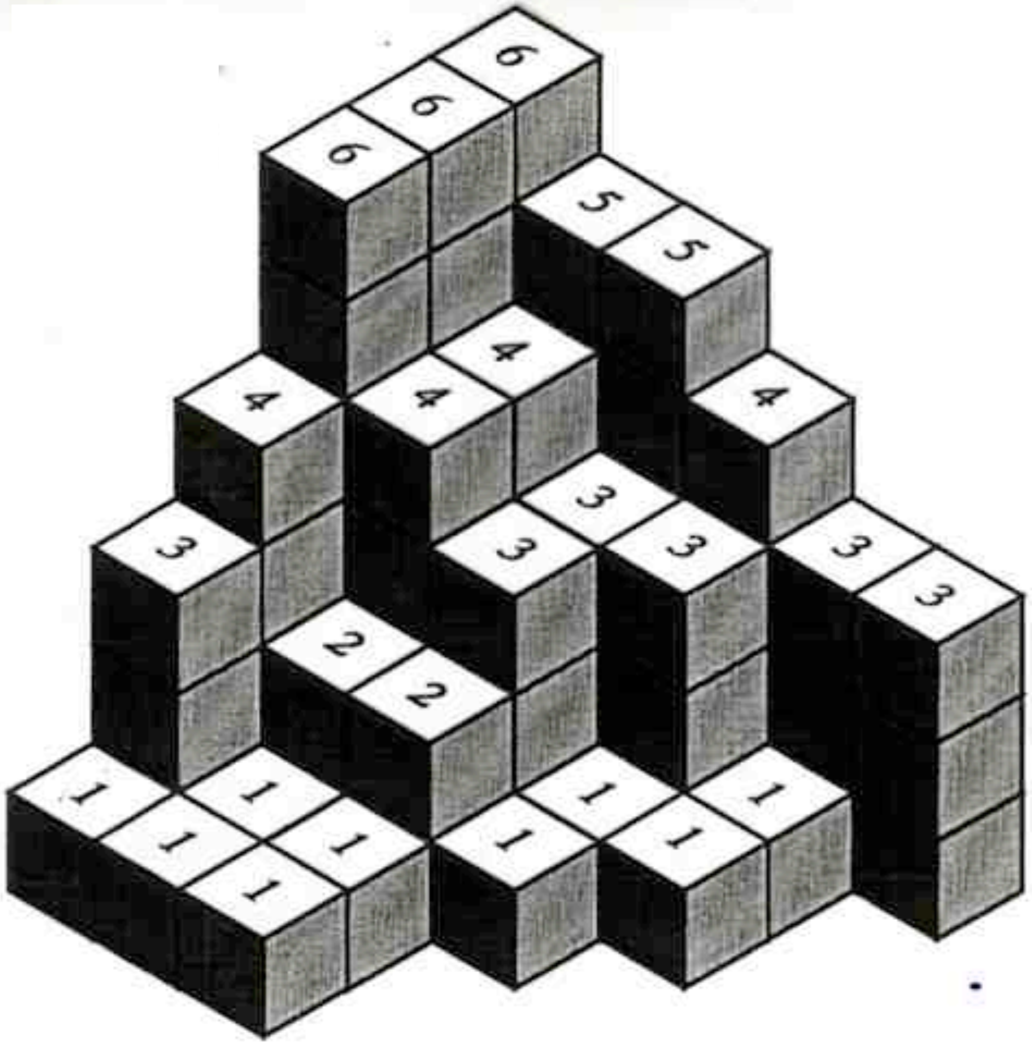
Robbins

The Mathematical Intelligencer (1991)

“These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain of living without understanding why they are true”

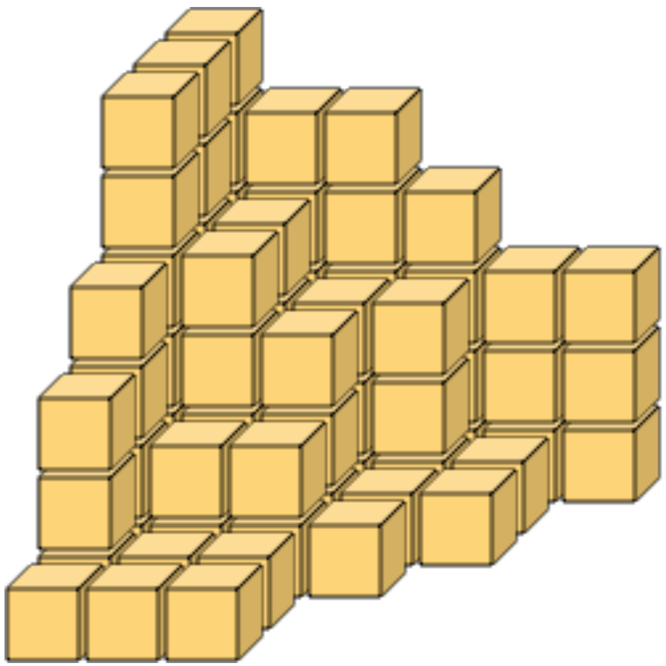
3D Ferrers diagram





plane partition

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

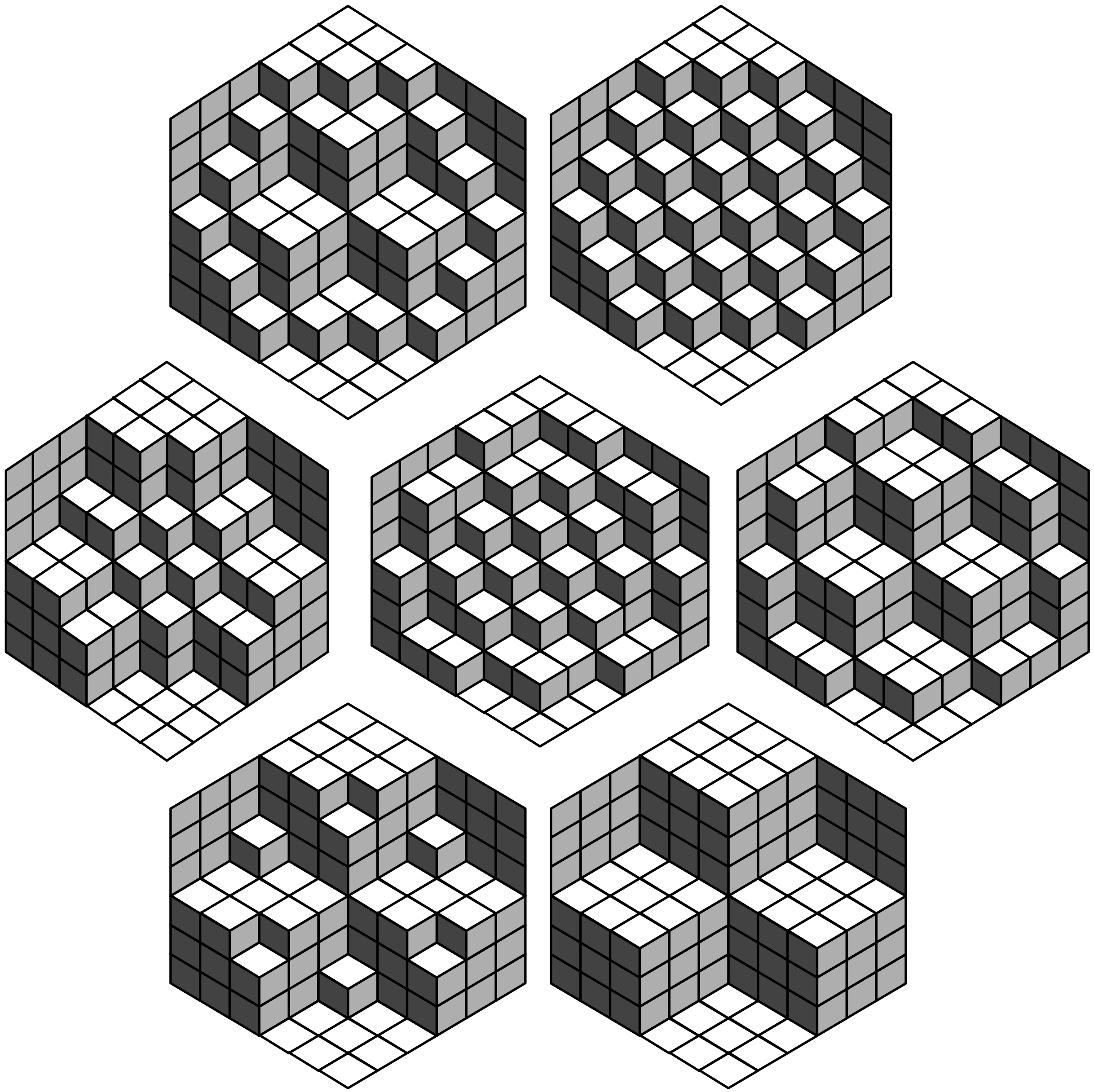


3D Ferrers diagram

TSSCPP

totaly symmetric
self complementary
plane partitions

1 , 2 , 7 , 42 , 429 , ...



1, 2, 7, 42, 429, 7436,

G. Andrews (1979)

- descending plane partitions

M. Mills, D.P. Robbins, H. Rumsey (1983)

- alternating sign matrices

- totally symmetric self-complementary plane partition (T.S.S.C.P.P.)

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = \frac{1!\,4!\,\cdots\,(3n-2)!}{n!\,(n+1)!\,\cdots\,(2n-1)!}$$

D. Zeilberger (1992- 1995)

(+ 90 checkers)

Proof of the A. S. M. conj.

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

Dedicated to my Friend, Mentor, and Guru, Dominique Foata.

Two stones build two houses. Three build six houses. Four build four and twenty houses. Five build hundred and twenty houses. Six build Seven hundreds and twenty houses. Seven build five thousands and forty houses. From now on, [exit and] ponder what the mouth cannot speak and the ear cannot hear.

(Sepher Yetsira IV,12)

Abstract: The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row- and column- sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\ldots(3n-2)!]/[n!(n+1)!\ldots(2n-1)!]$, as conjectured by Mills, Robbins, and Rumsey.

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[1] To appear in Electronic J. of Combinatorics (Foata's 60th Birthday issue). Version of July 31, 1995; original version written December 1992. The Maple package ROBBINS, accompanying this paper, can be downloaded from the www address in footnote 2 below.

[2] Department of Mathematics, Temple University, Philadelphia, PA 19122, USA.
E-mail:`zeilberg@math.temple.edu`. WWW:`http://www.math.temple.edu/~zeilberg`. Anon. ftp: `ftp.math.temple.edu`, directory `/pub/zeilberg`. Supported in part by the NSF.

[3] See the Exodion for affiliations, attribution, and short bios.

Subsublemma 1.1.3:

$$\sum_{\pi\in\mathcal{S}_k} \mathrm{sgn}(\pi)\cdot\pi\left[\frac{x_1x_2^2\ldots x_k^k}{(1-x_k)(1-x_kx_{k-1})\cdots(1-x_kx_{k-1}\ldots x_1)}\right] = \frac{x_1\cdots x_k\prod_{1\le i<j\le k}(x_j-x_i)}{\prod_{i=1}^k(1-x_i)\prod_{1\le i<j\le k}(1-x_ix_j)} \quad . \qquad (Issai)$$

[Type 'S113(k);' in ROBBINS, for specific k.]

Proof : See [PS], problem VII.47. Alternatively, (Issai) is easily seen to be equivalent to Schur's identity that sums all the Schur functions ([Ma], ex I.5.4, p. 45). This takes care of subsublemma 1.1.3. □

Inserting $(Issai)$ into $(Stanley)$, expanding $\prod_{1\le i<j\le k}(x_j-x_i)$ by Vandermonde's expansion,

$$\sum_{\pi\in\mathcal{S}_k} \mathrm{sgn}(\pi)\cdot\pi(x_1^0x_2^1\cdots x_k^{k-1}) \quad ,$$

using the antisymmetry of Δ_k once again, and employing crucial fact $\aleph_4$, we get the following string of equalities:

$$b_k(n) = \frac{1}{k!}CT_{x_1,\ldots,x_k}\left\{\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n}x_i^{n+k-1}}\left(\frac{x_1\cdots x_k\prod_{1\le i<j\le k}(x_j-x_i)}{\prod_{i=1}^k(1-x_i)\prod_{1\le i<j\le k}(1-x_ix_j)}\right)\right\}$$

$$= \frac{1}{k!}CT_{x_1,\ldots,x_k}\left\{\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-2}\prod_{1\le i<j\le k}(1-x_ix_j)}\left(\sum_{\pi\in\mathcal{S}_k}\mathrm{sgn}(\pi)\cdot\pi(x_1^0x_2^1\ldots x_k^{k-1})\right)\right\}$$

$$= \frac{1}{k!}\sum_{\pi\in\mathcal{S}_k} CT_{x_1,\ldots,x_k}\left\{\pi\left[\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-2}\prod_{1\le i<j\le k}(1-x_ix_j)}\left(\prod_{i=1}^k x_i^{i-1}\right)\right]\right\}$$

$$= \frac{1}{k!}\sum_{\pi\in\mathcal{S}_k} CT_{x_1,\ldots,x_k}\left\{\pi\left[\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right]\right\}$$

$$= \frac{1}{k!}\sum_{\pi\in\mathcal{S}_k} CT_{x_1,\ldots,x_k}\left\{\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}$$

$$= \frac{1}{k!}\left(\sum_{\pi\in\mathcal{S}_k}1\right) CT_{x_1,\ldots,x_k}\left\{\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\}$$

$$= CT_{x_1,\ldots,x_k}\left\{\frac{\Delta_k(x_1,\ldots,x_k)}{\prod_{i=1}^k(\bar{x}_i)^{k+n+1}x_i^{n+k-i-1}\prod_{1\le i<j\le k}(1-x_ix_j)}\right\} , \qquad (George''')$$

where in the last equality we have used Levi Ben Gerson's celebrated result that the number of elements in $\mathcal{S}_k$ (the symmetric group on k elements,) equals $k!$. The extreme right of $(George''')$ is exactly the right side of $(MagogTotal)$. This completes the proof of sublemma 1.1. □

"EXTREME UGLYNESS CAN BE BEAUTIFUL"

Doron Zeilberger

Bordeaux, May, 1991

3rd FPSAC

Kuperberg (1995)

6-vertex model

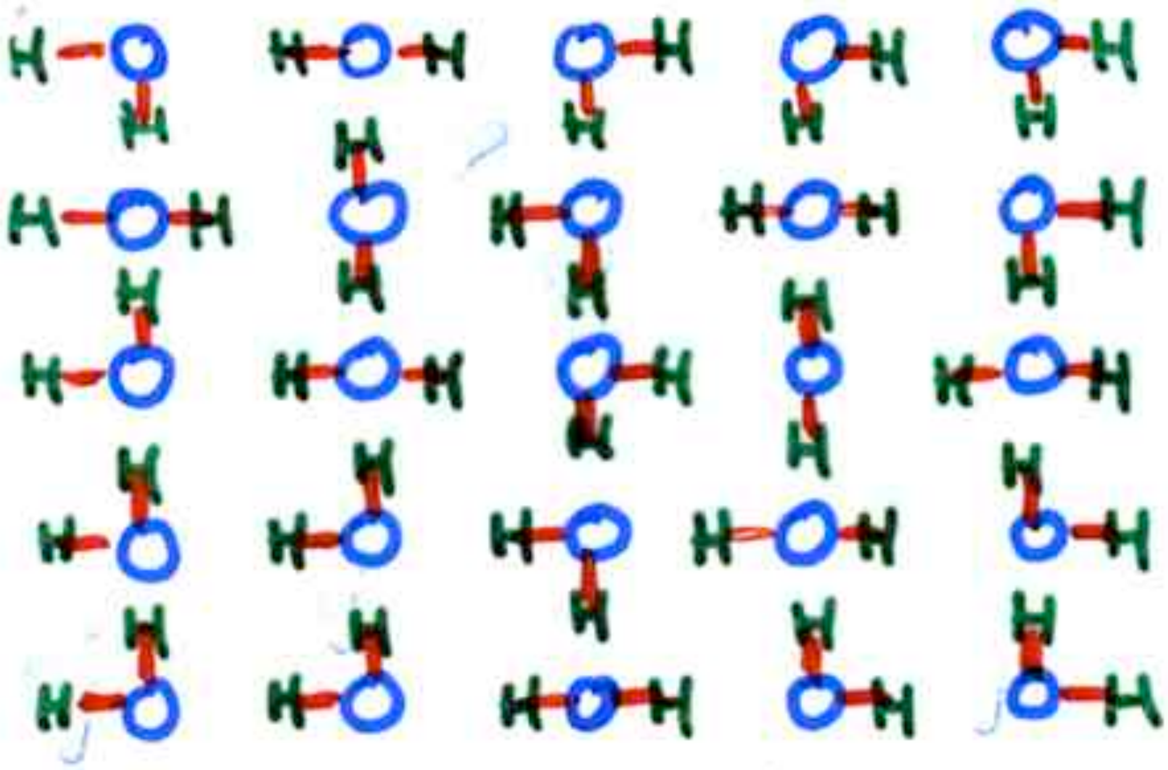
(ice model)

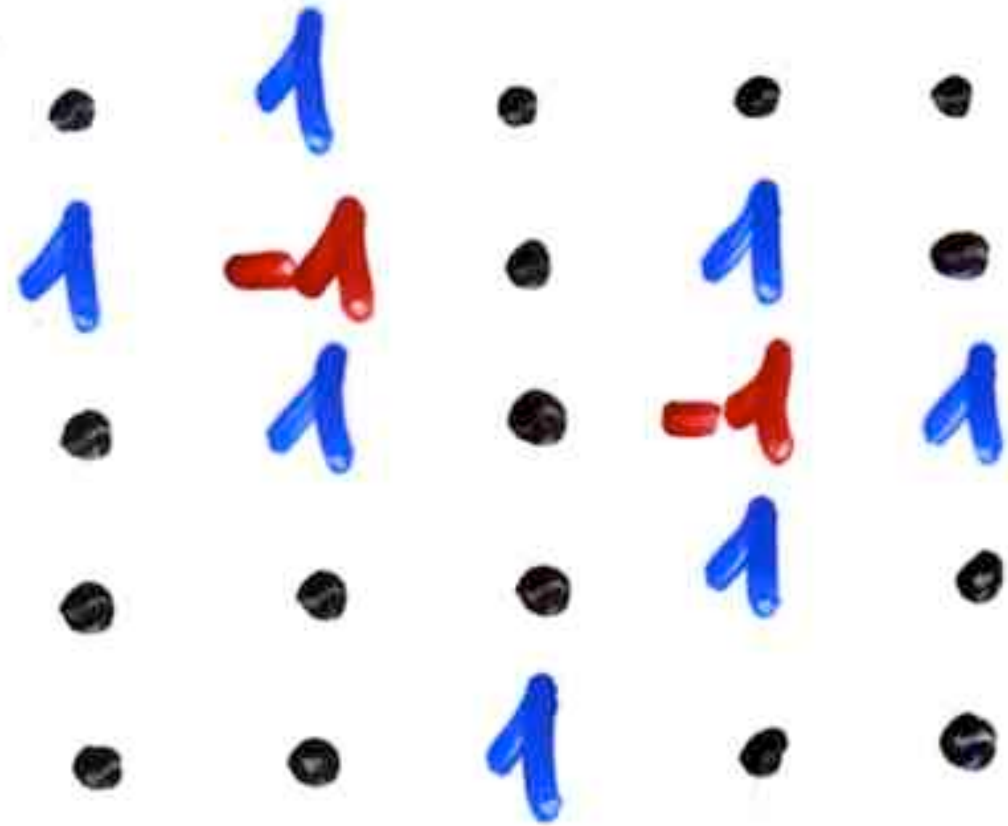
with domain wall boundary condition

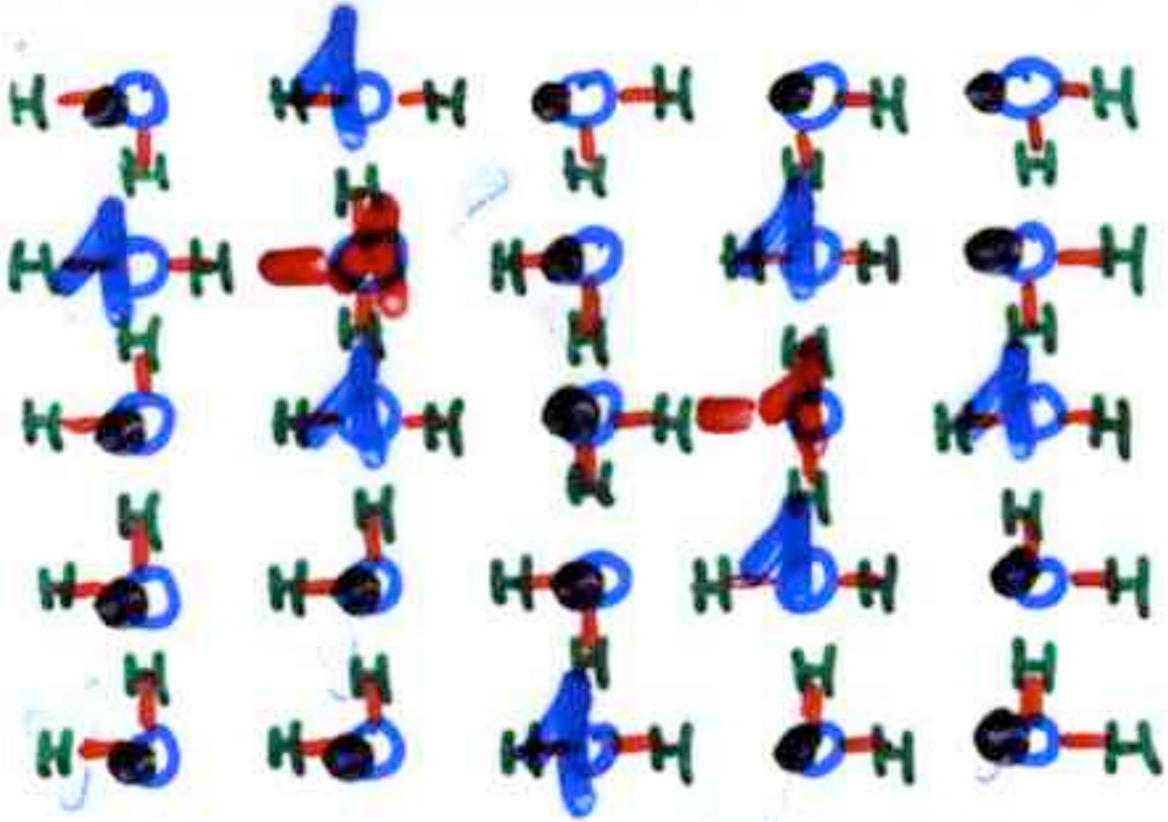
ice model

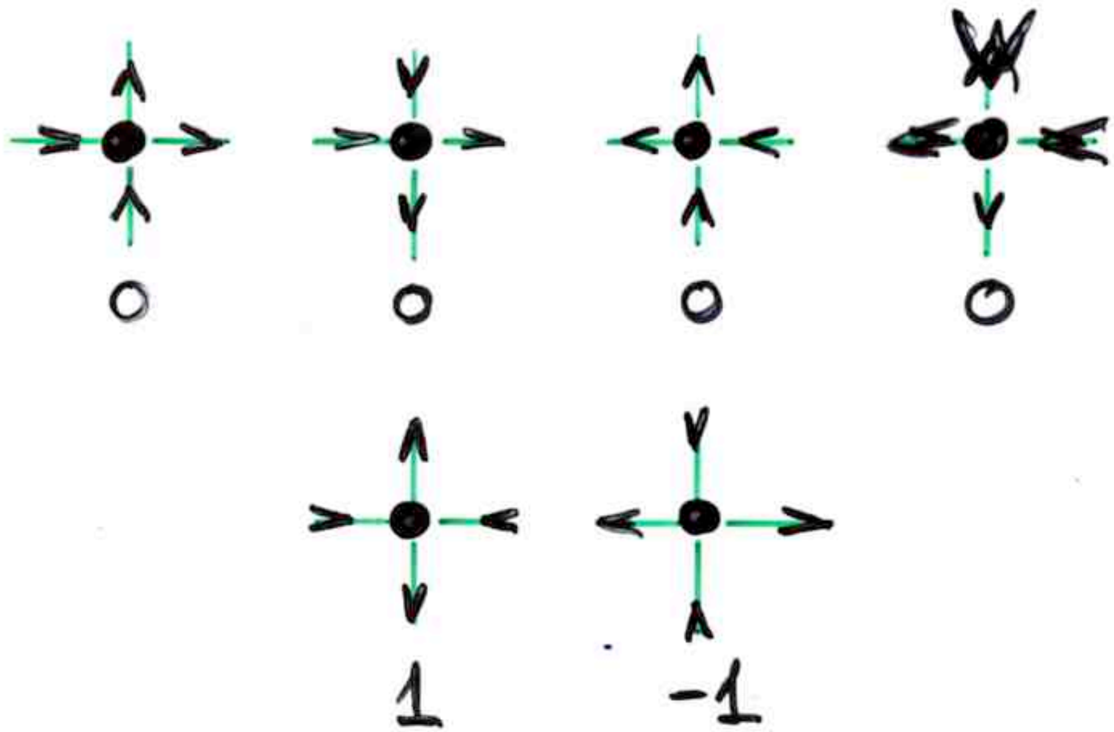
or

six-vertex model

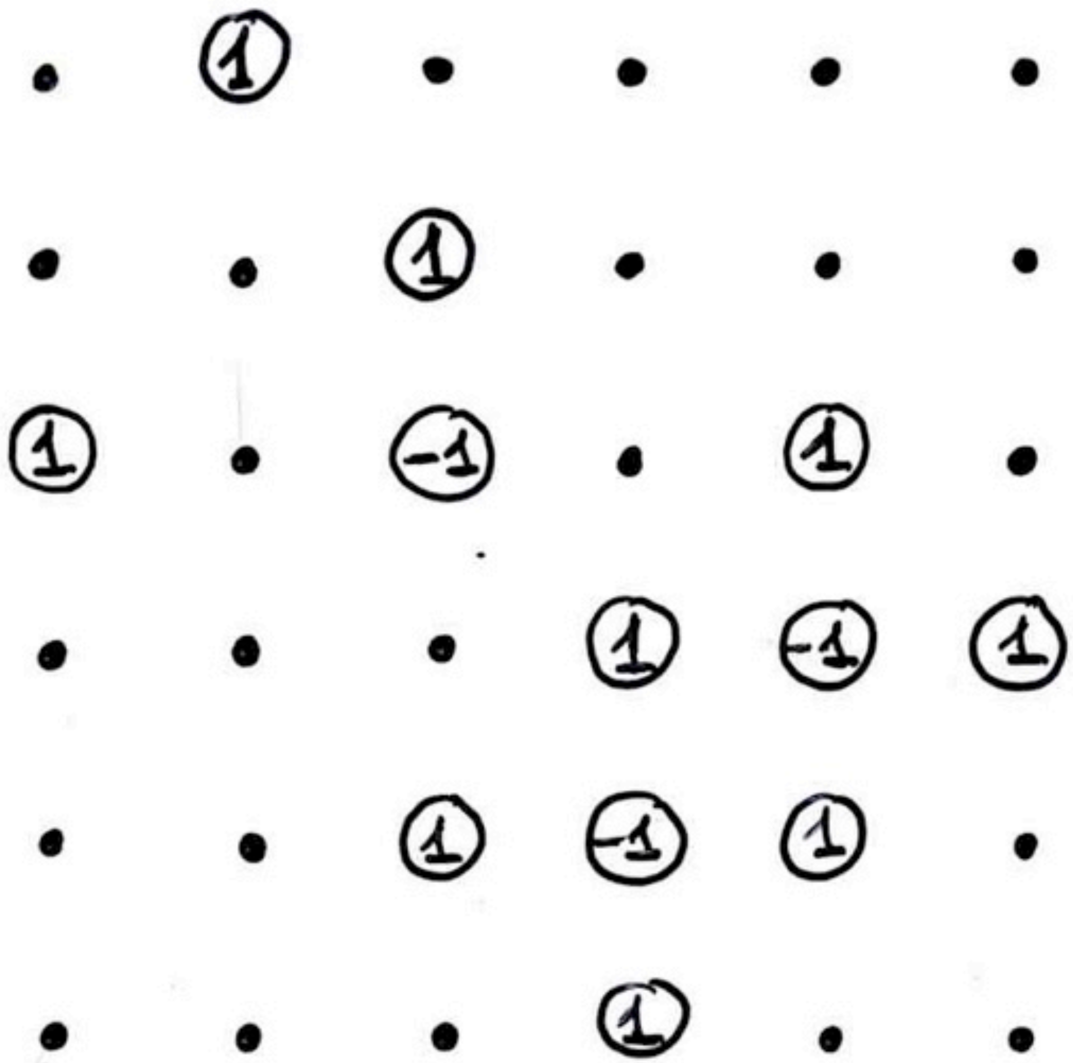








6-vertex model



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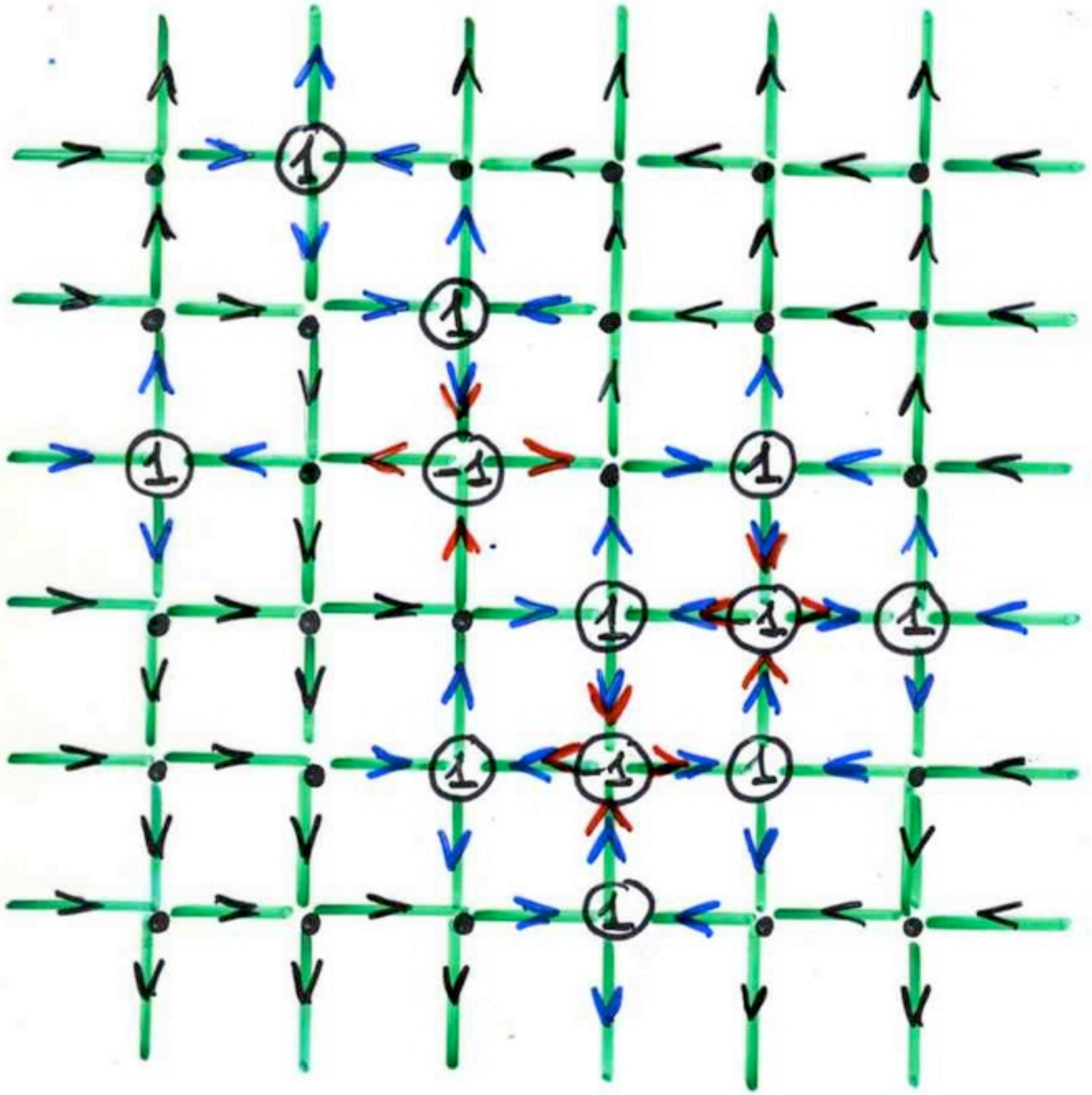
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fonction de partition

Gaudin

Korepin, Bogoliubov, Isergin

"Quantum Inverse Scattering Method and Correlation Functions" (1993)

$$Z_n(\vec{x};\vec{y};a) = \frac{\prod_{i=1}^{n} x_i/y_i \prod_{1\le i,j\le n} (x_i/y_j)(ax_i/y_j)}{\prod_{1\le i<j\le n} (x_i/x_j)(y_j/y_i)} \det(M)$$

$$M = \frac{1}{(x_i/y_j)(ax_i/y_j)}$$

équation Yang-Baxter

Proofs and Confirmations
The story of the alternating sign matrix conjecture

David M. Bressoud
Macalester College
Saint Paul, MN

July 28, 1997

Razumov - Stroganov conjecture 2000-2001

Spin chains and combinatorics

A. V. Razumov, Yu. G. Stroganov

Institute for High Energy Physics
142284 Protvino, Moscow region, Russia

(J. R. R. Tolkien)

The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$N = 3: \quad \psi_{001} = 1;$$
$$N = 5: \quad \psi_{00011} = 1, \quad \psi_{00101} = 2;$$
$$N = 7: \quad \psi_{0000111} = 1, \quad \psi_{0001101} = \psi_{0001011} = 3, \quad \psi_{0010011} = 4 \quad \psi_{0010101} = 7.$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$\psi_{000001111} = 1, \quad \psi_{000010111} = 4, \quad \psi_{000011011} = 6, \quad \psi_{000100111} = 7,$$
$$\psi_{000101011} = 17, \quad \psi_{000101101} = 14, \quad \psi_{000110011} = 12, \quad \psi_{001001011} = 21,$$
$$\psi_{001010011} = 25, \quad \psi_{001010101} = 42.$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101} = \psi_{000010111} = 4.$$

Razumov - Stroganov
(ex)- conjecture 2000-2001

proof by :
L. Cantini and A.Sportiello (March 2010)
arXiv: 1003.3376 [math.CO]
completely combinatorial proof

The "cellular Ansatz"

Heisenberg operators

U, D

$UD = DU + 1$

$$UD = DU + I$$

Lemme - Tout mot $w \in \{U, D\}^*$ s'écrit

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

normal ordering

$$UD = DU + Id$$

$$U^n D^n = \ ?$$

$$UUUDDD = UU(DU + Id)DD$$

$$= UUDUDD + UUDD$$

$$= UDUUDD + 2\,UUDD$$

$$= DUUUDD + 3\,UUDD$$

$$UUDD = UDUD + UD$$

$$= DUUD + 2\,UD$$

$$= DUDU + DU + 2\,(DU + I_d)$$

$$= DD\,UU + 2\,DU$$

$$= DDUU + 4\,DU + 2\,I_d$$

$$U^3D^3 = DU\,(DDUU + 4DU + 2I_d) + 3\,(DDUU + 4DU + 2I_d)$$

$$= DD\,UDUU + DDUU + 4\,(DD\,UU + DU) + 2\,DU + 3\,DDUU + 12\,DU + 6\,I_d$$

$$= D^3U^3 + 9\,D^2U^2 + 18\,DU + 6\,I_d$$

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

$$c_{n,i} = \binom{n}{i}^2 (n-i)!$$

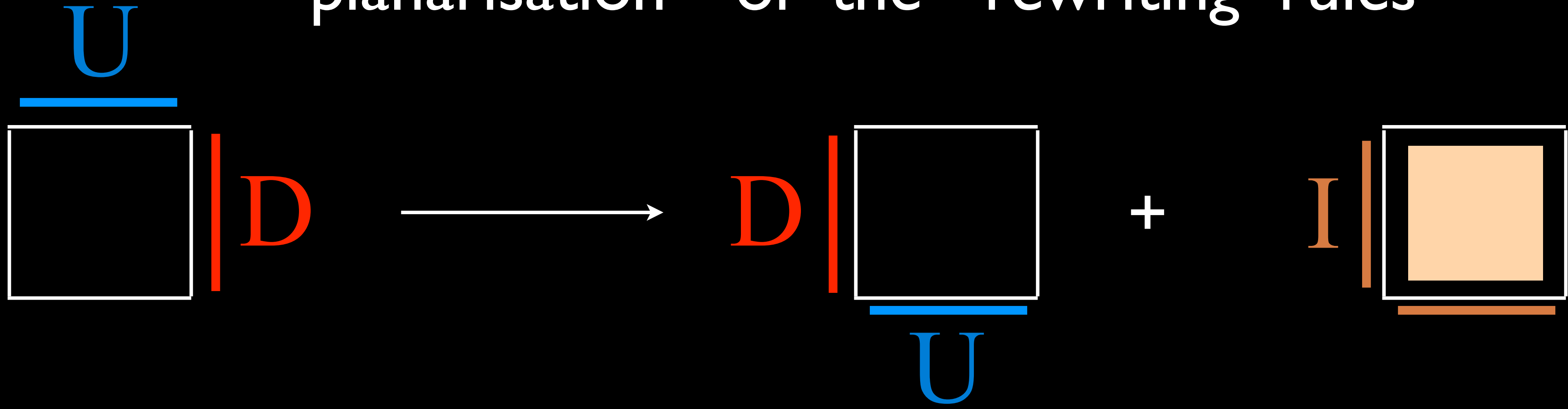
The cellular Ansatz
for $UD = DU + 1$

first step

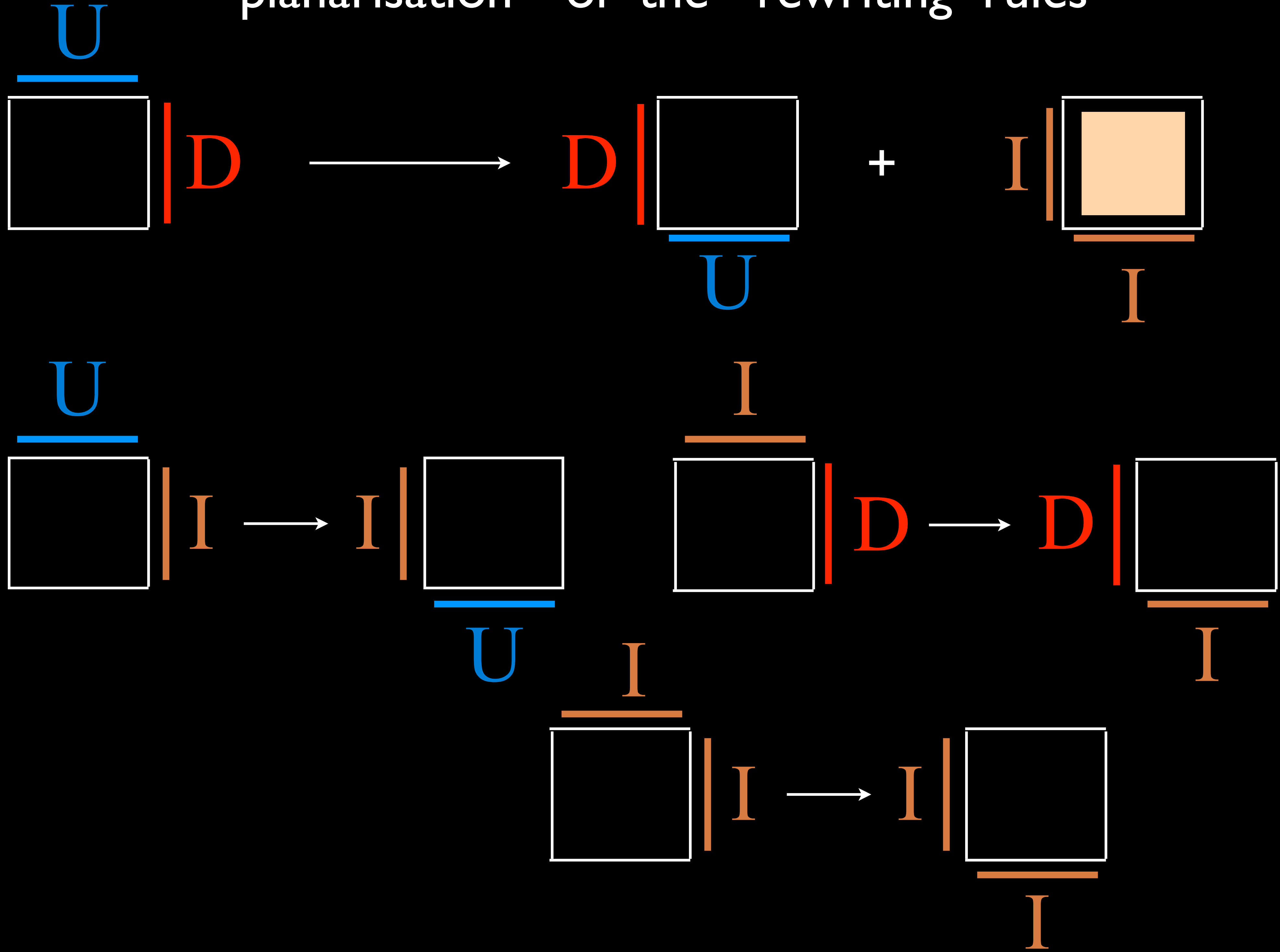
$$UD = DU + I$$

$$UD \to DU \qquad UD \to I$$

“planarisation” of the “rewriting rules”



“planarisation” of the “rewriting rules”

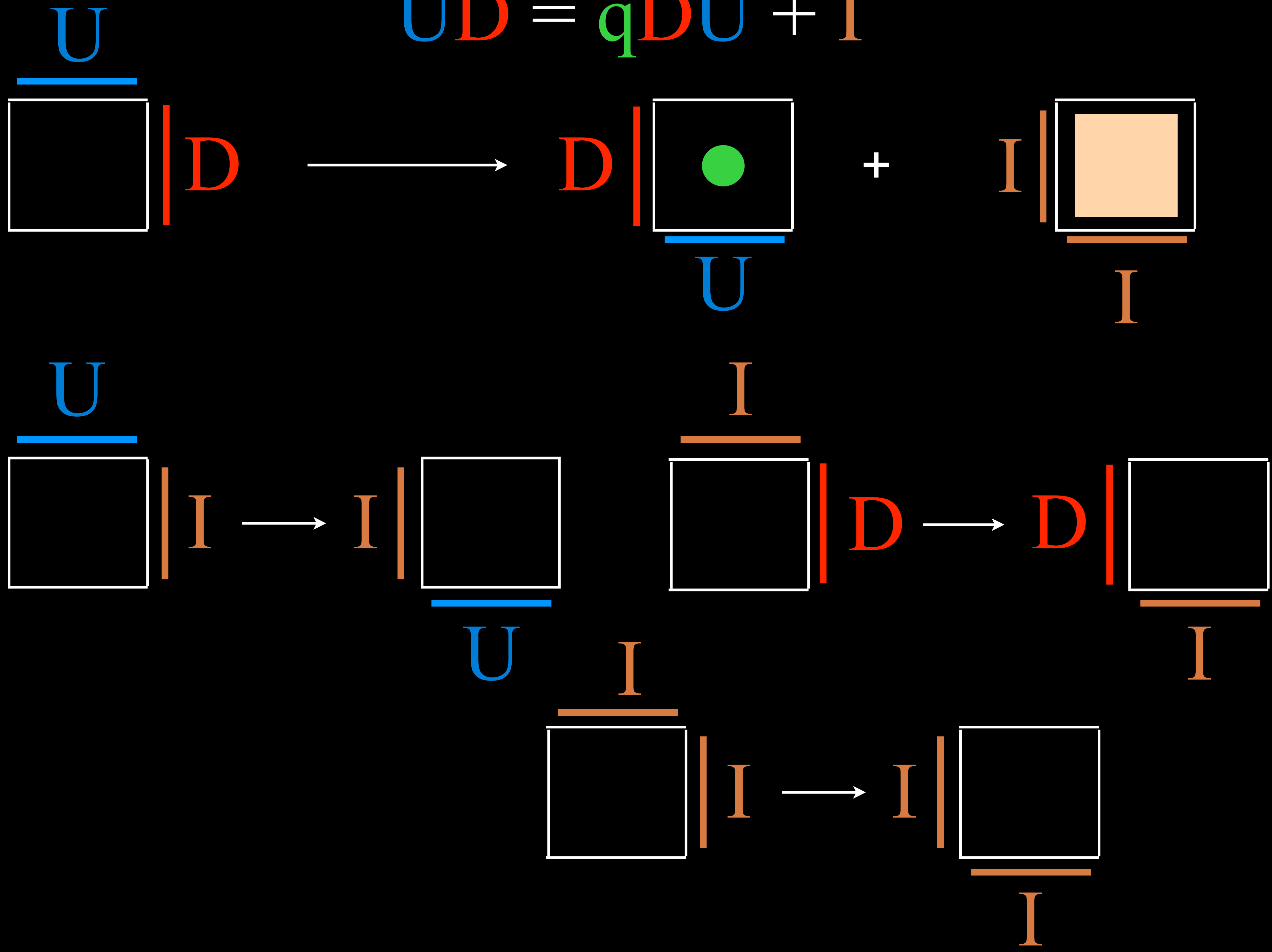


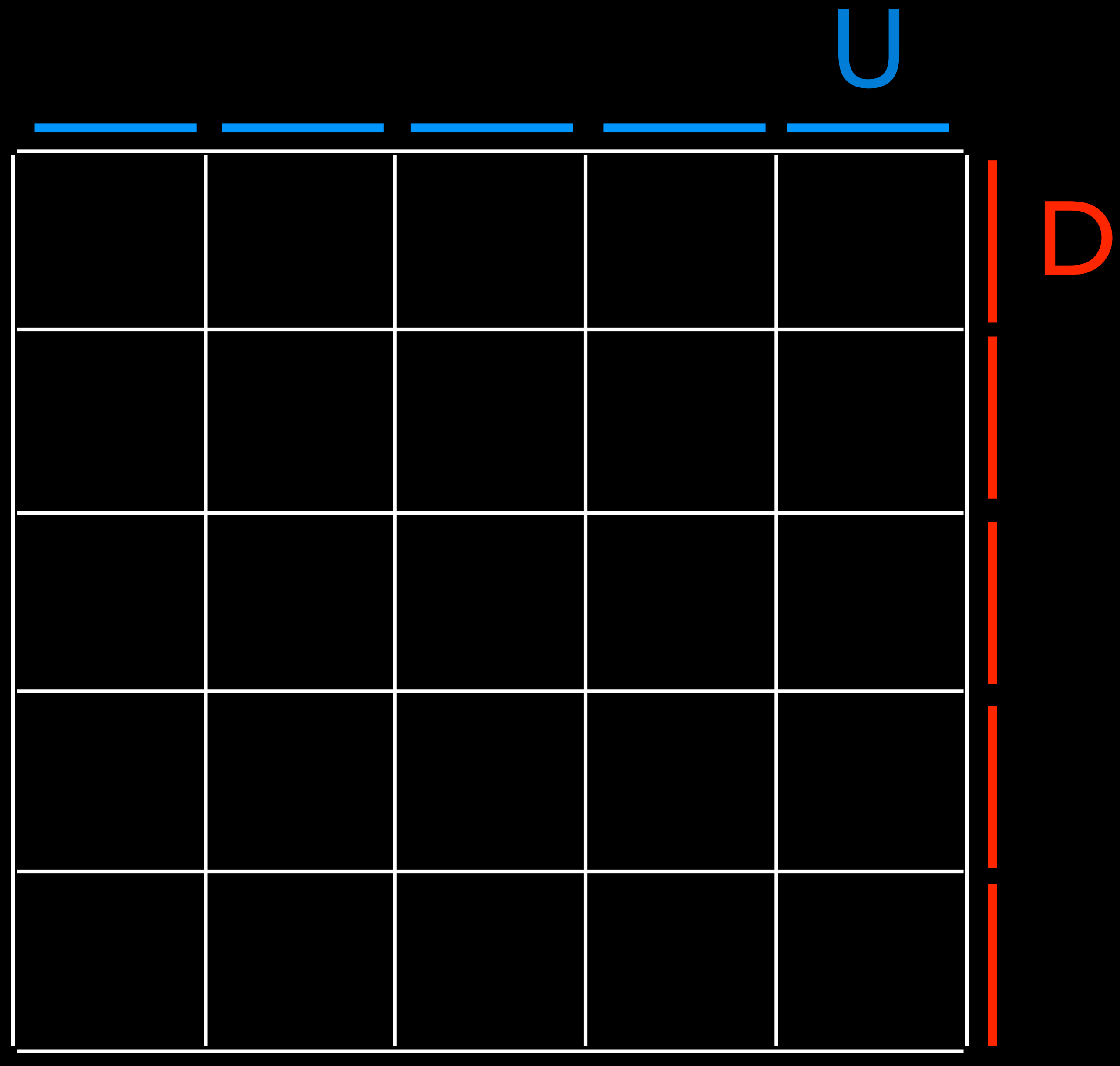
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

rewriting rules

$$UD = qDU + I$$

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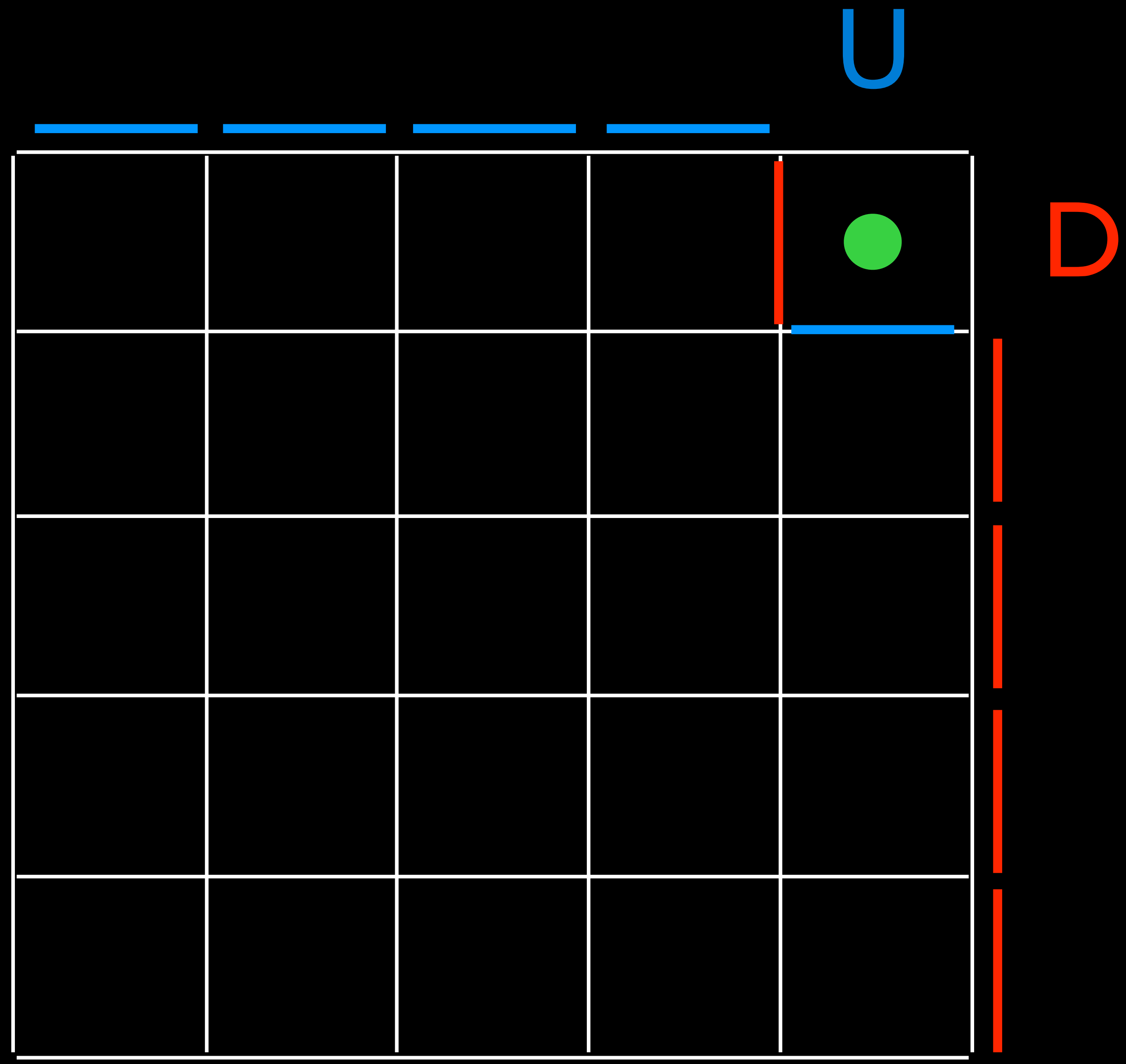
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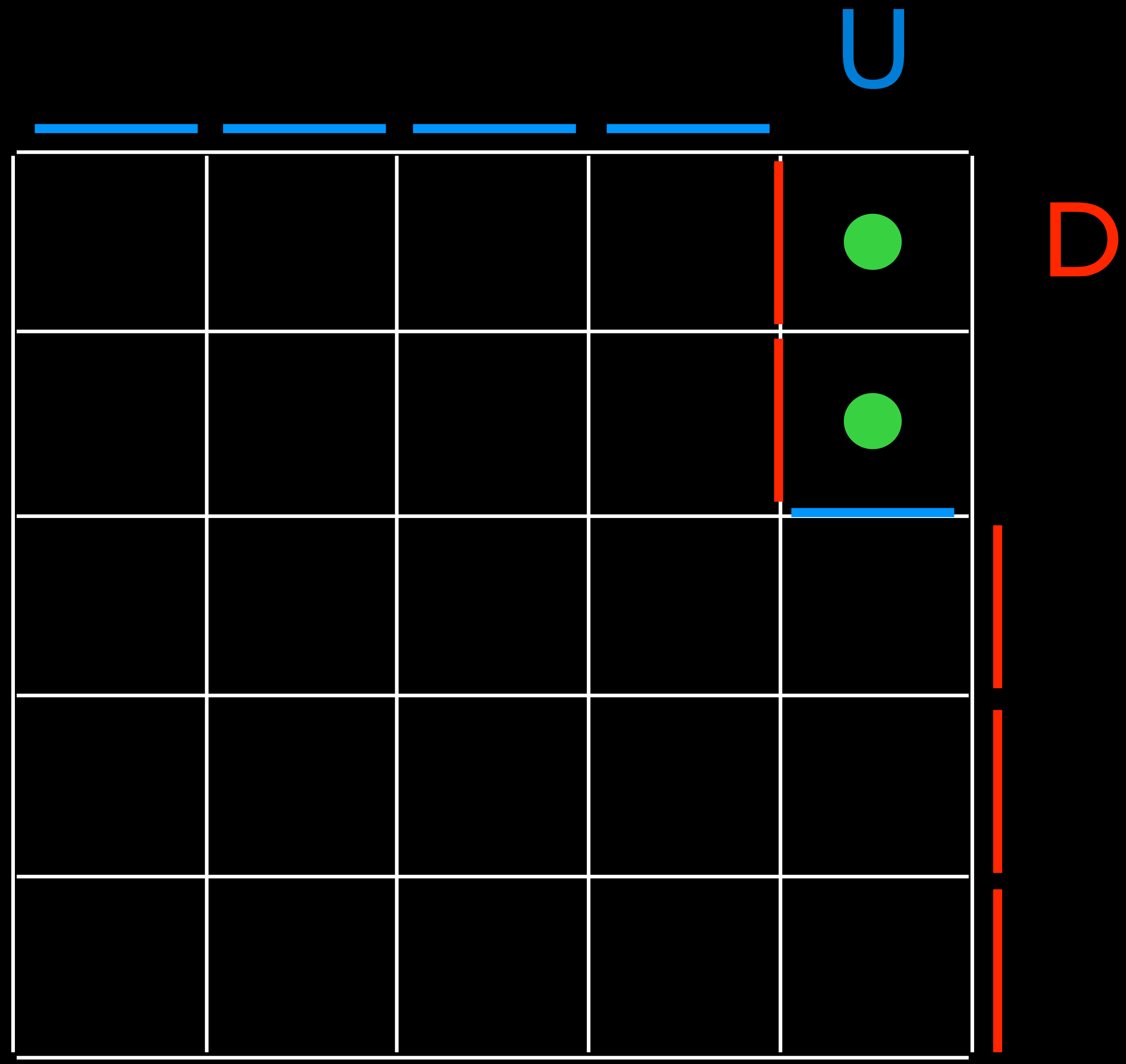
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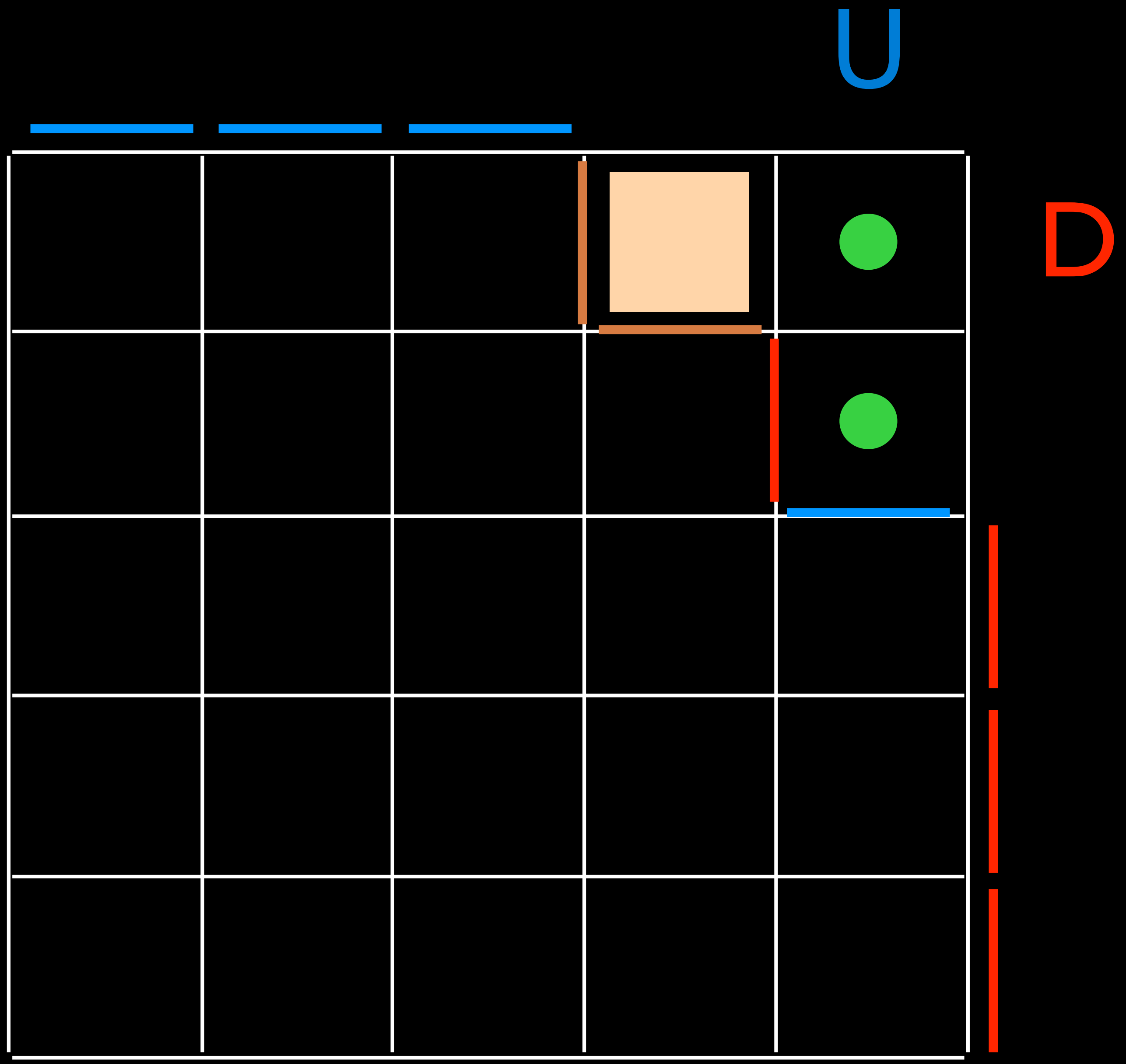
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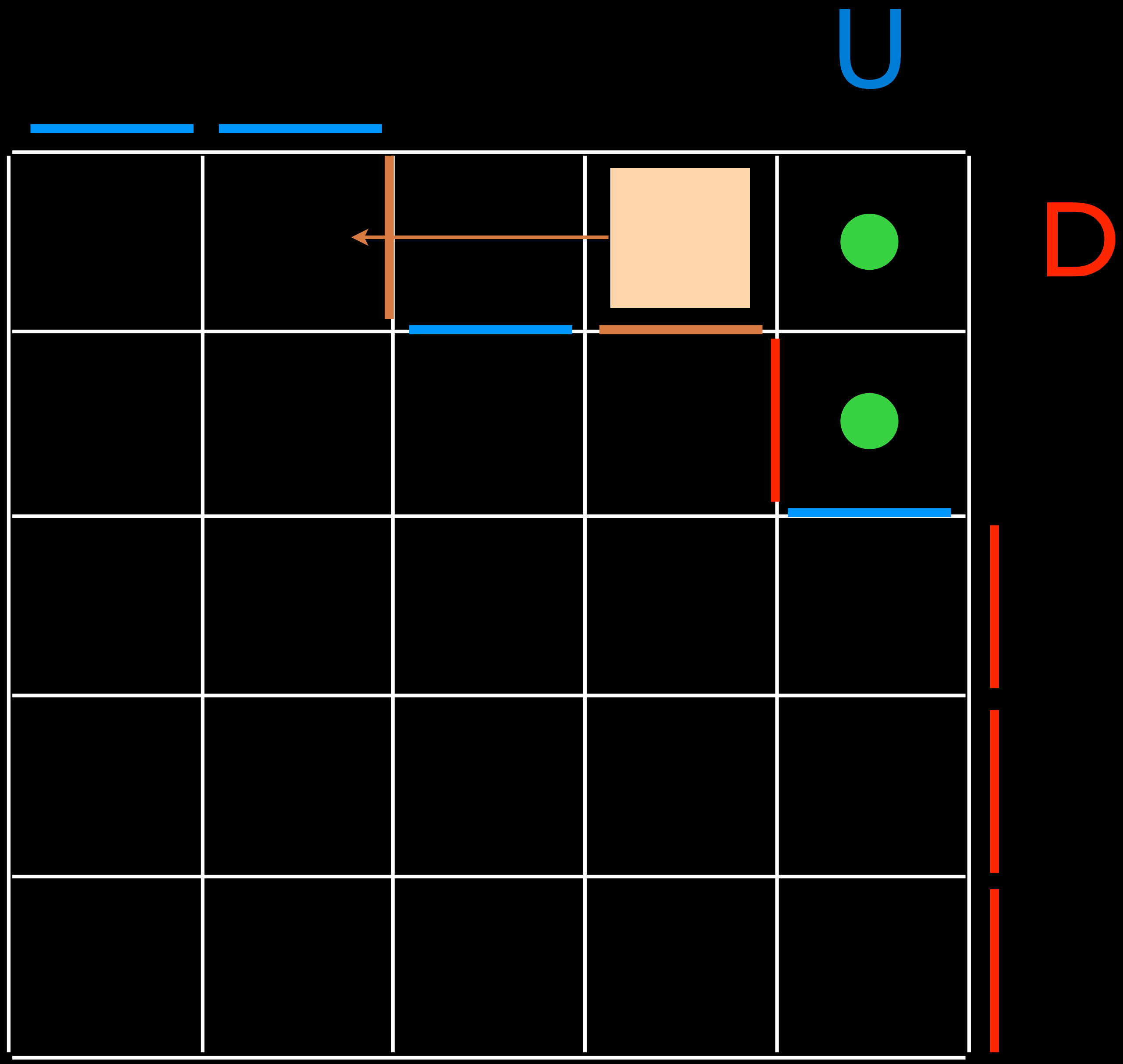
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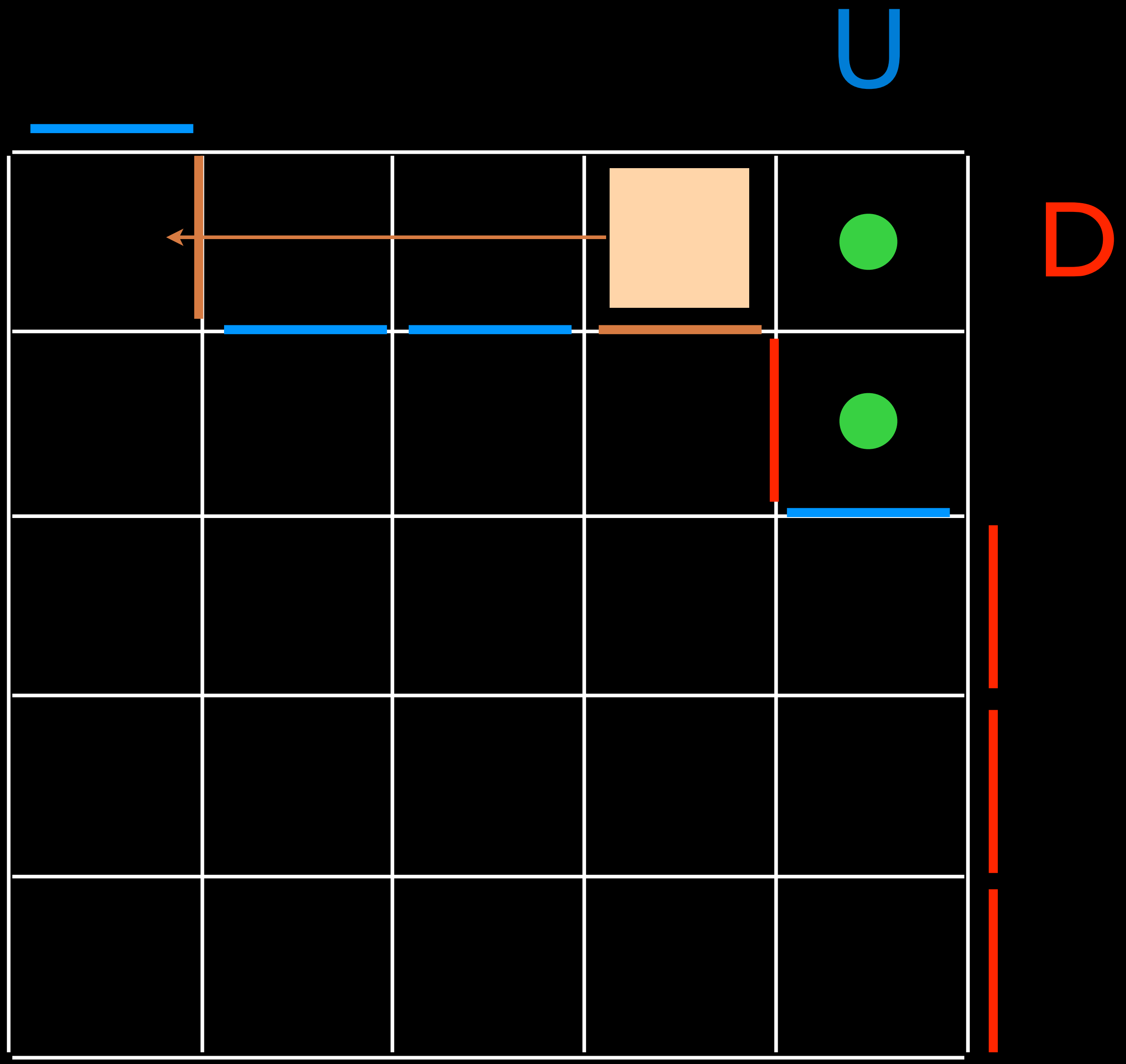
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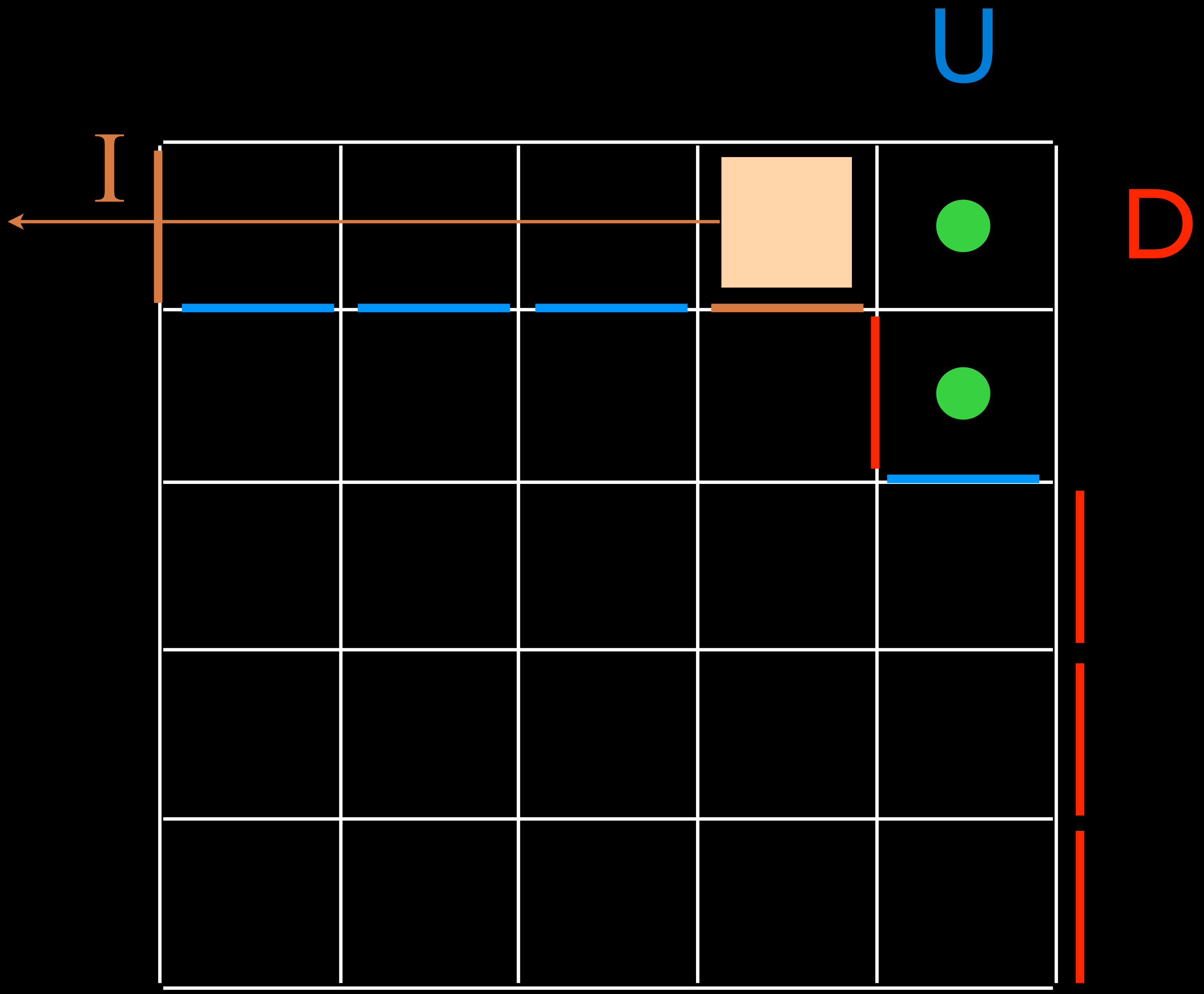
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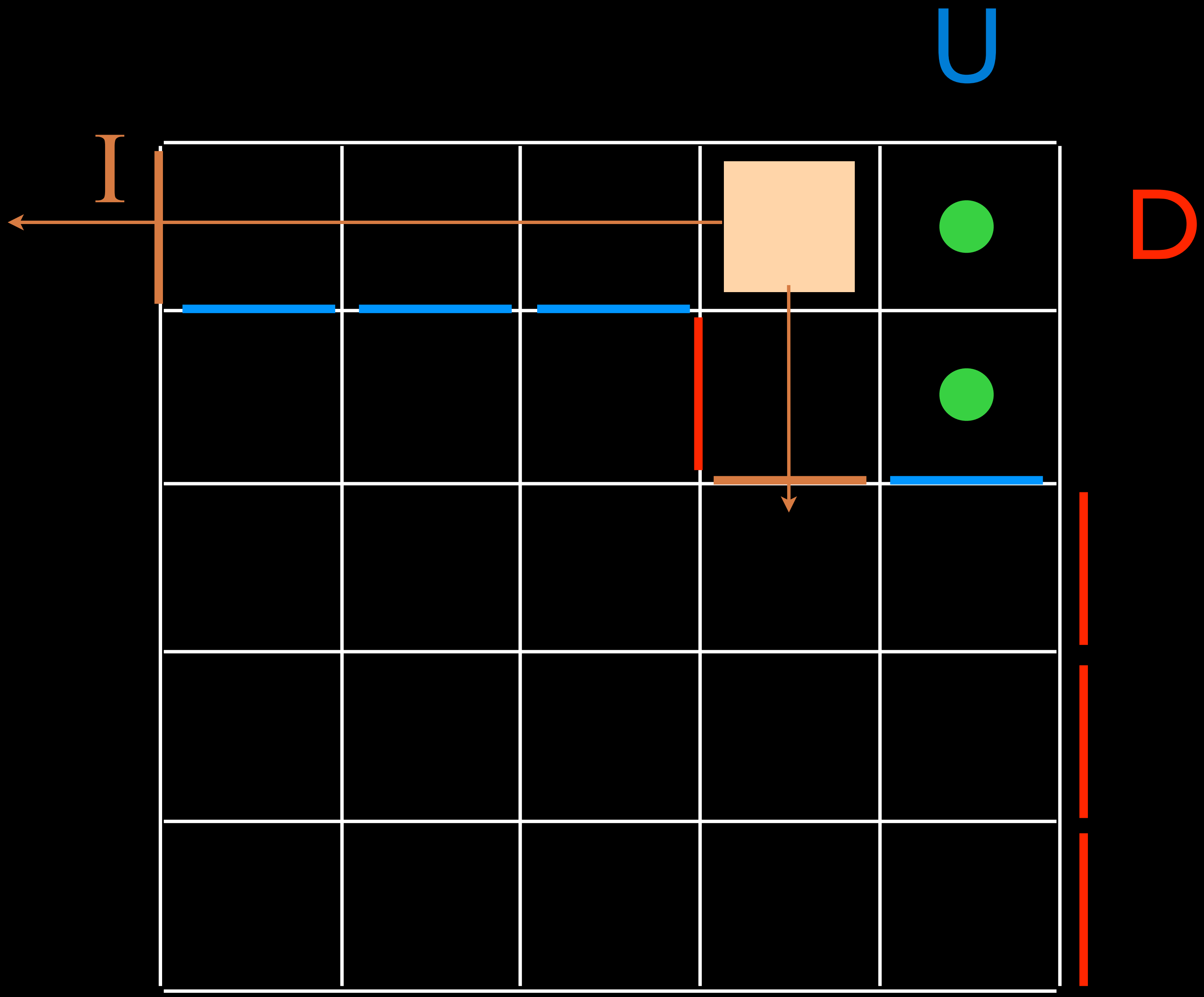
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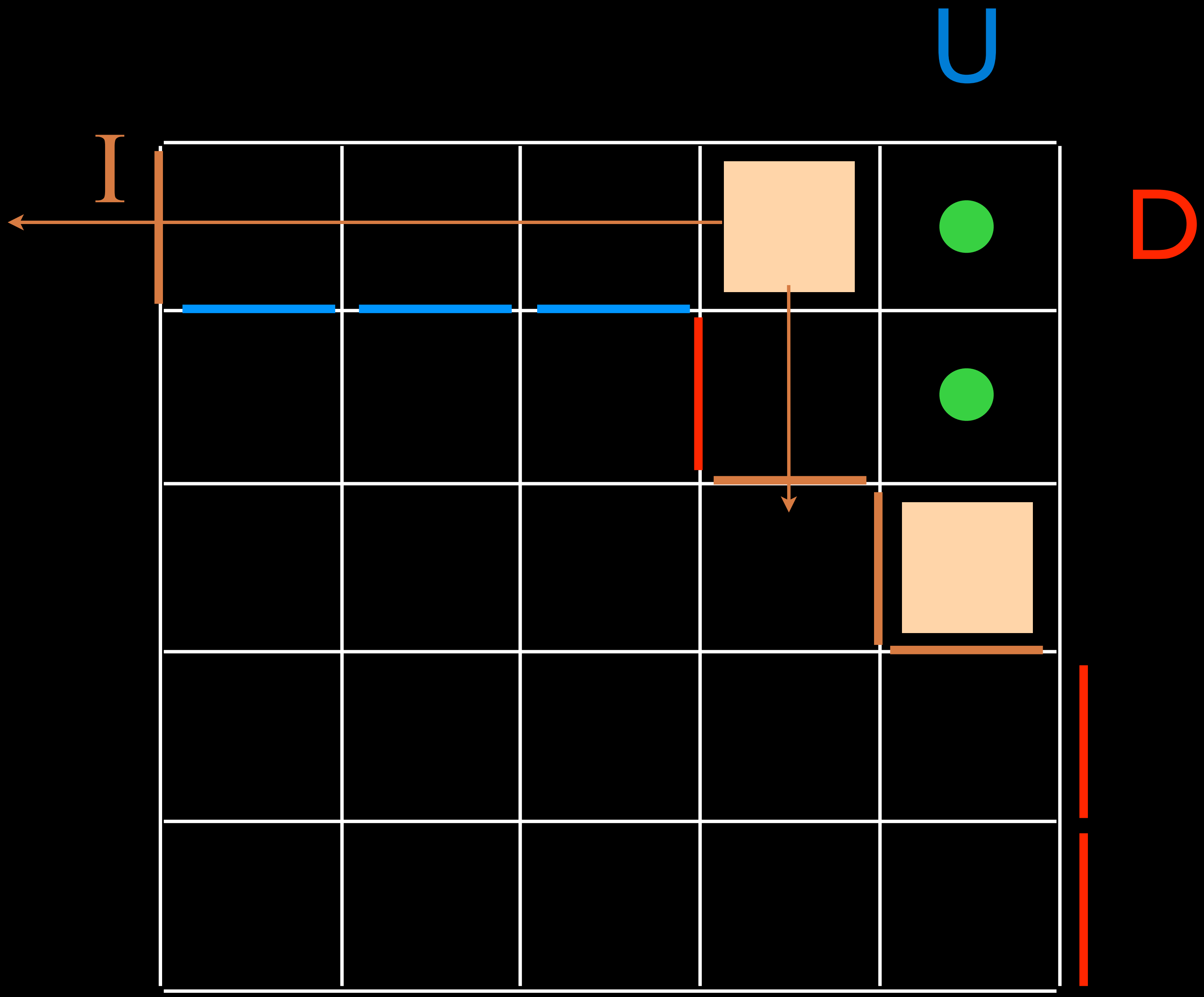
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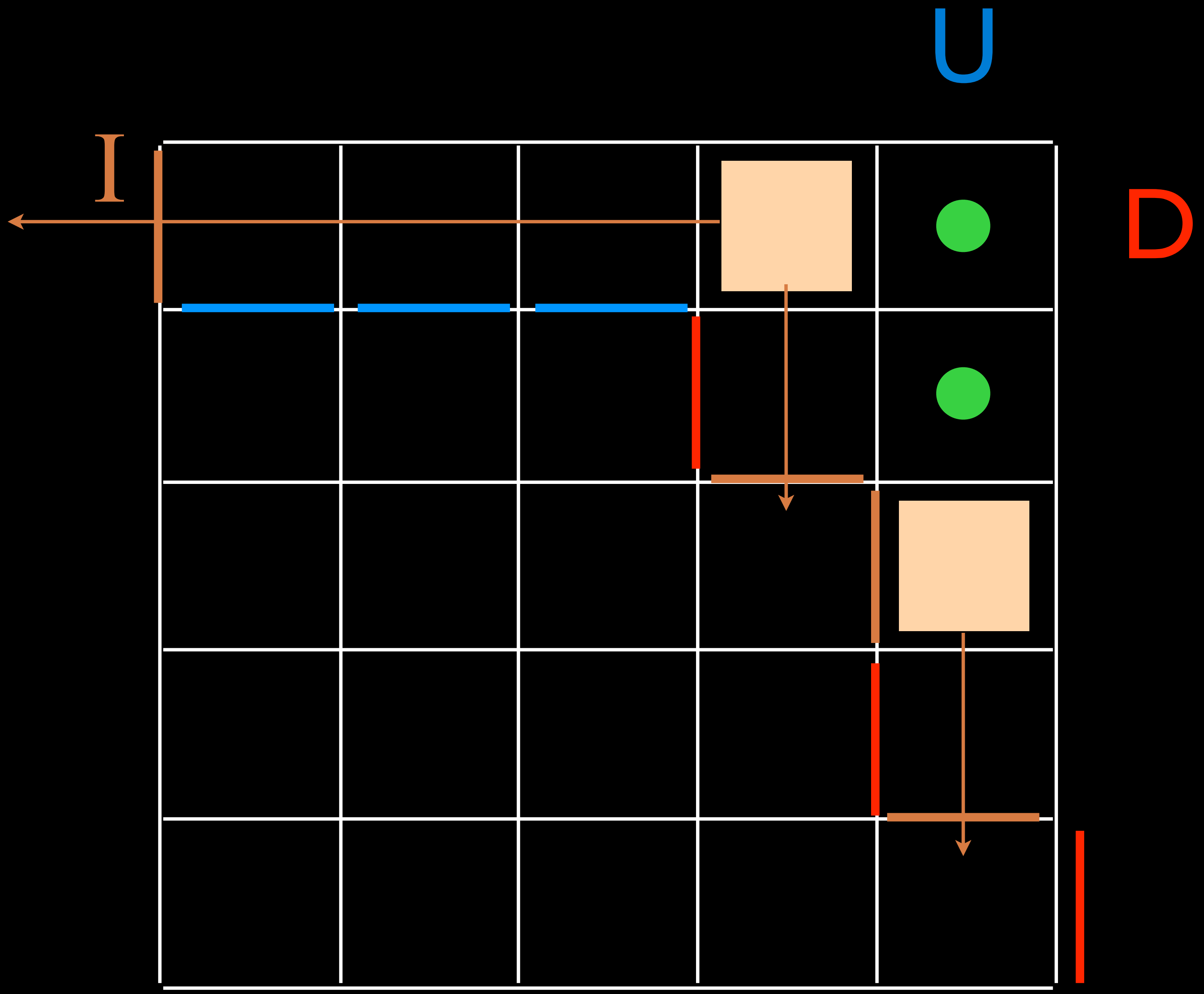
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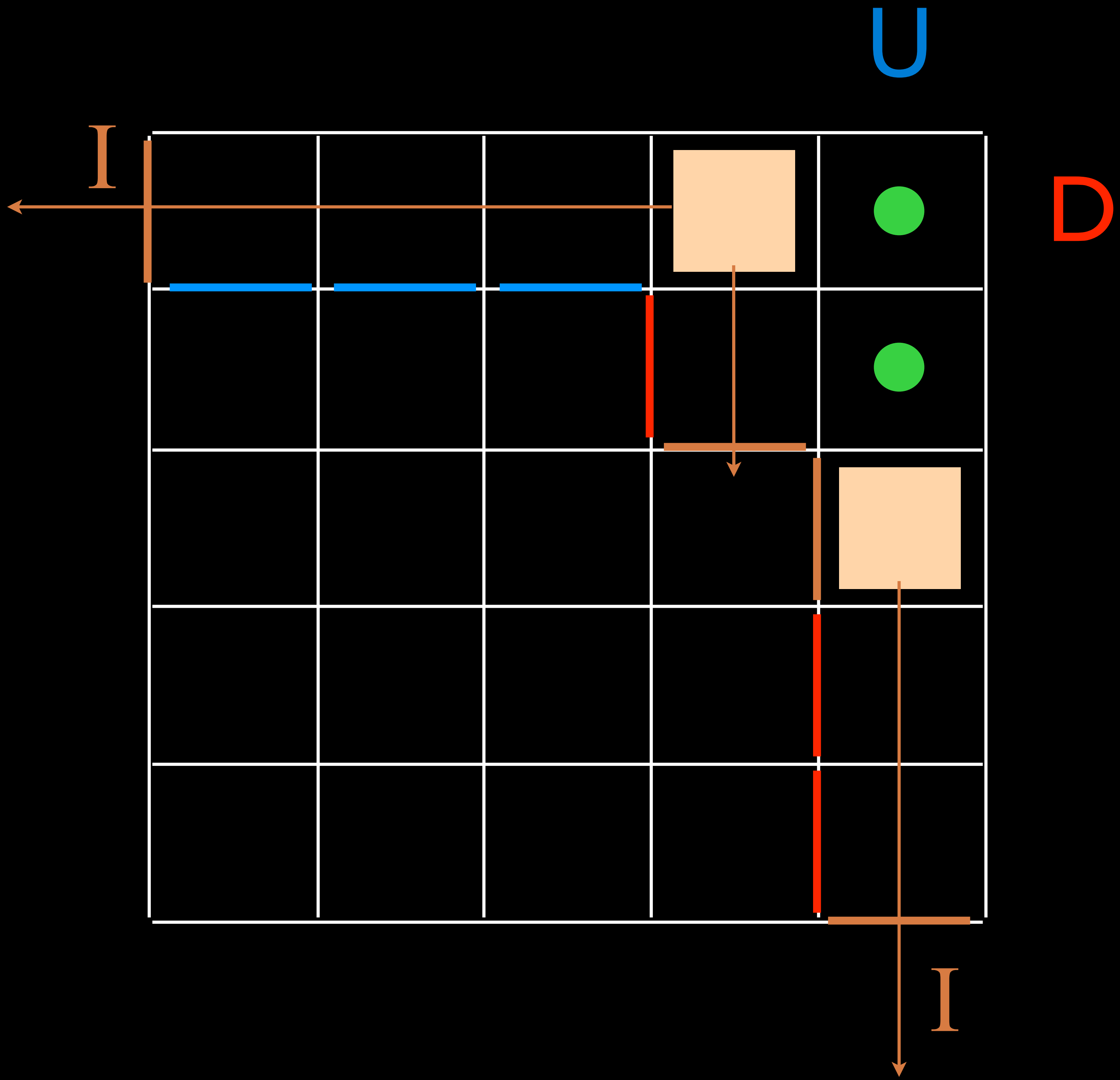
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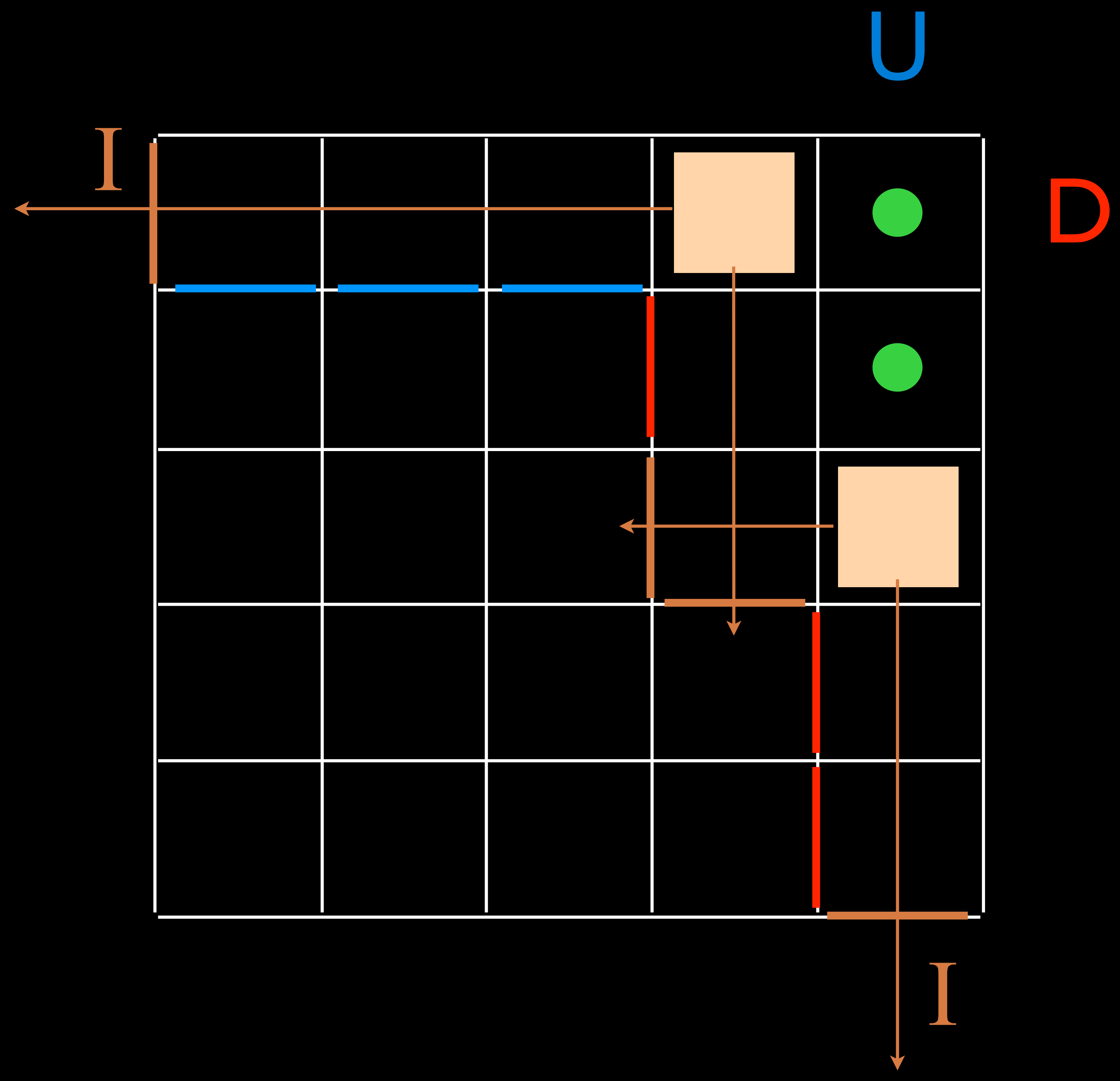
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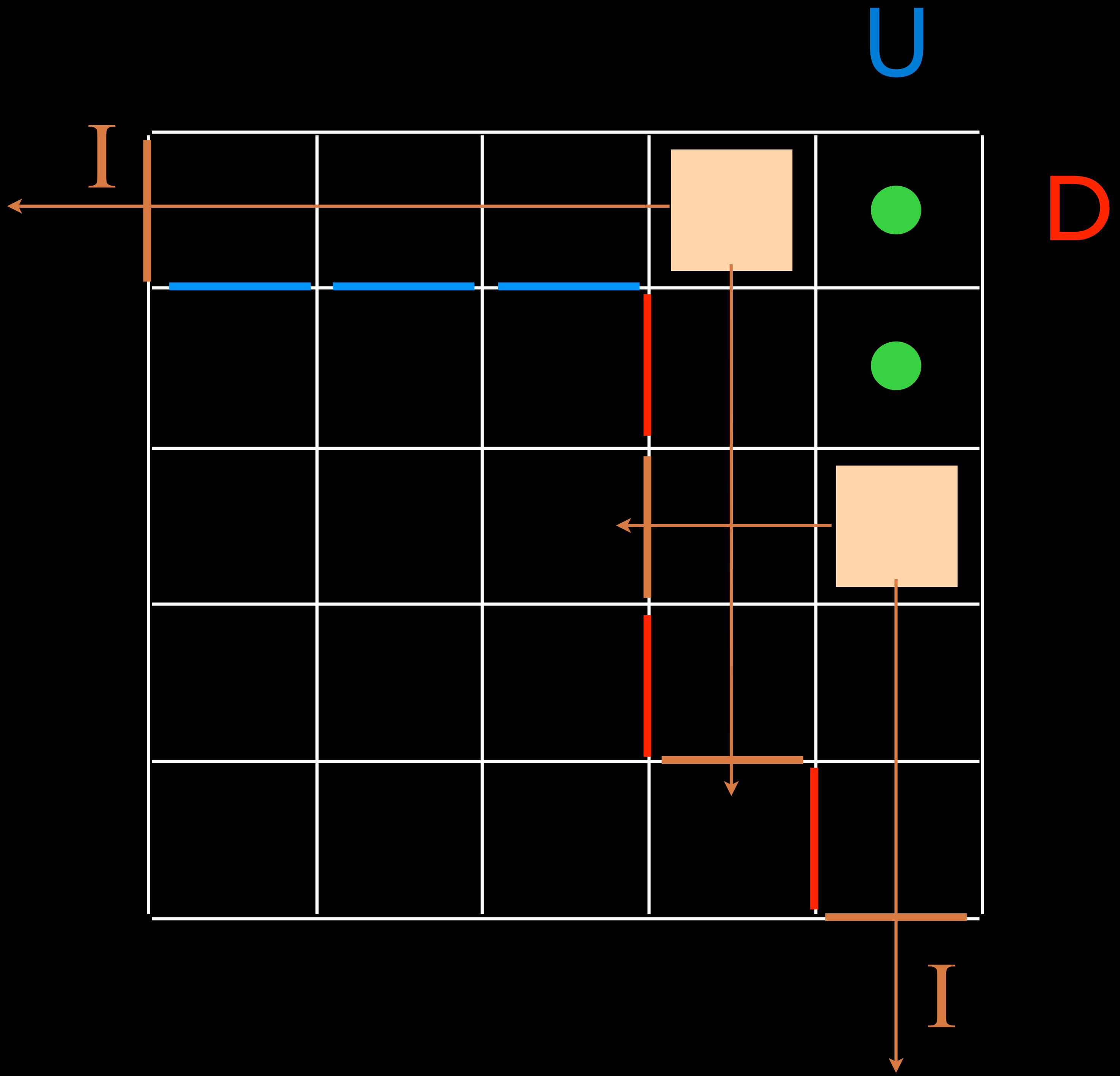

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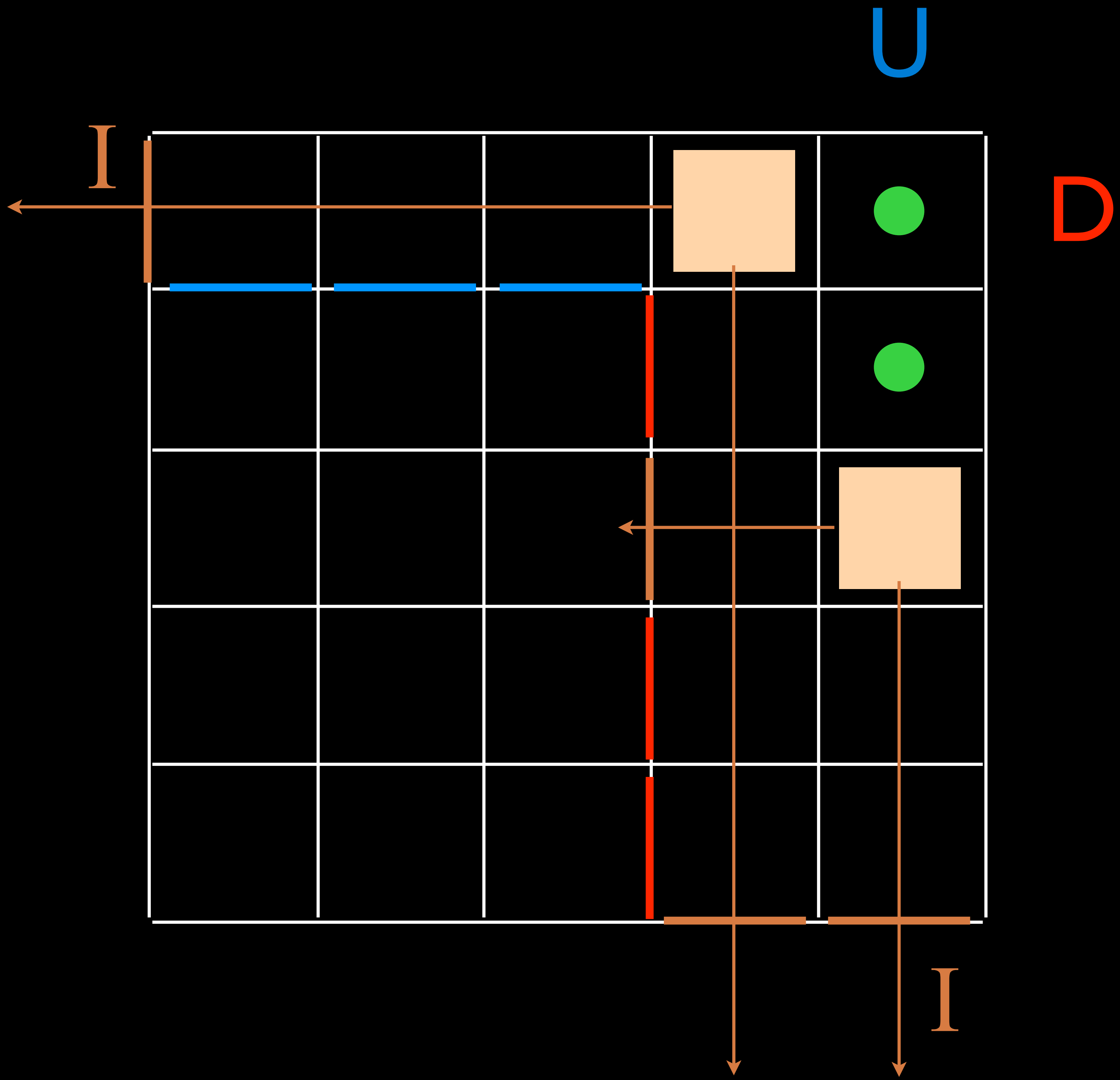

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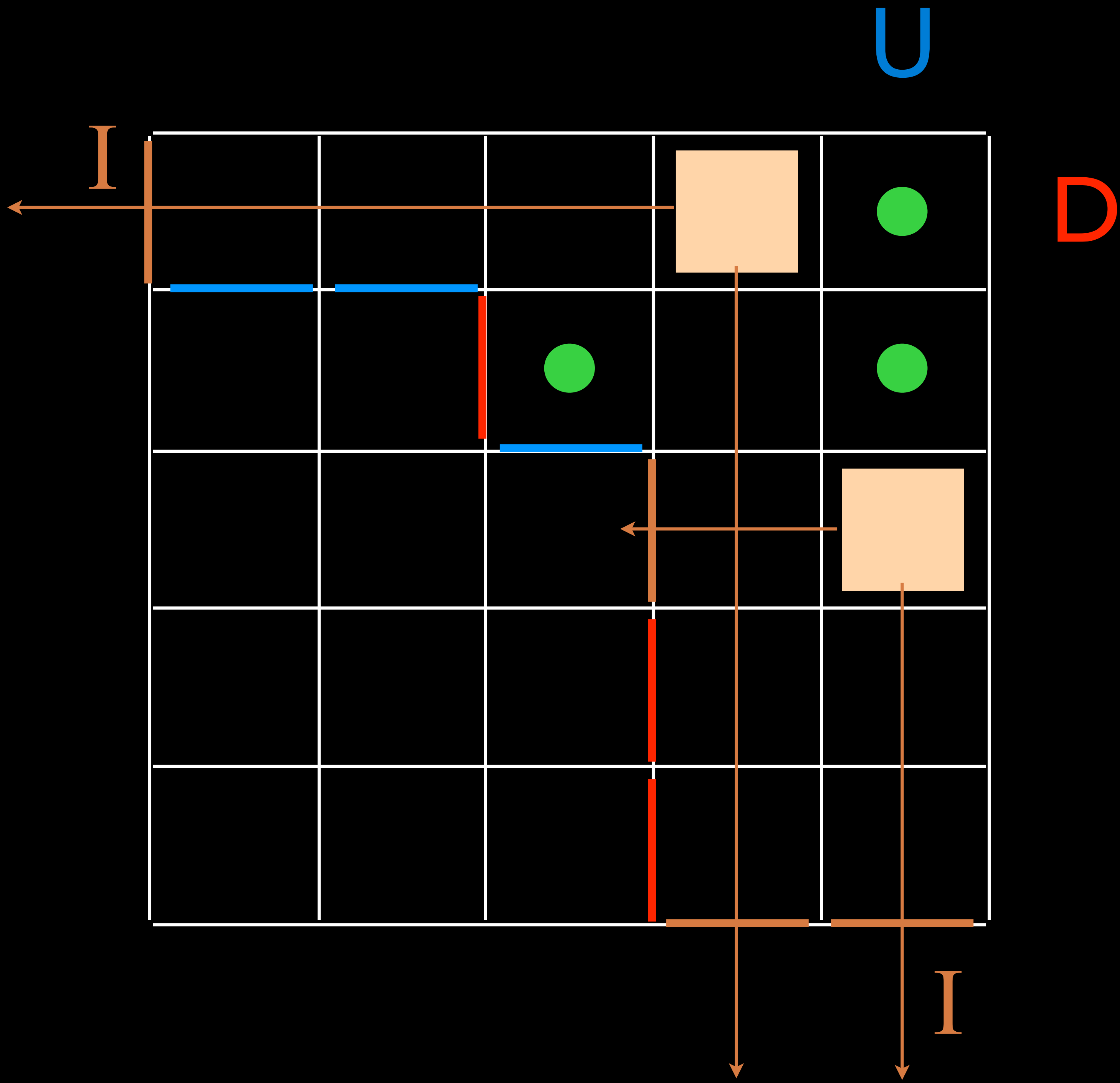
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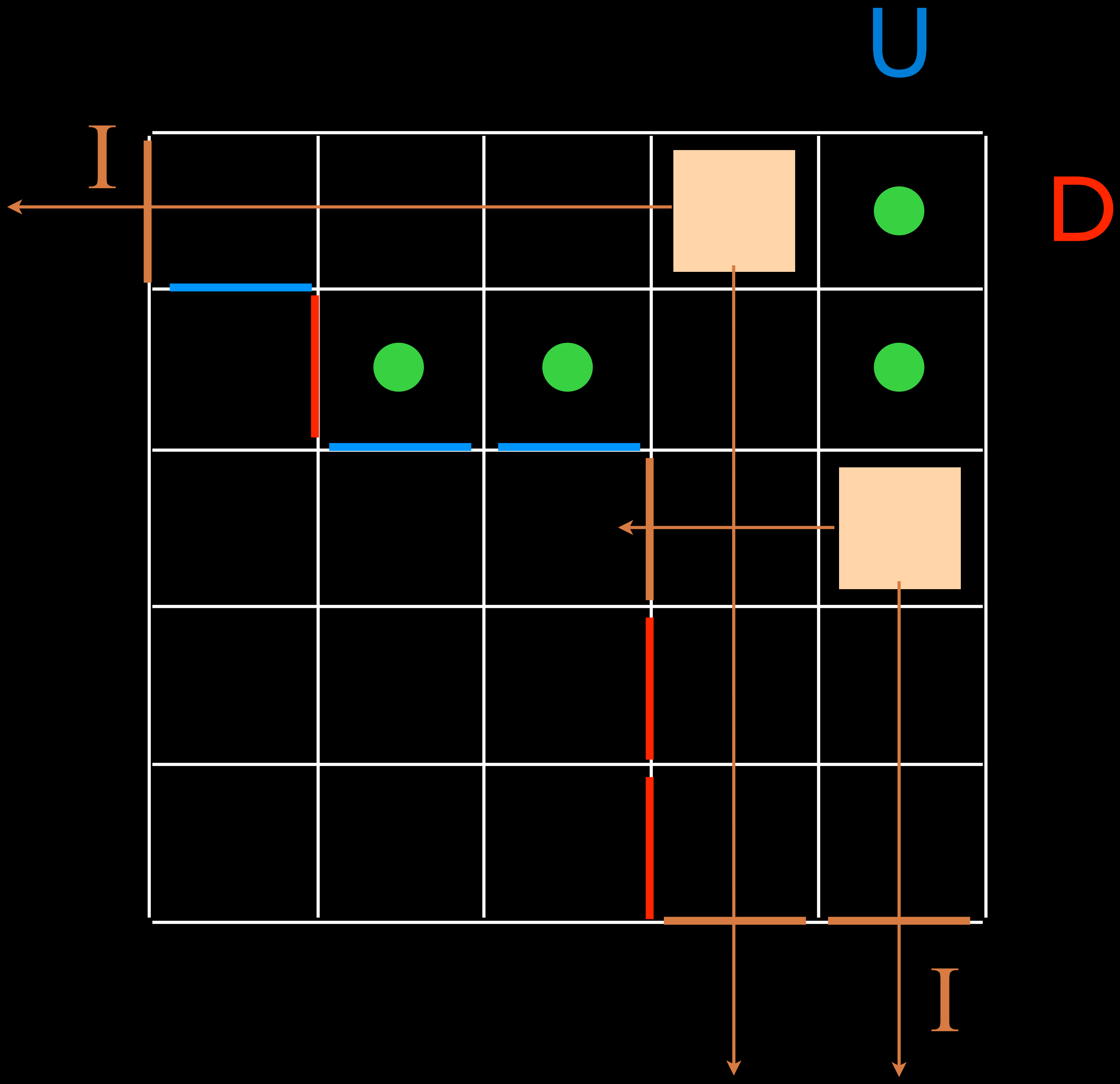

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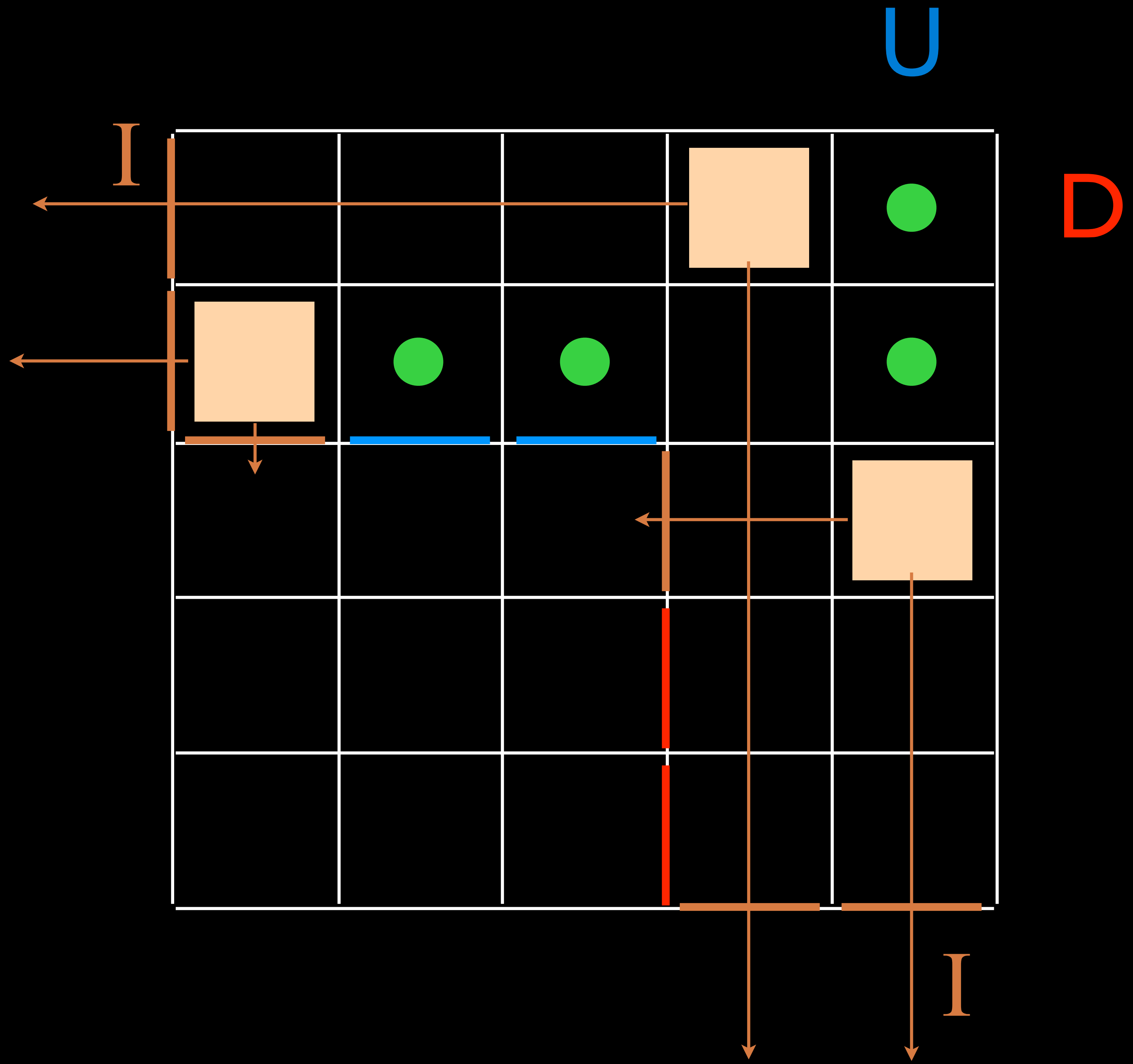
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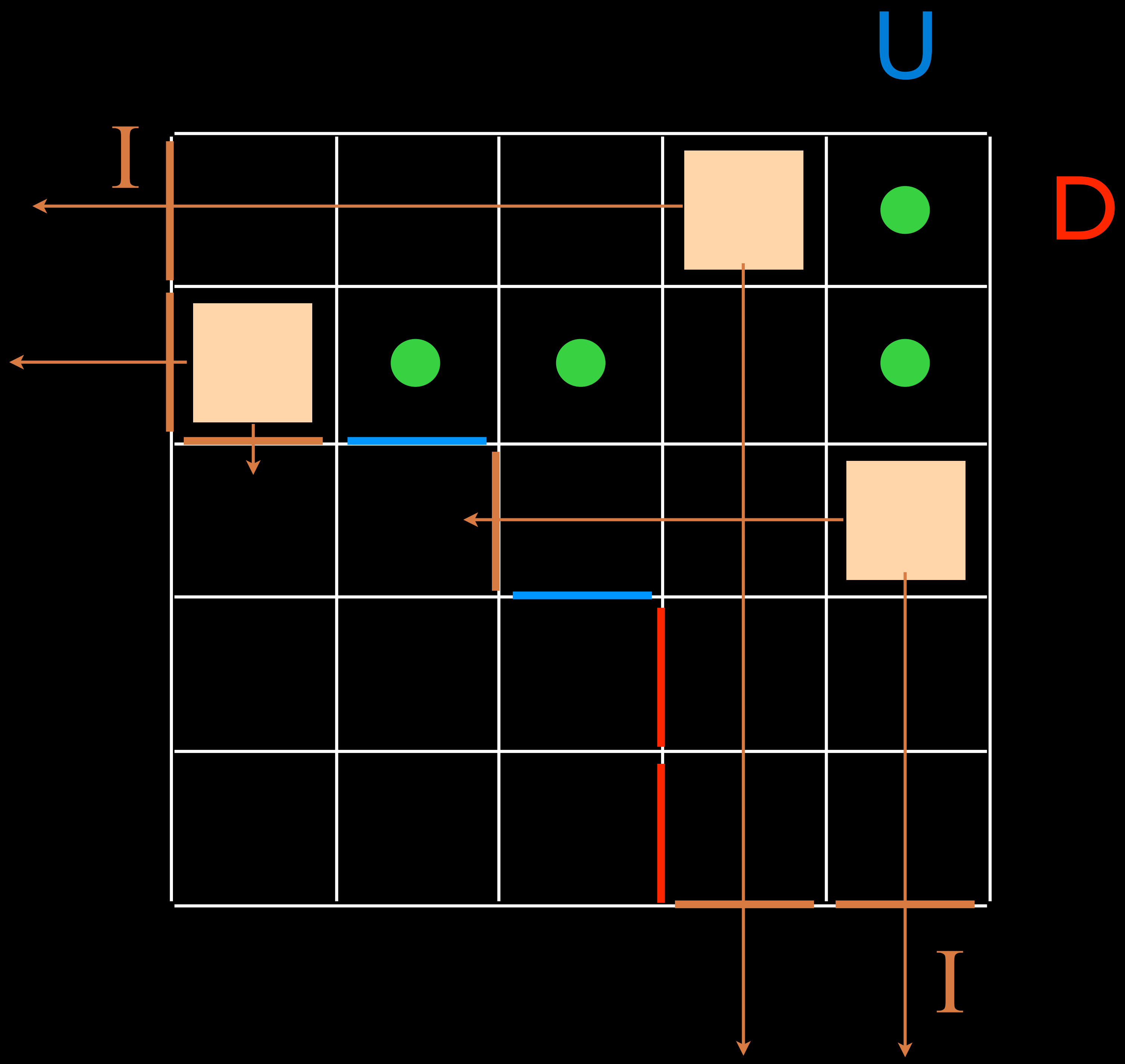
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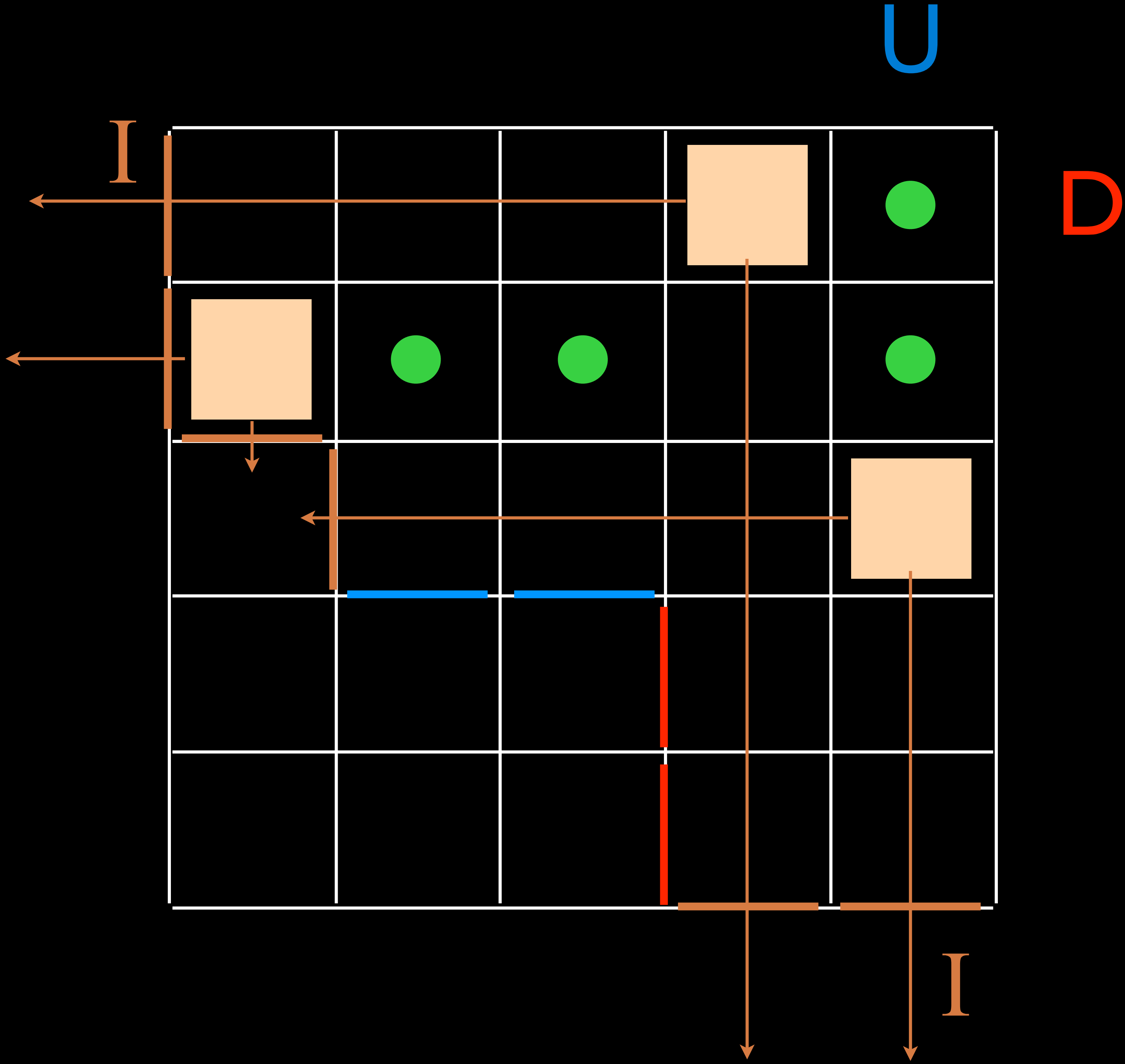
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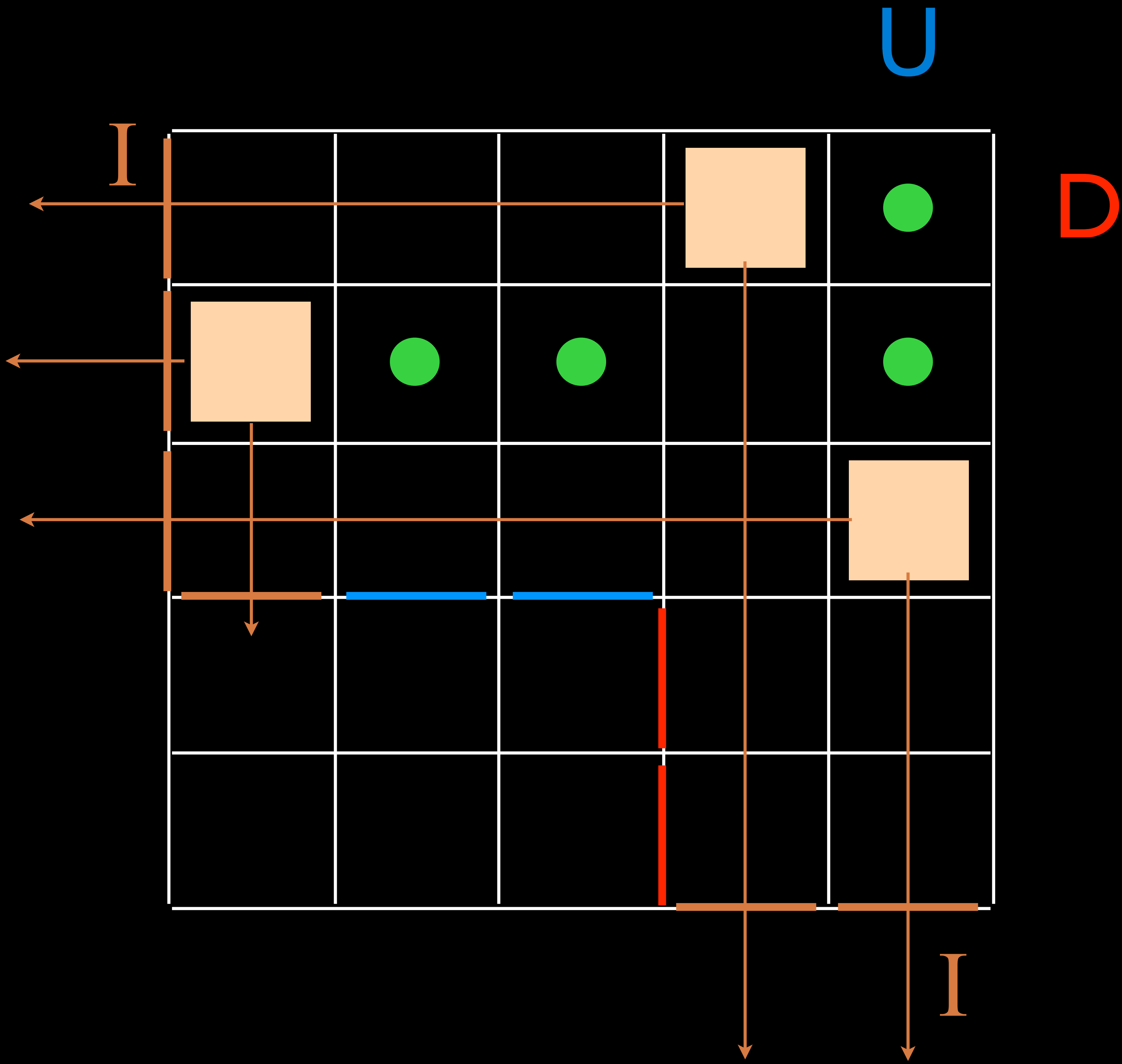
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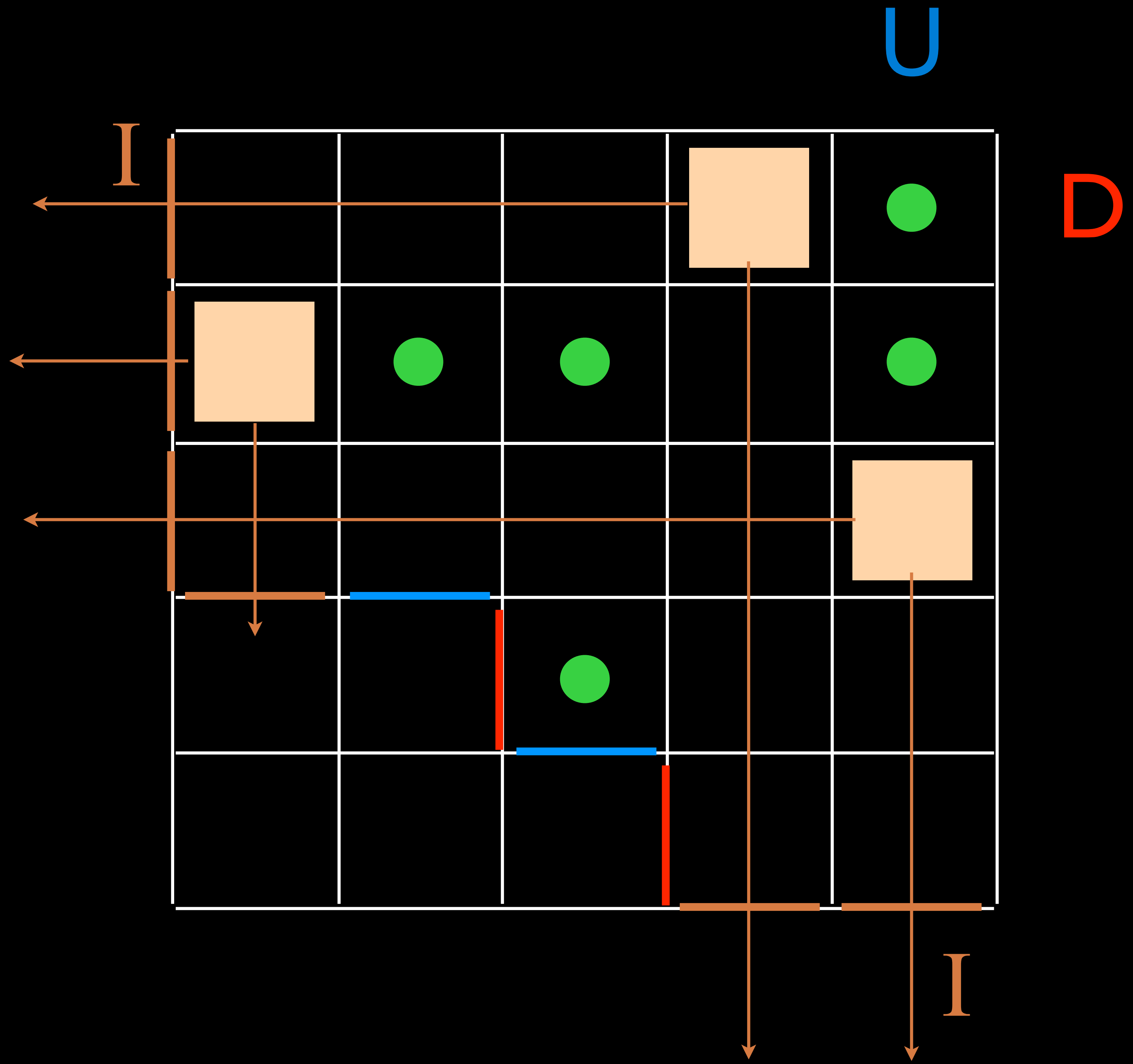
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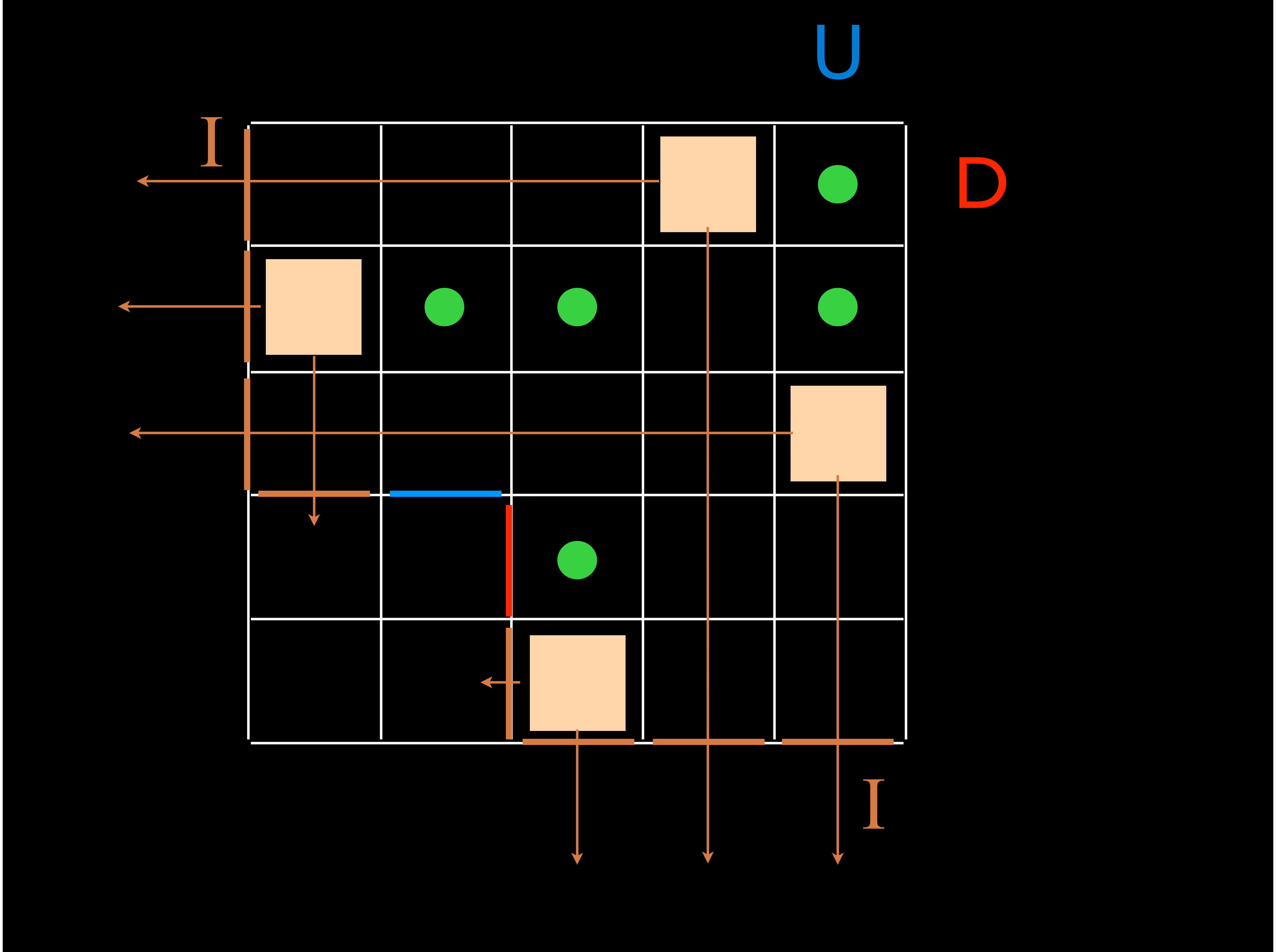
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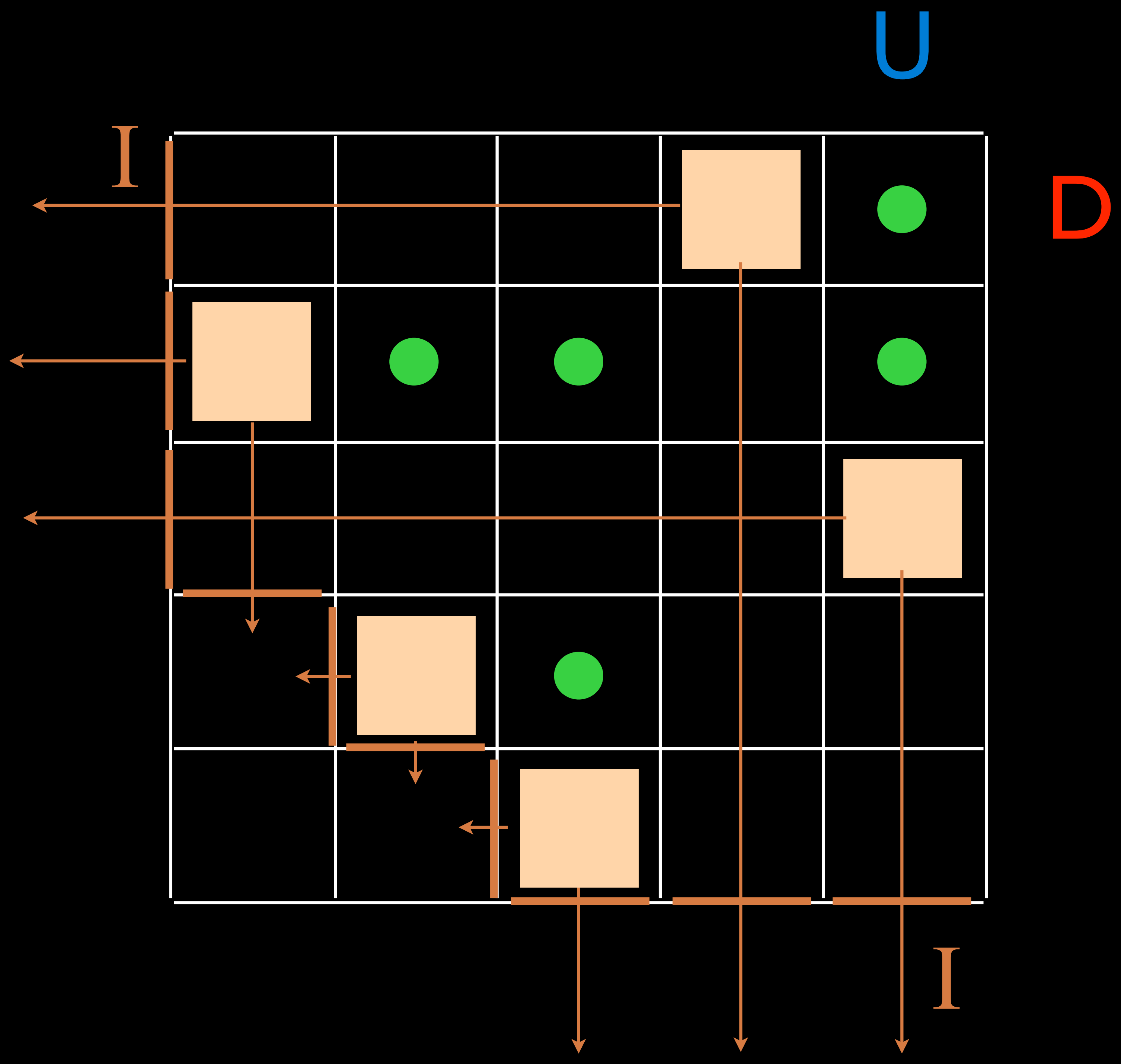
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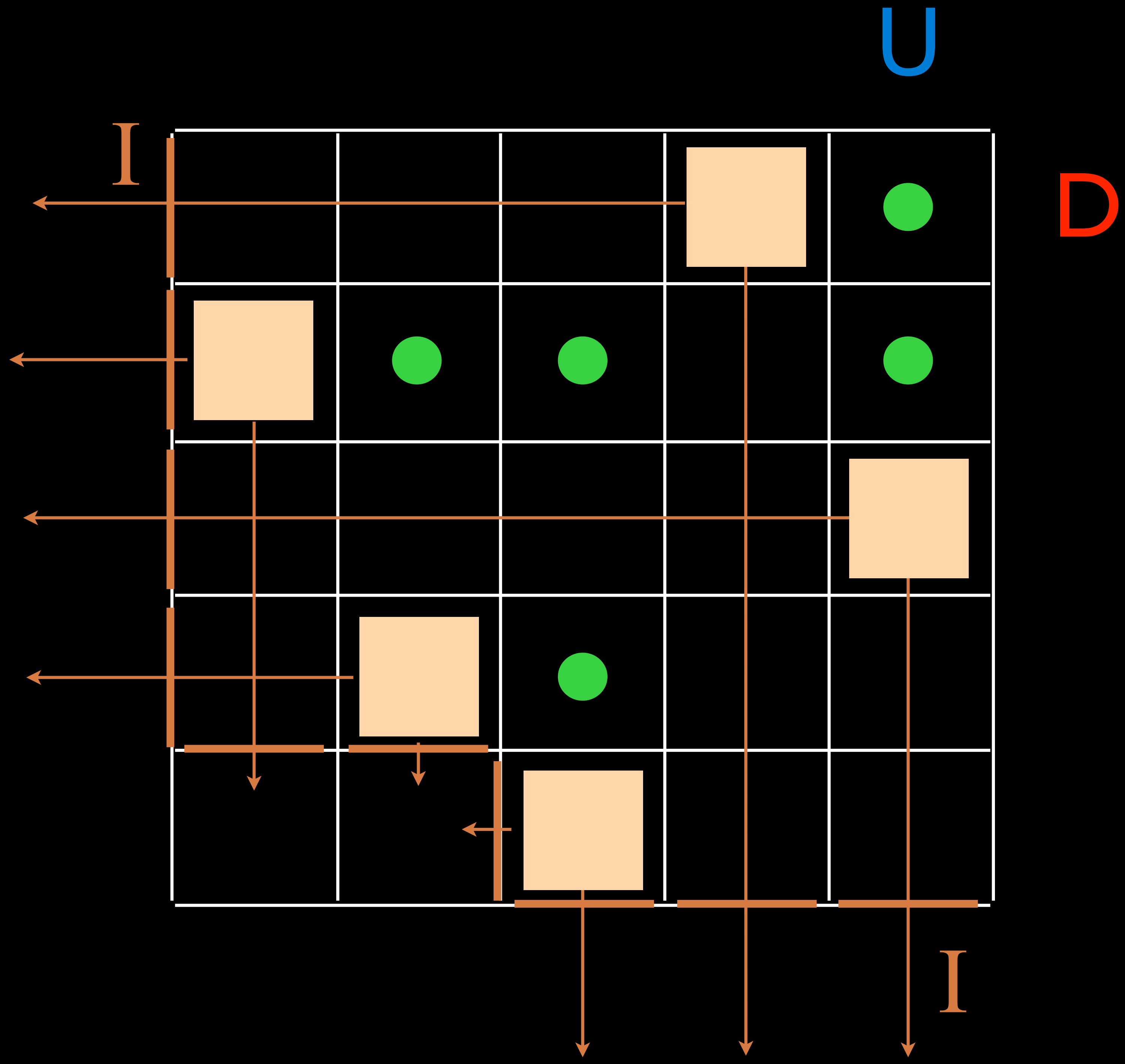
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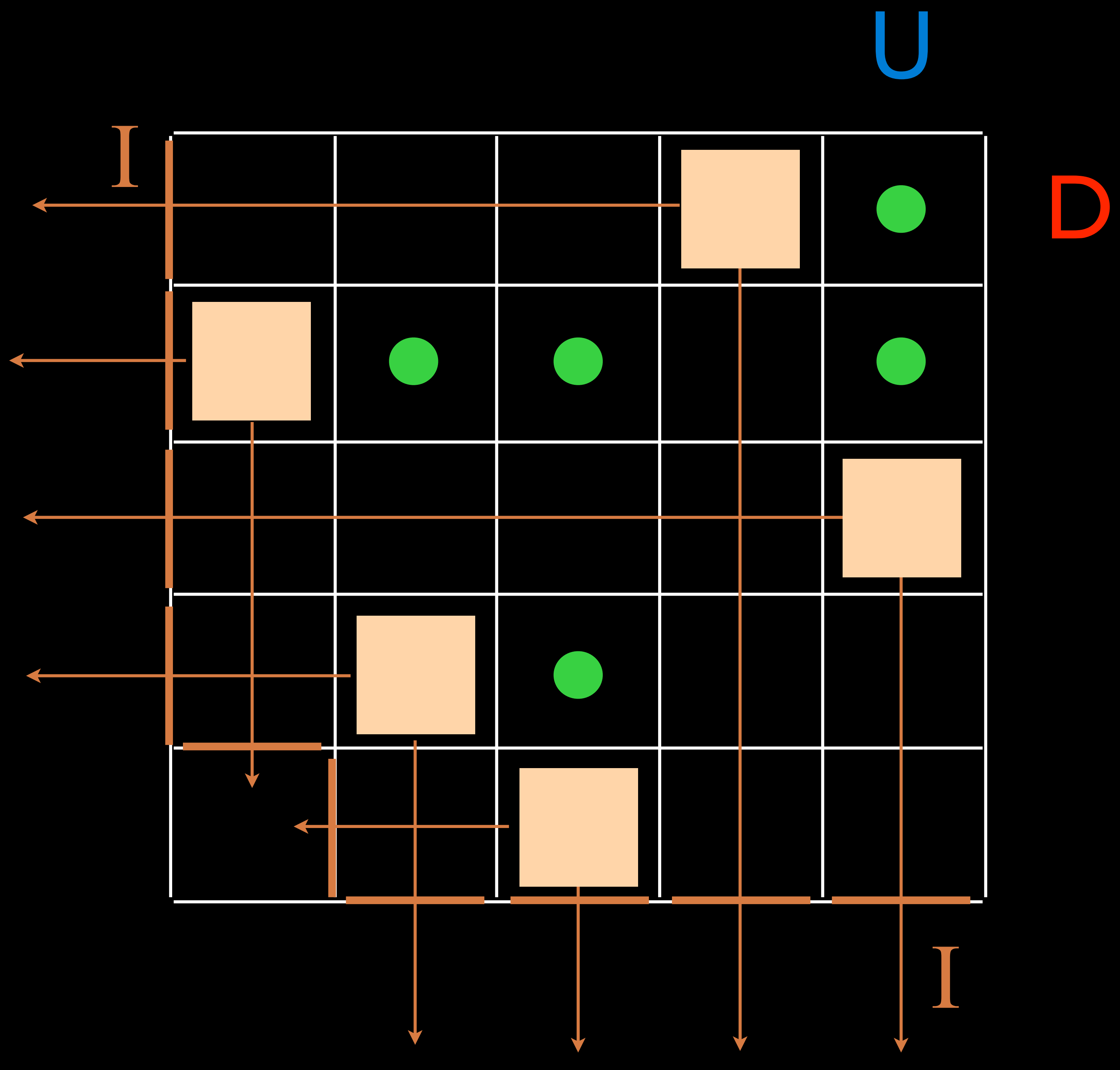
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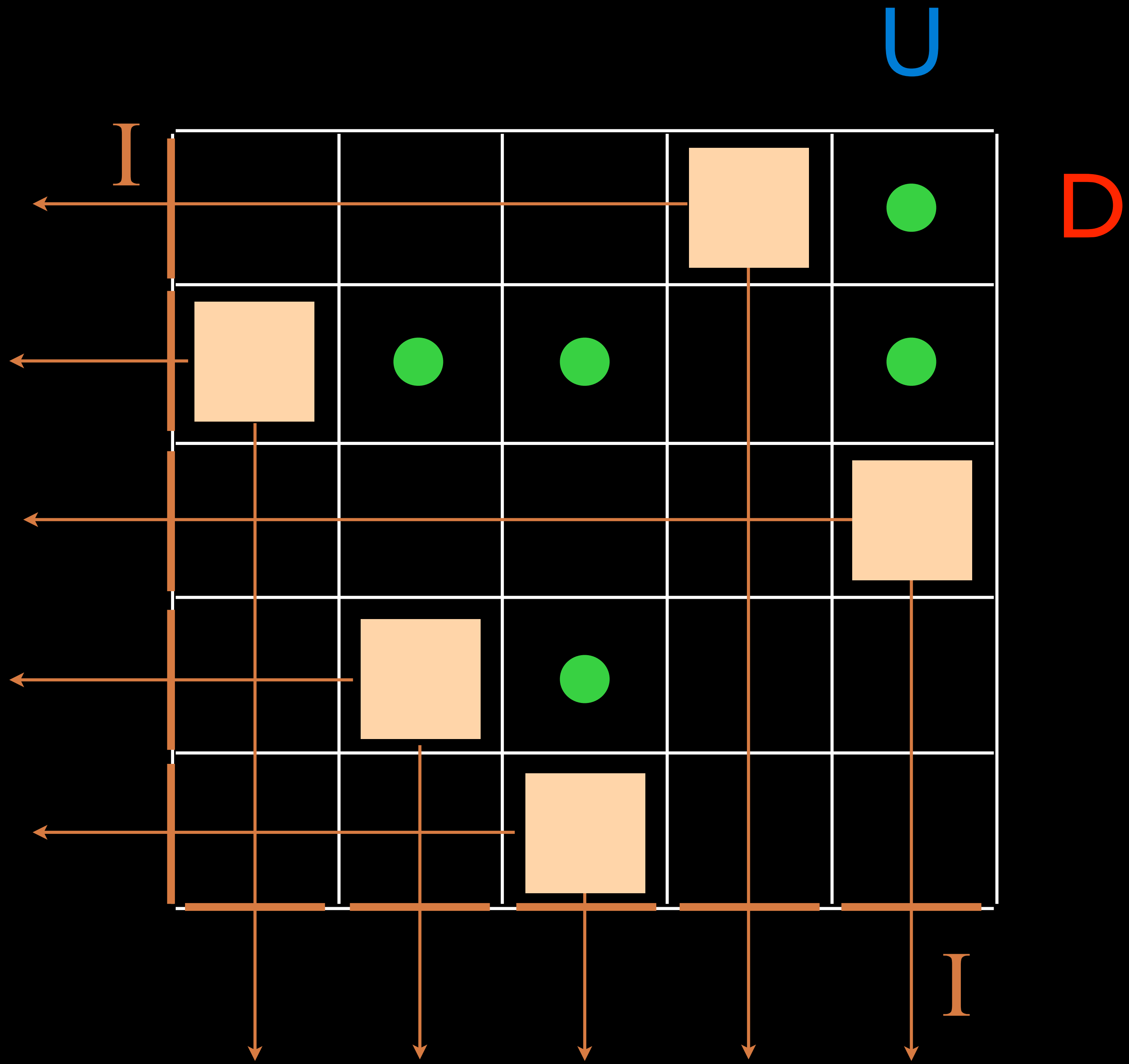
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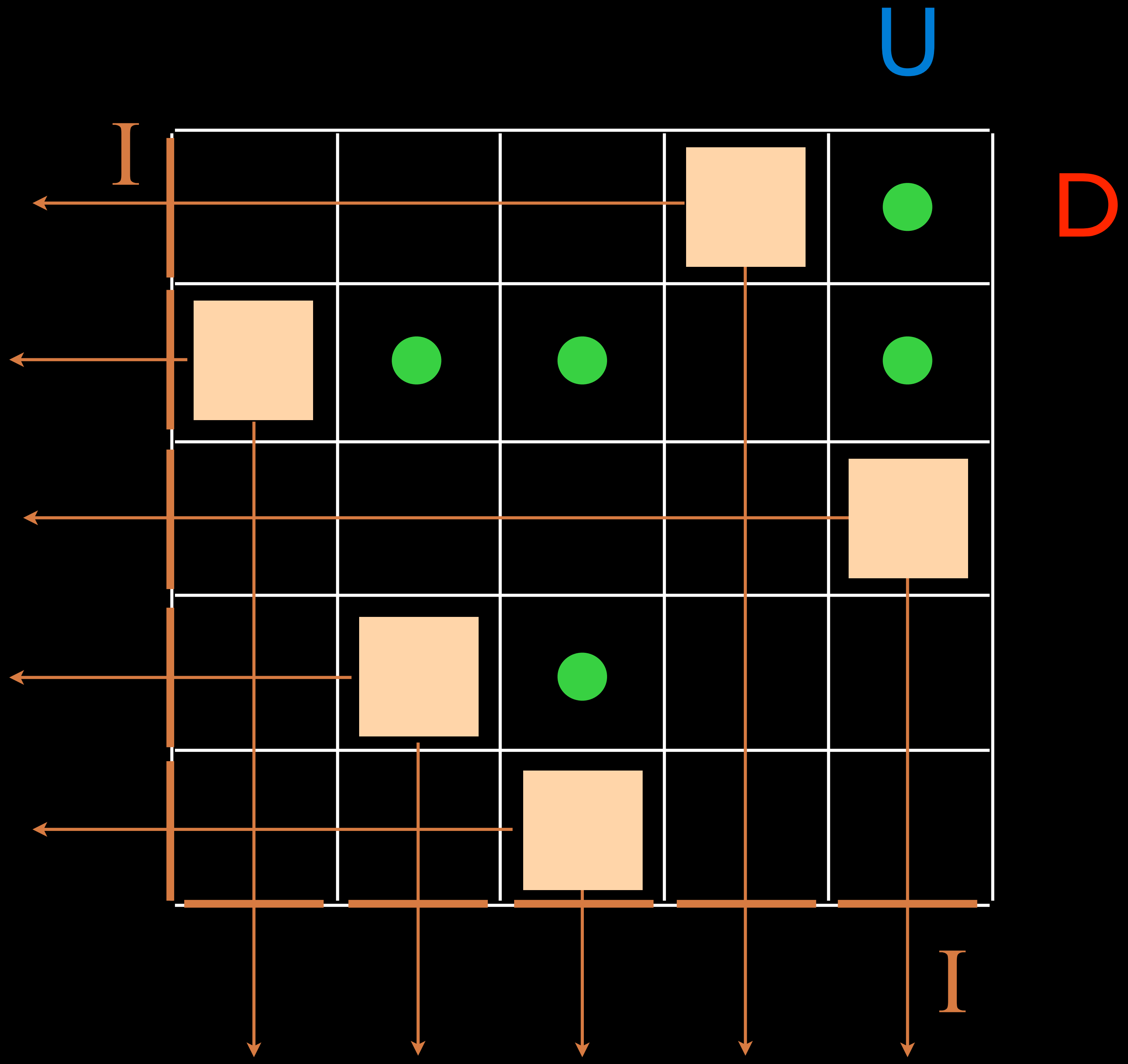
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

Quadratic algebra Q

5 rewriting rules

"complete" Q-tableau (5 labels)

↓ Q-tableau (2 labels)



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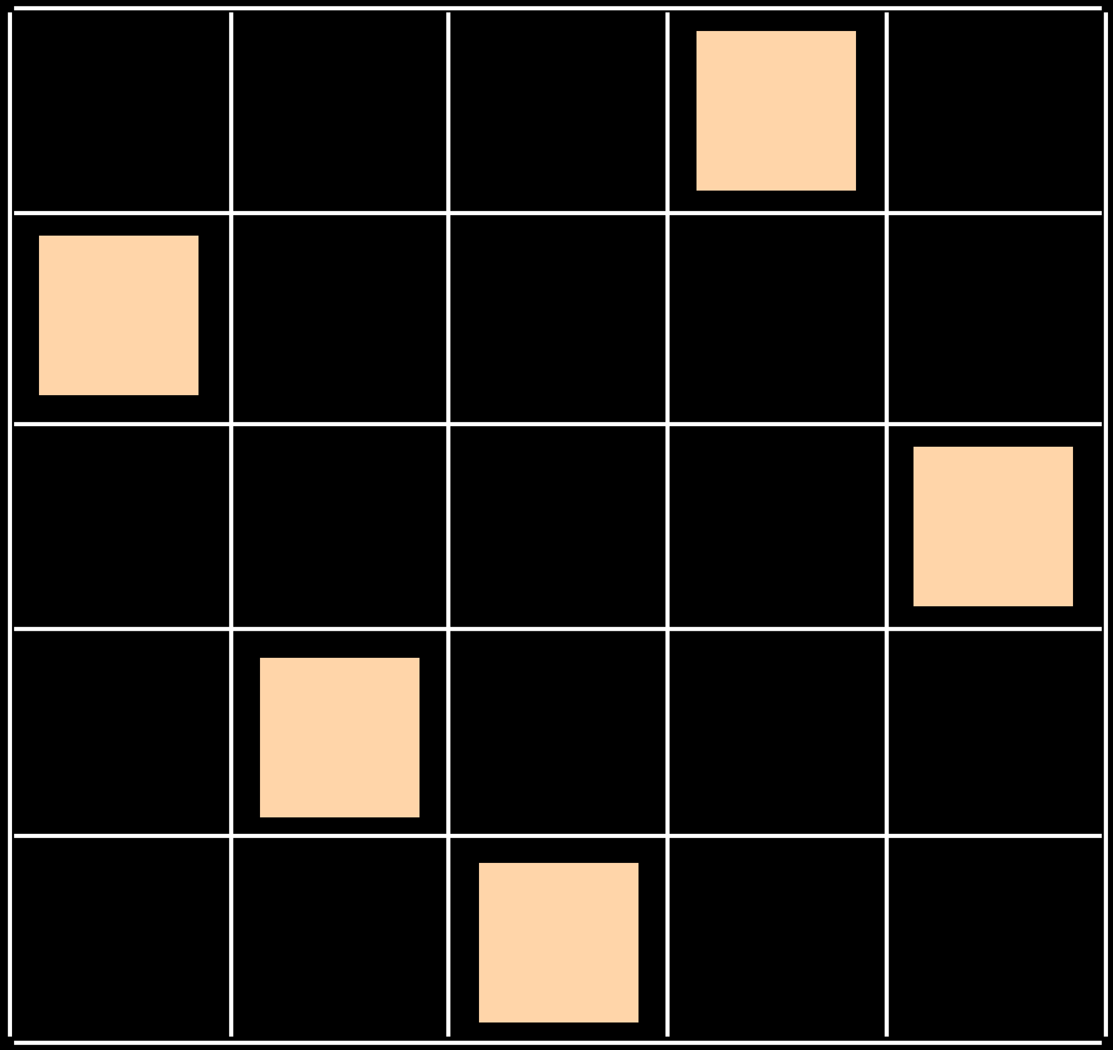
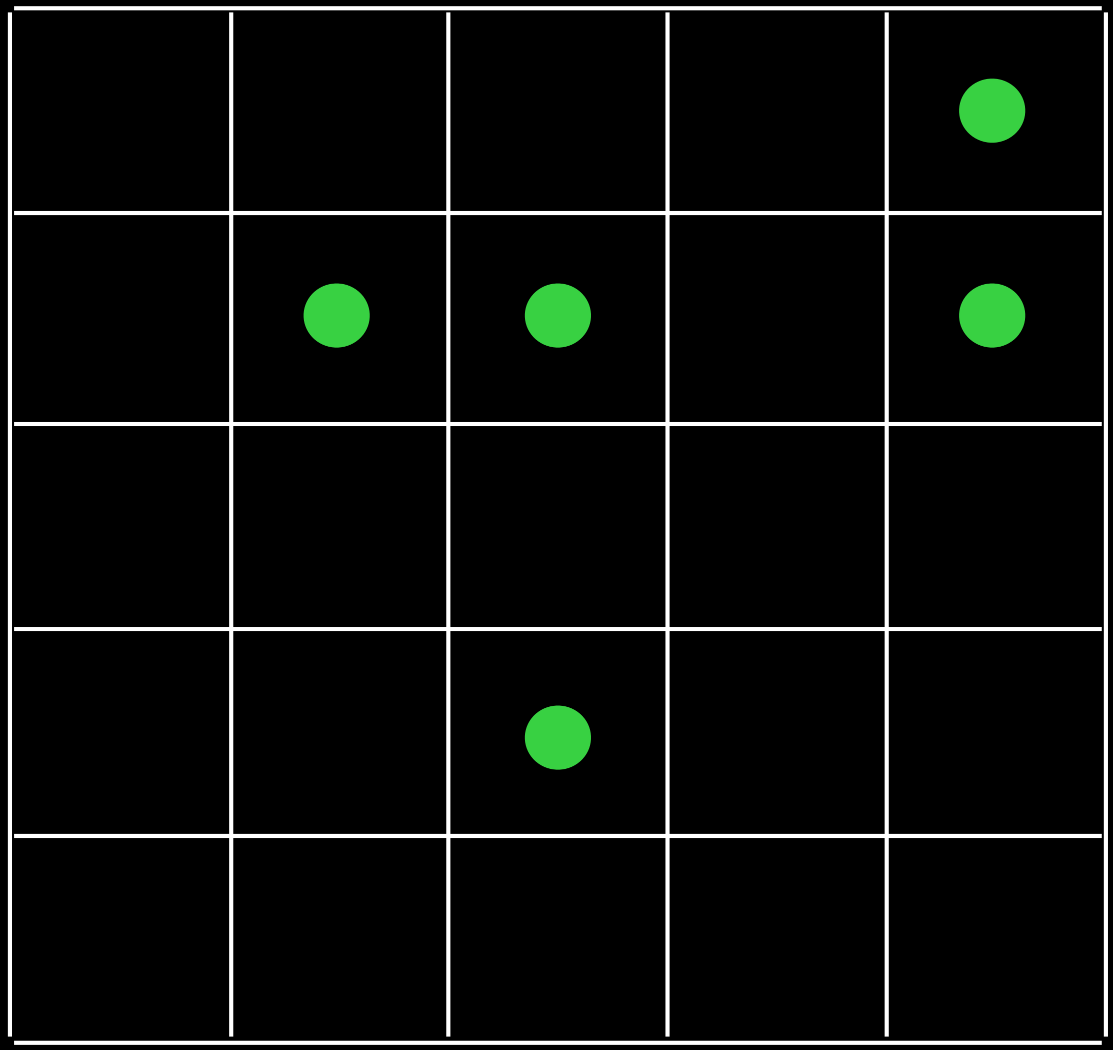


permutation as a Q- tableau



another Q-tableau:
Rothe diagram of a permutation

alternating sign matrices (ASM)
and a quadratic algebra

$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

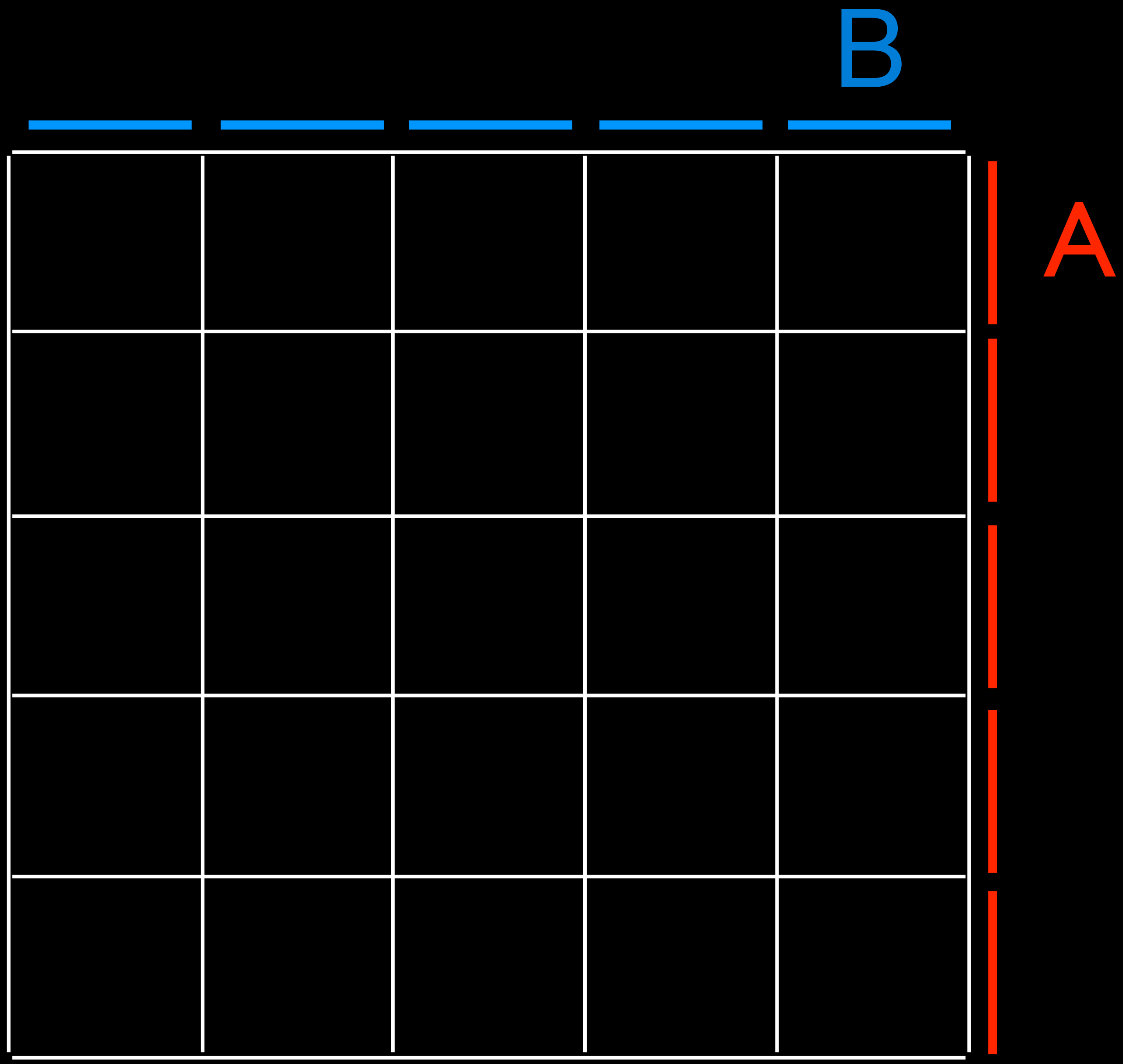
Lemma. Any word $w(A, A', B, B')$ in letters A, A', B, B', can be uniquely written

$$\sum c(u, v; w)\ \underbrace{u(A, A')}_{\text{word in } A, A'}\ \underbrace{v(B, B')}_{\text{word in } B, B'}$$

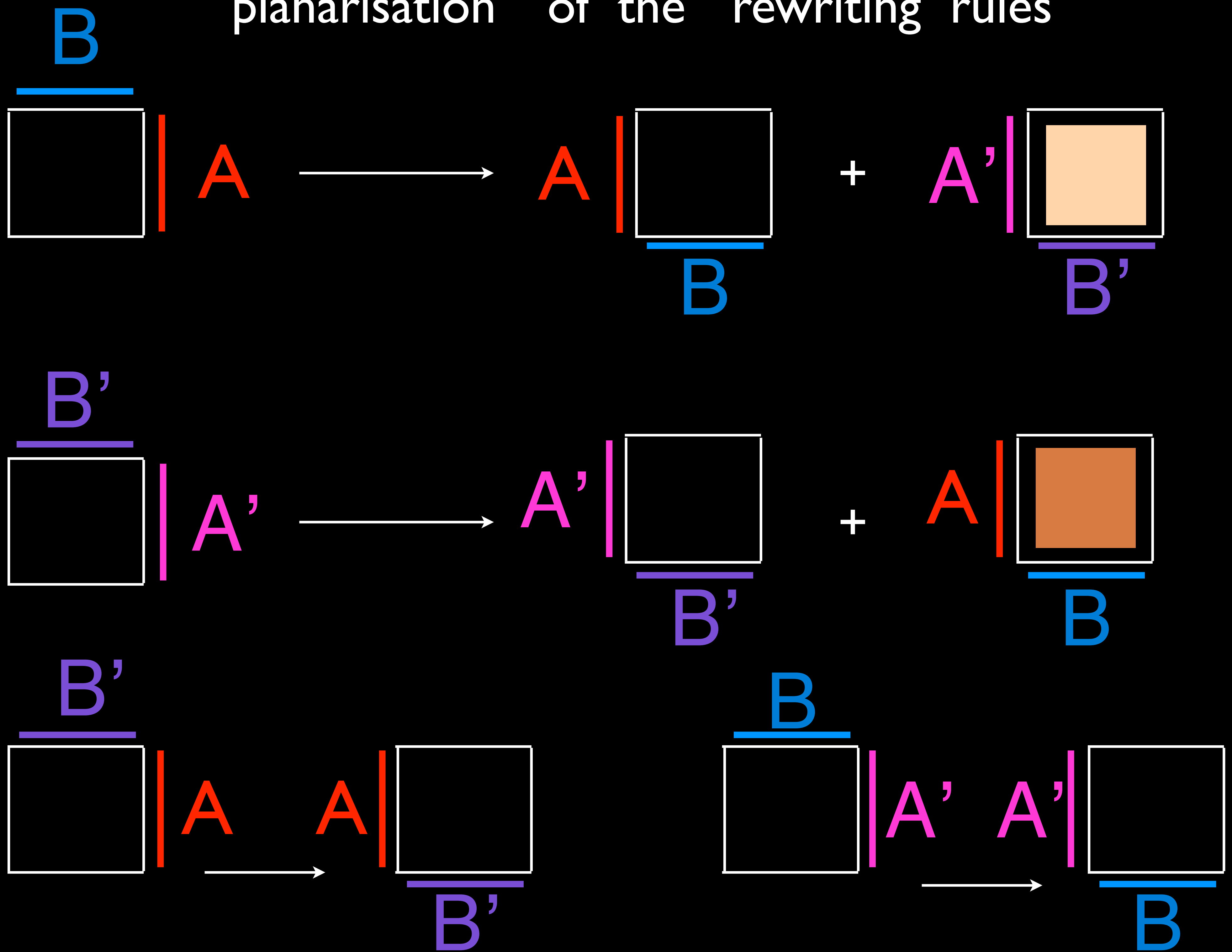
Prop. For $w = B^n A^n$

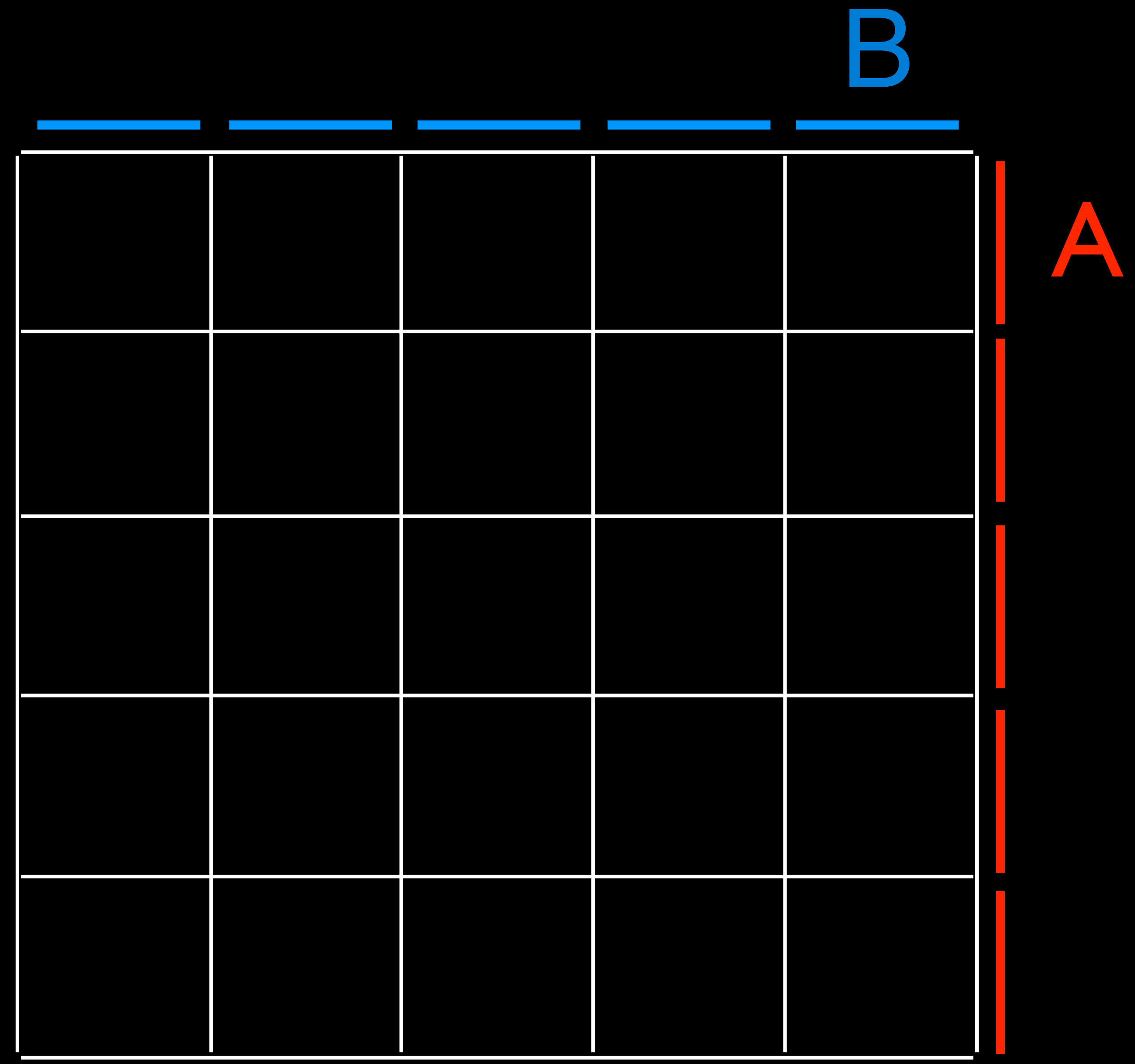
$u = A'^n$, $v = B'^n$

$c(u, v; w) =$ the number of $n \times n$ ASM (alternating sign matrices) “planar” proof:



“planarisation” of the “rewriting rules”





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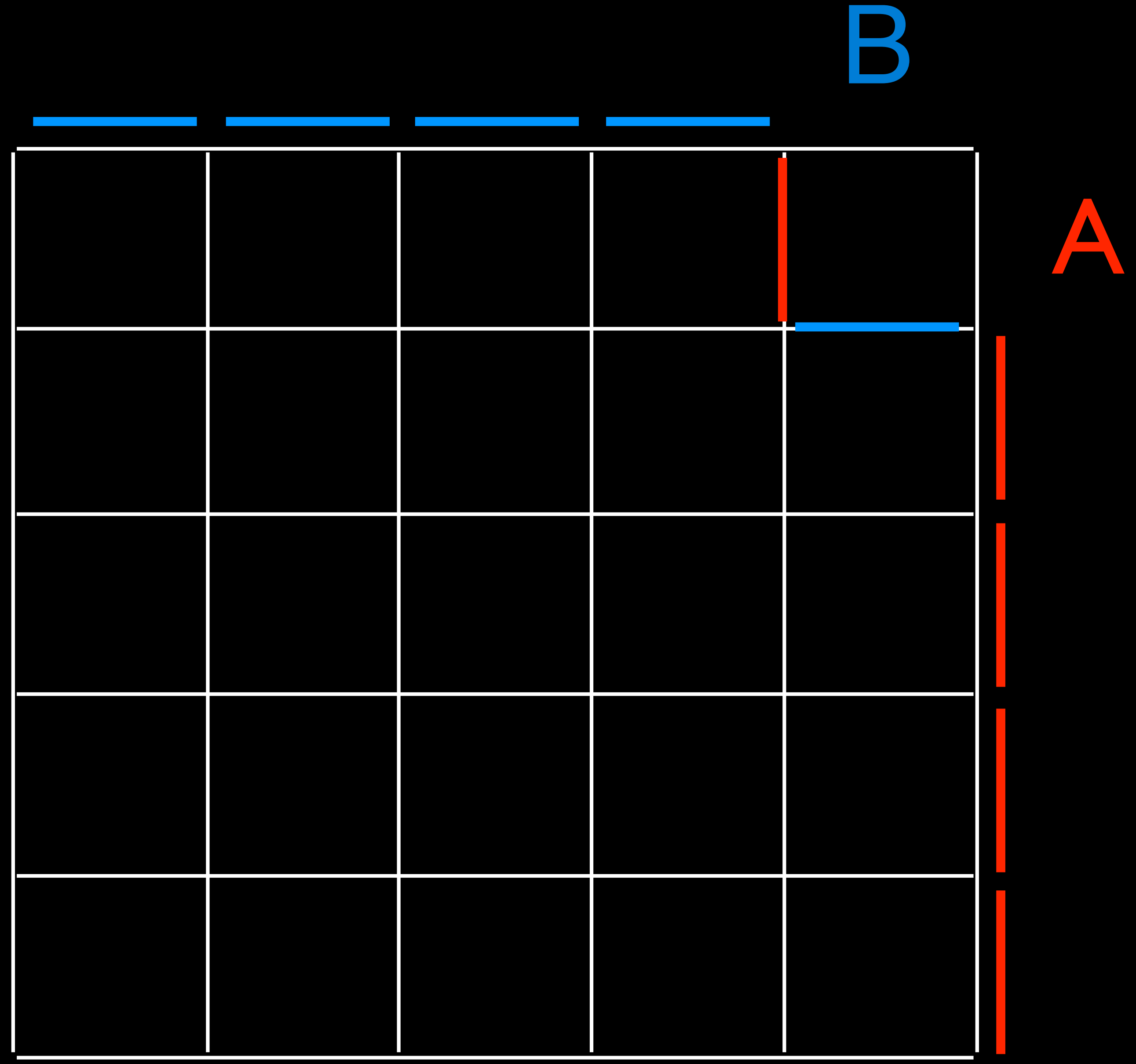
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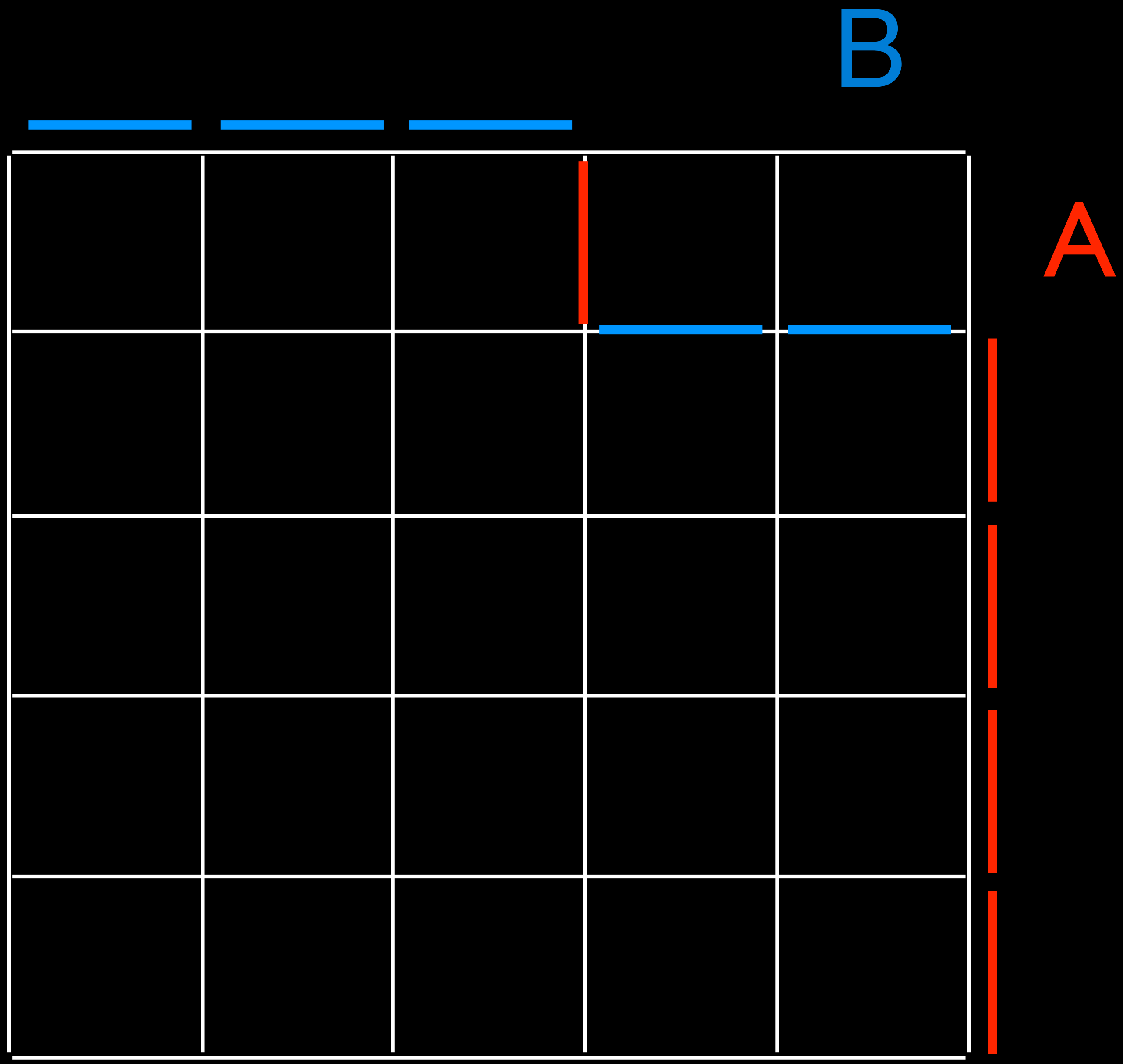
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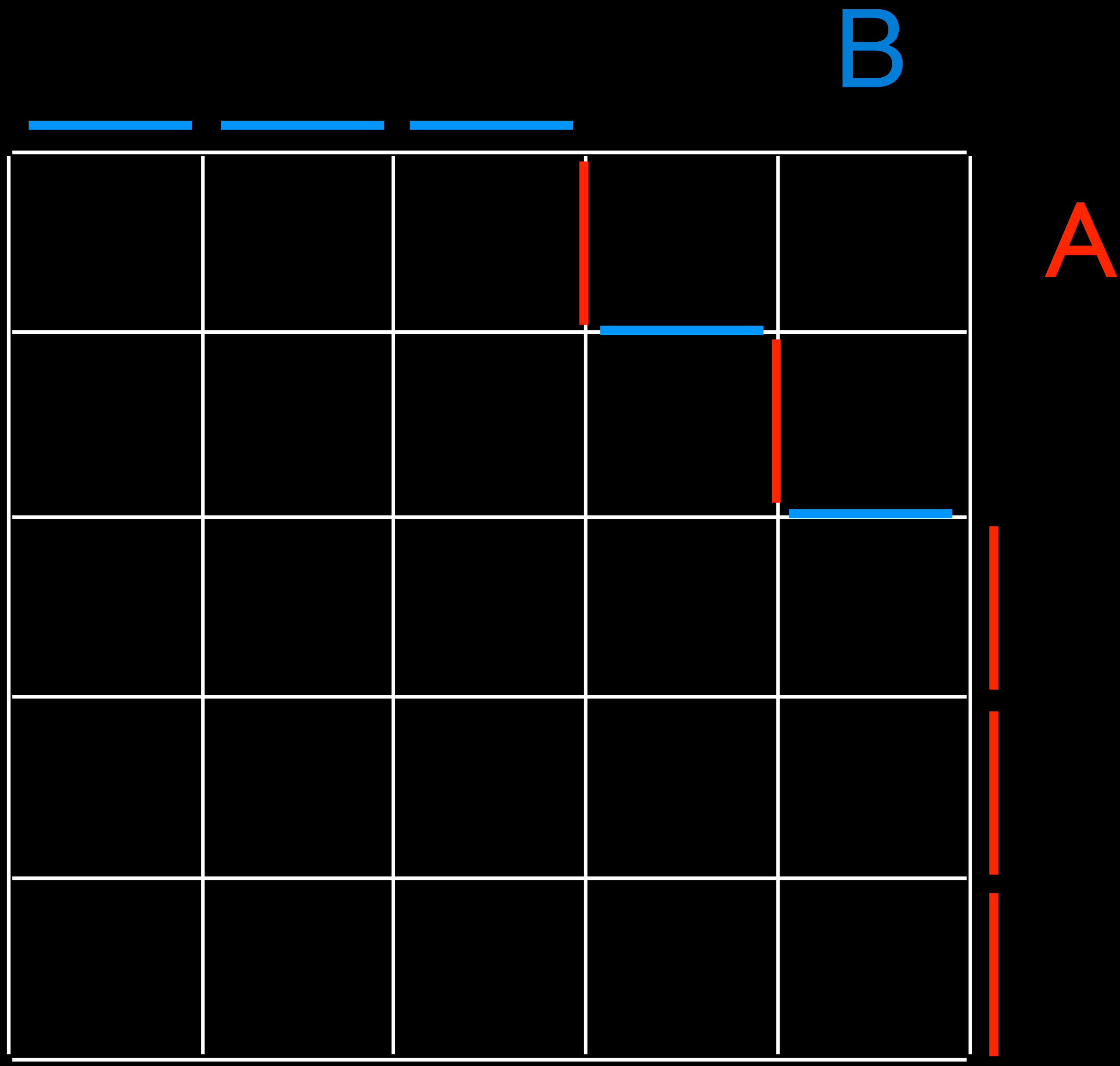
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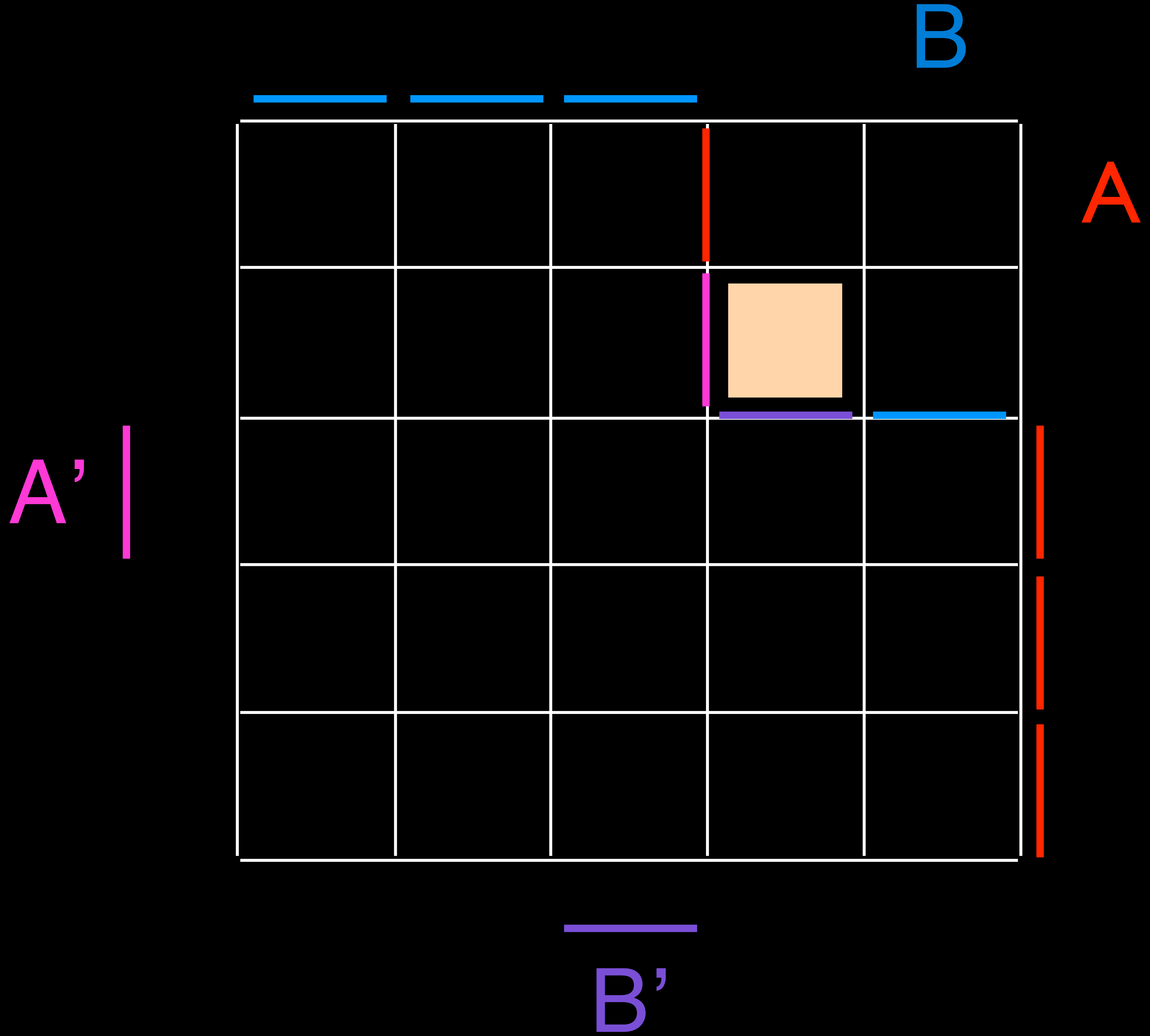
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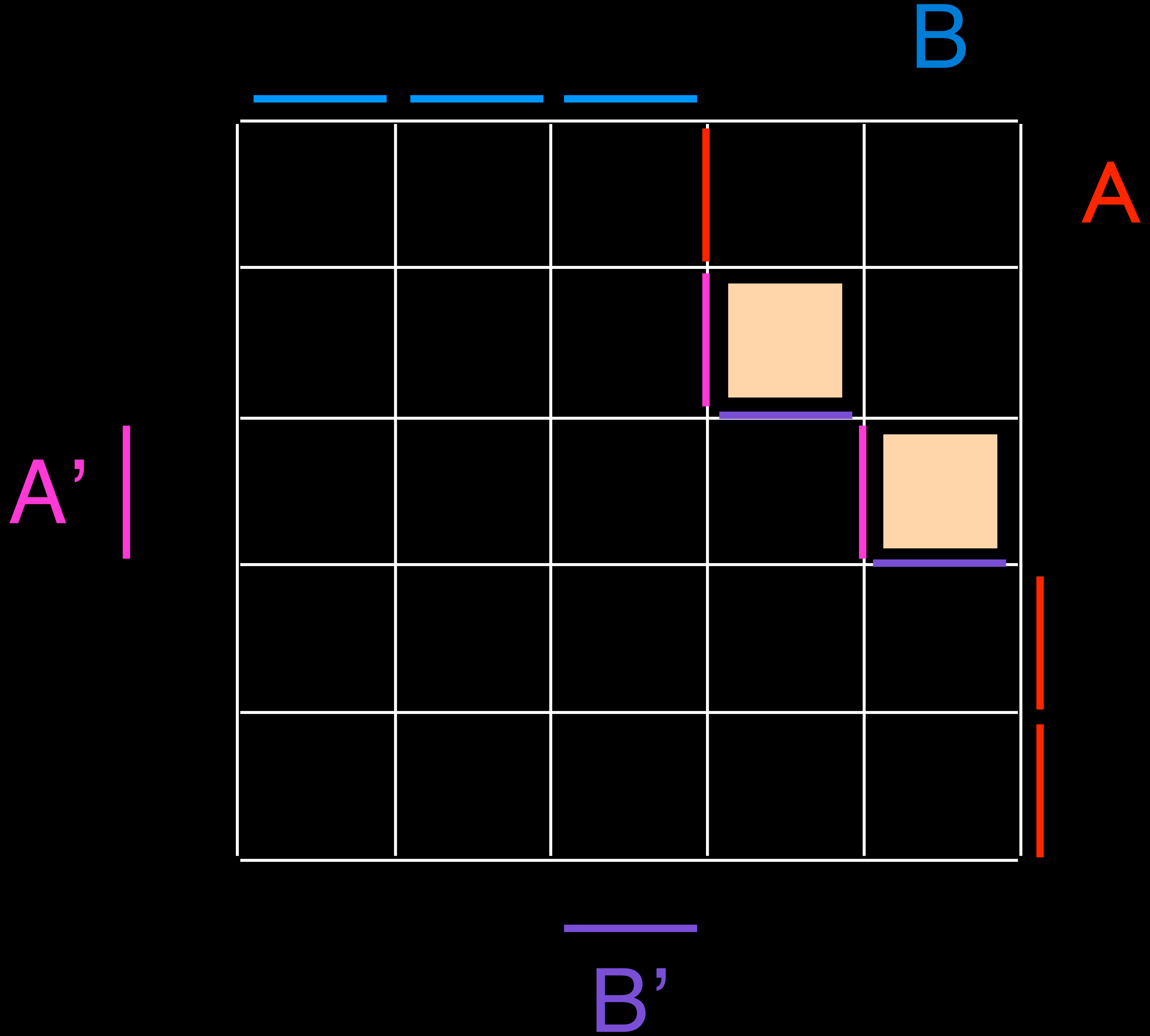
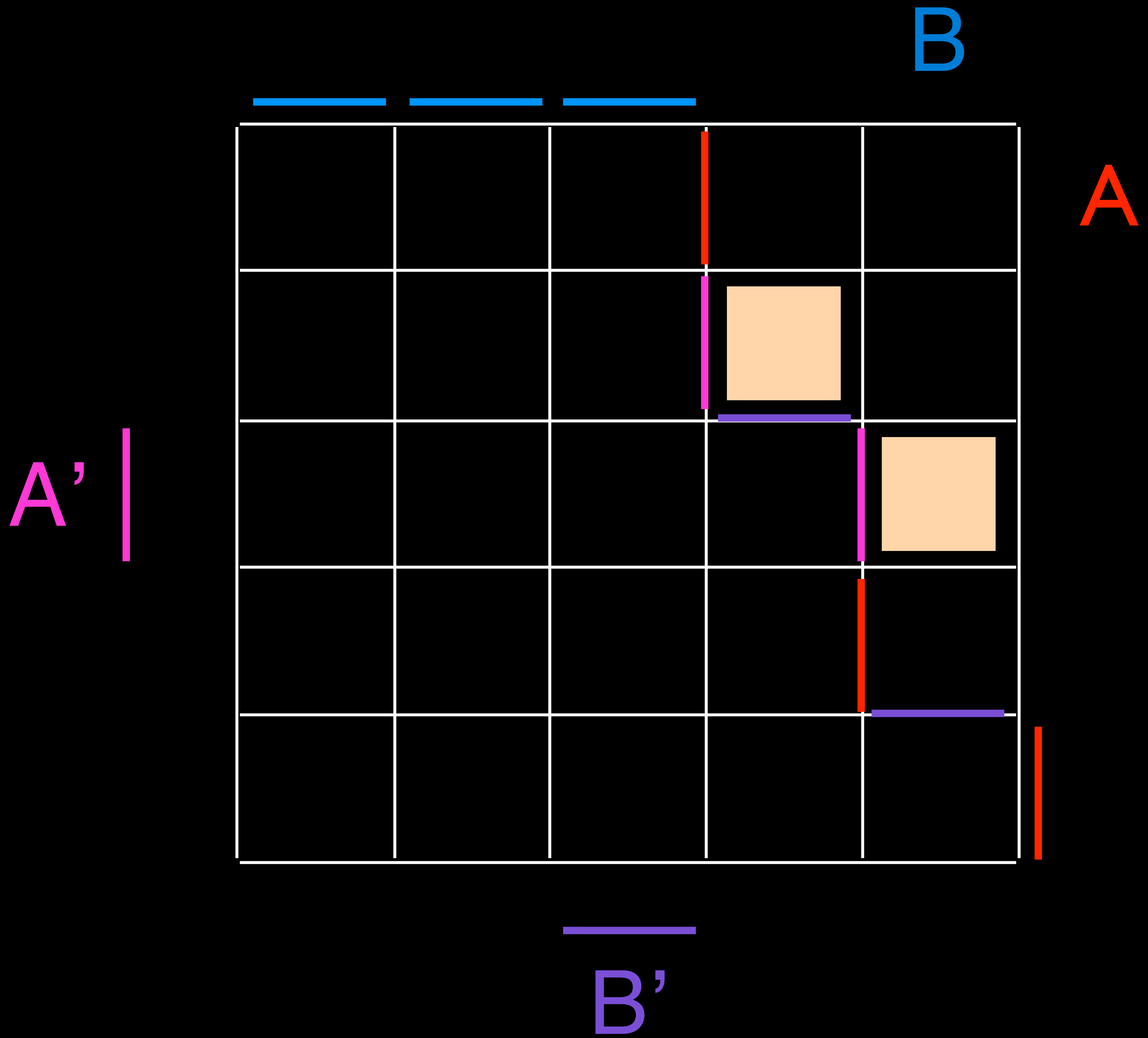
A’
B’



B
A
A’
B’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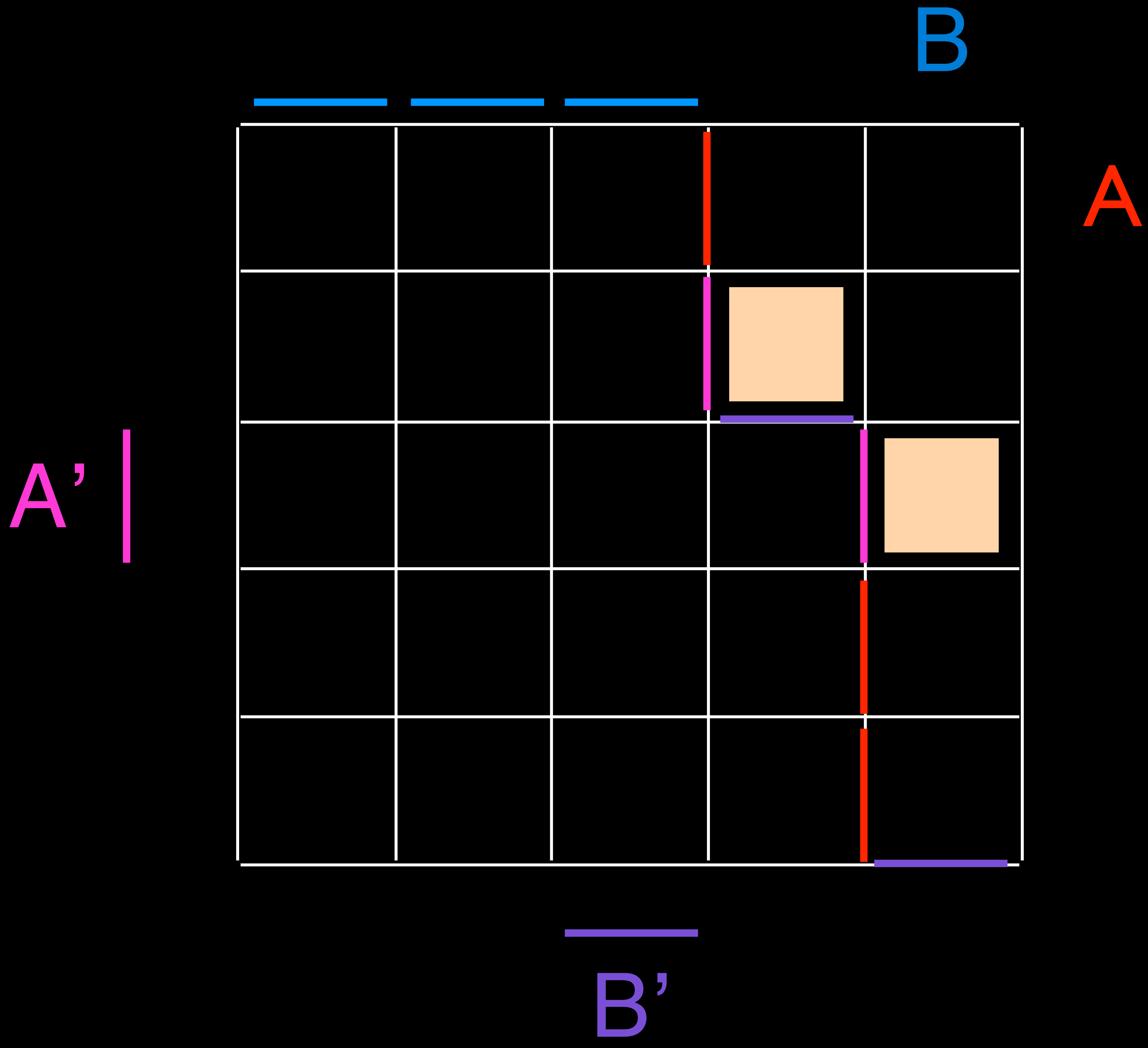

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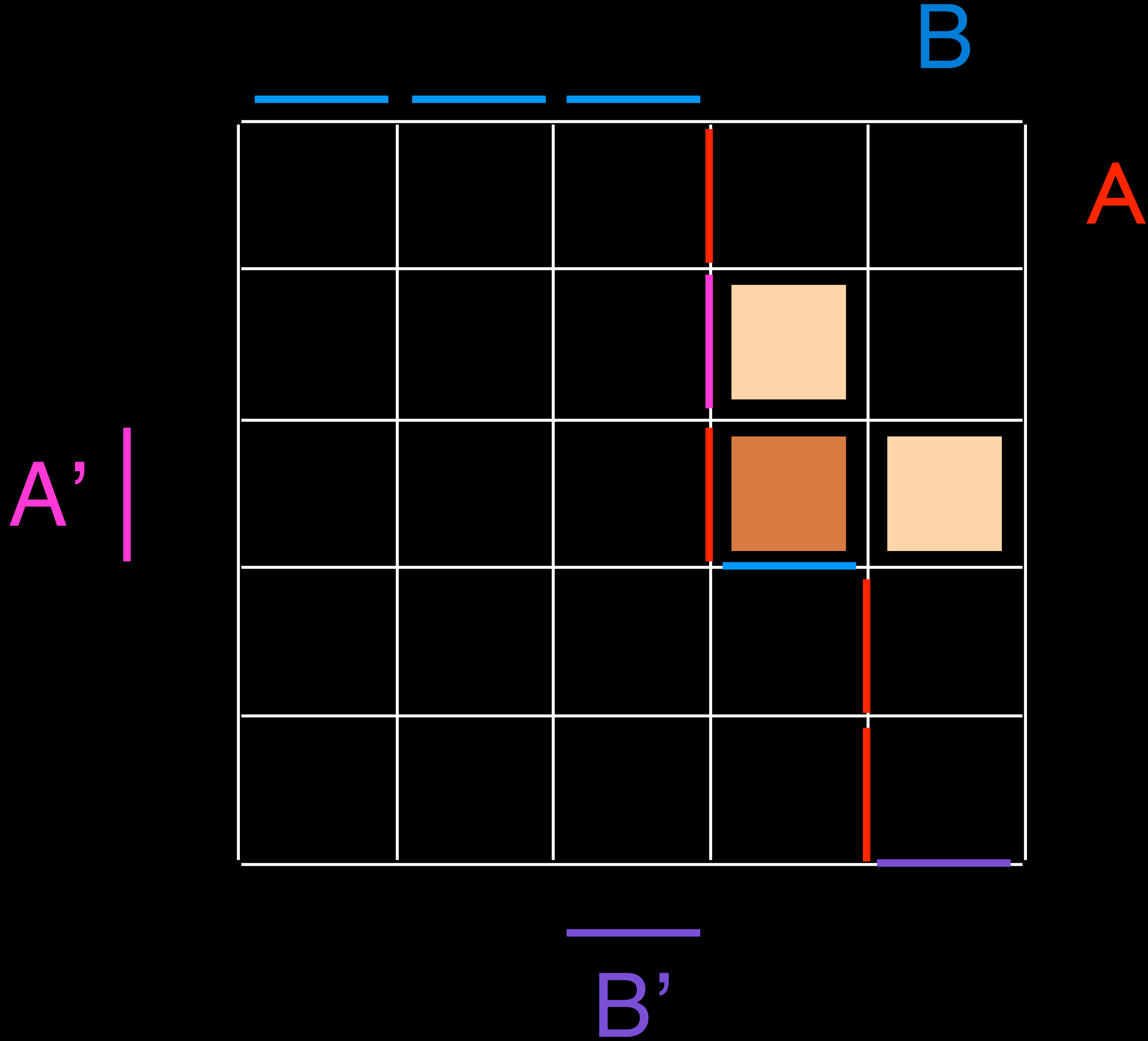
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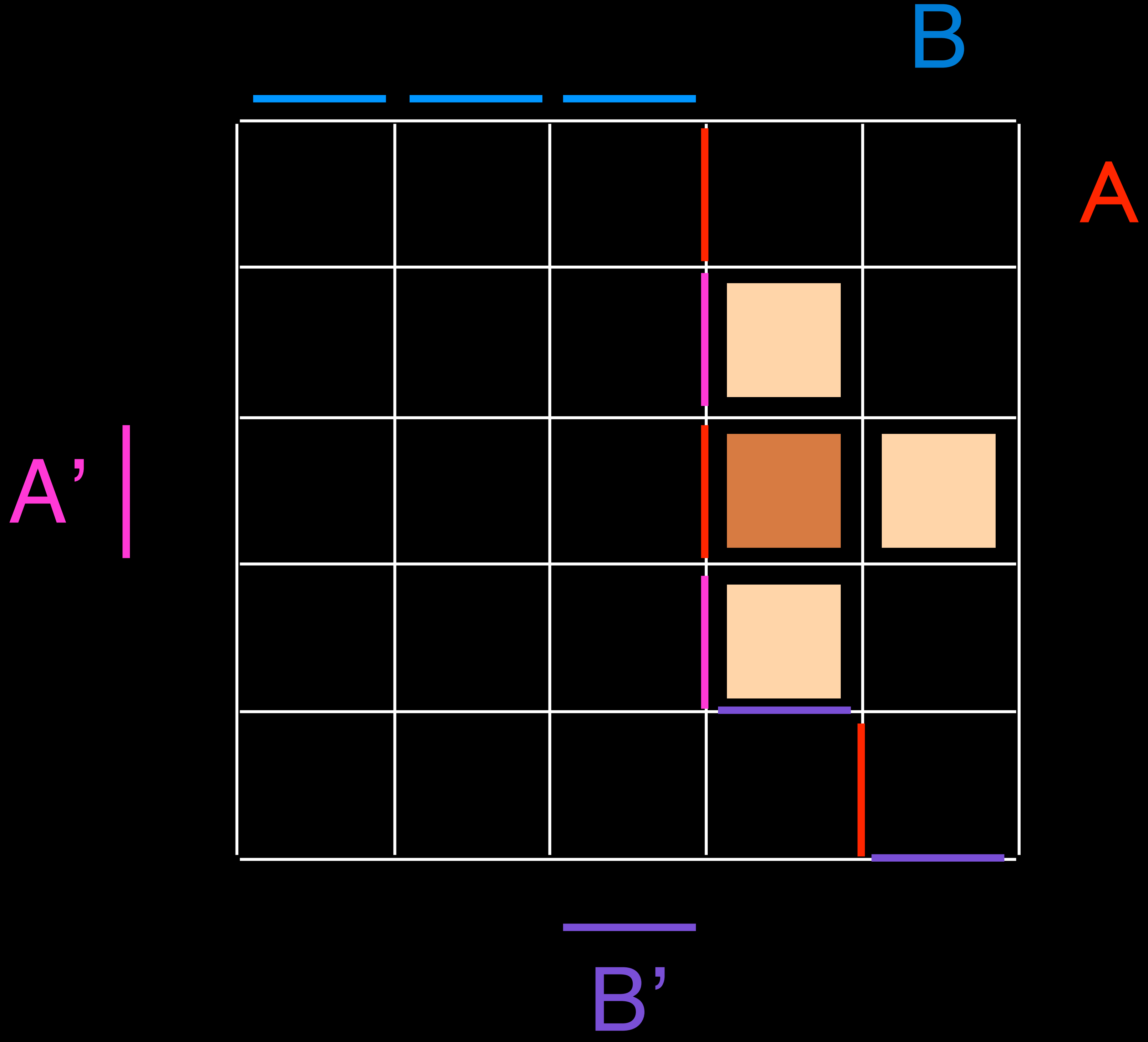
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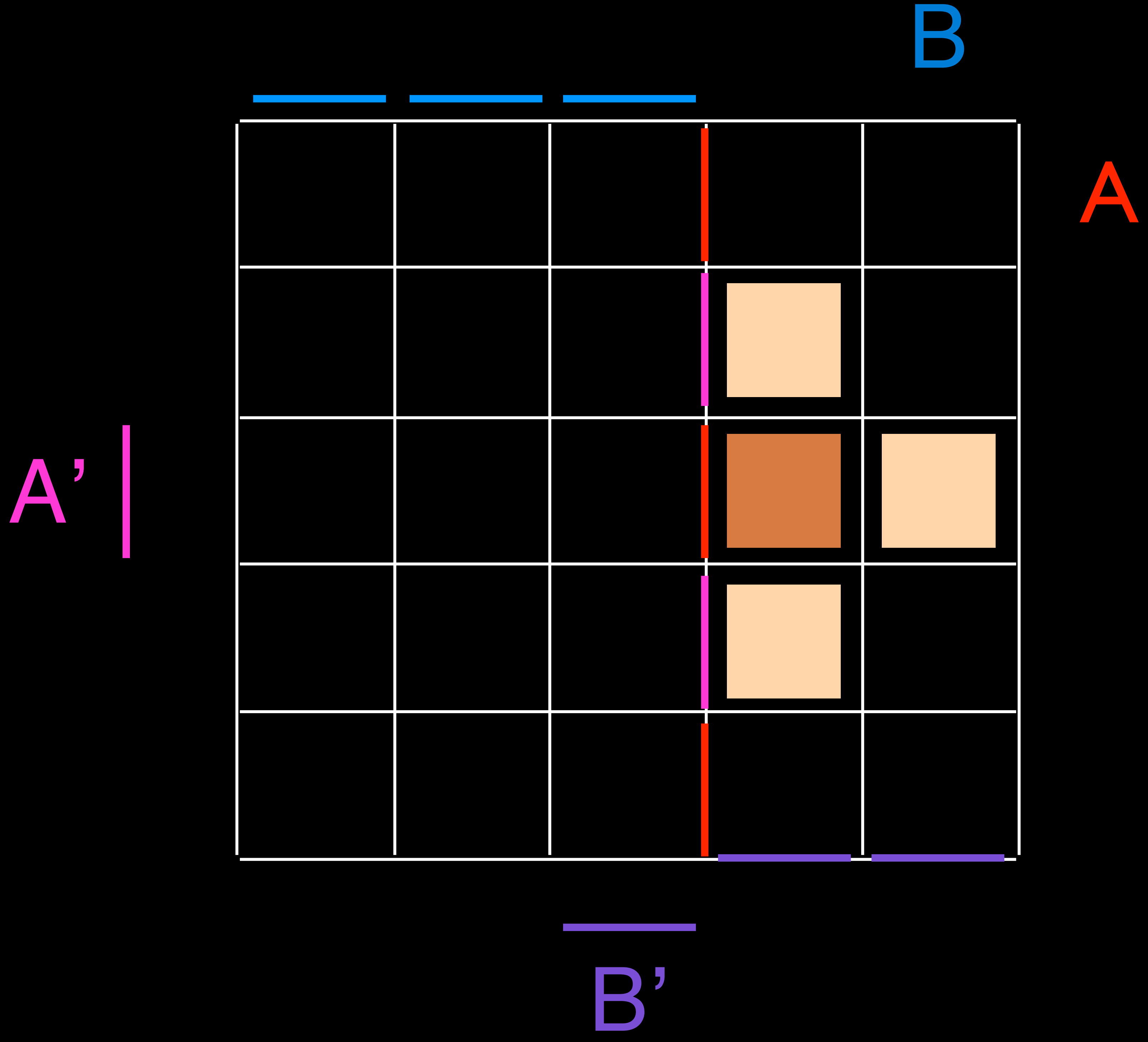
B
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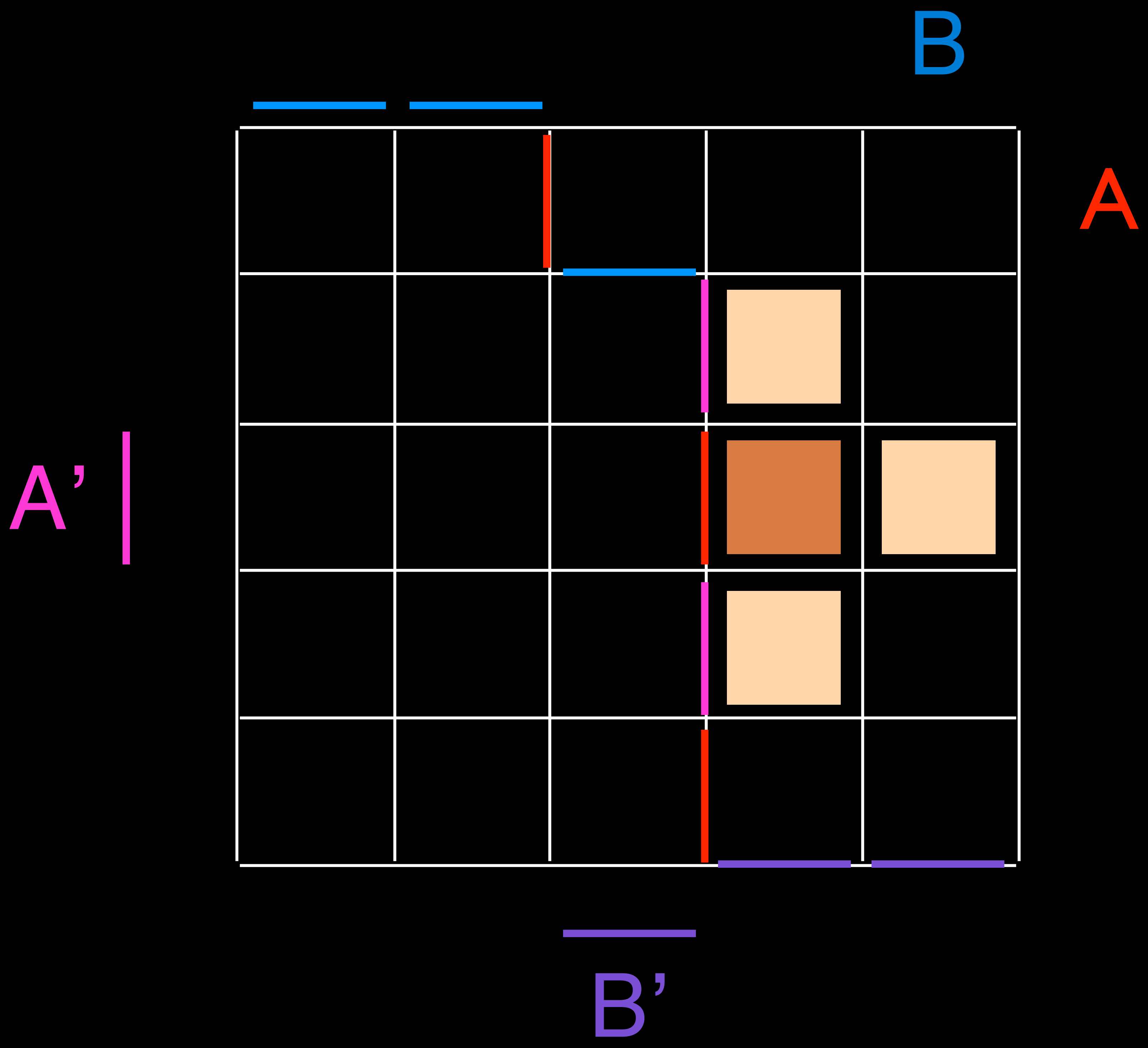
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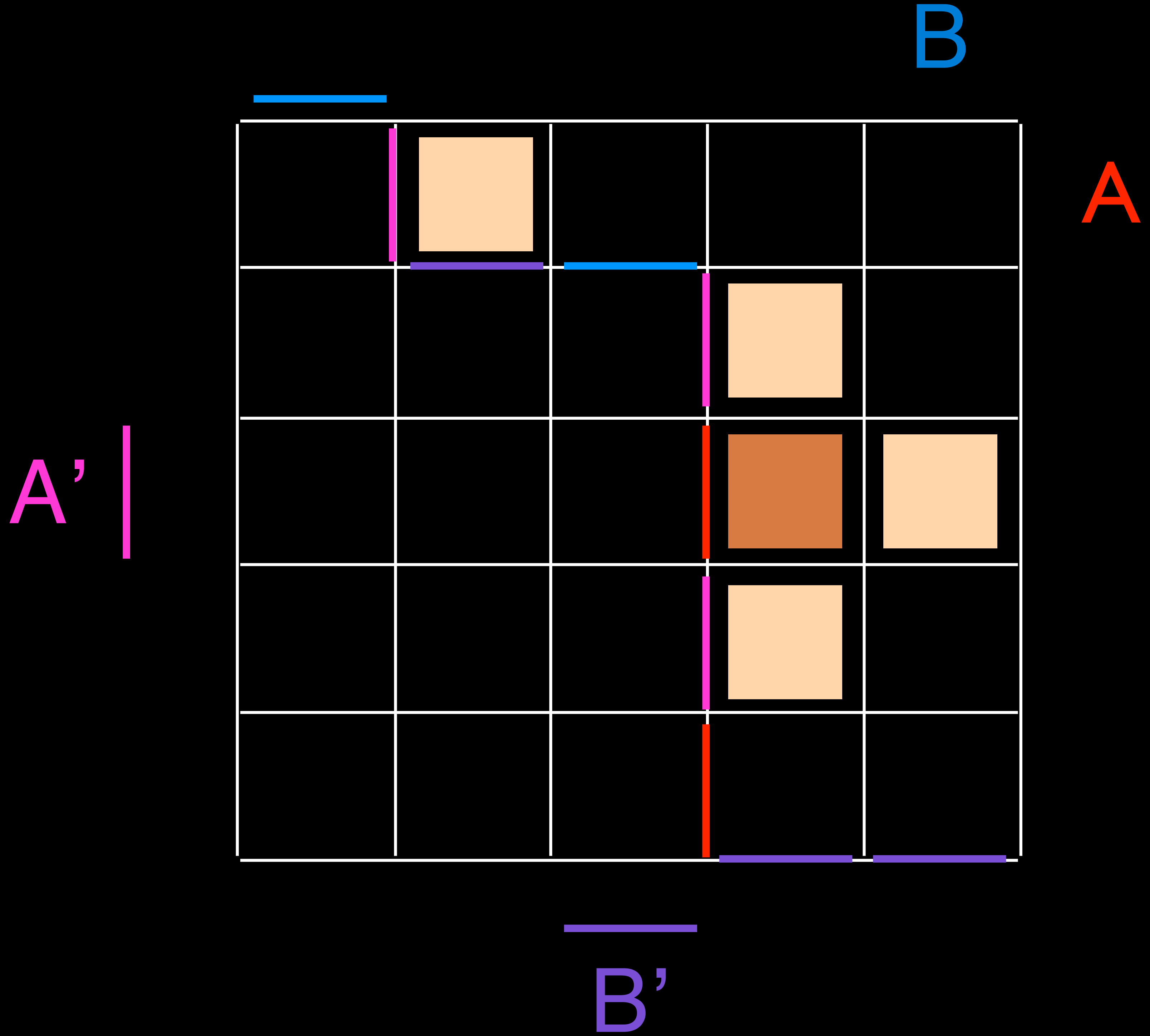
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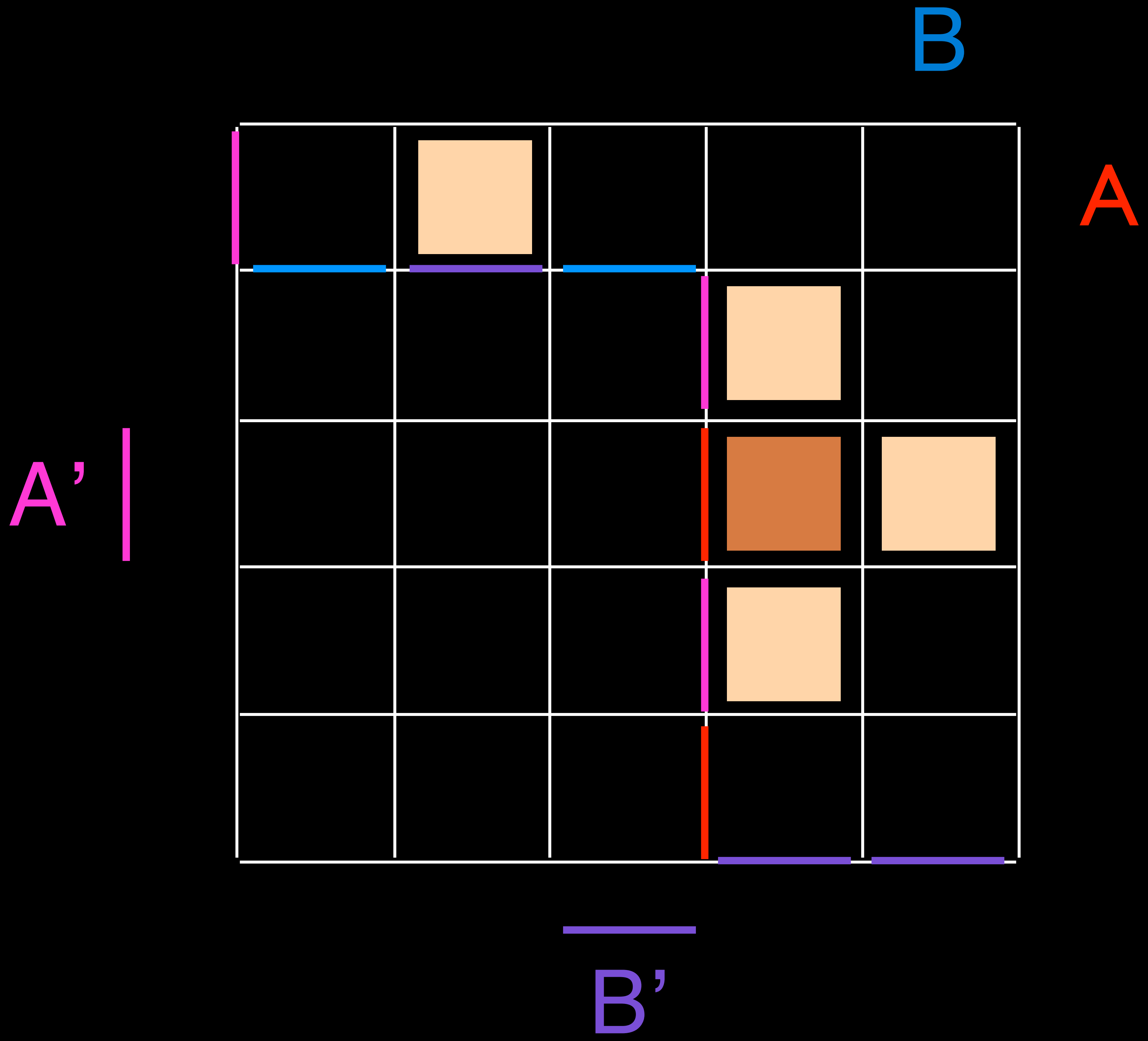

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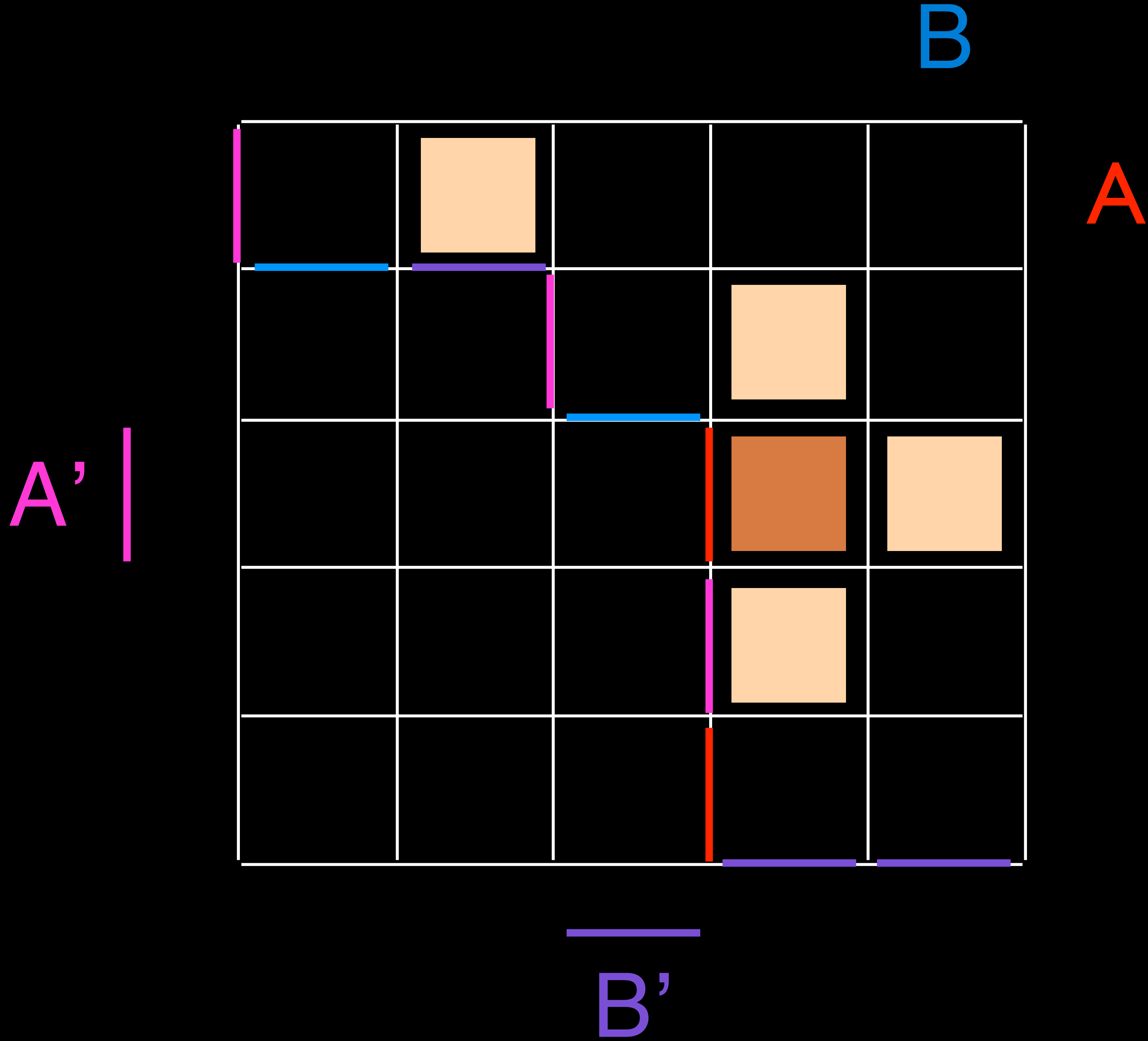
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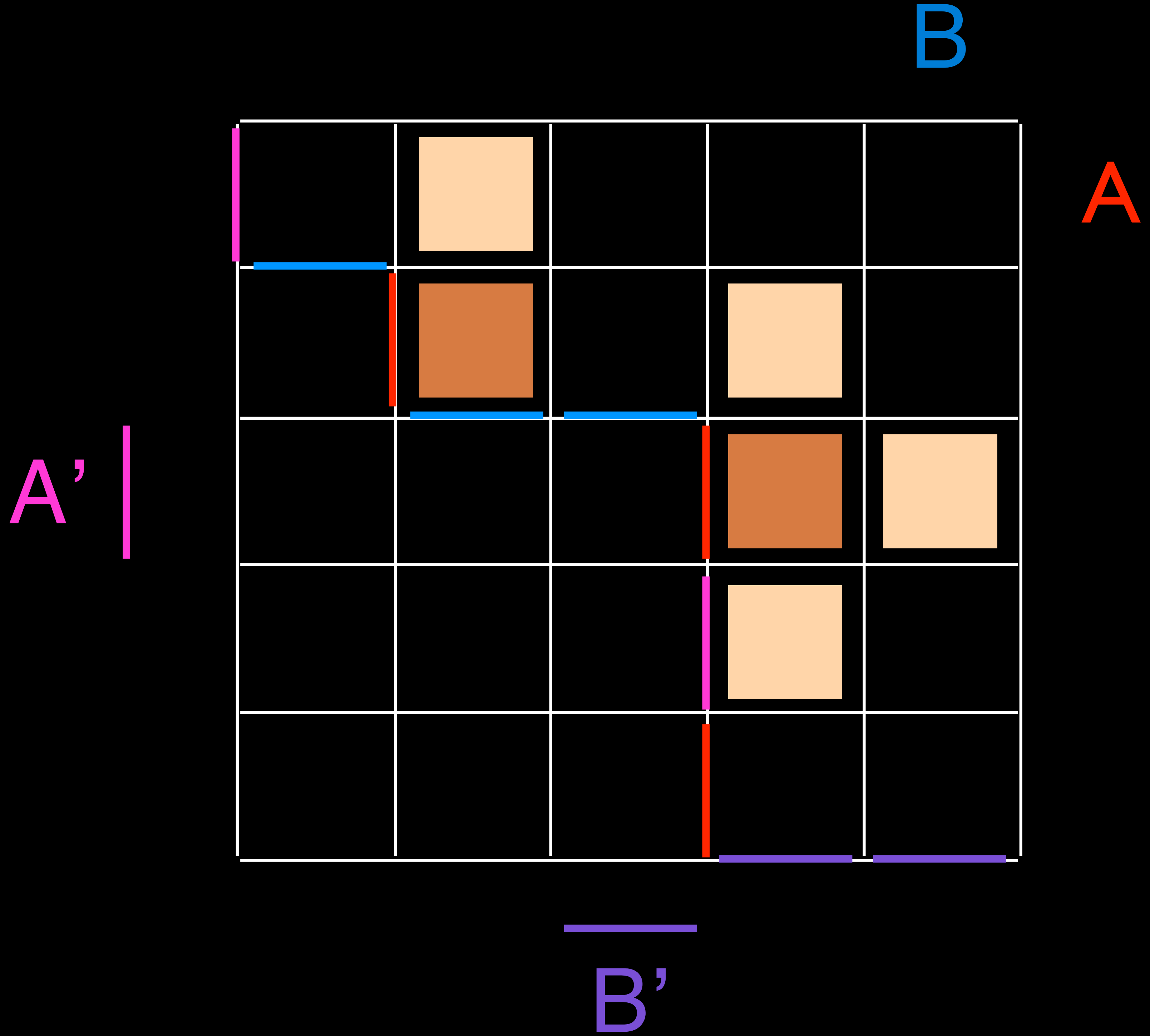

B
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A'
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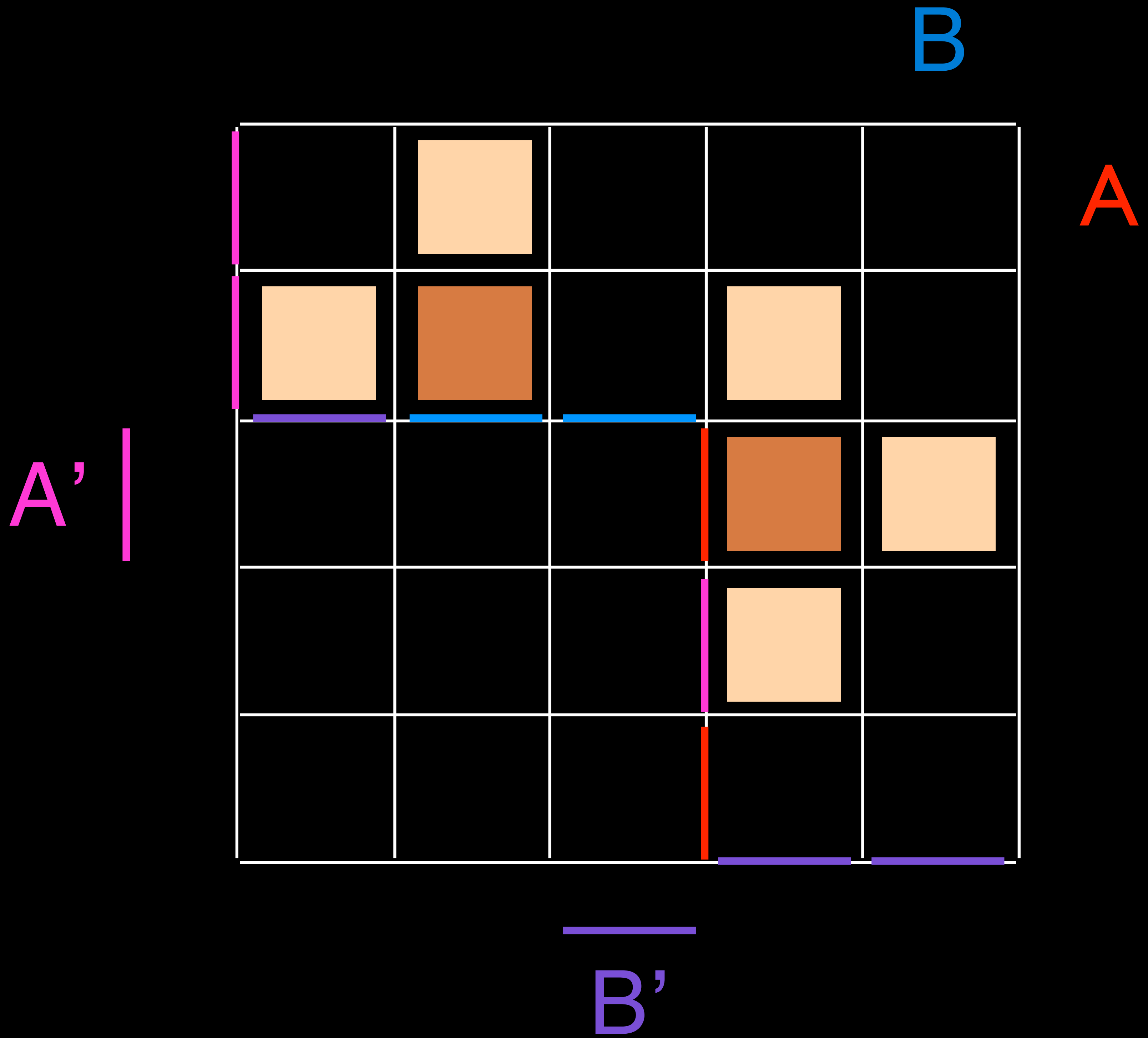

B
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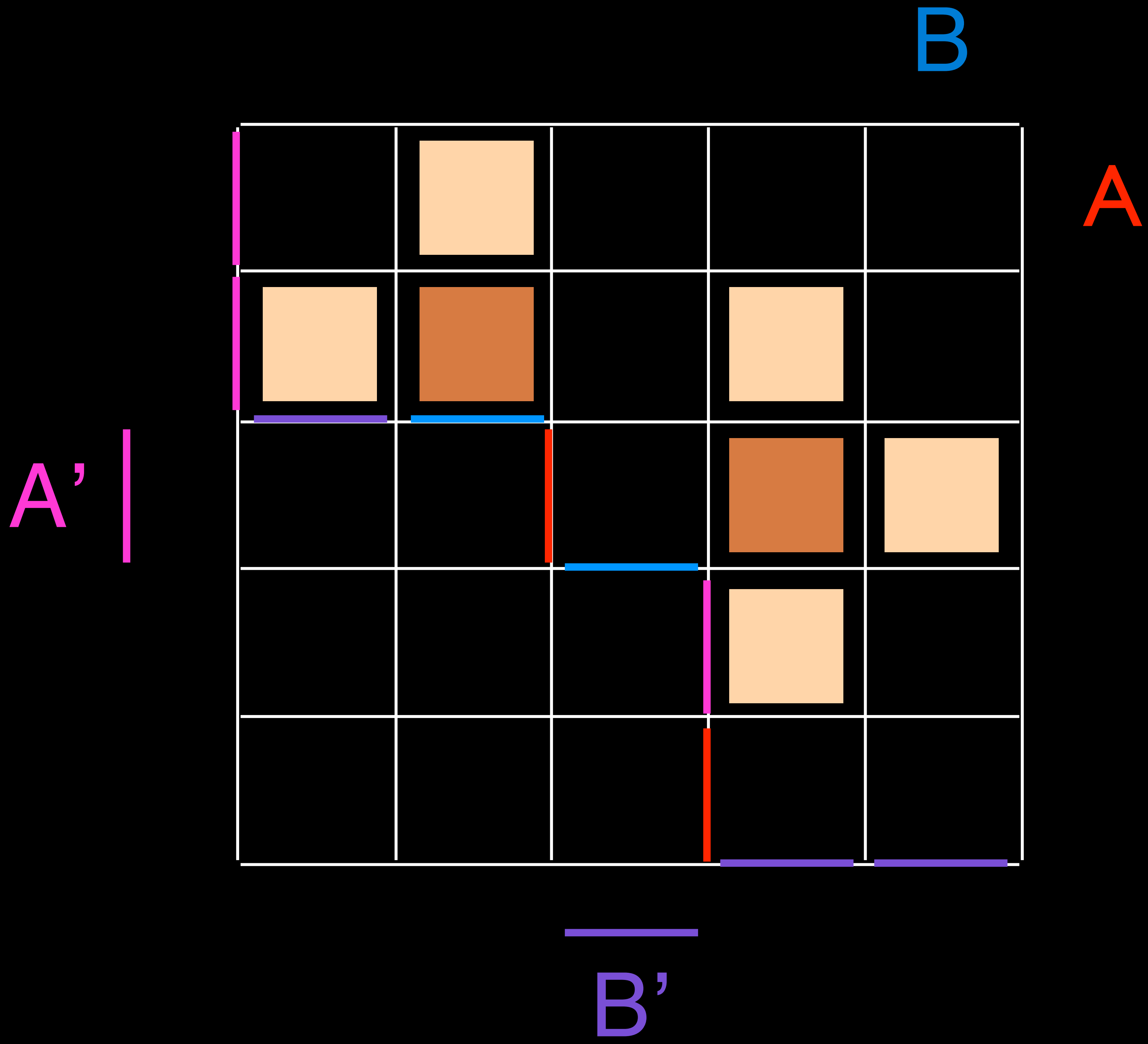
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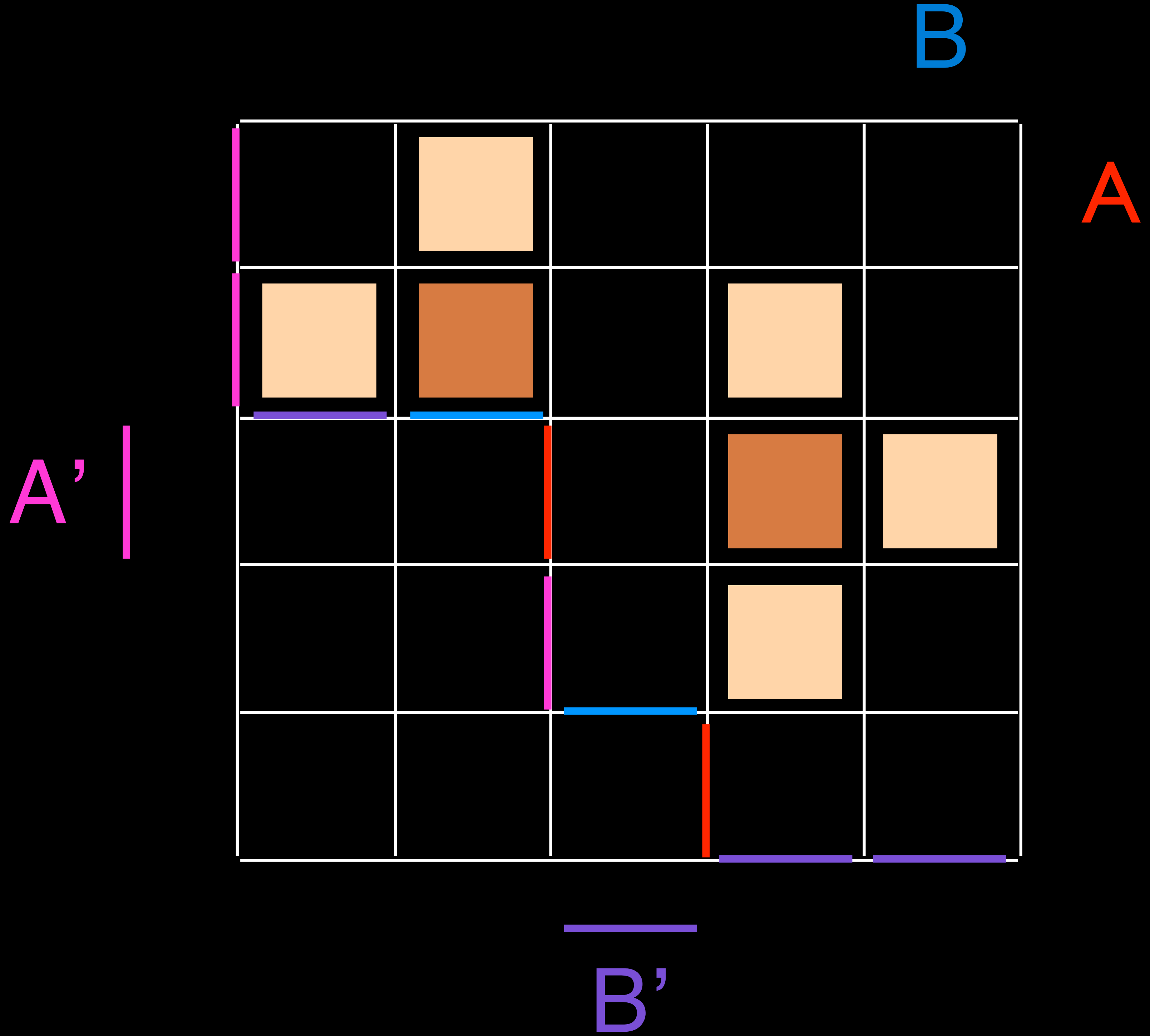
B
A
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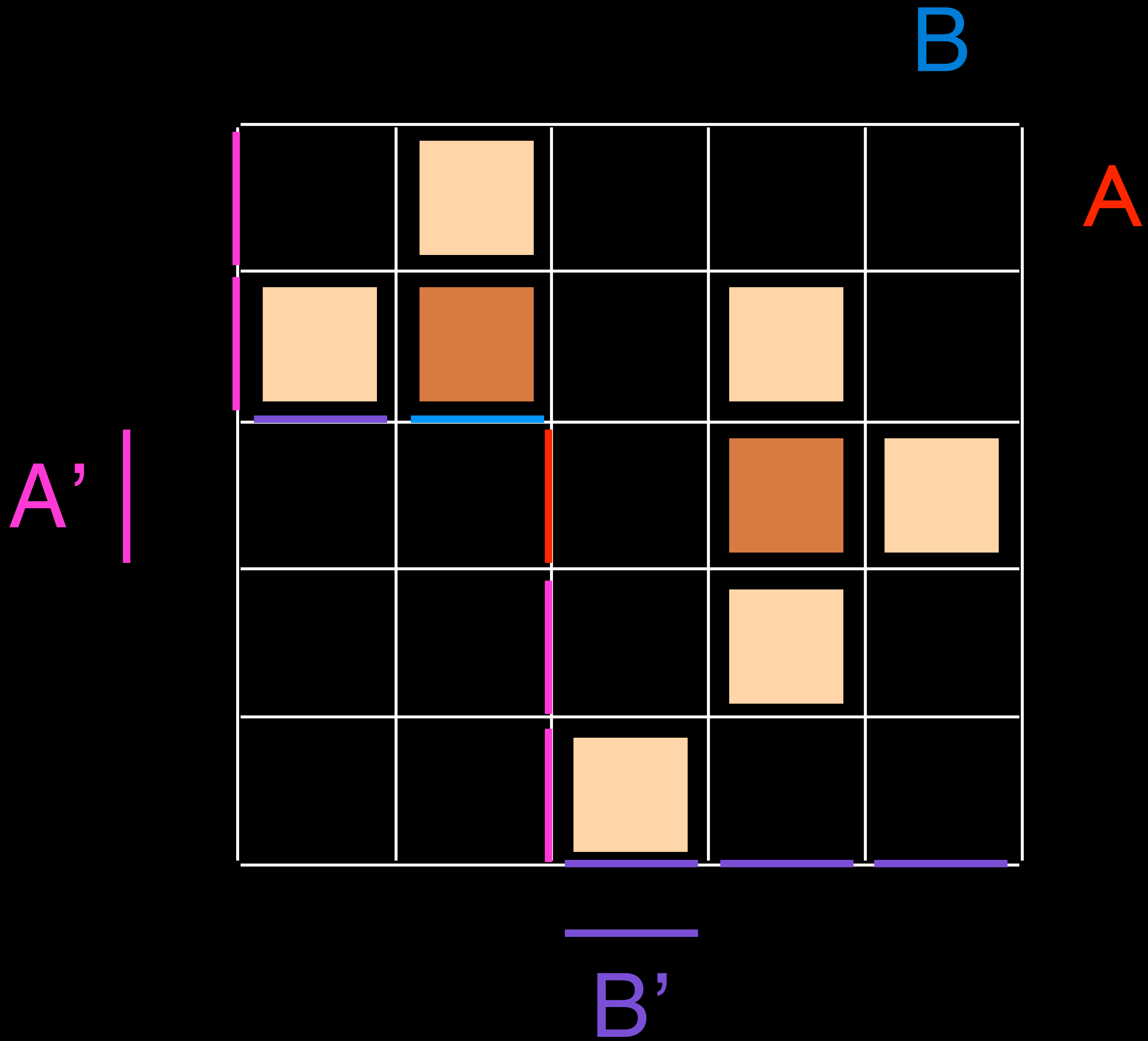
B
A
A'
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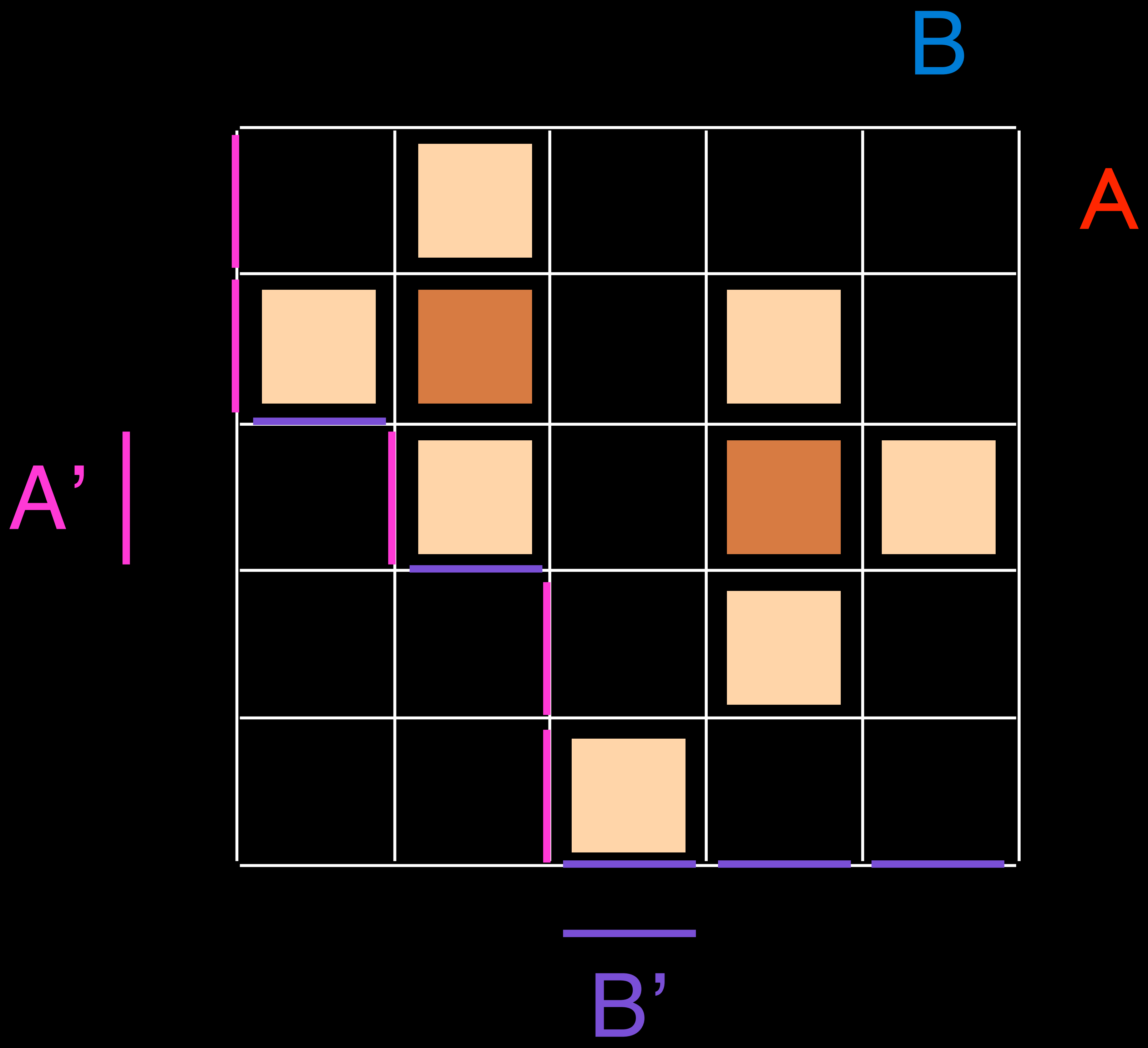
B
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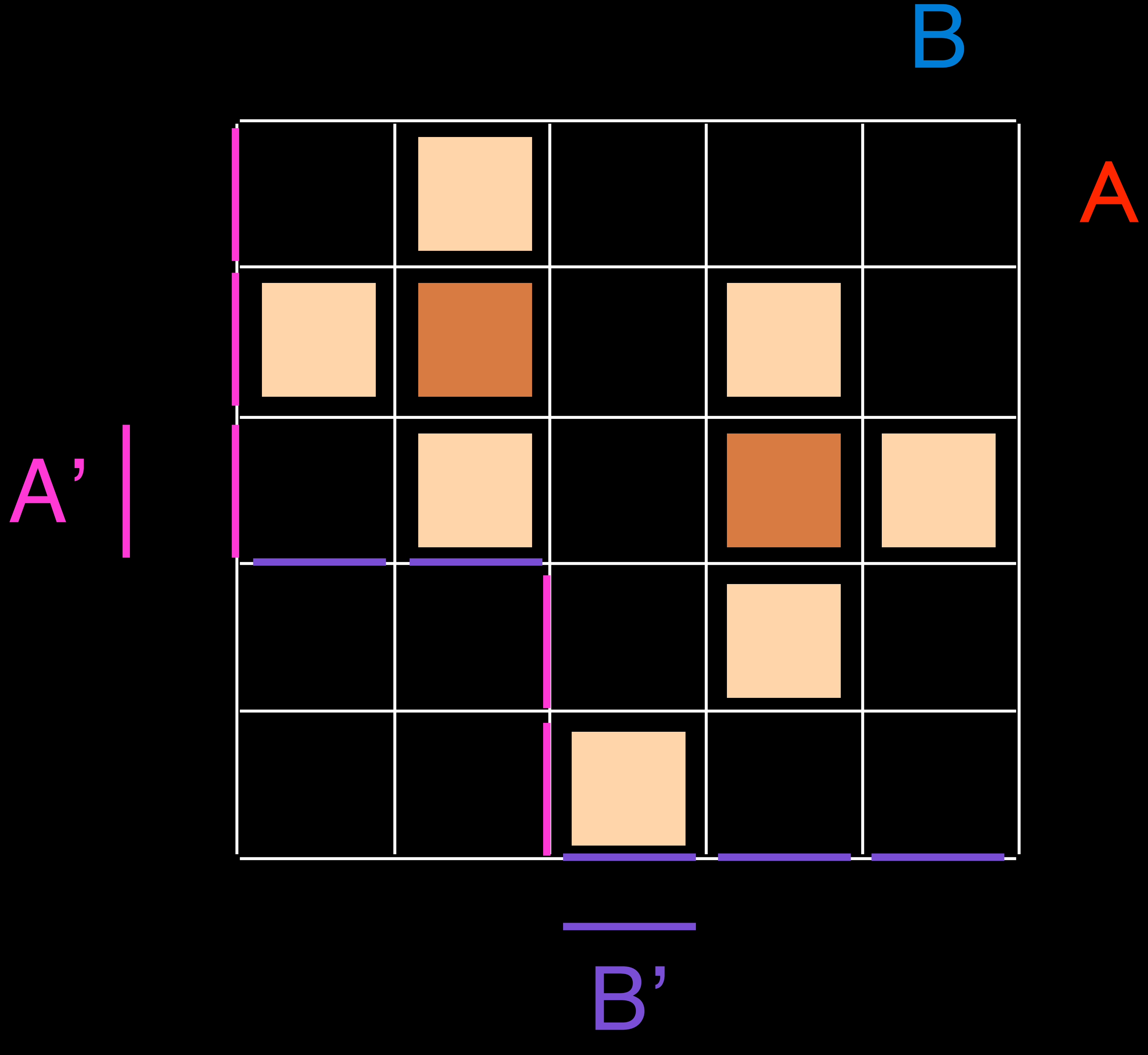
B
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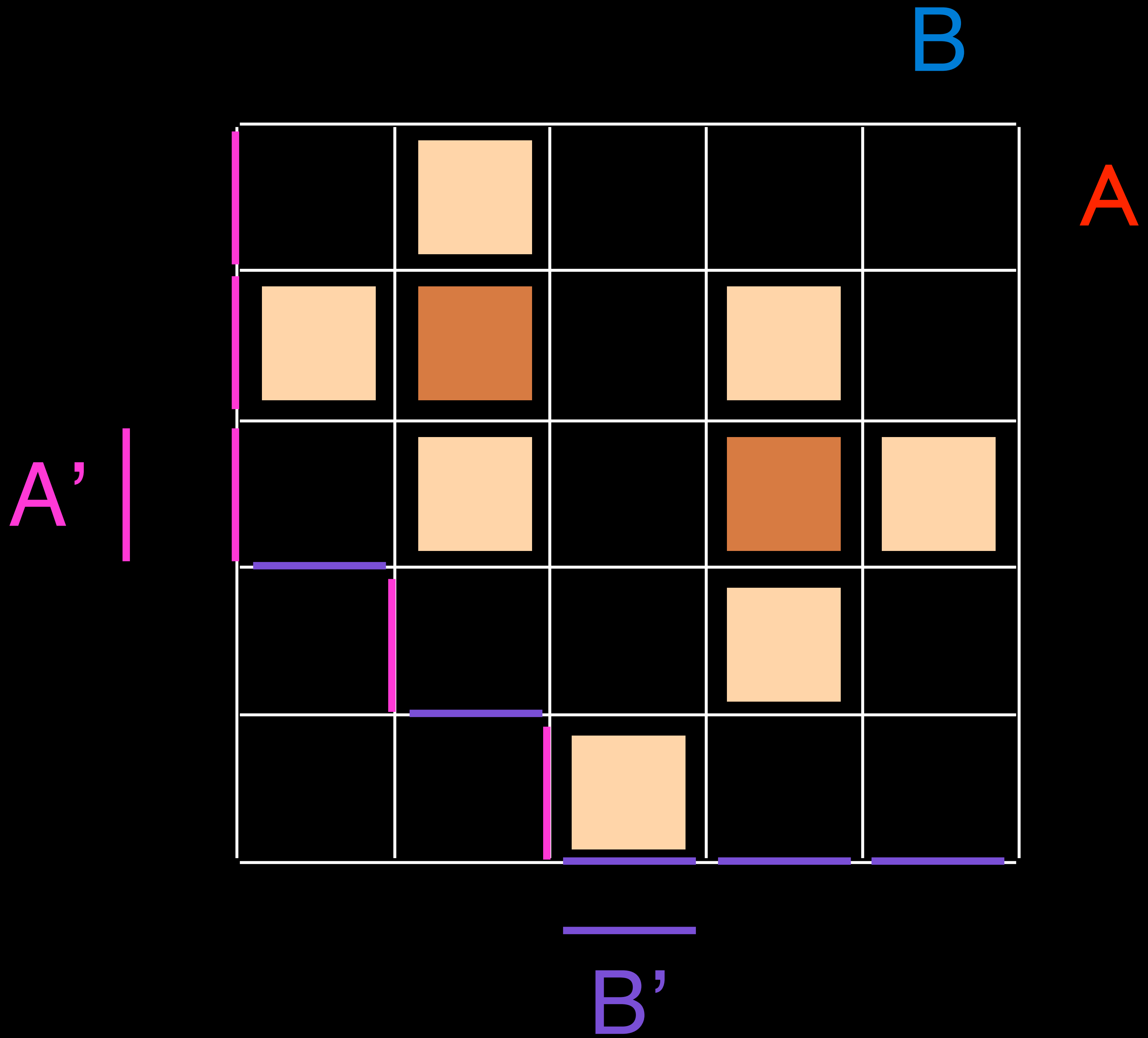
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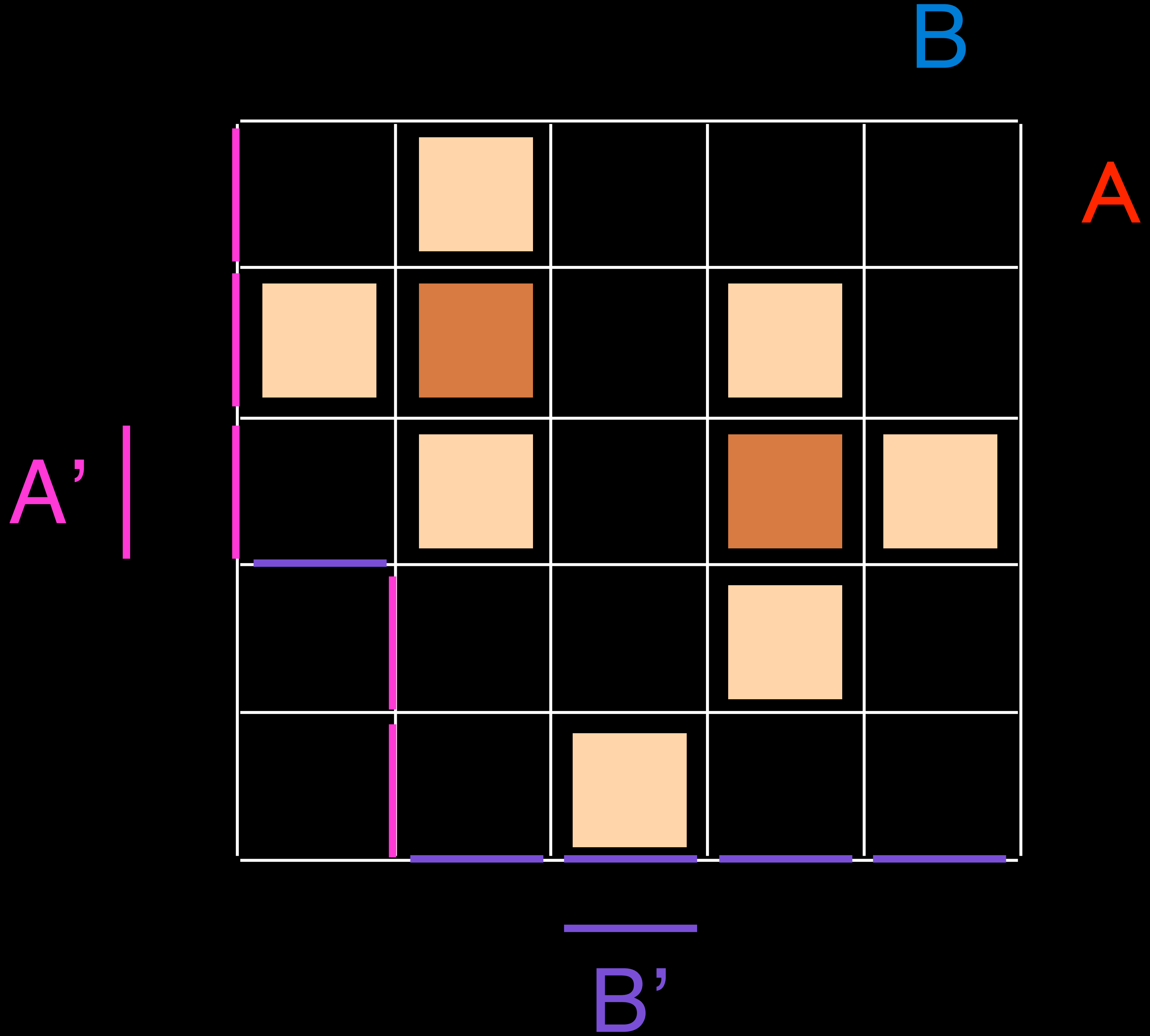
B
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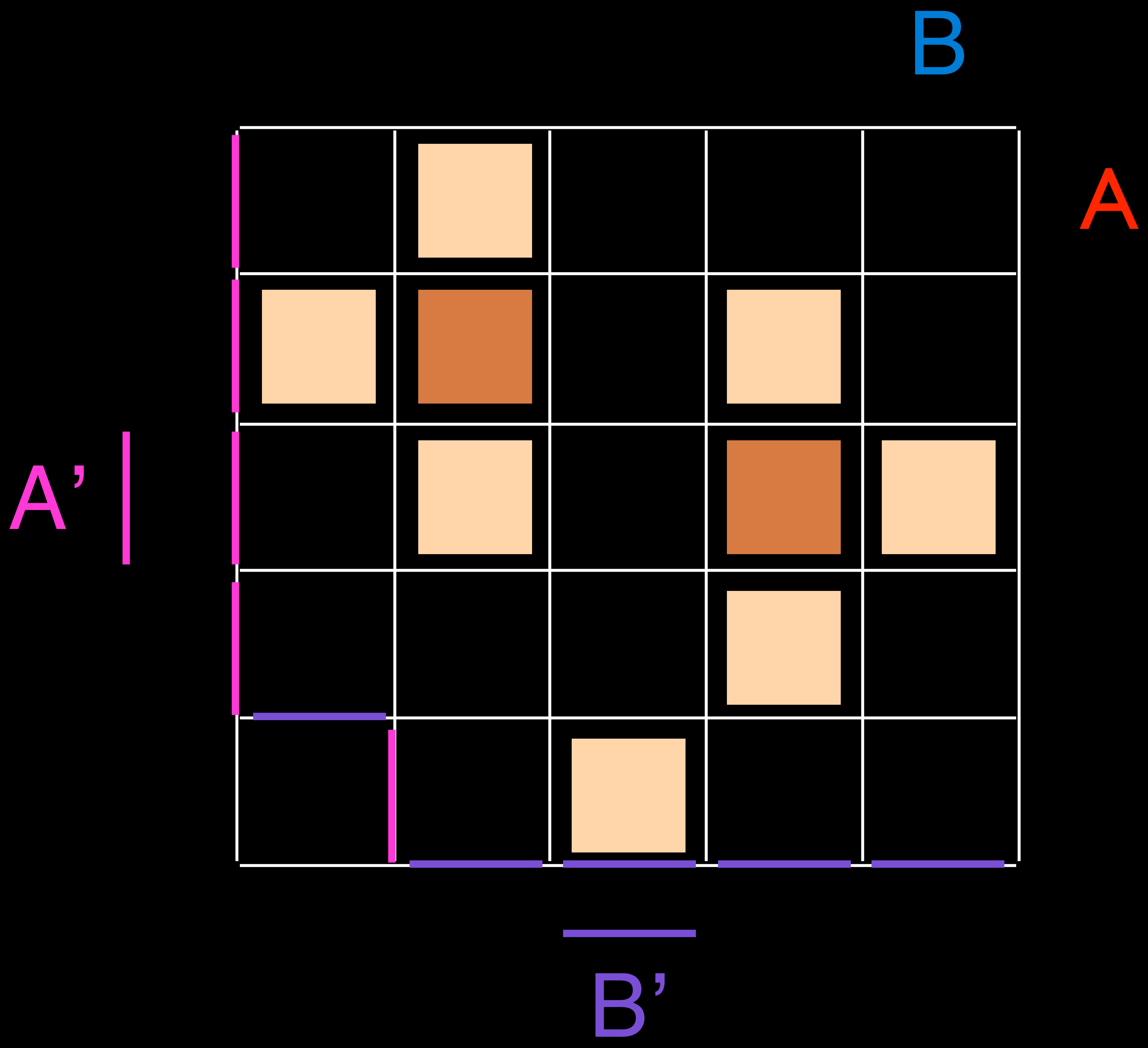
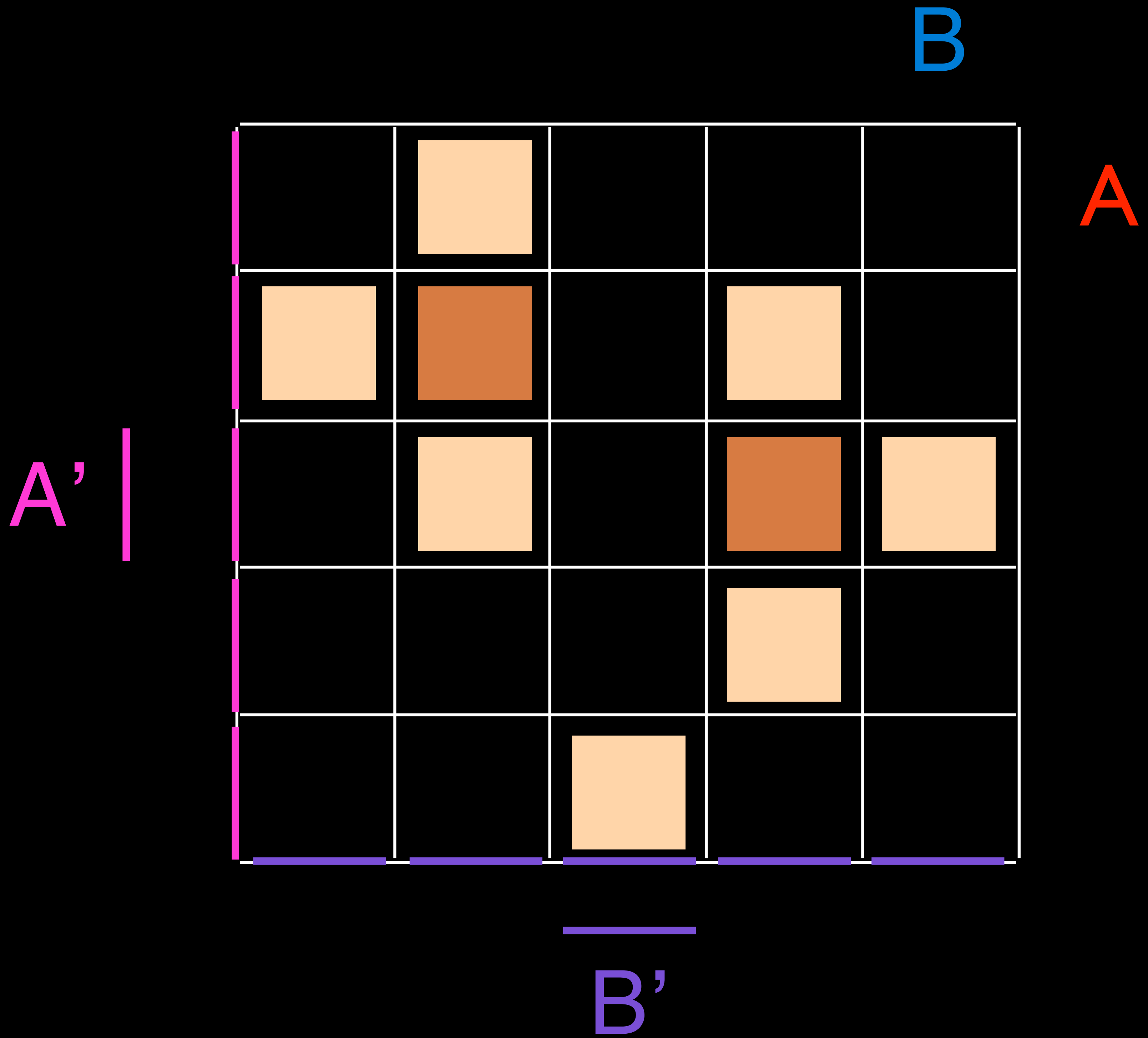
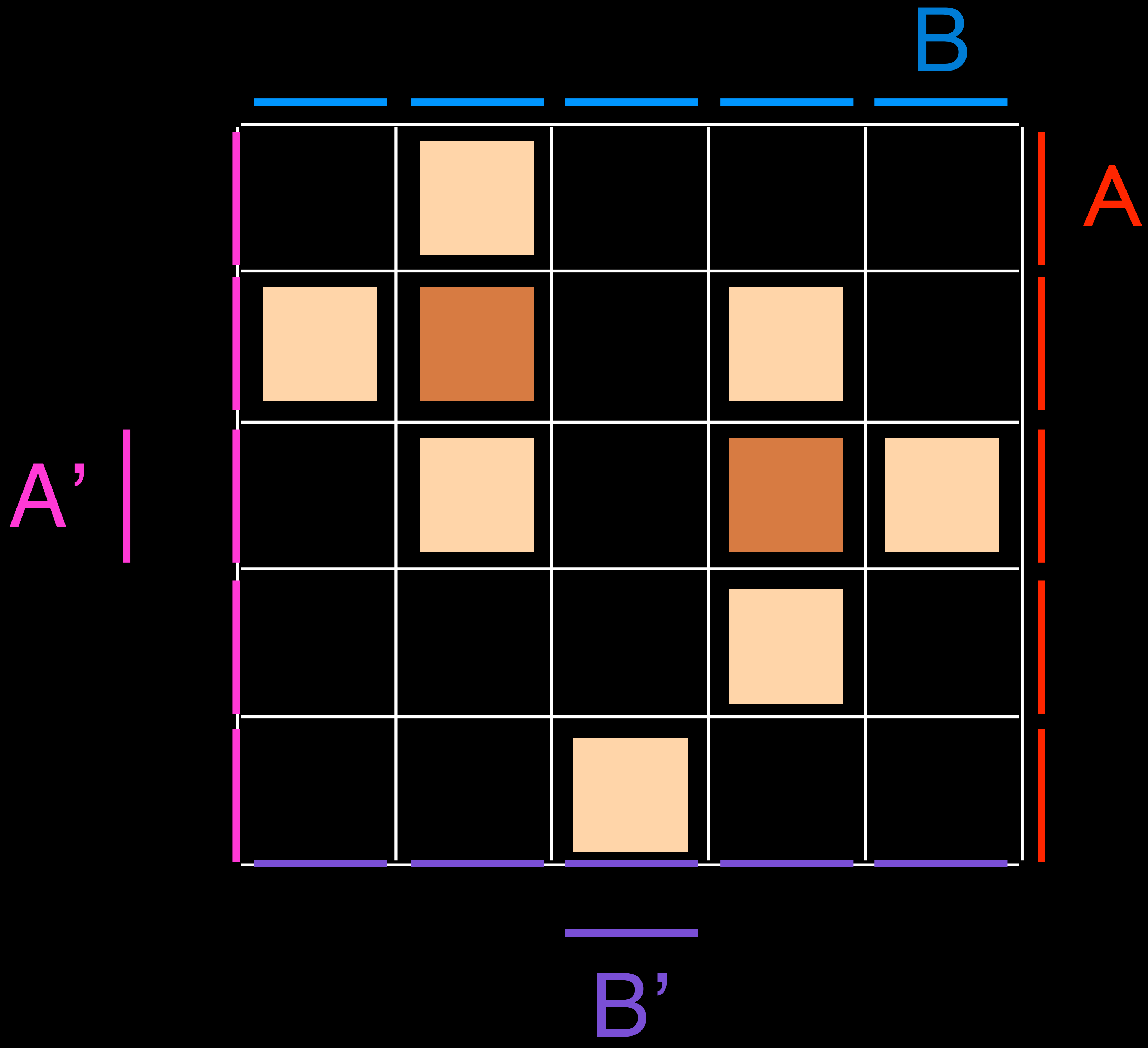
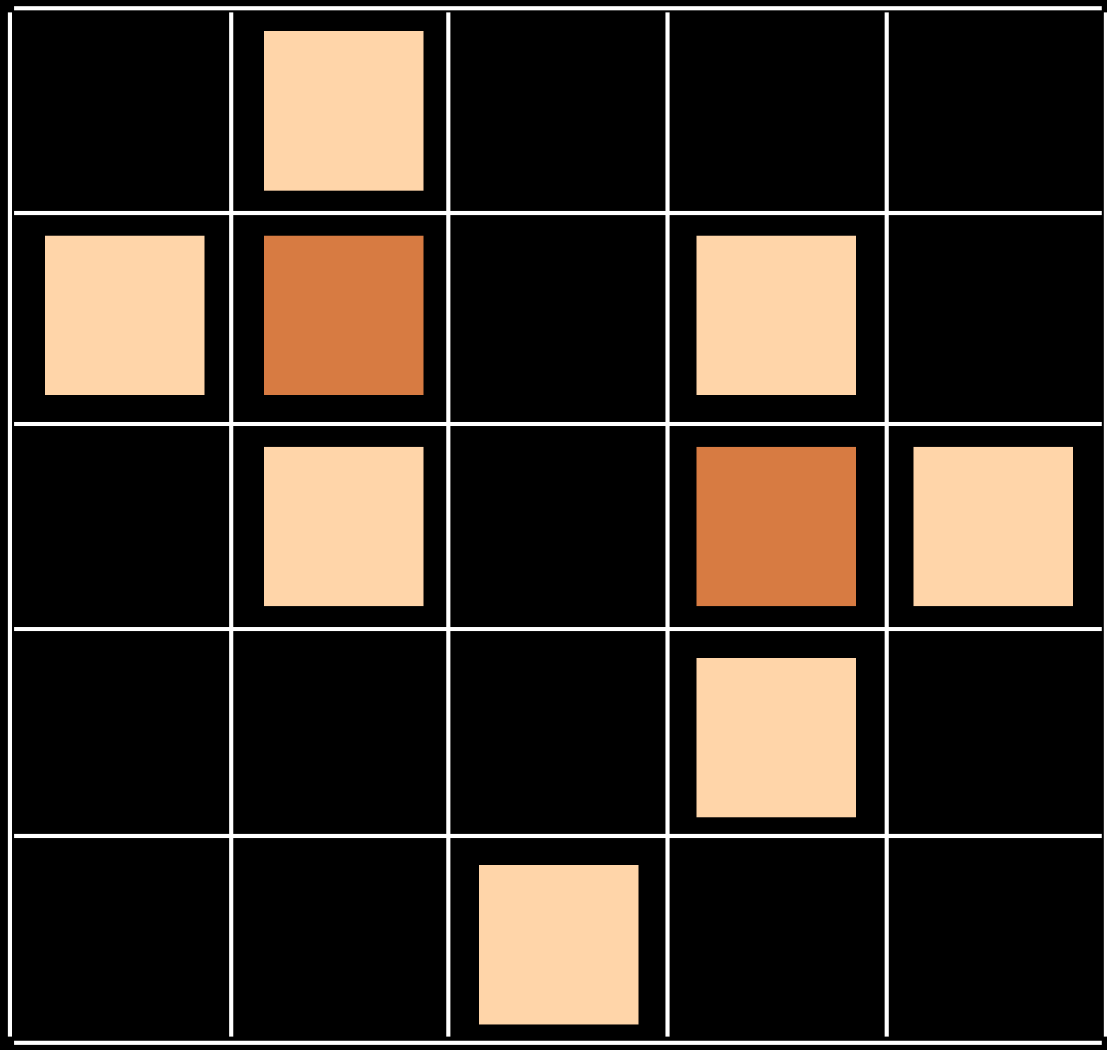
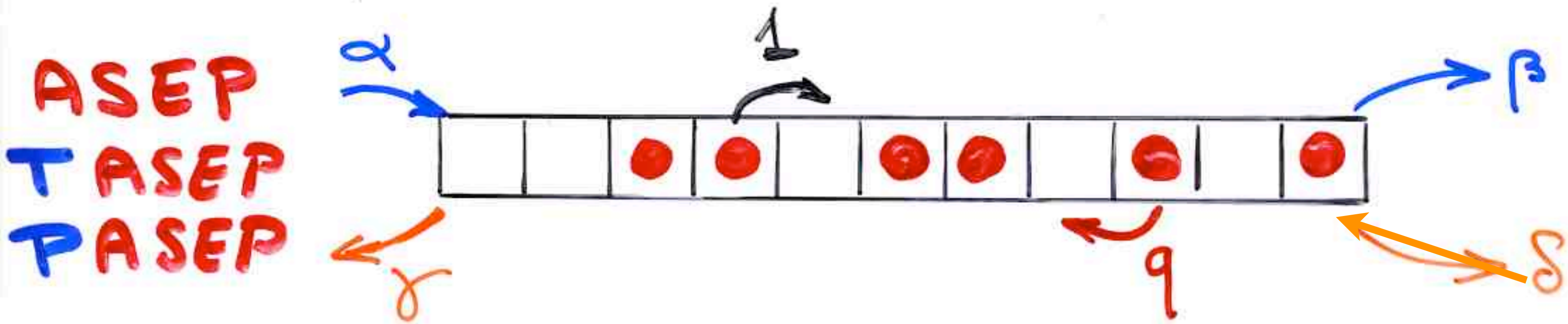

B
A
A'
B'

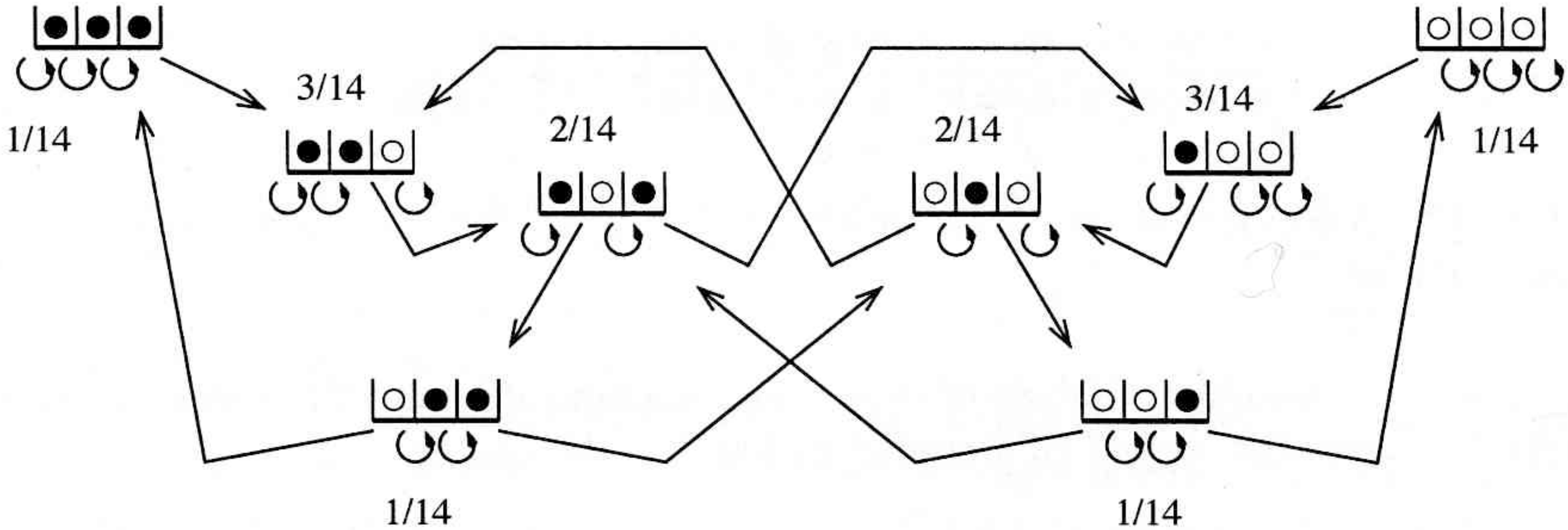

B
A
A'
B'


B
A
A'
B'



The PASEP

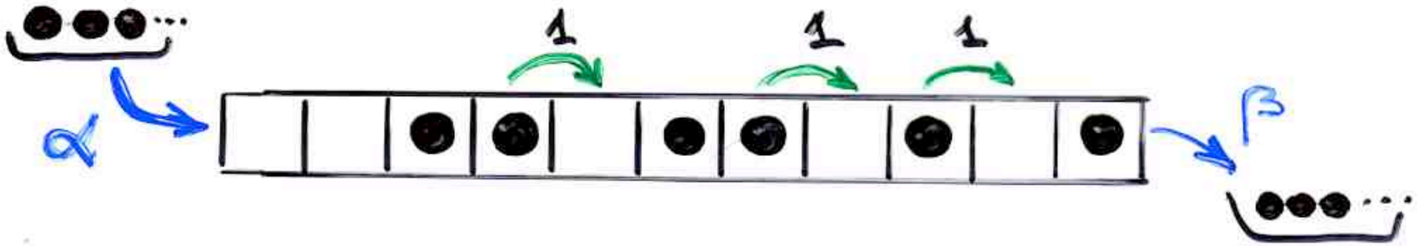

ASEP
TASEP
PASEP
α
1
β
γ
q
δ



stationary
probabilities

TASEP

"totally asymmetric exclusion process"



Combinatorics of the PASEP

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), XGV, (2007)

(P) ASEP

Brak, Corteel, Essam, Parviainen, Rechnitzer (2006)
Corteel, Williams (2006) (2008) (2009) XGV, (2008)
Corteel, Stanton, Stanley, Williams (2010)

Derrida, ...
Mallick, ... Golinelli, Mallick (2006)

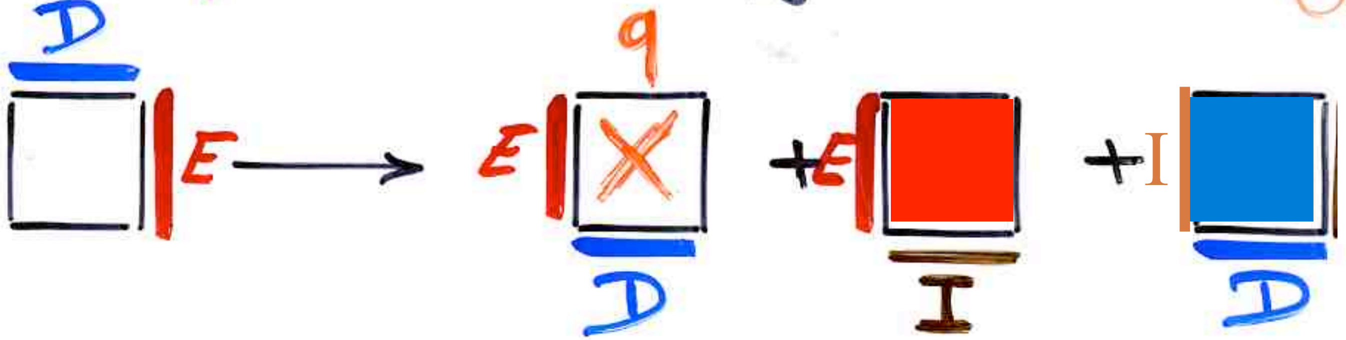
The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier (1993)

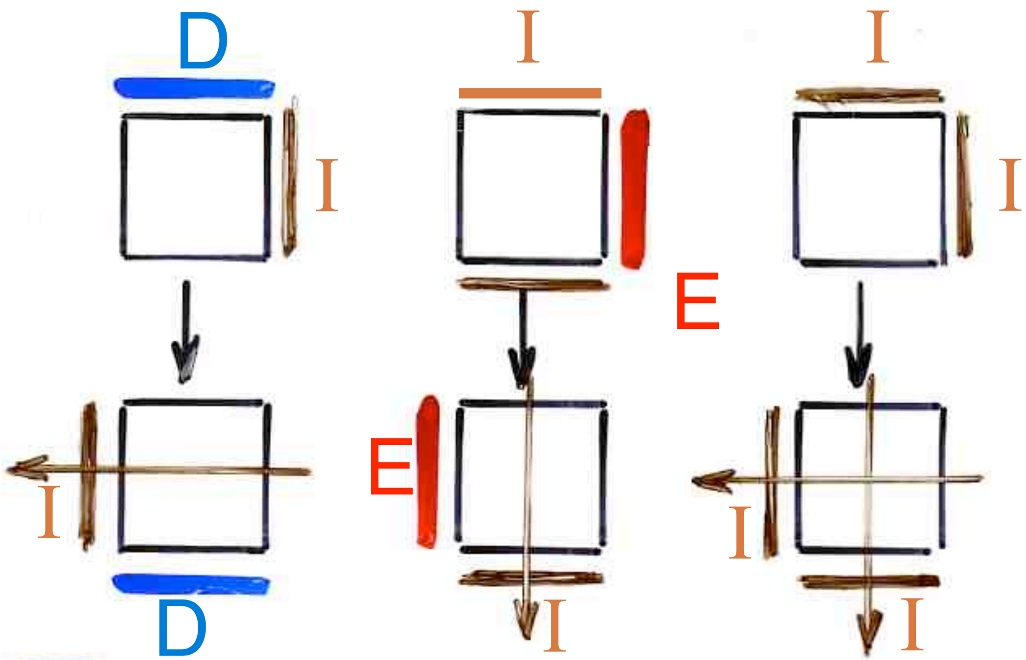
The PASEP algebra

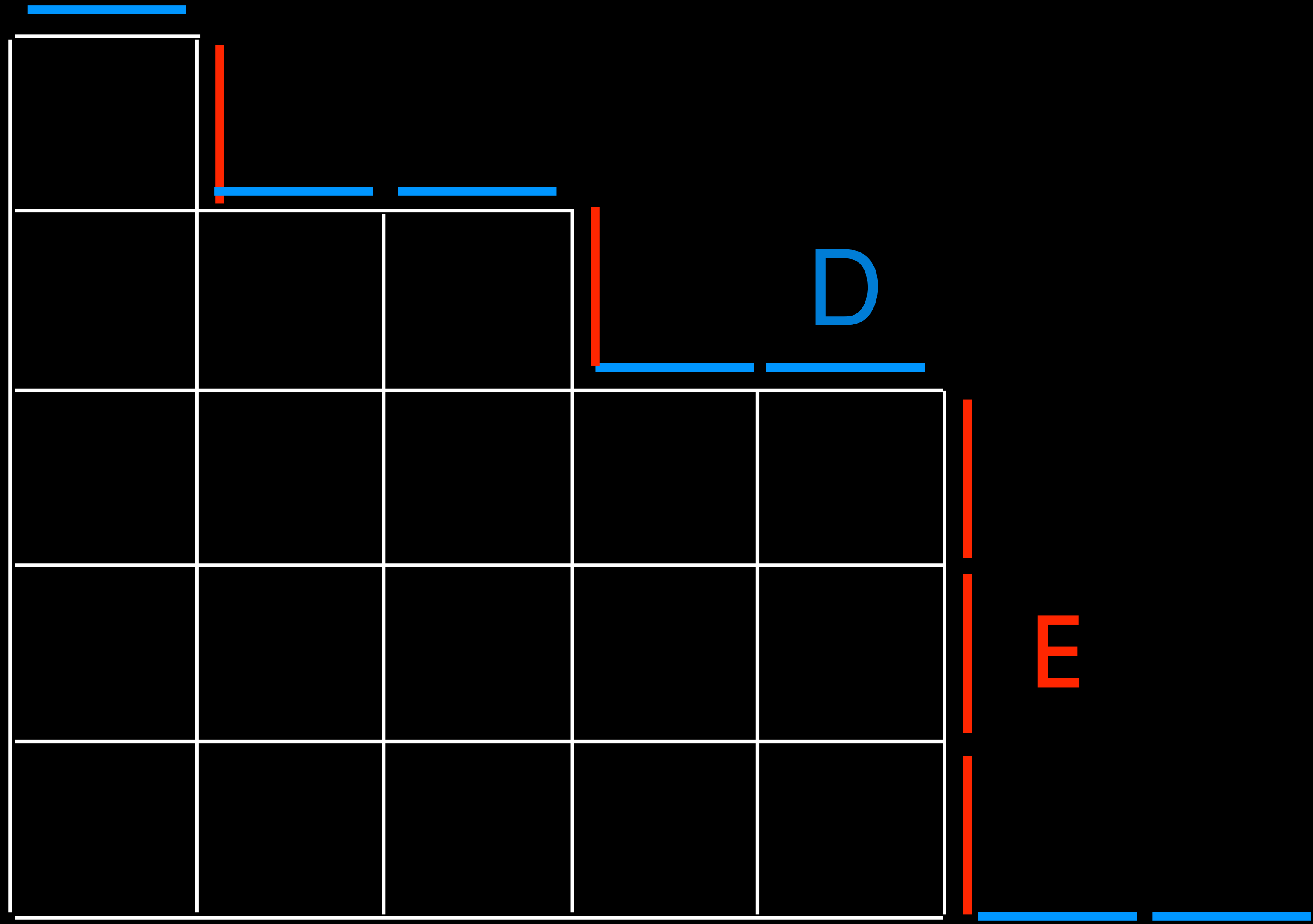
$$DE = qED + E + D$$

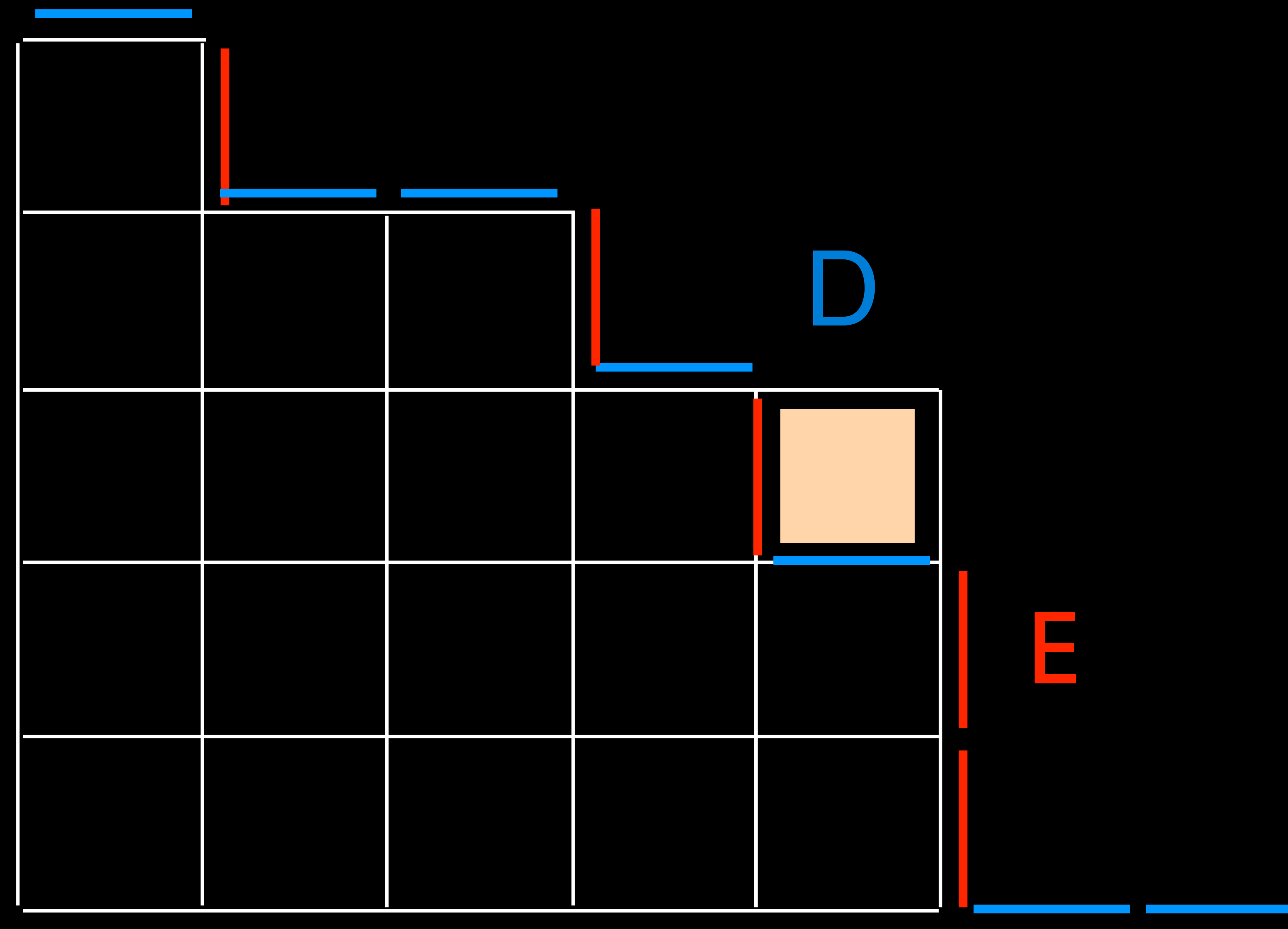
Proof: "planarization" of the rewriting rules



I identity




D
E


D
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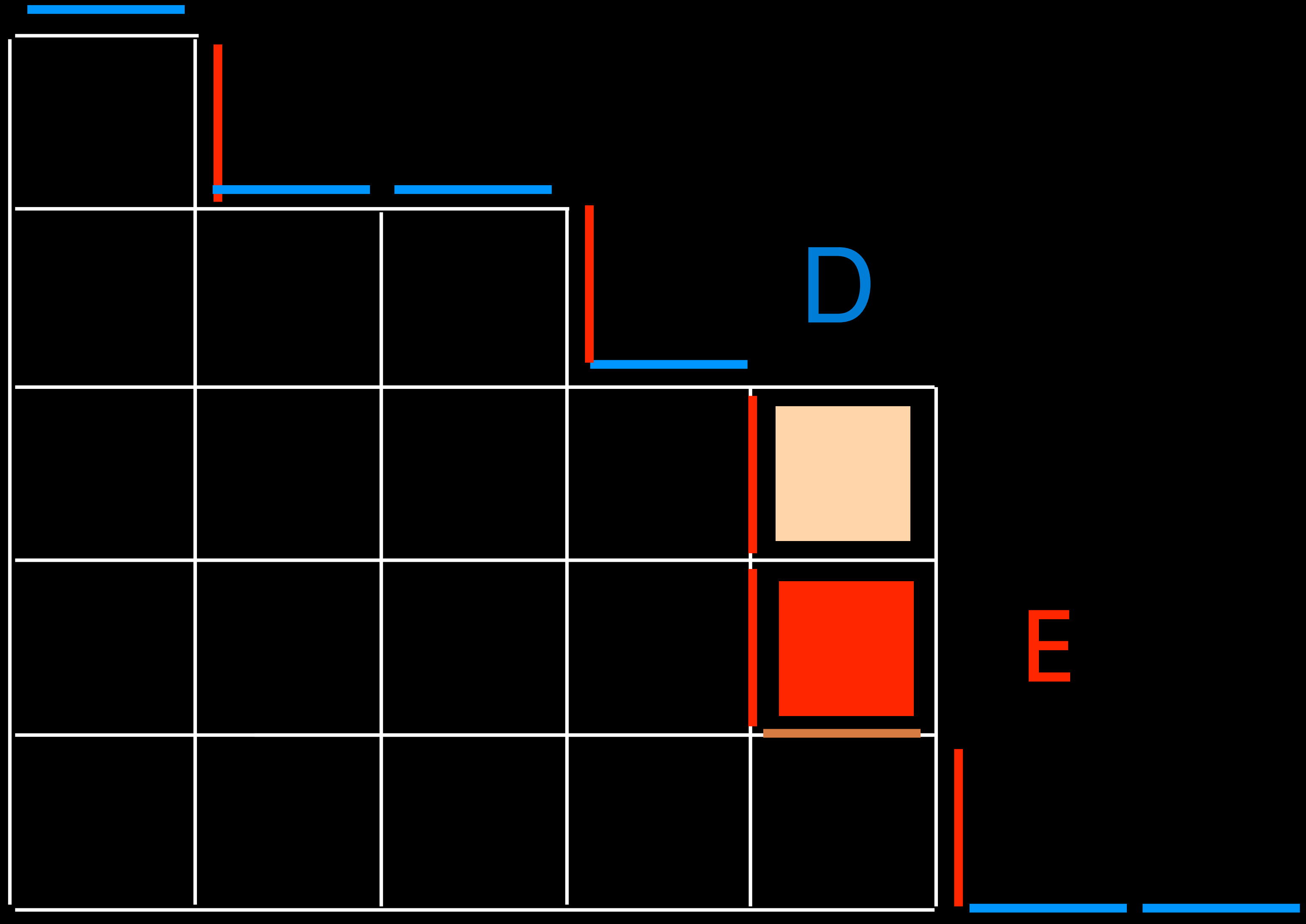

D
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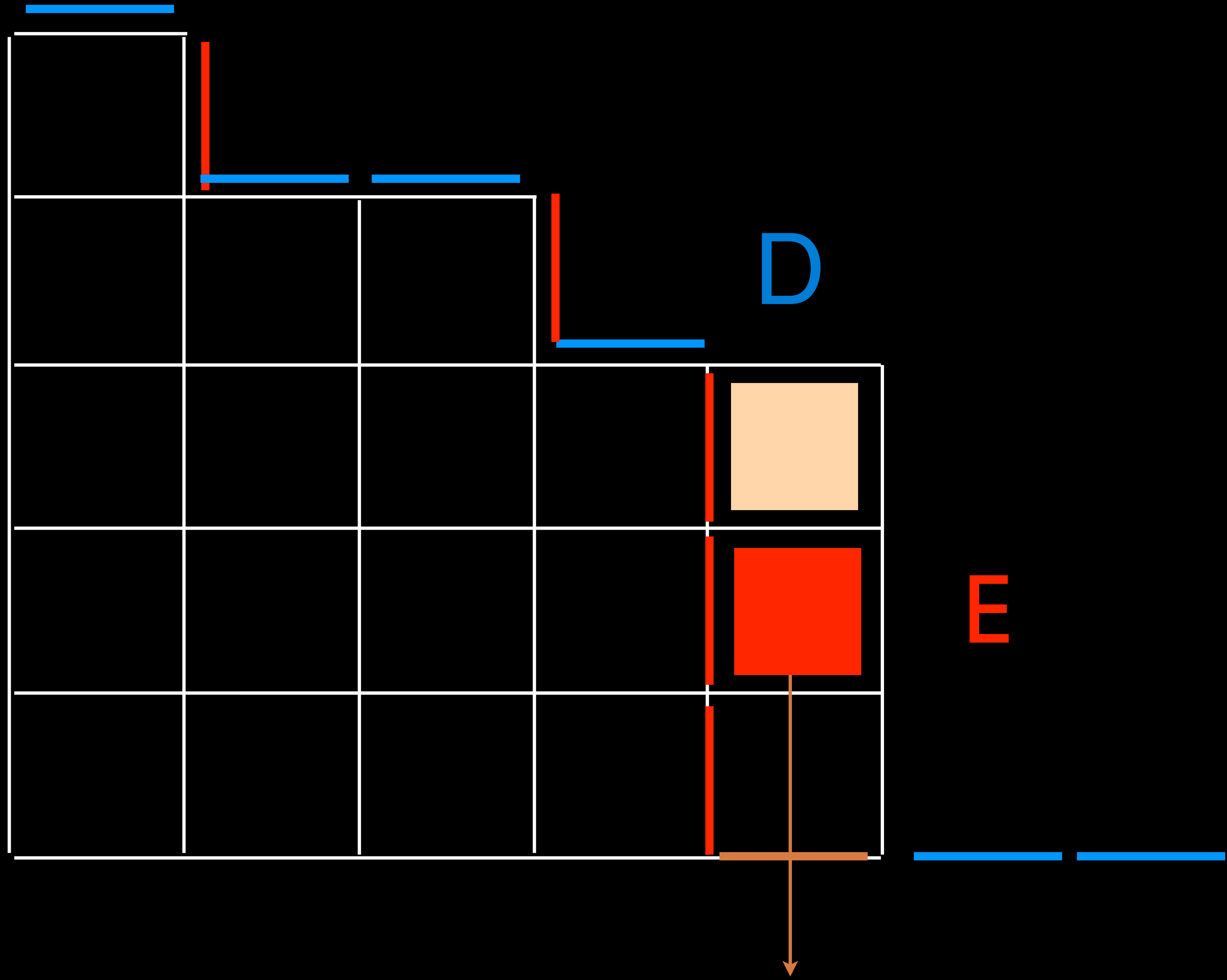
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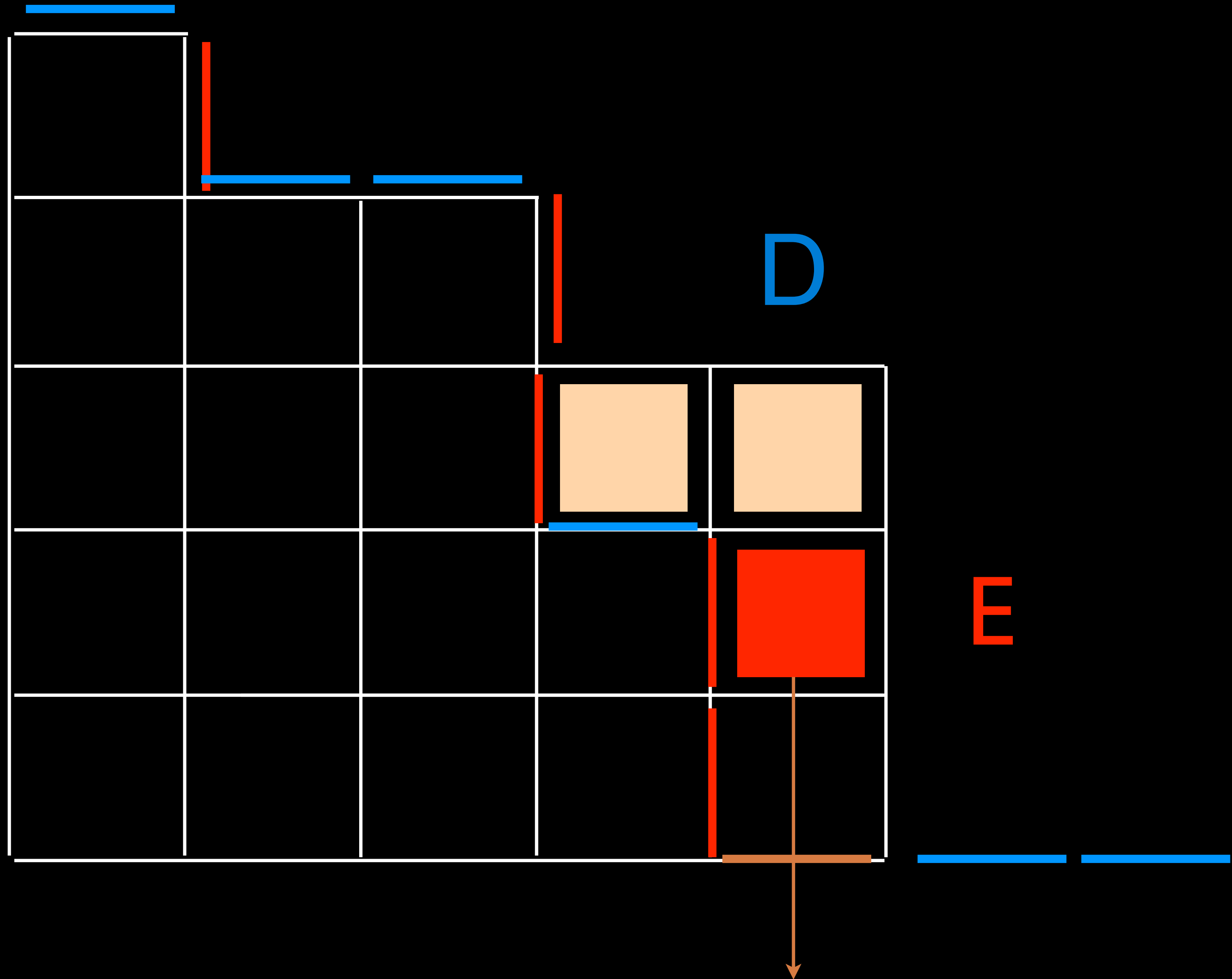
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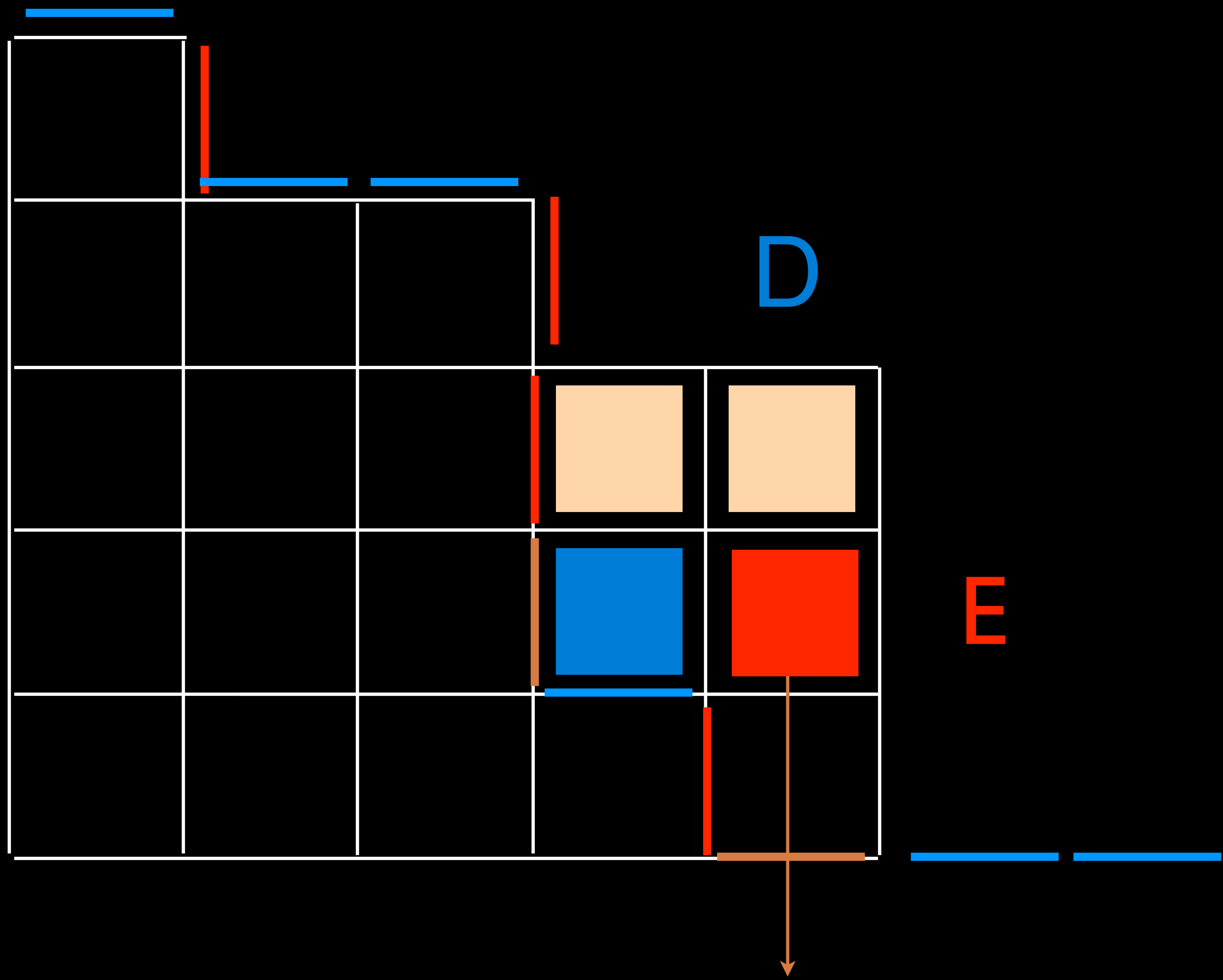
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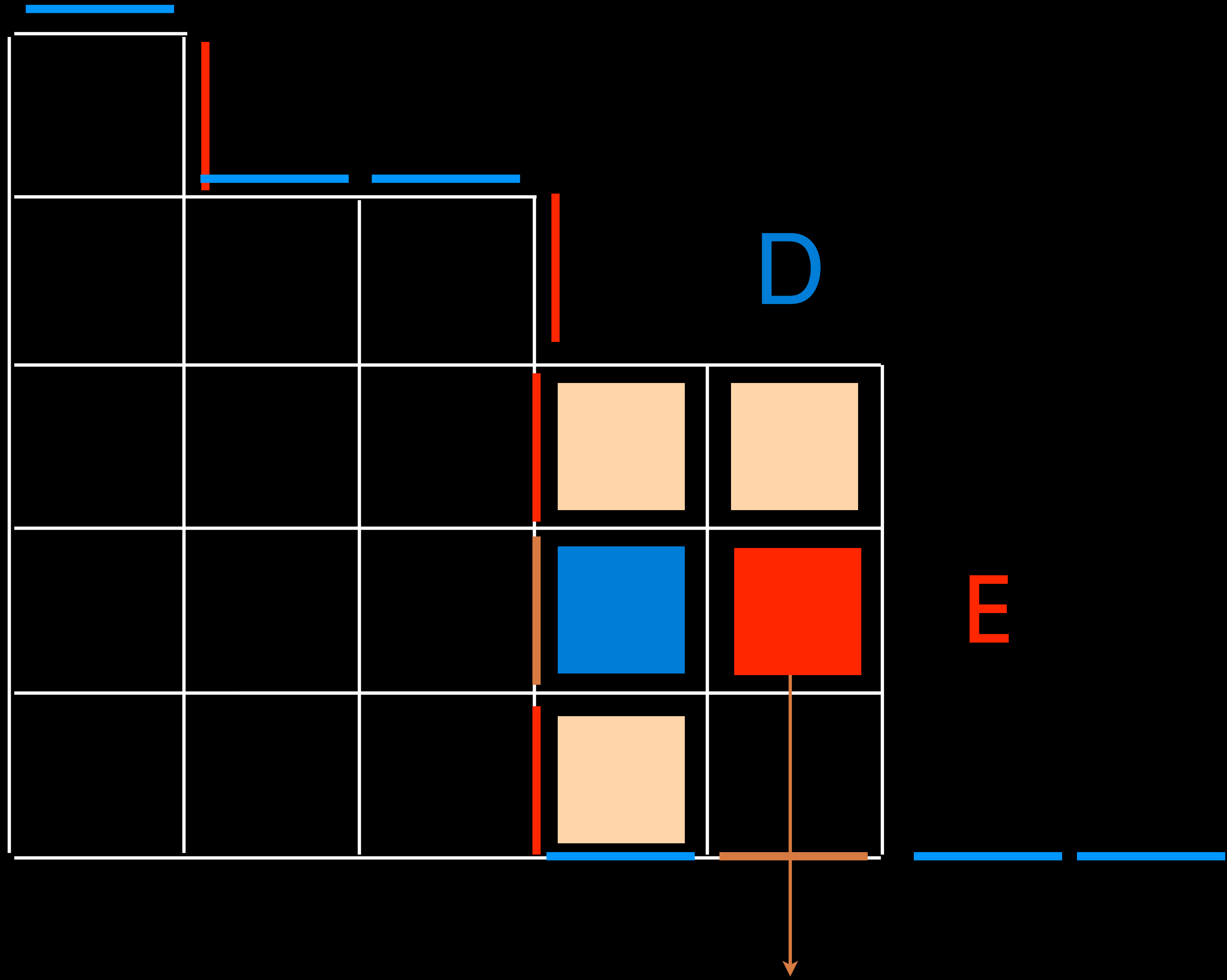
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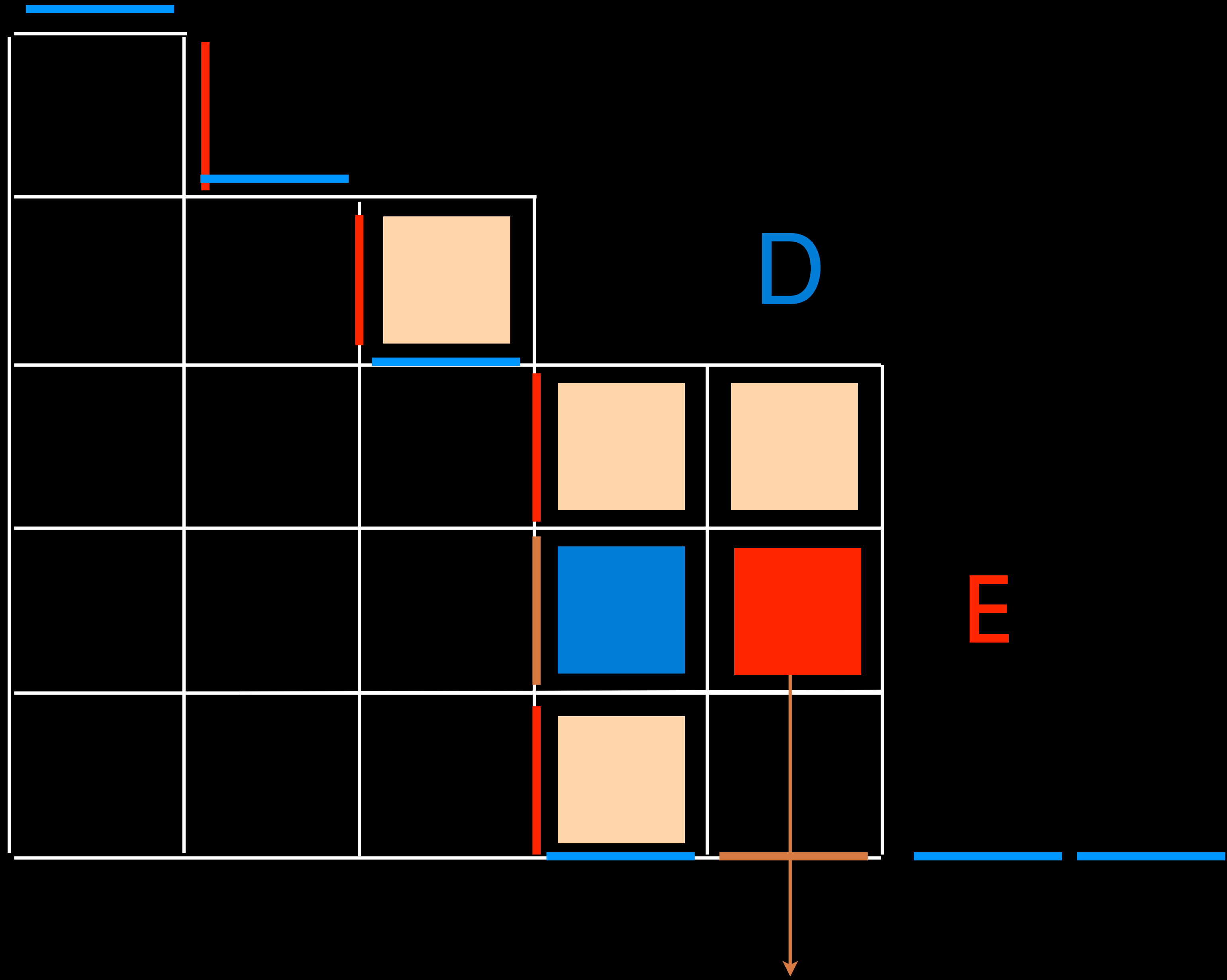
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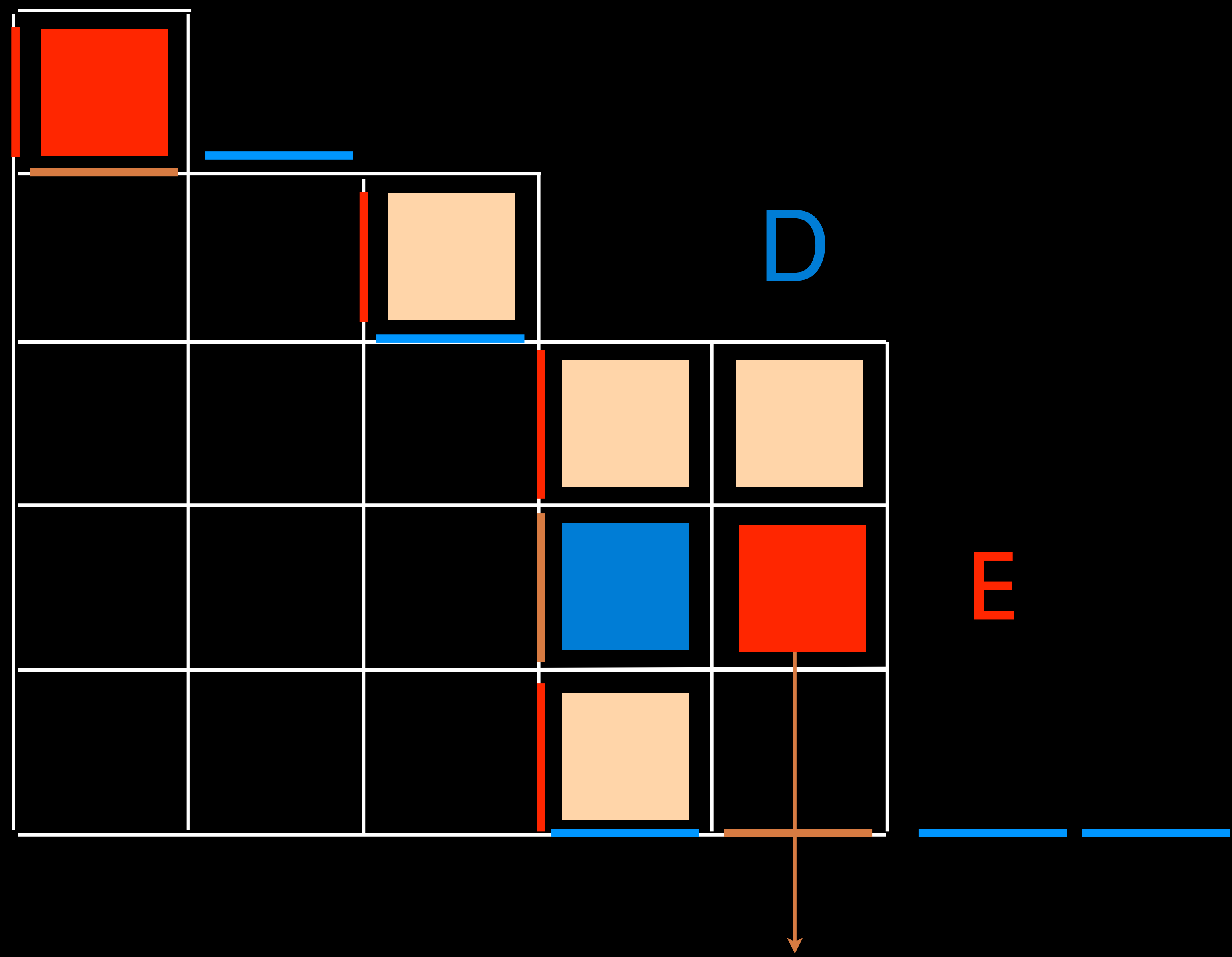

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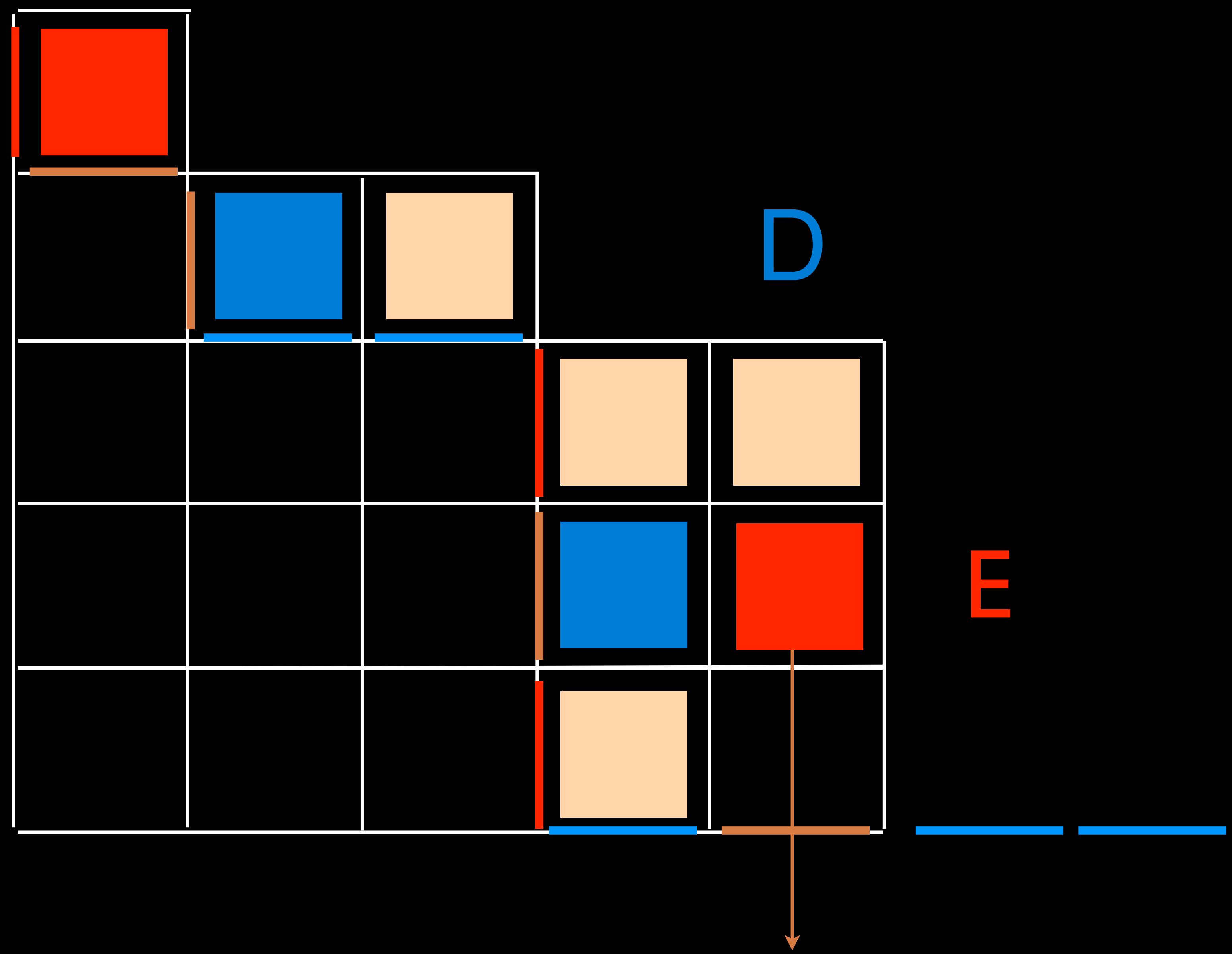
D
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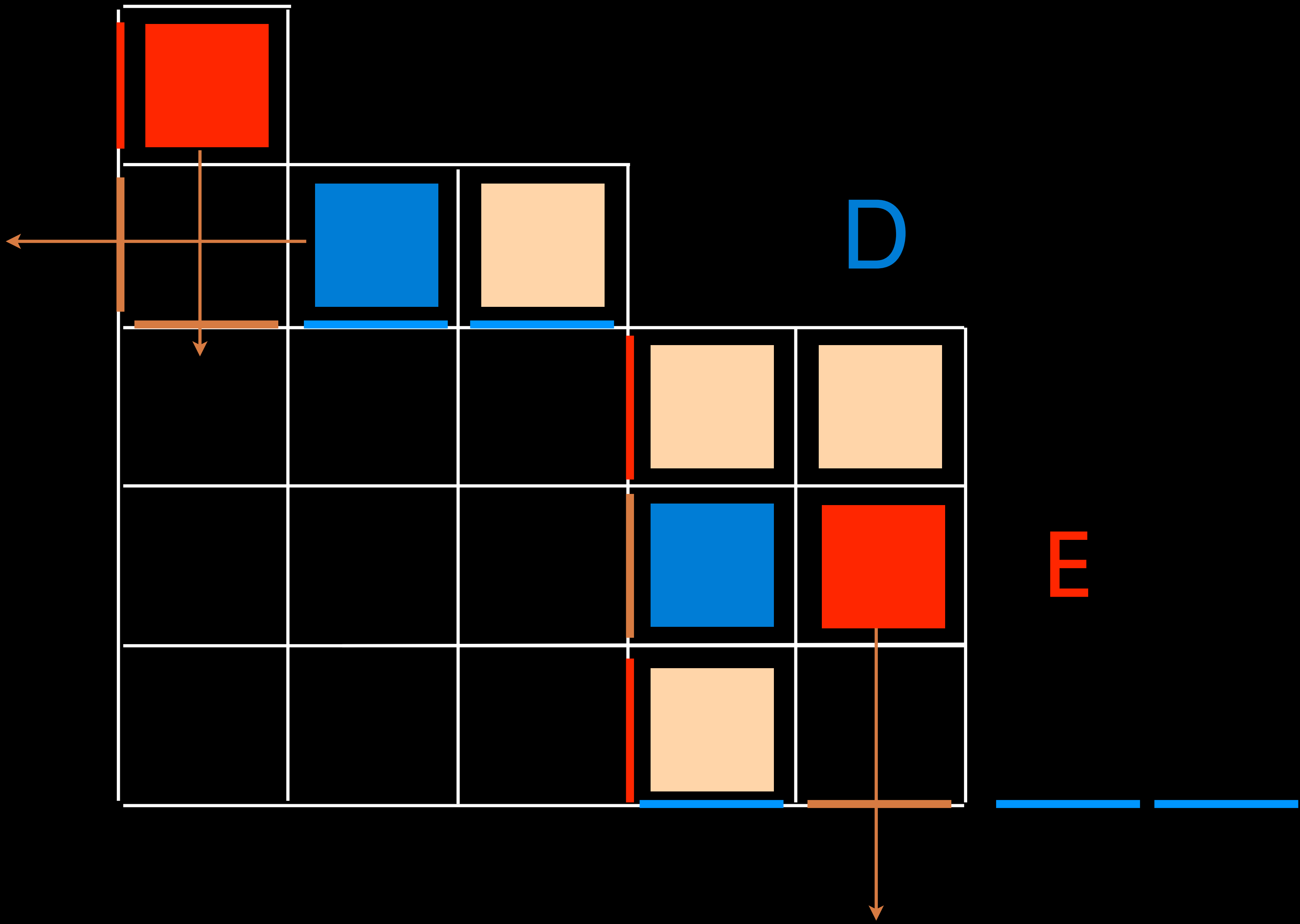
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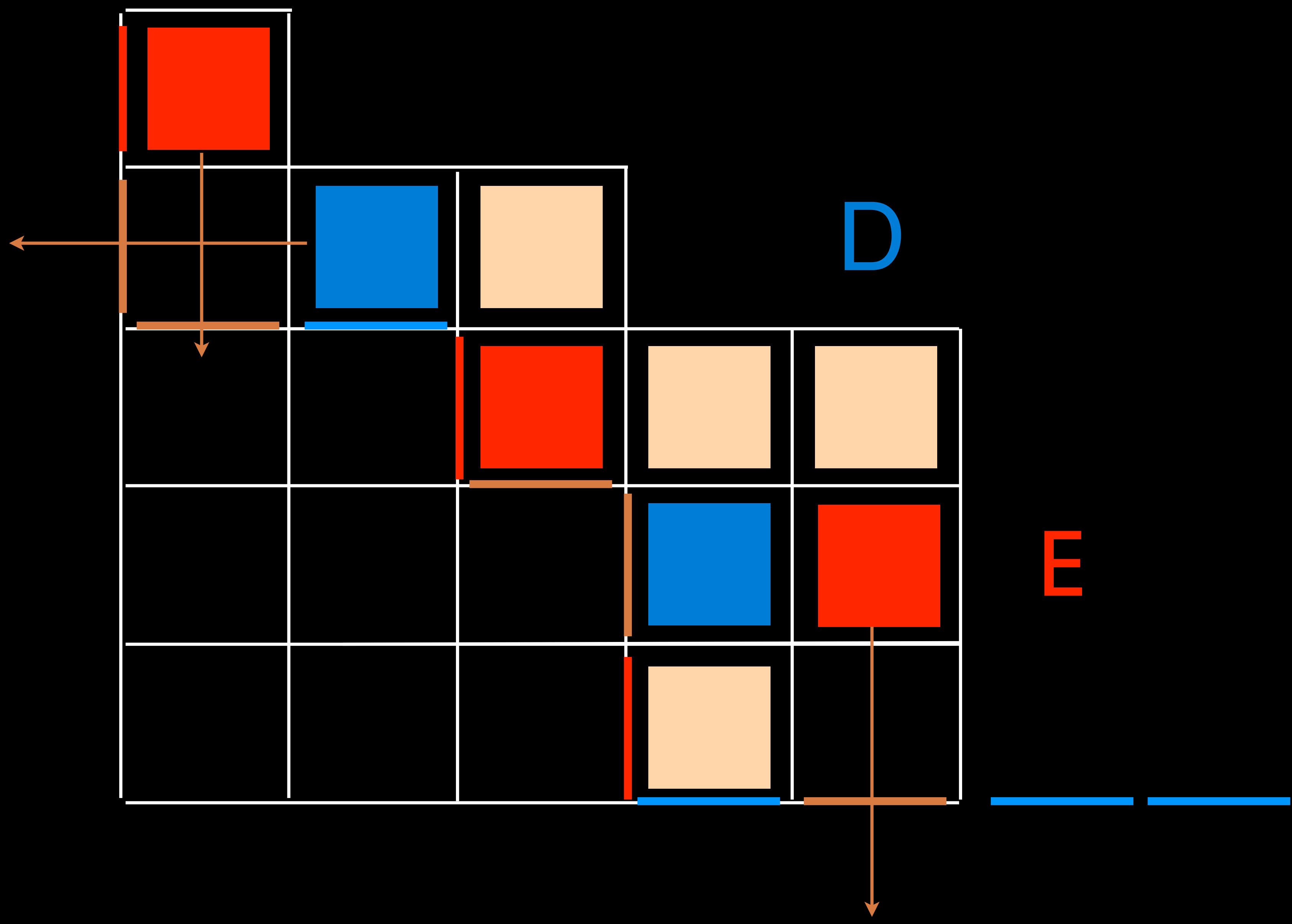
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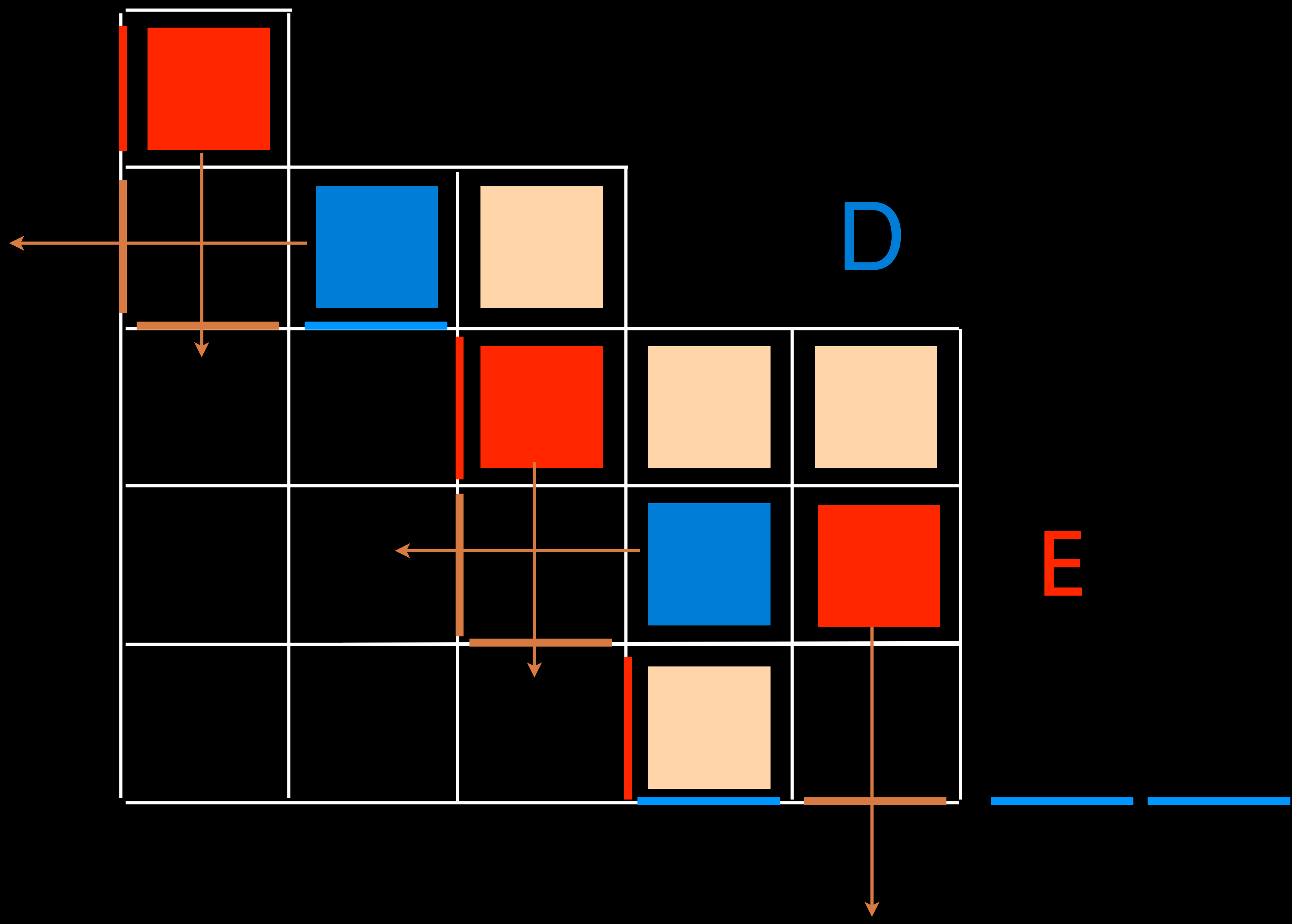
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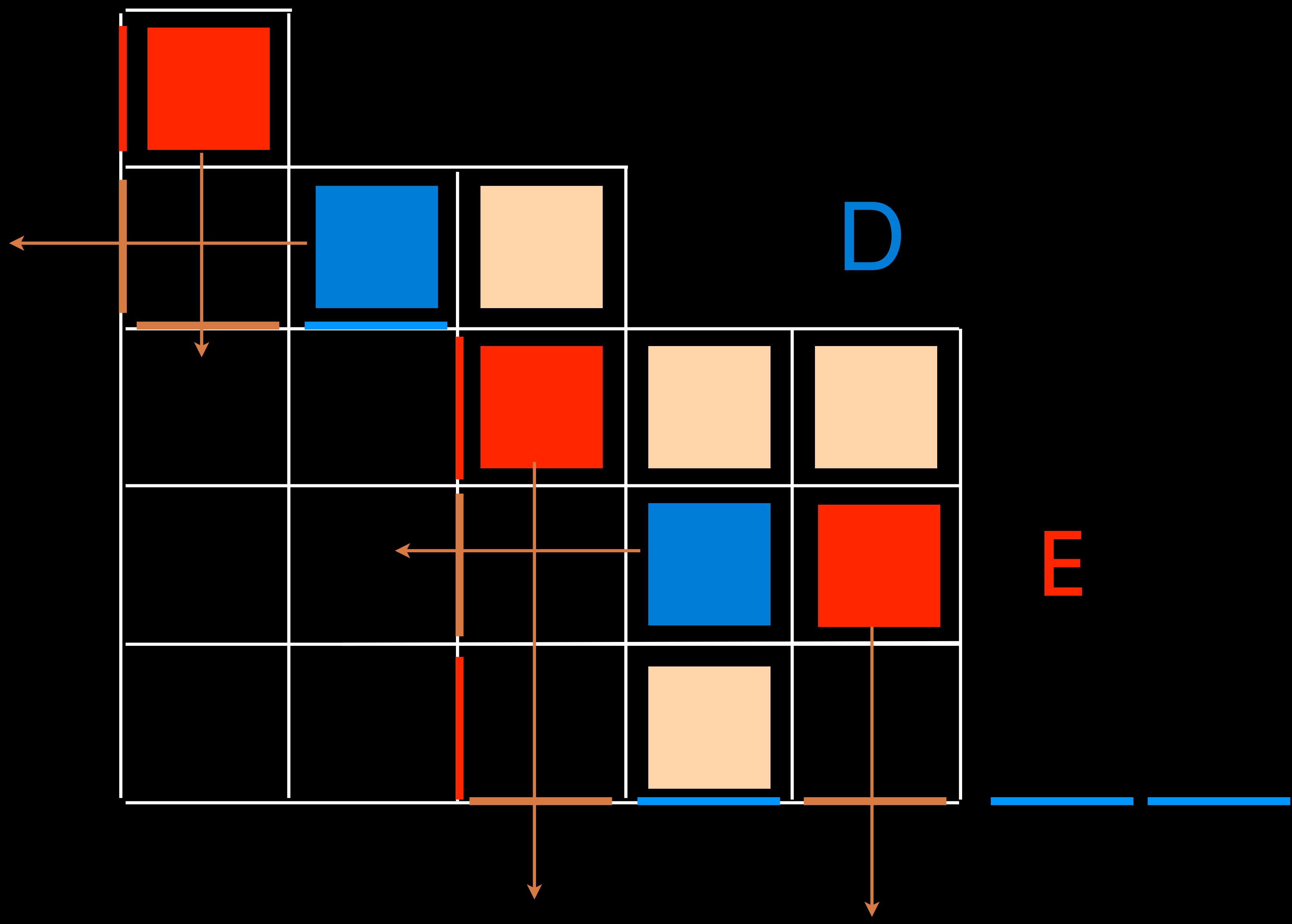
D
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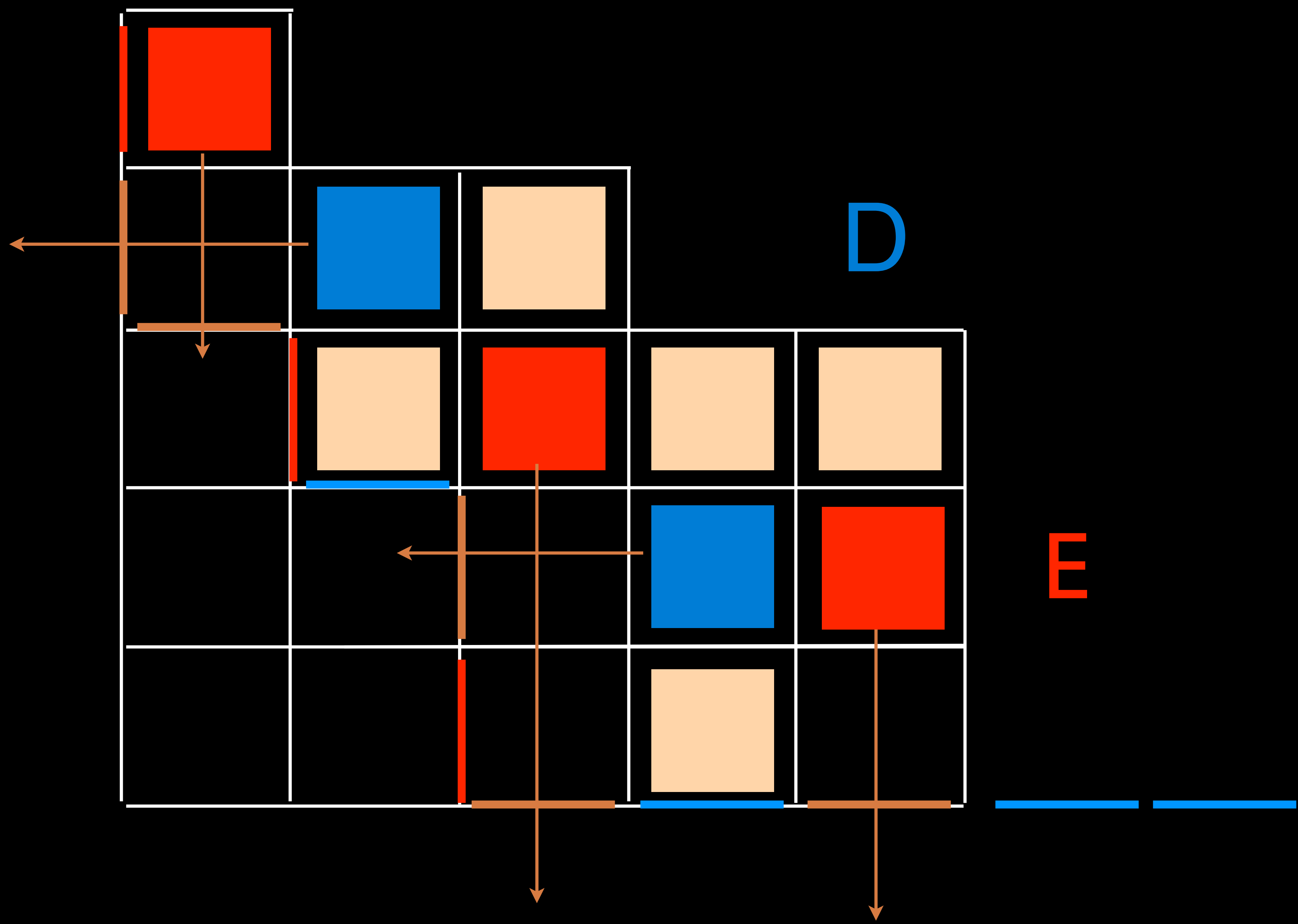
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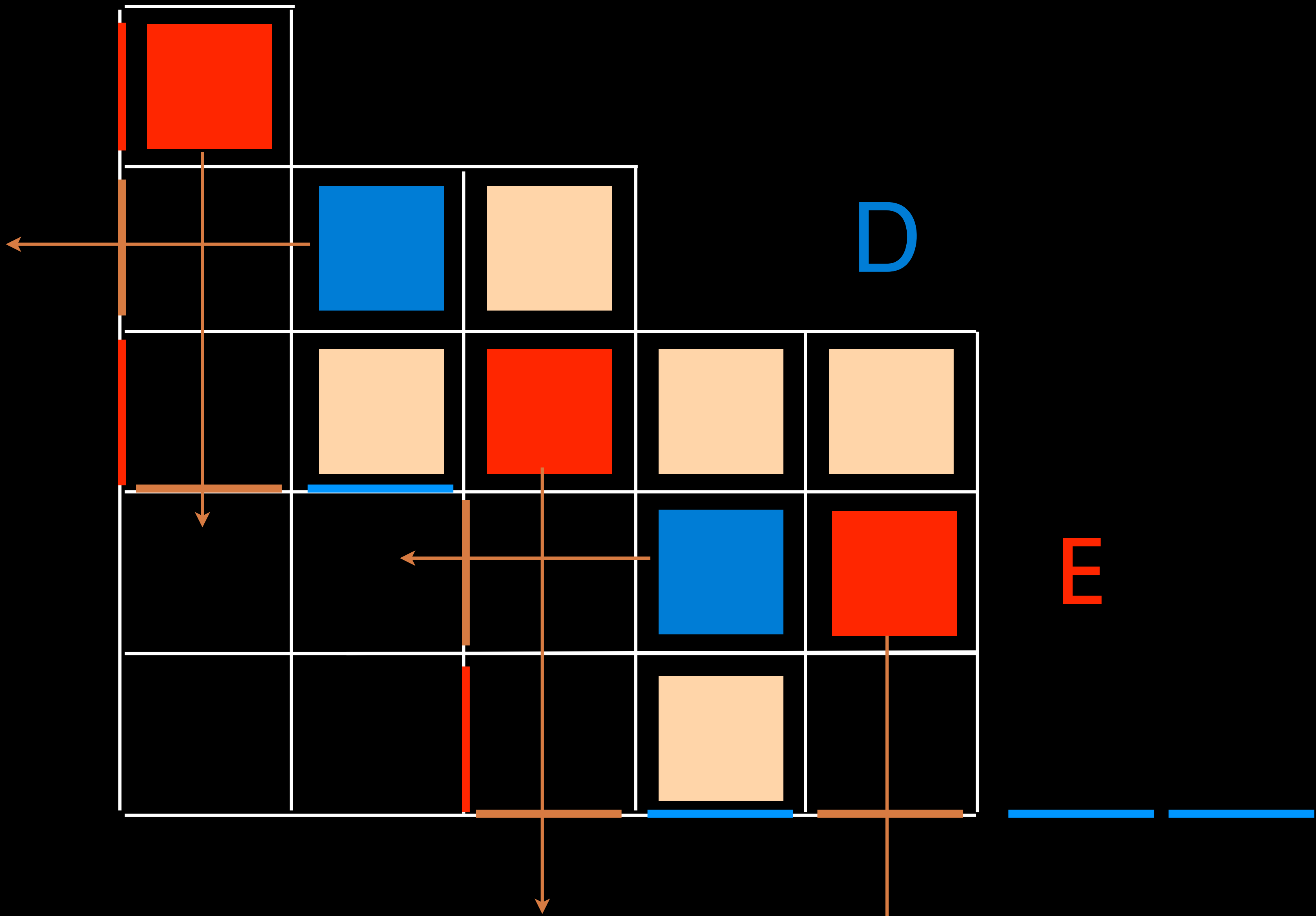
D
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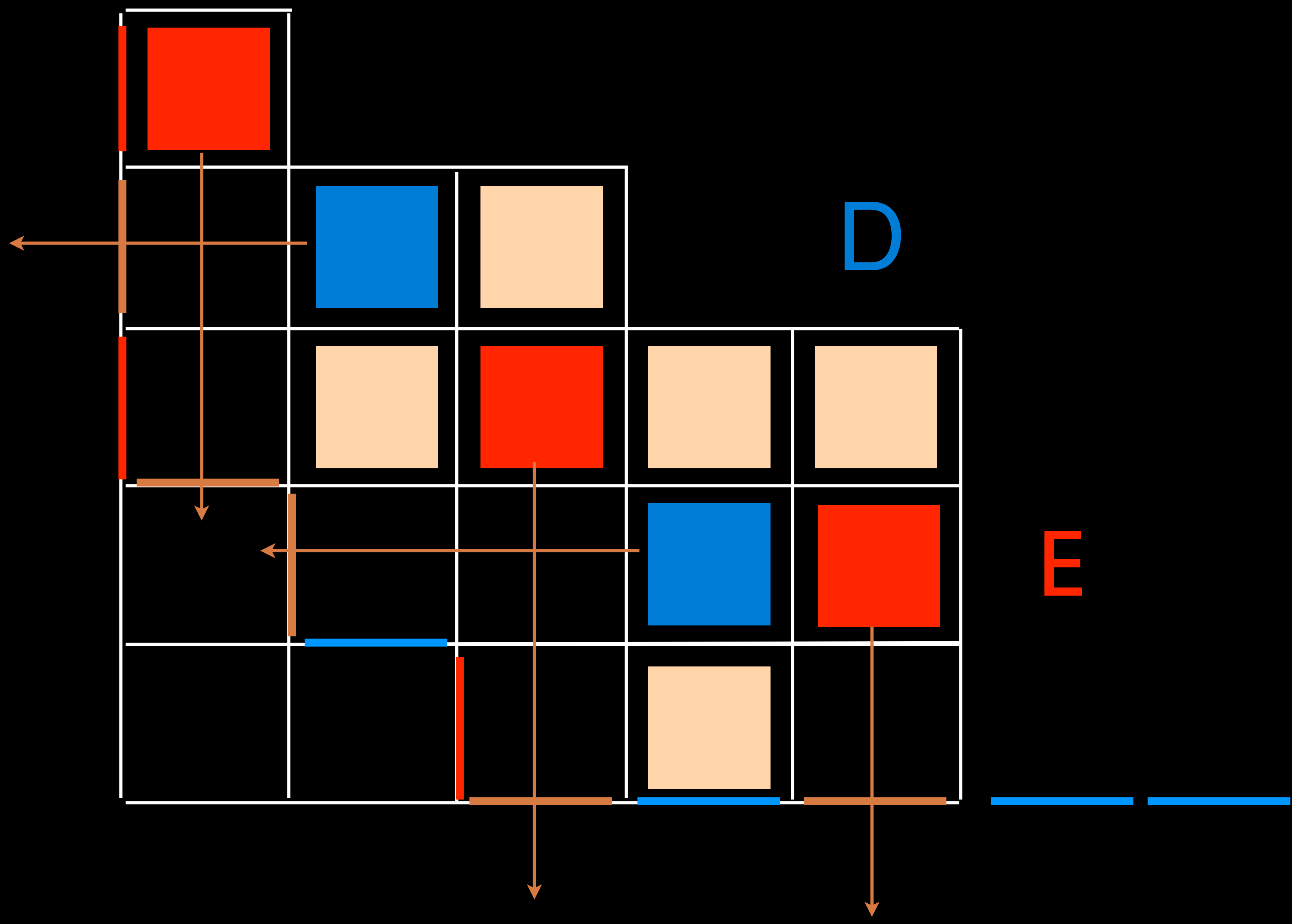
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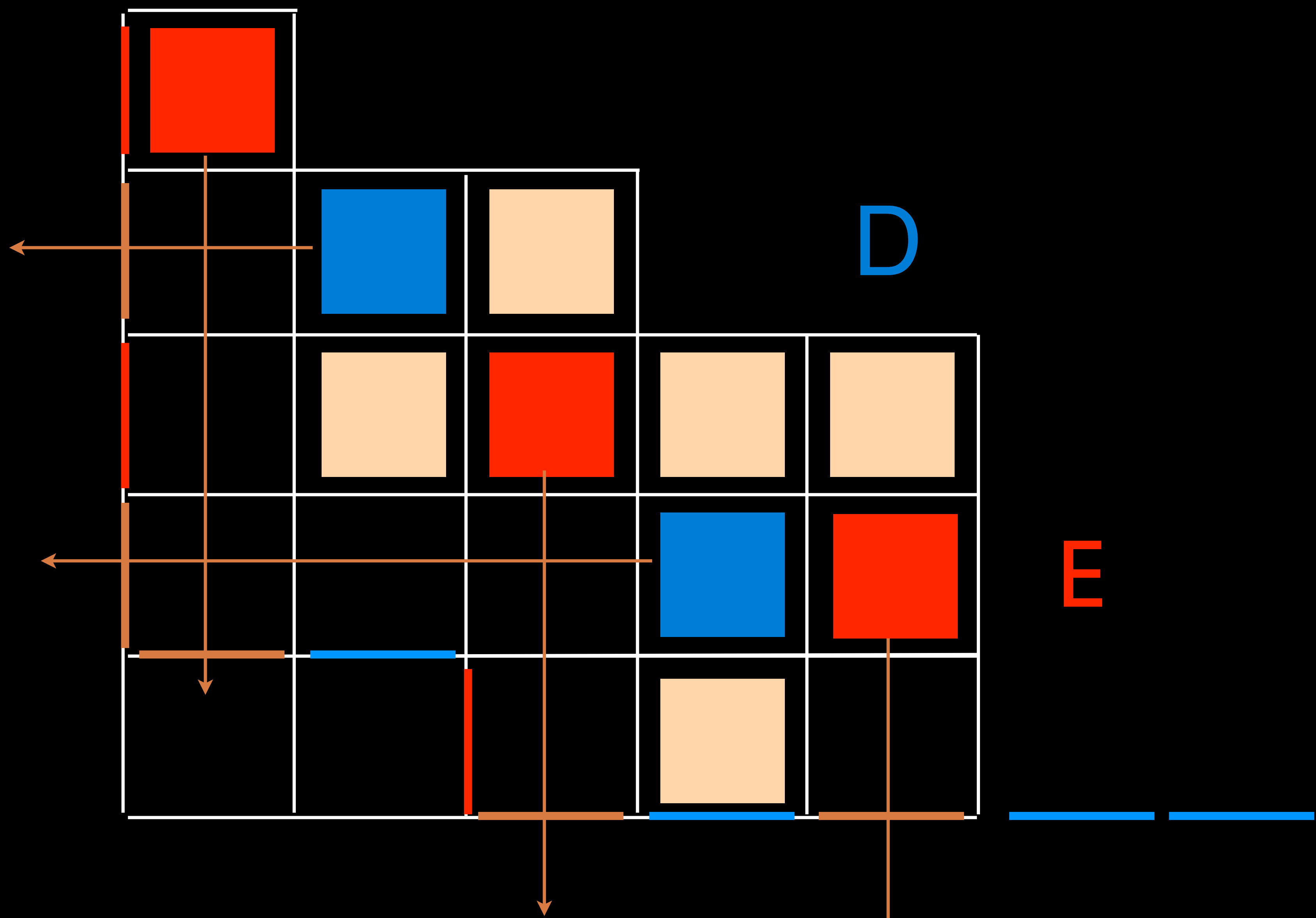
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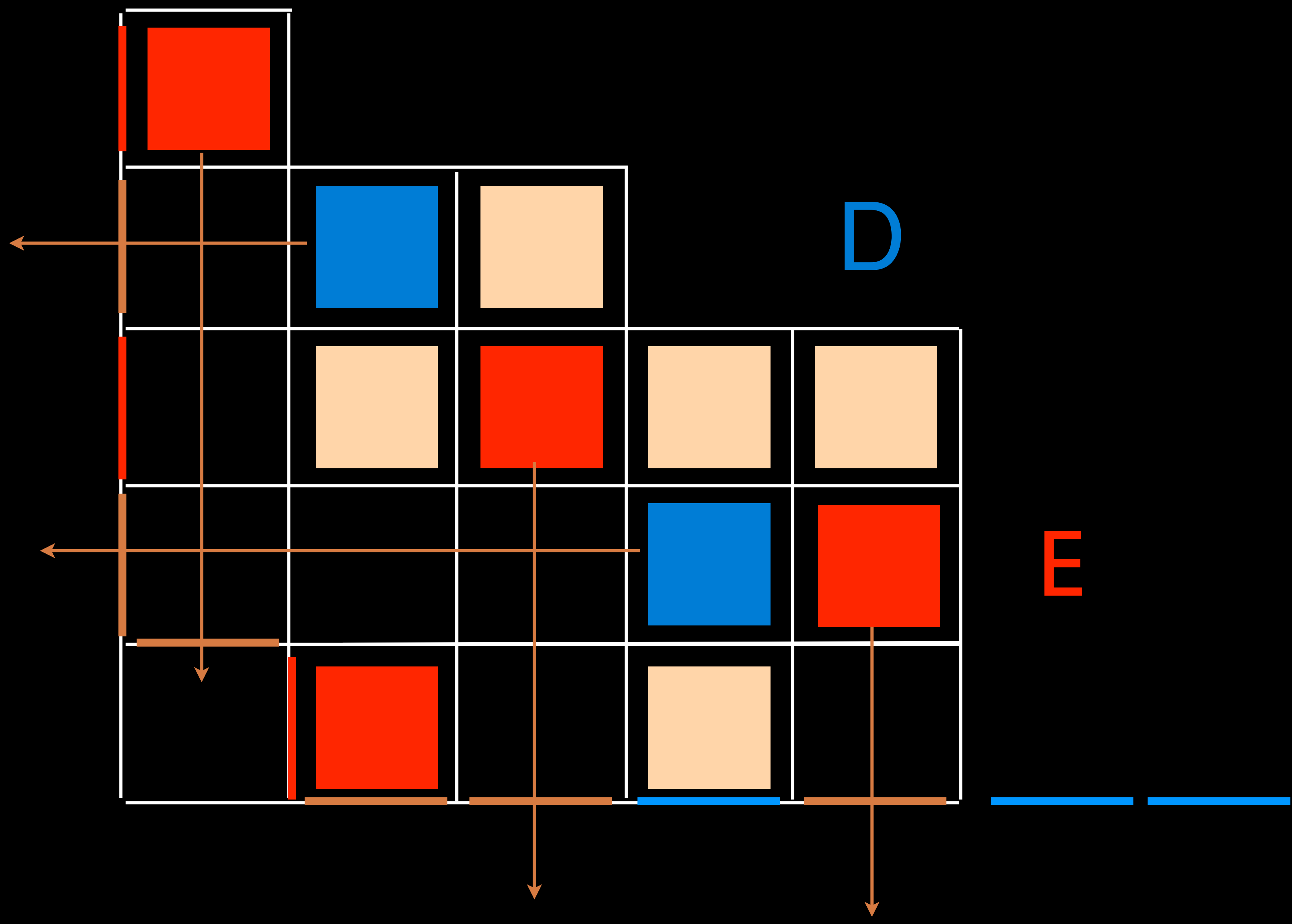
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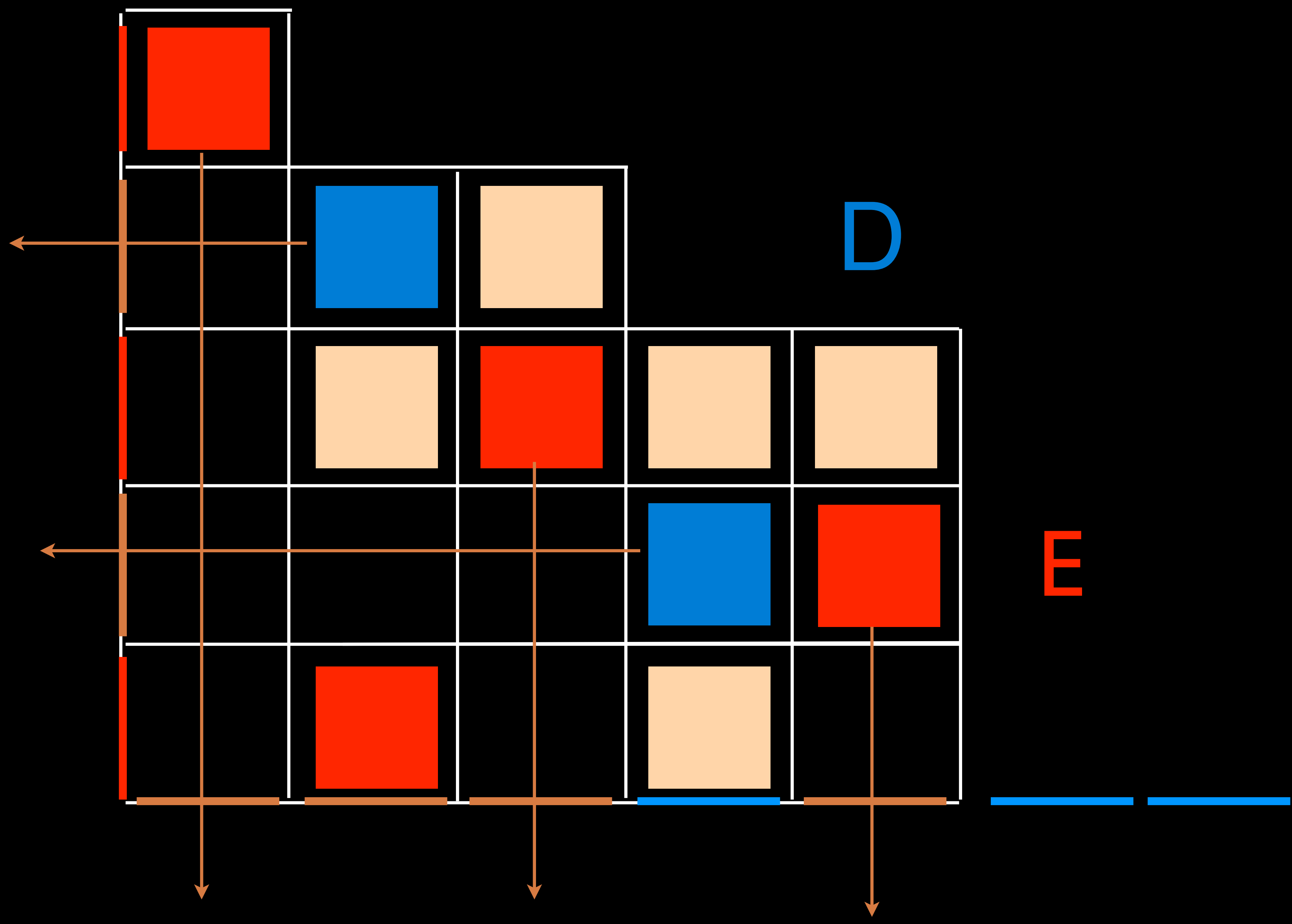
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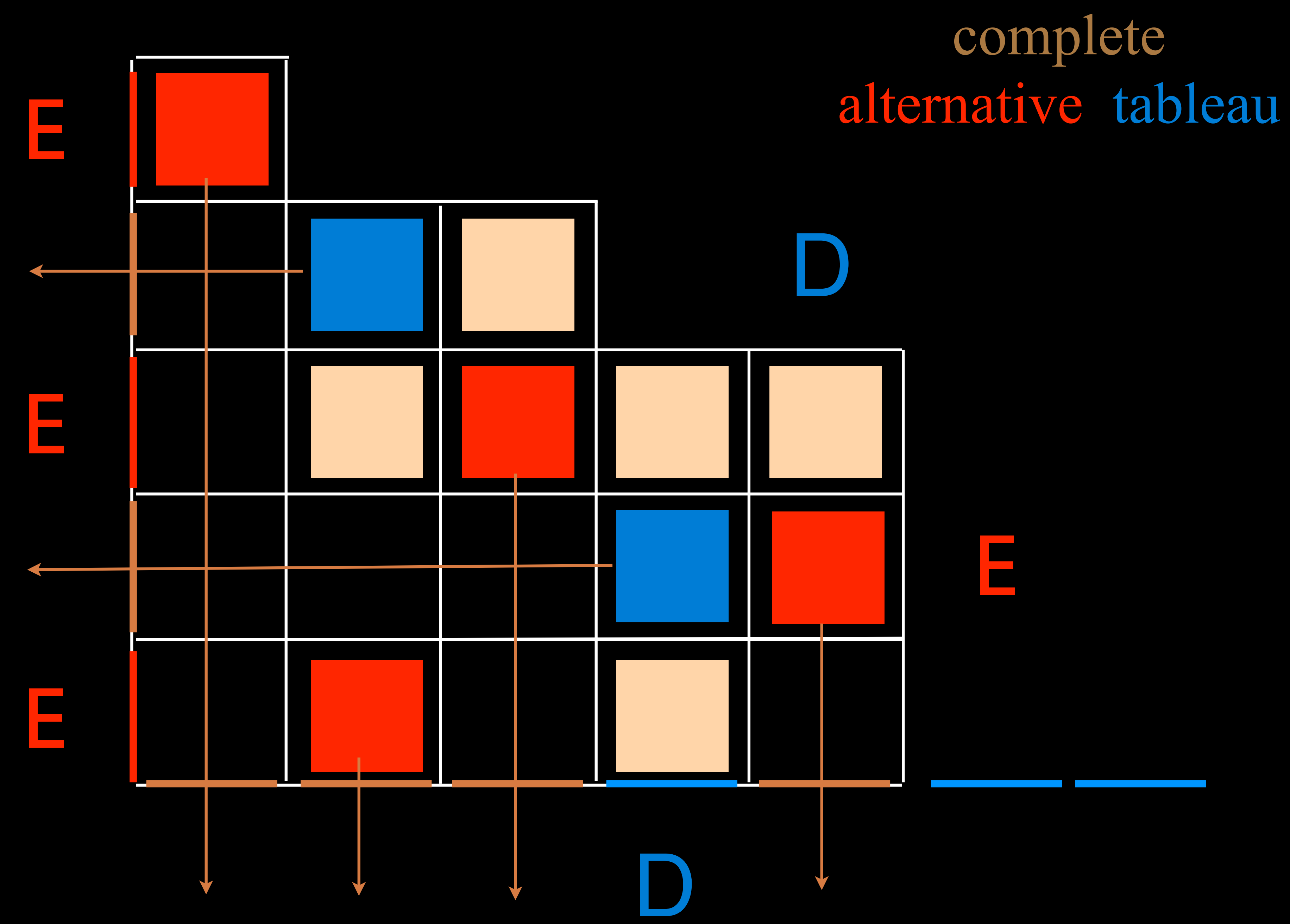
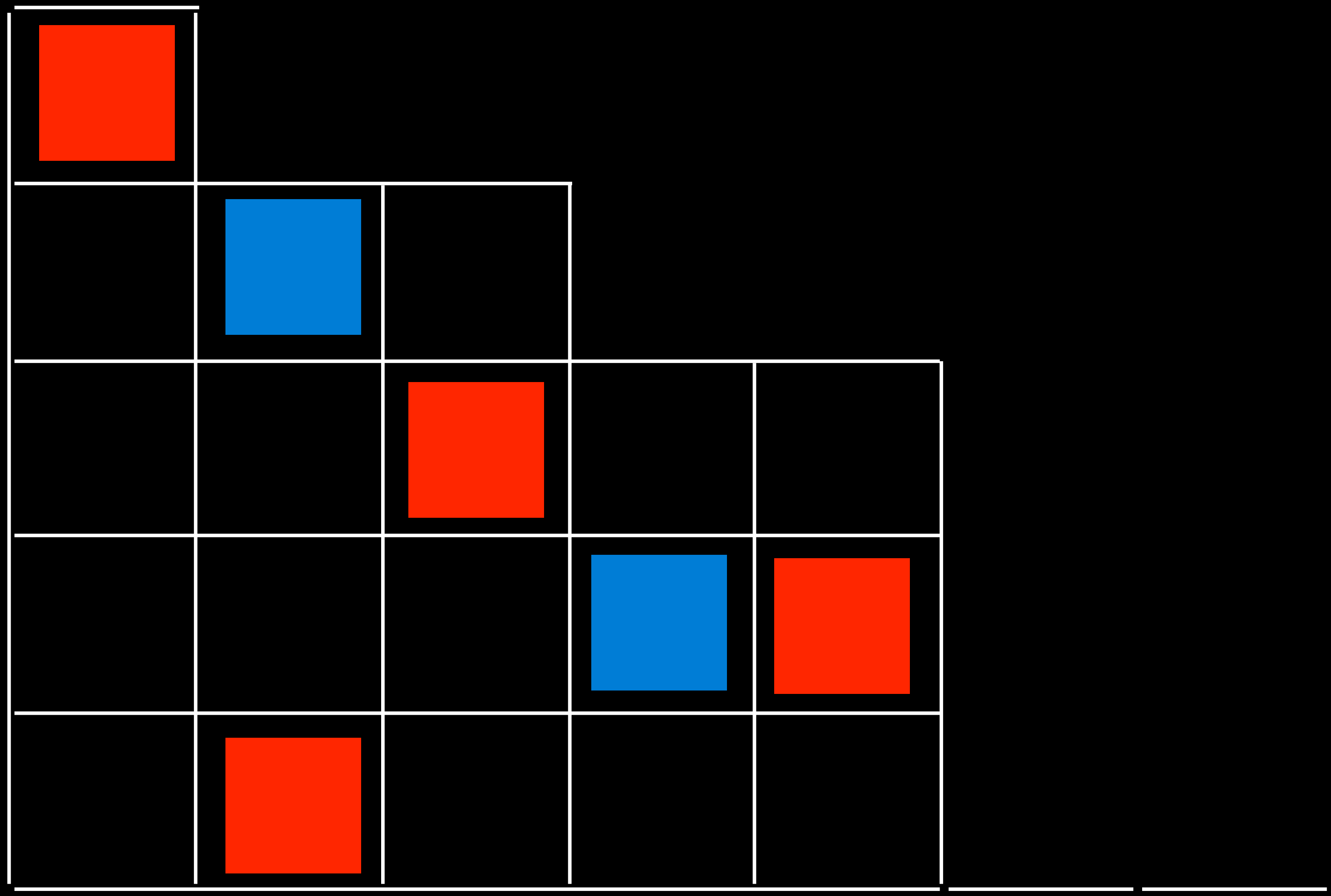

complete
alternative tableau
E
D
E
E
E
D

alternative tableau



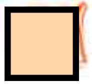
Def - profile of an alternative tableau

word $w \in \{E, D\}^*$

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

T alternative tableau with profile w

$k(T)$ = nb of 

$i(T)$ = nb of rows without blue cell

$j(T)$ = nb of columns without red cell

stationary probabilities

$$\mathbb{P}_n(\tau_1, \dots, \tau_n) = f_n(\tau_1, \dots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \dots, \tau_n)$$

partition function

Cor. The stationary probability associated to the state $\tau = (\tau_1, \ldots, \tau_n)$ (PASEP) is

$$\mathrm{prob}_{\tau}(q;\alpha,\beta) = \frac{1}{Z_n} \sum_{T} q^{L(T)} \alpha^{-f(T)} \beta^{-u(T)}$$

T: alternative tableaux, profile τ

$f(T)$ nb of rows without cell

$u(T)$ nb of columns without cell

$L(T)$ nb of cells

q-Laguerre

orthogonal polynomials

permutation tableau

S. Corteel, L. Williams
(2007) (2008) (2009)

permutation tableau

A. Postnikov (2001, ...)
totally nonnegative part of the Grassmannian

E. Steingrímsson, L. Williams (2005)

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

α, β, q $\qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$

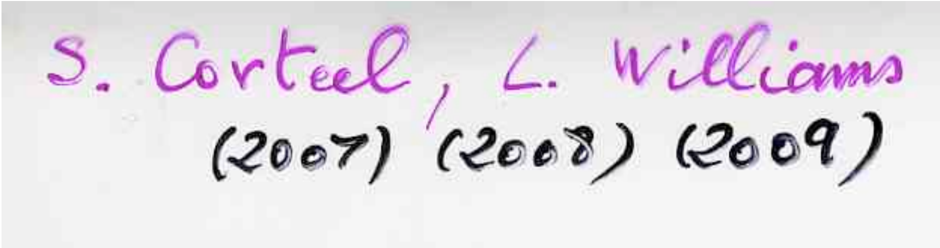
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$

$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \quad q$

Askey-Wilson polynomials



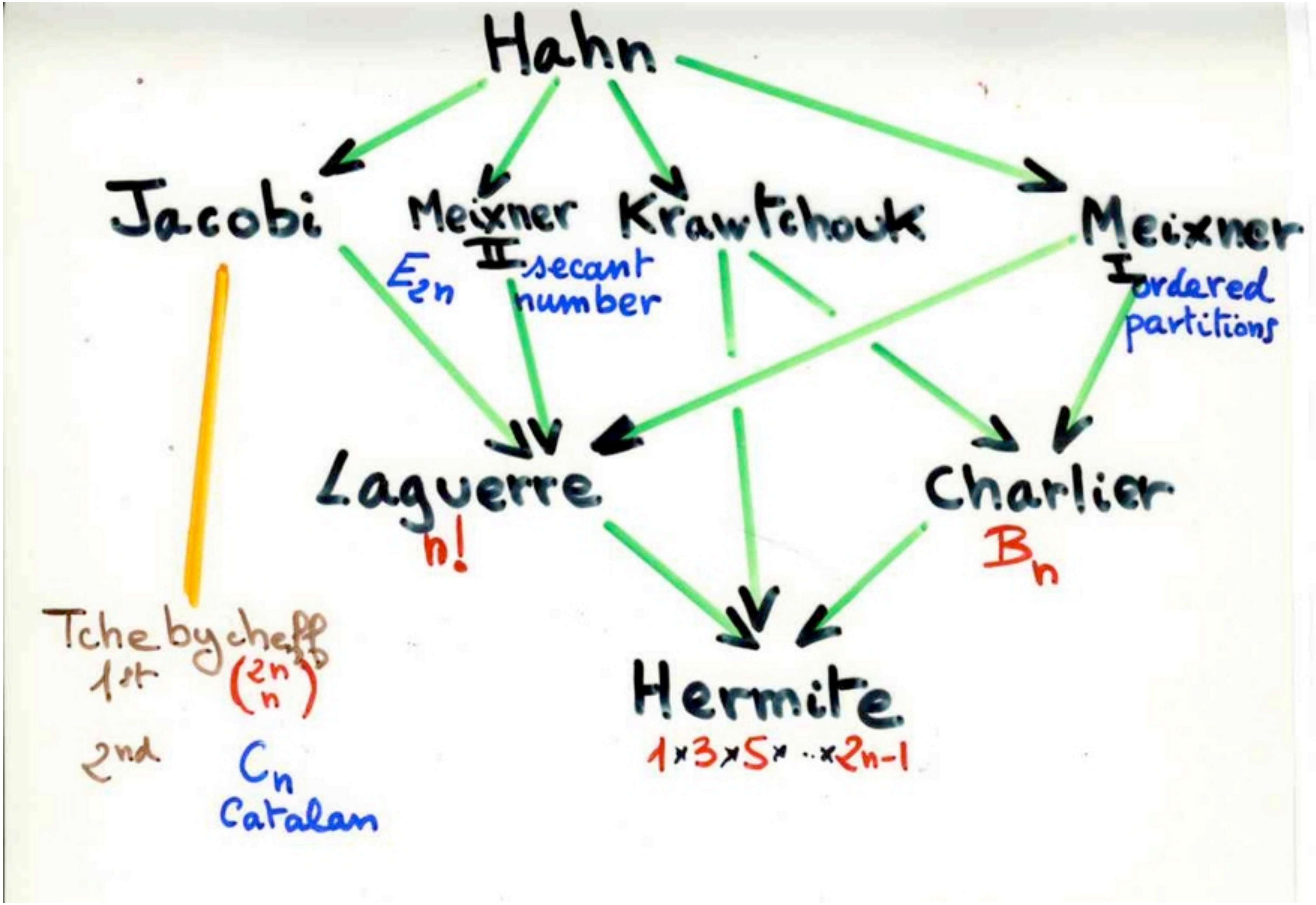
(2010)

S. Corteel, R. Stanley, D. Stanton, L. Williams (2010)

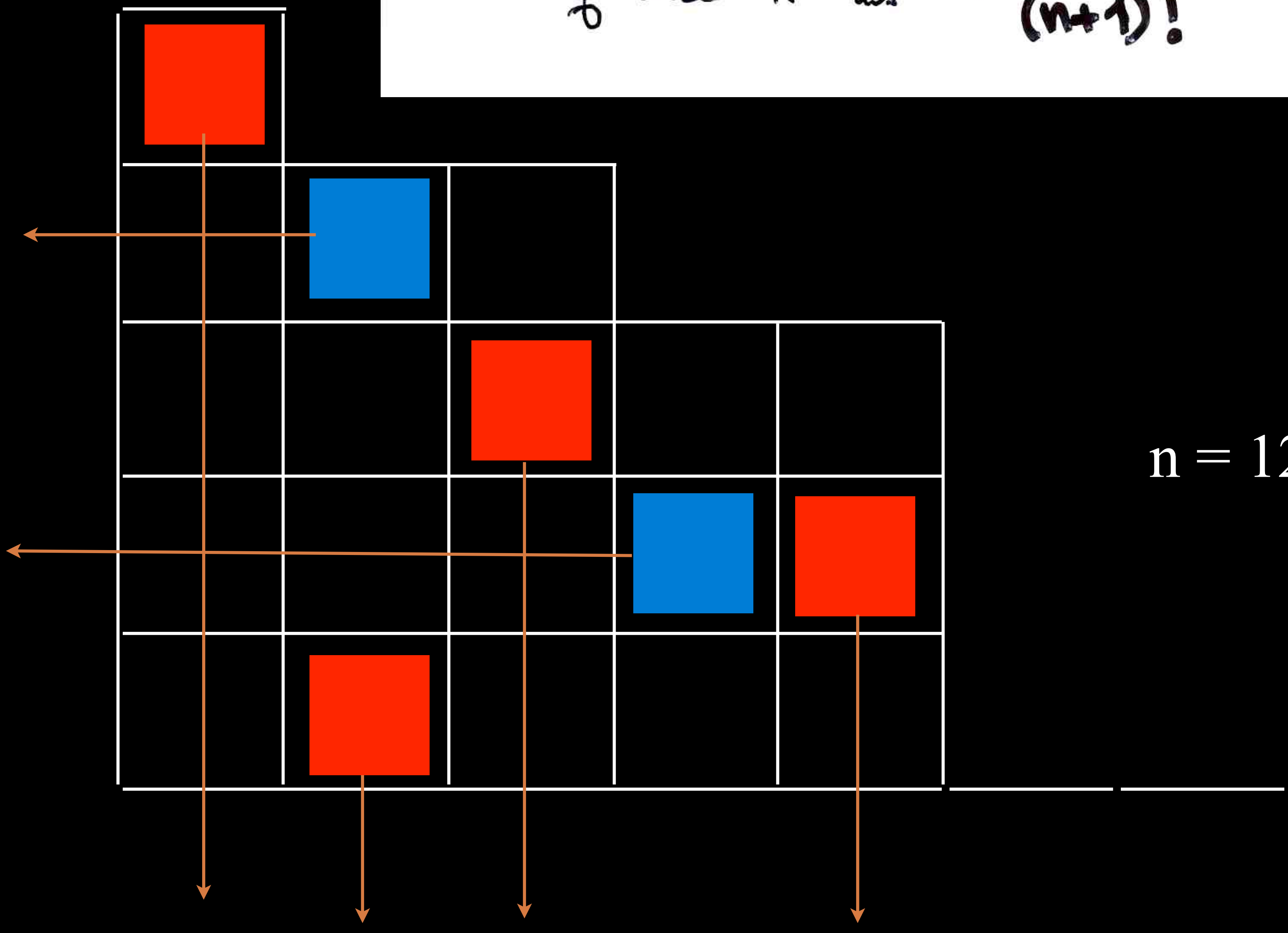
partition function Z
as a sum of certain weights of «staircase tableaux»
or as the moments of the Askey-Wilson polynomials

alternative tableaux with two colors for the blue and for the red cells,
plus two colors for each edges of the border of the Ferrers diagram

Askey-Wilson



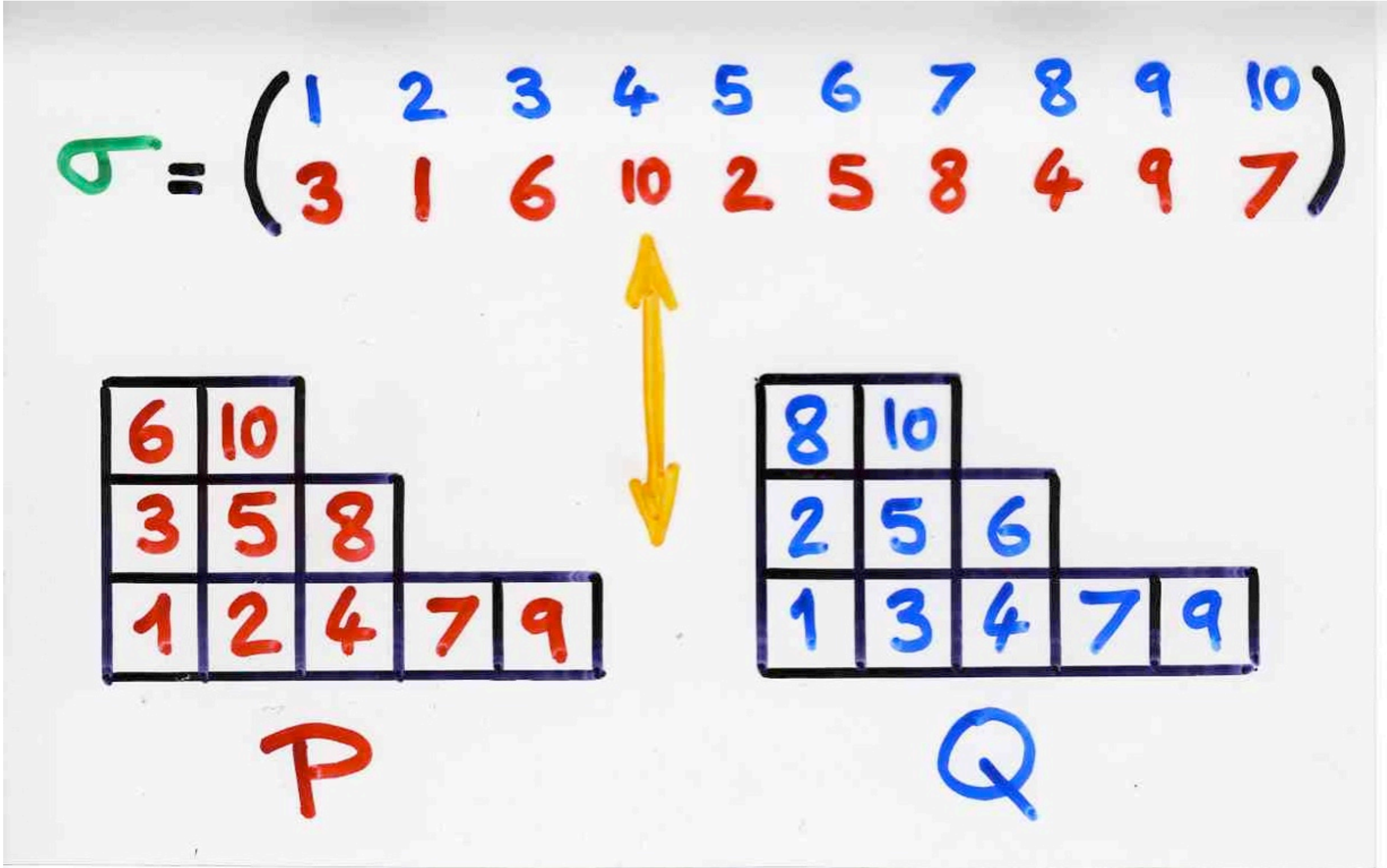
Prop. The number of alternative tableaux of size n is (n+1)!



n = 12

The cellular Ansatz
second part:

guided construction
of a bijection
(from a representation of the quadratic
algebra Q with "combinatorial operators")



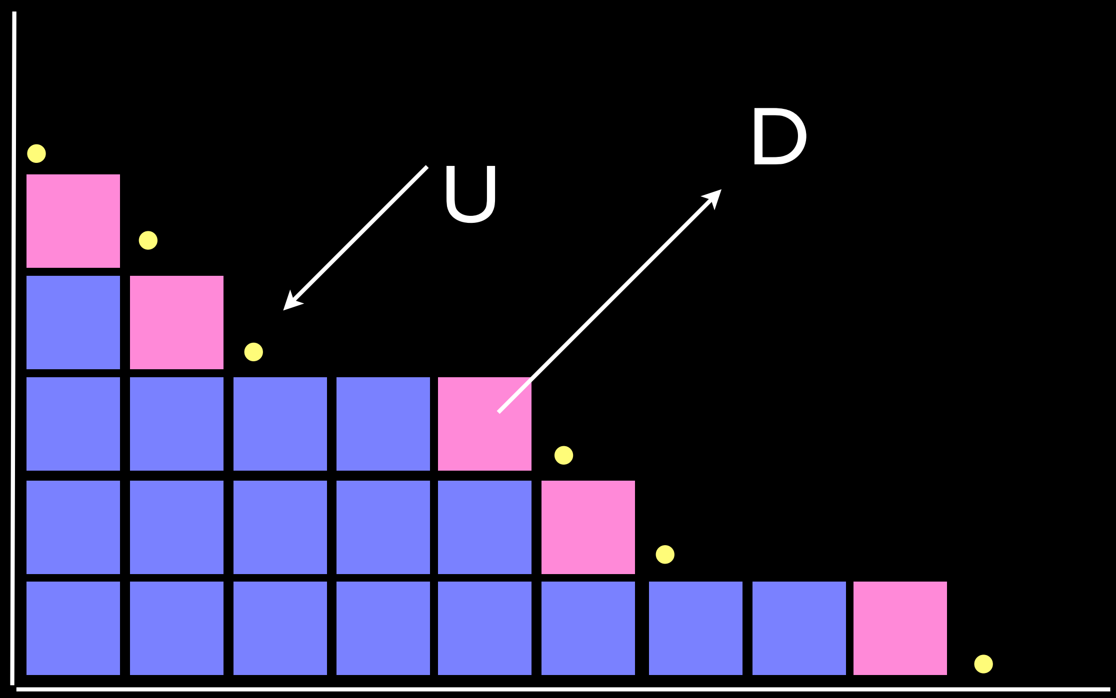
The Robinson-Schensted correspondence (RSK) between permutations and pair of (standard) Young tableaux with the same shape

representation of the operators U, D

Sergey Fomin

(with C. K.)

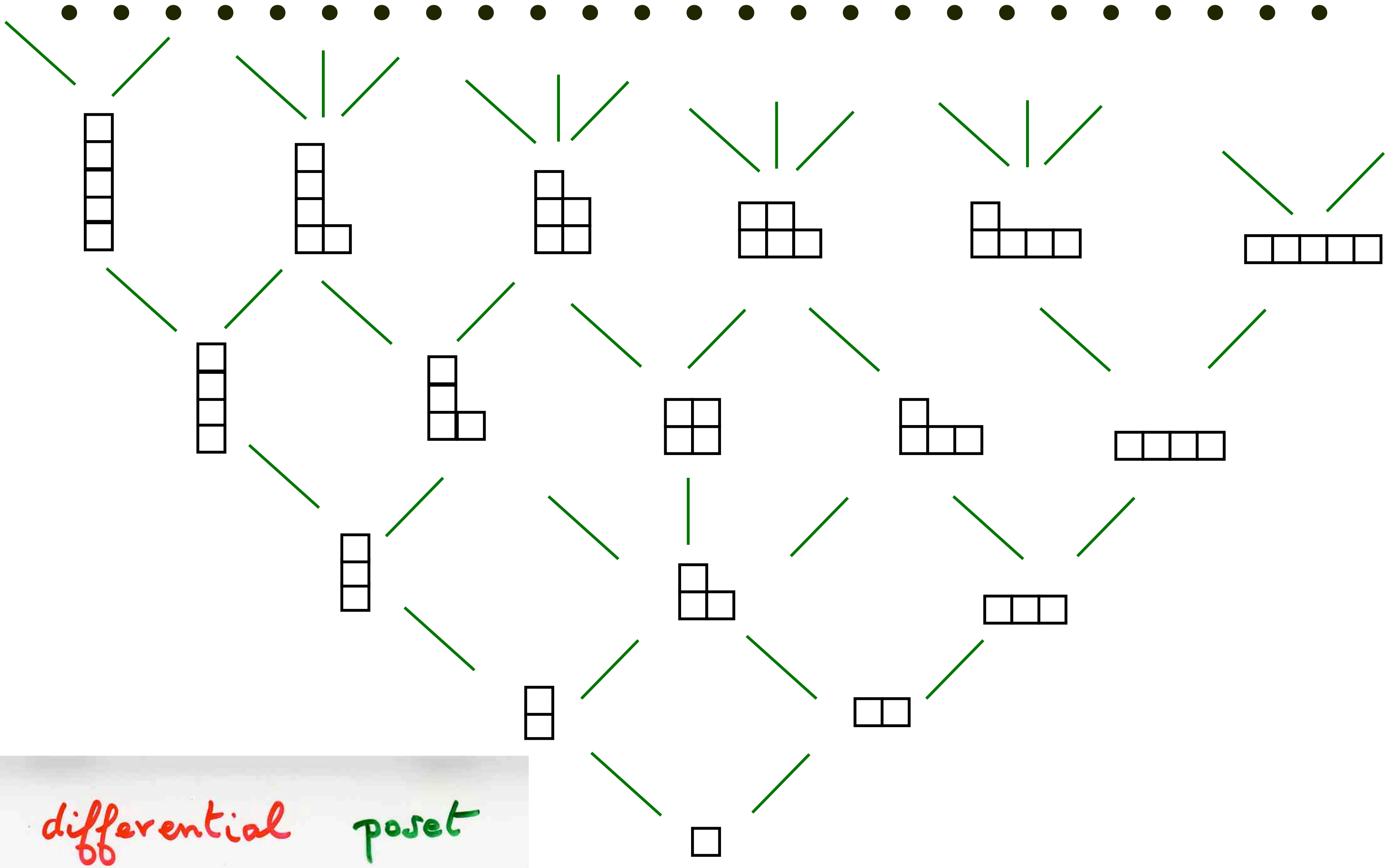
Operators U and D



adding
or deleting
a cell in
a Ferrers
diagram

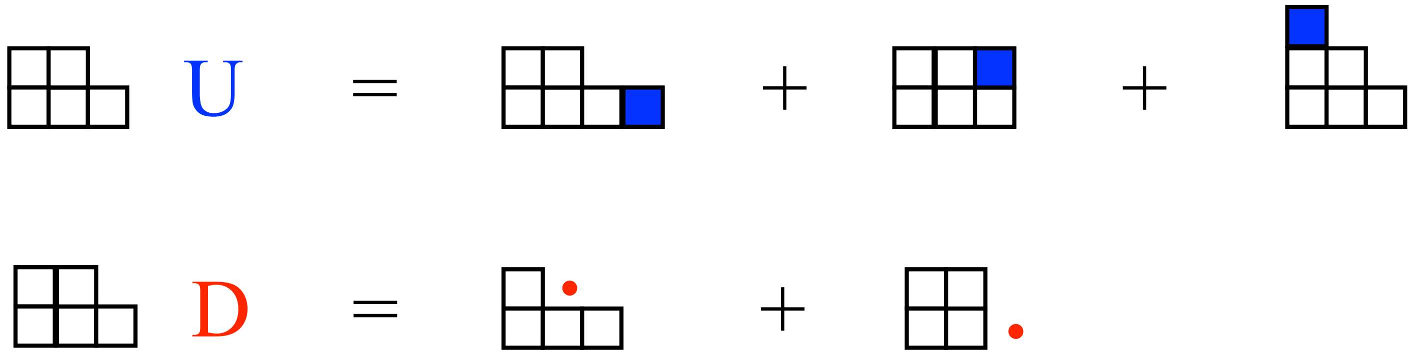
Young lattice

Young lattice



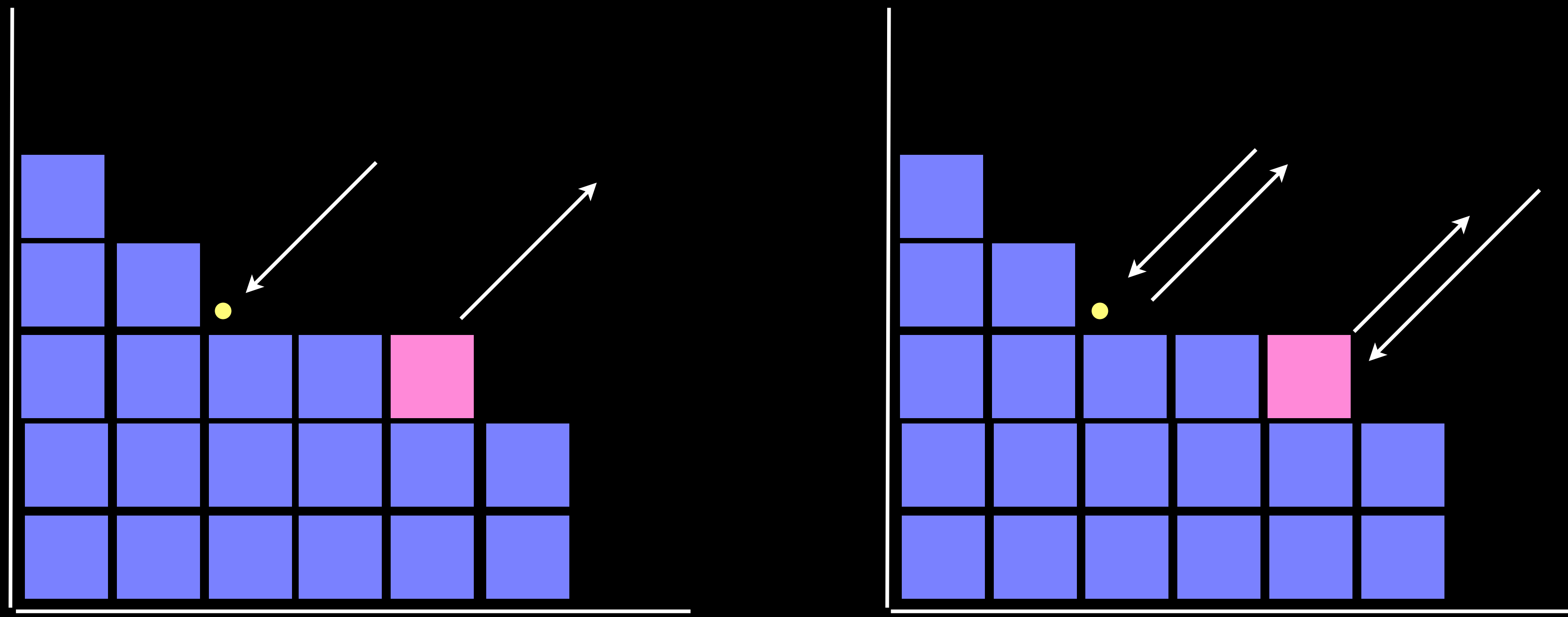
differential poset

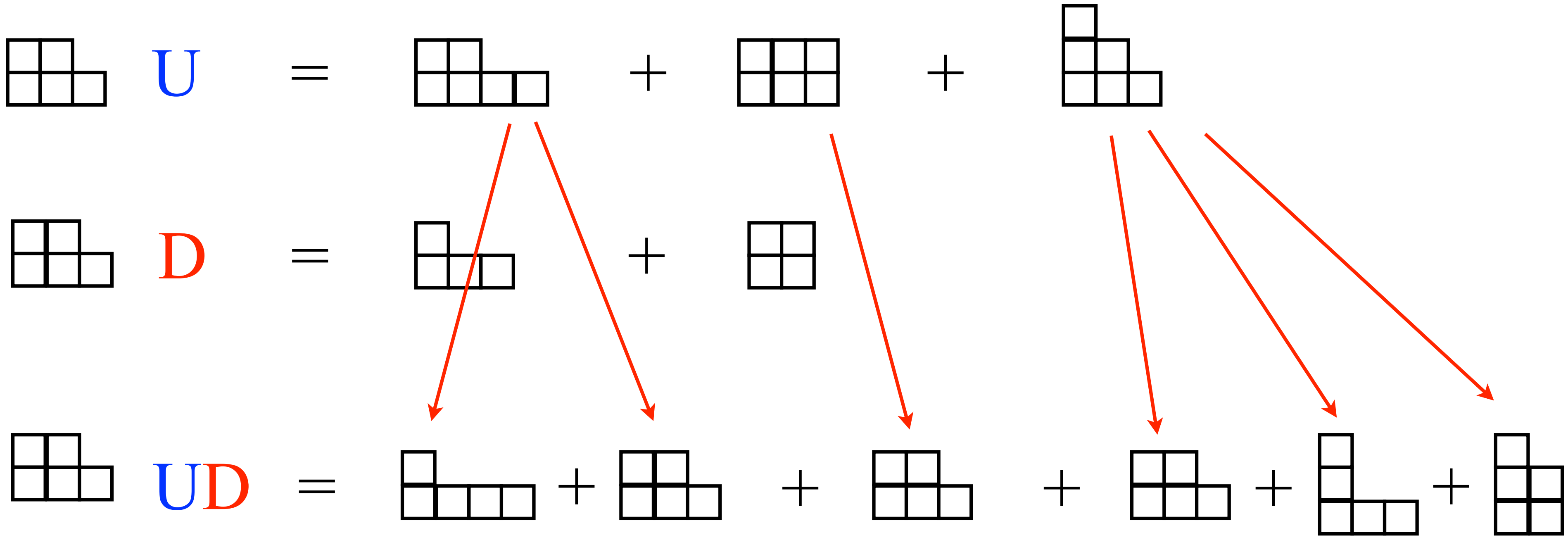
Fomin, Stanley

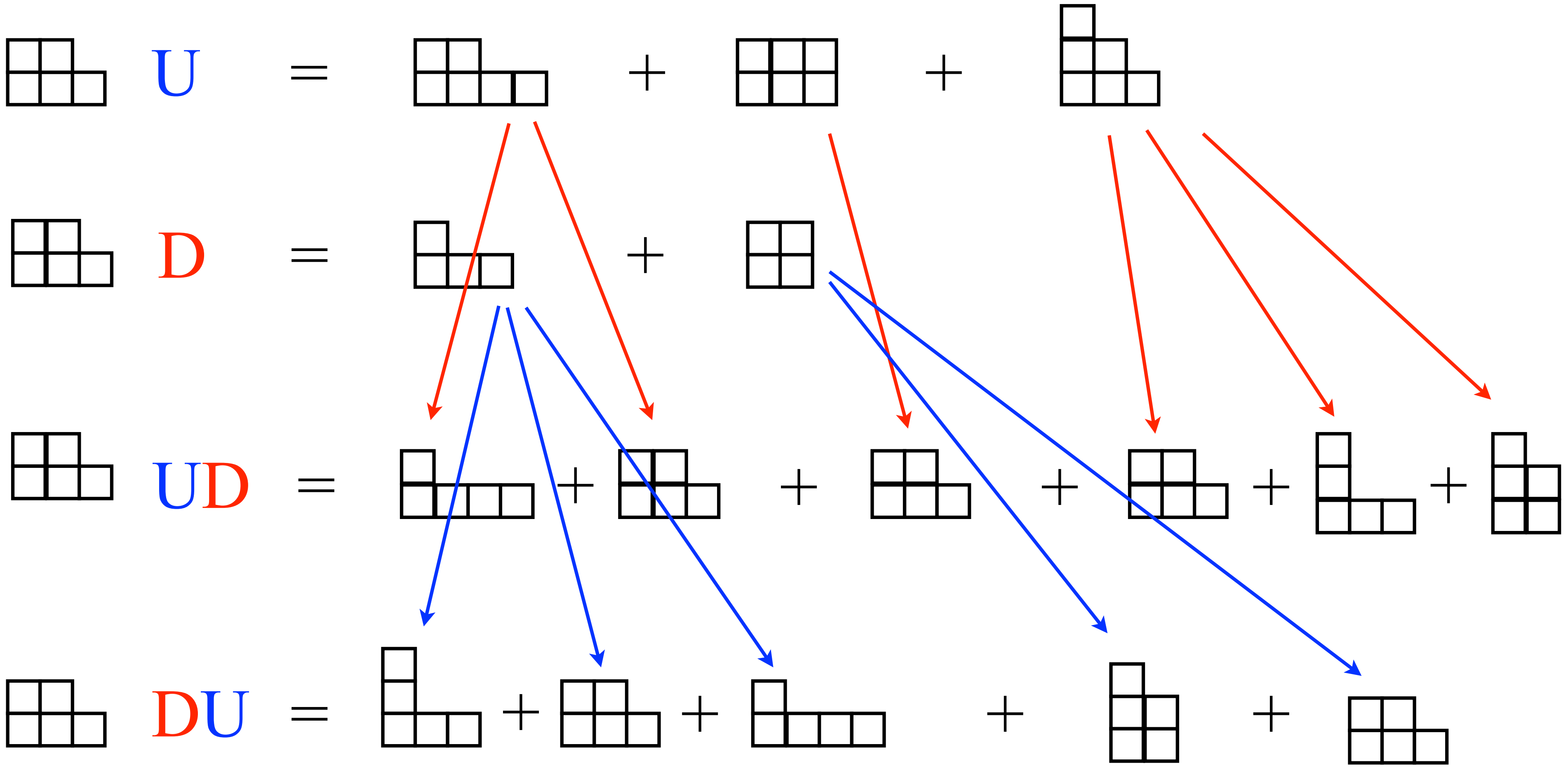

U
D

Heisenberg commutation relation

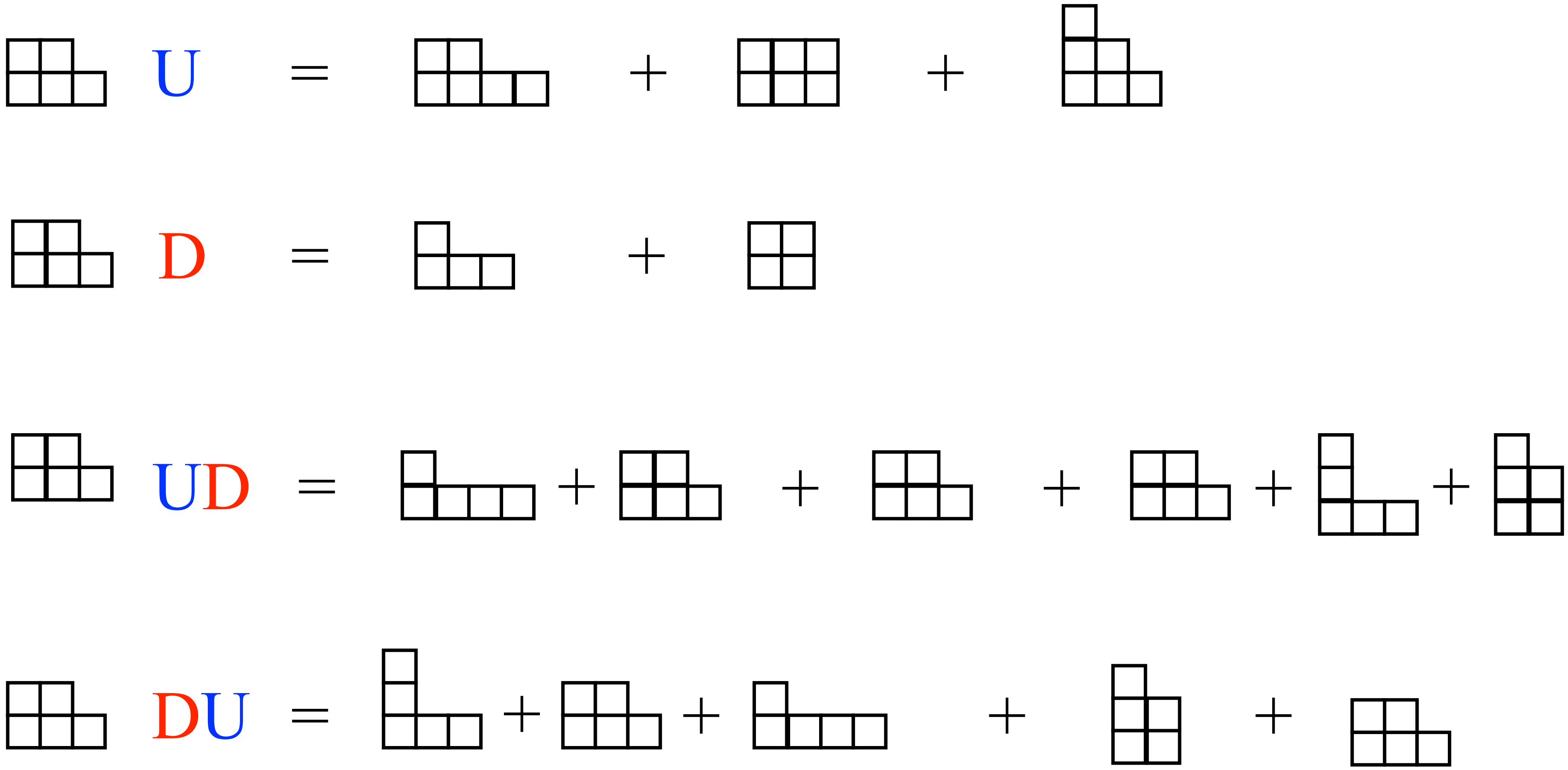
$$UD = DU + I$$

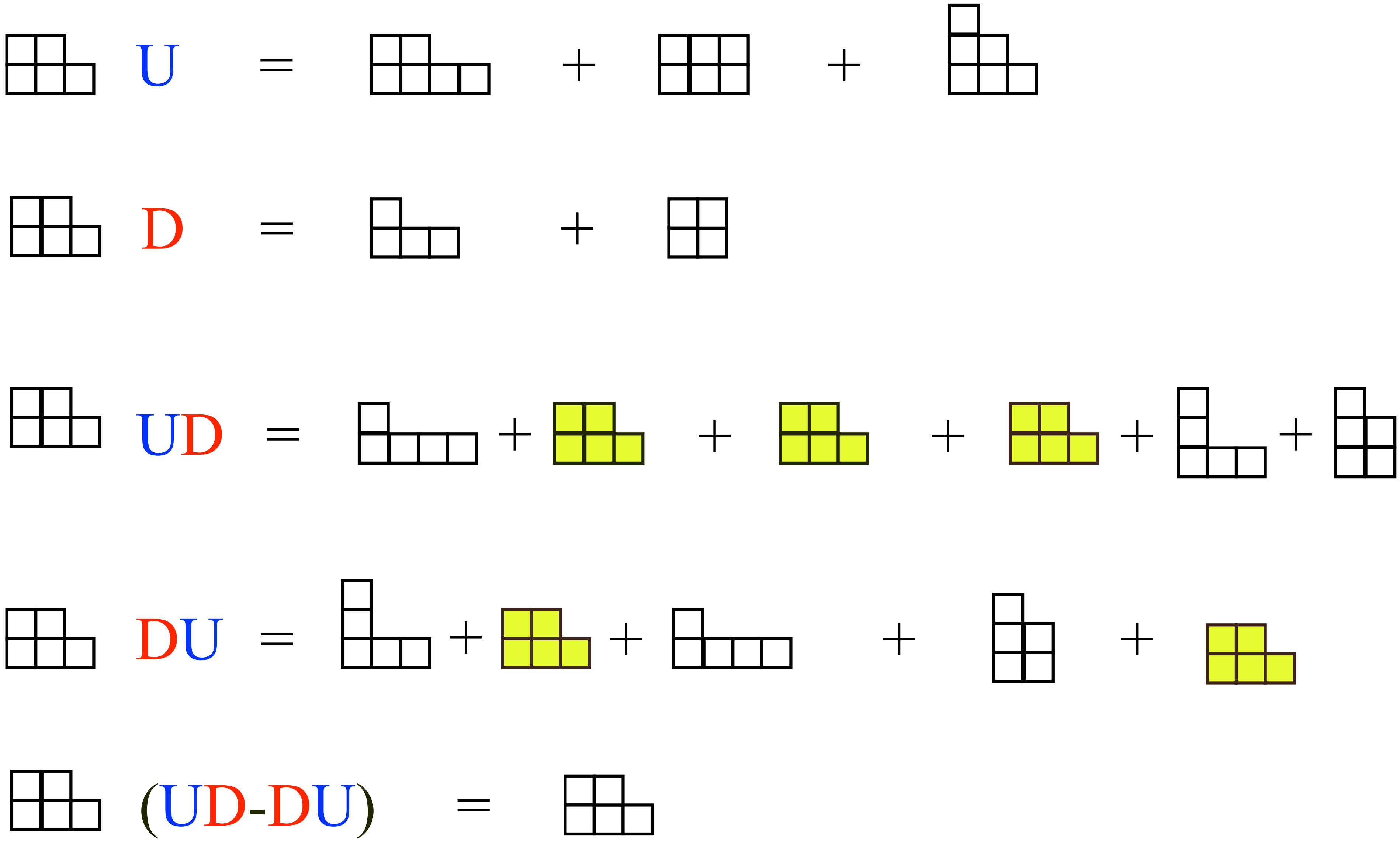



U
D
UD

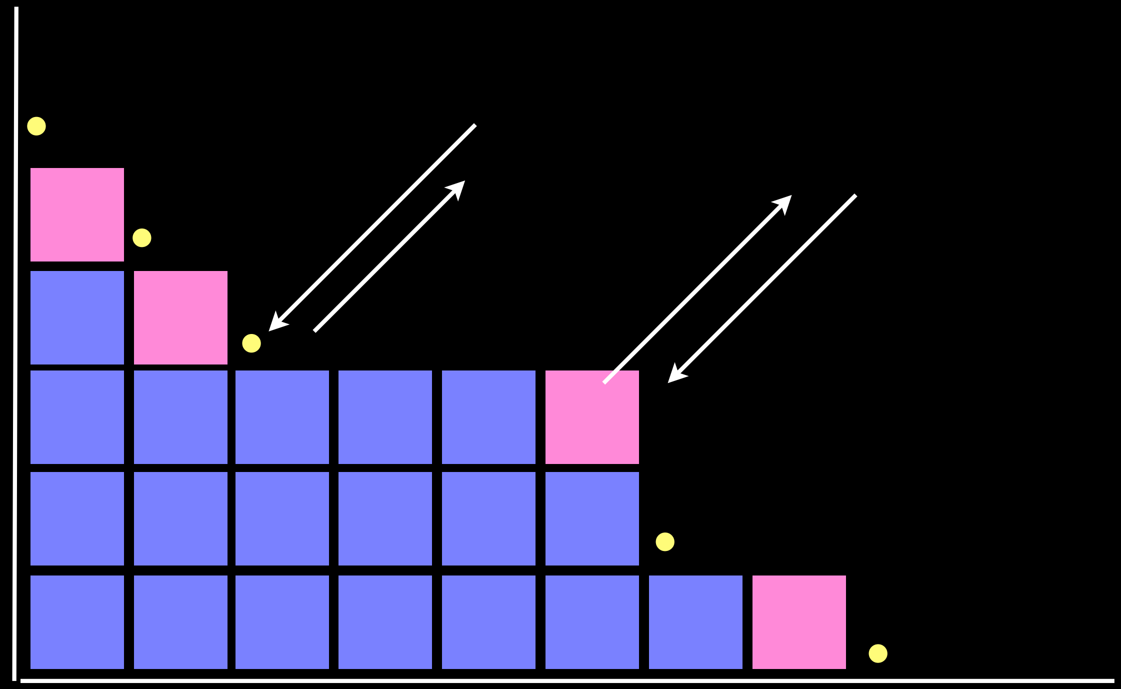


U =
D =
UD =
DU =


U
D
UD
DU


U
D
UD
DU
(UD-DU)

UD = DU + I



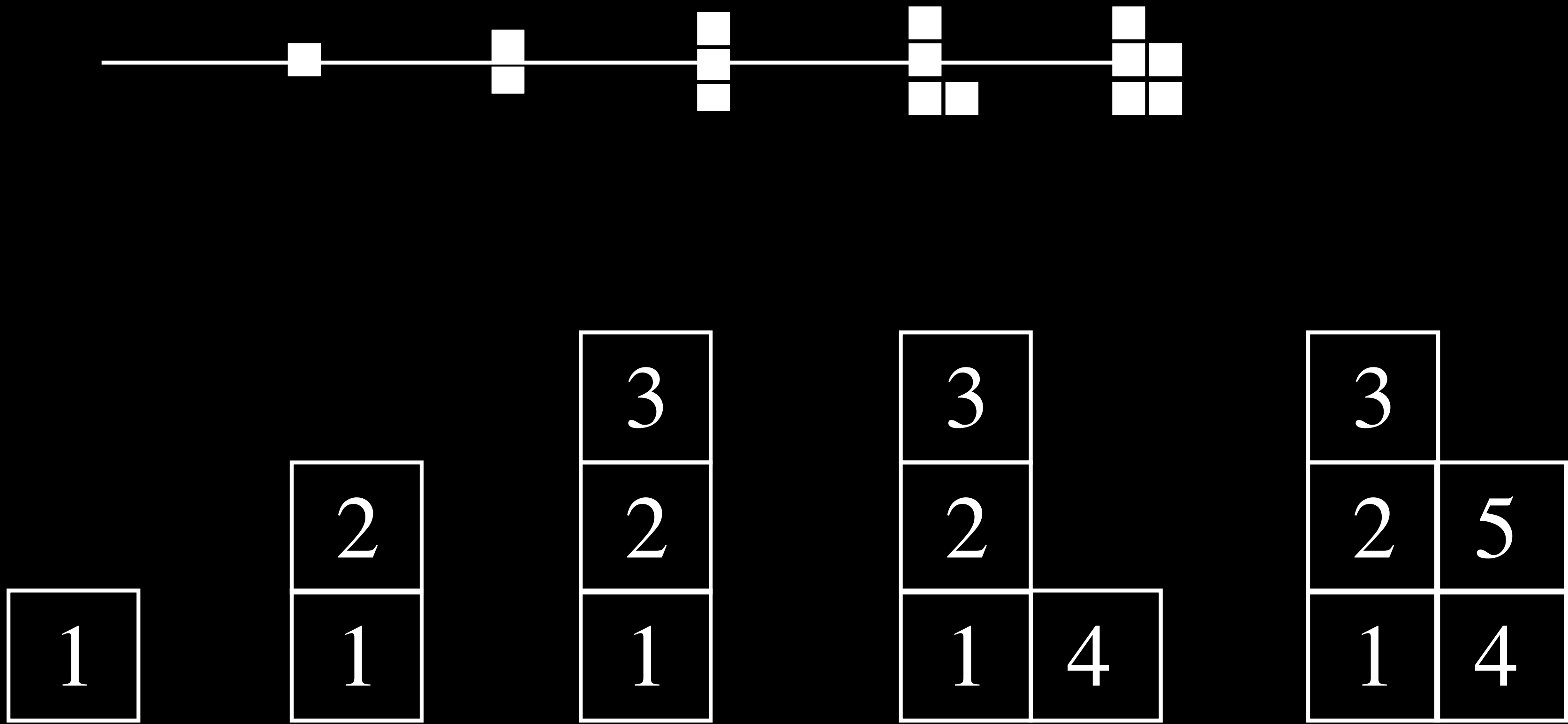
$$U^n D^n = \sum_{0 \leq i \leq n} c_{n,i} \, D^i U^i$$

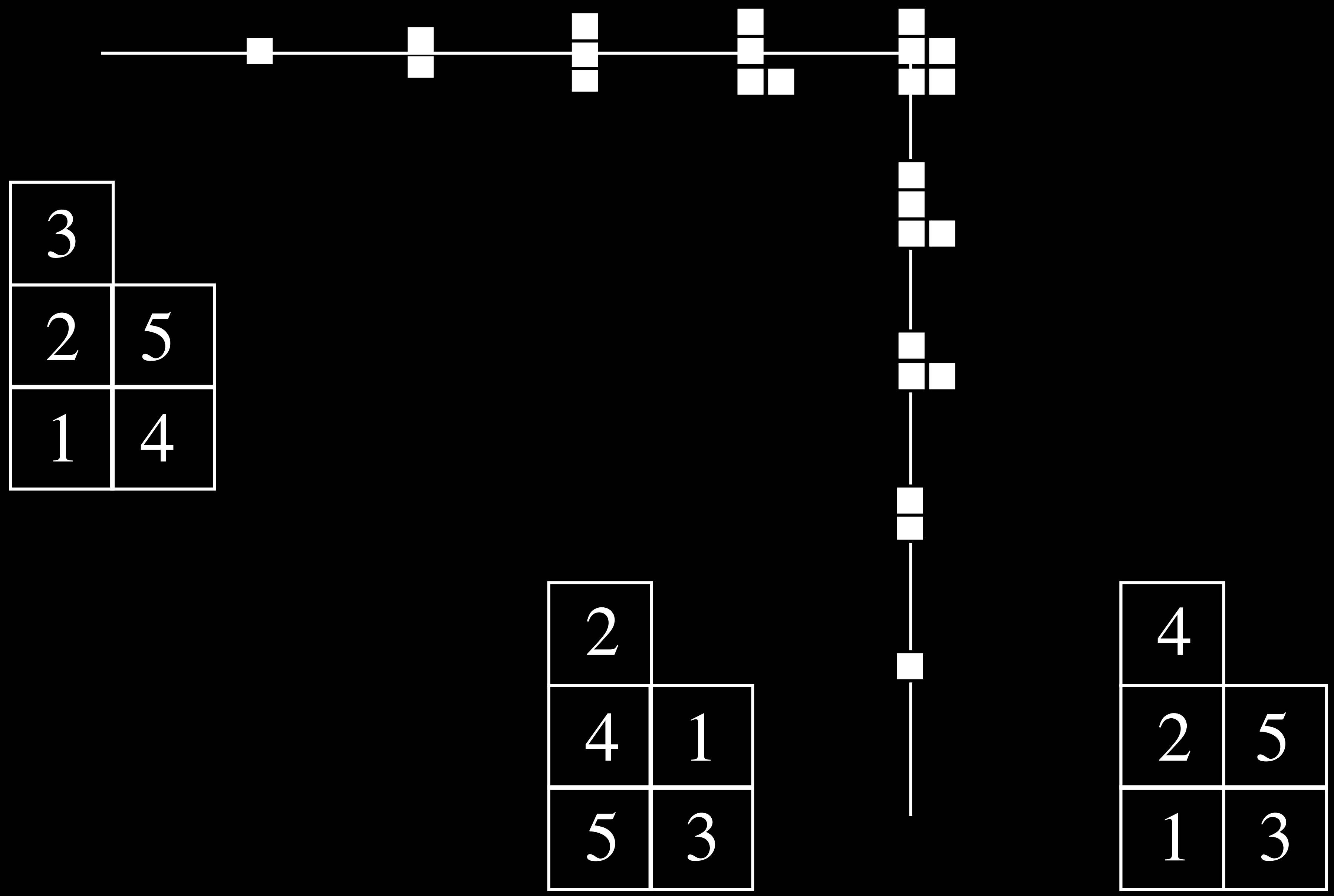
normal ordering

$$c_{n,0} = n!$$

$$\langle \varnothing | U^n D^n | \varnothing \rangle = \sum_{0 \leq i \leq n} c_{n,i} \langle \varnothing | D^i U^i | \varnothing \rangle$$

$$= c_{n,0}$$


1
2
1
3
2
1
3
2
1 4
3
2 5
1 4


3
2 5
1 4
2
4 1
5 3
4
2 5
1 3

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

$$\updownarrow$$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

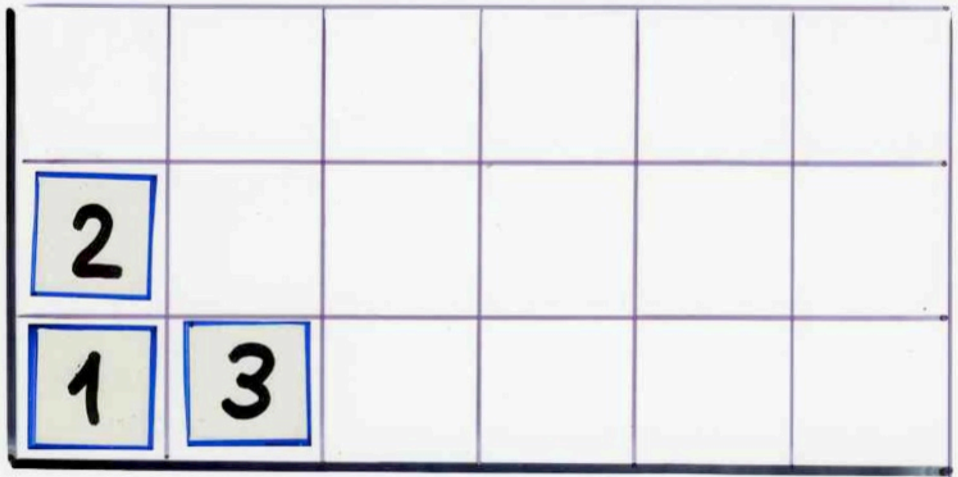
3					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

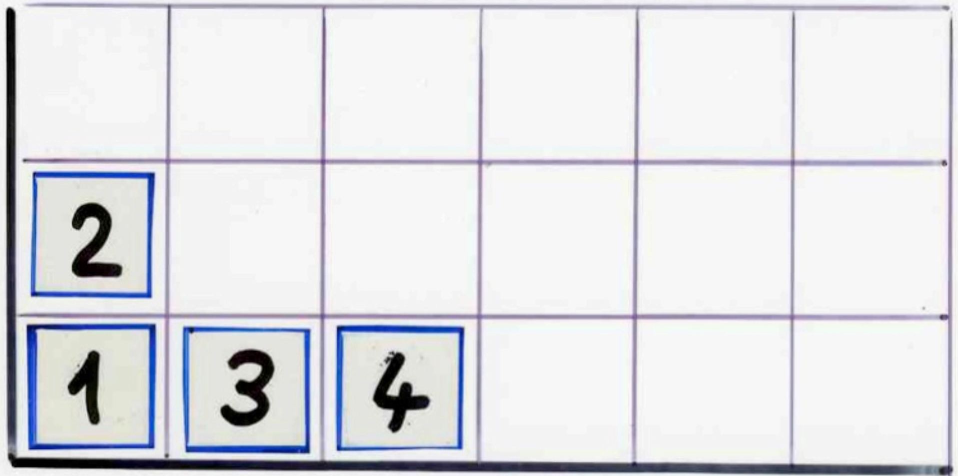
2					
1					

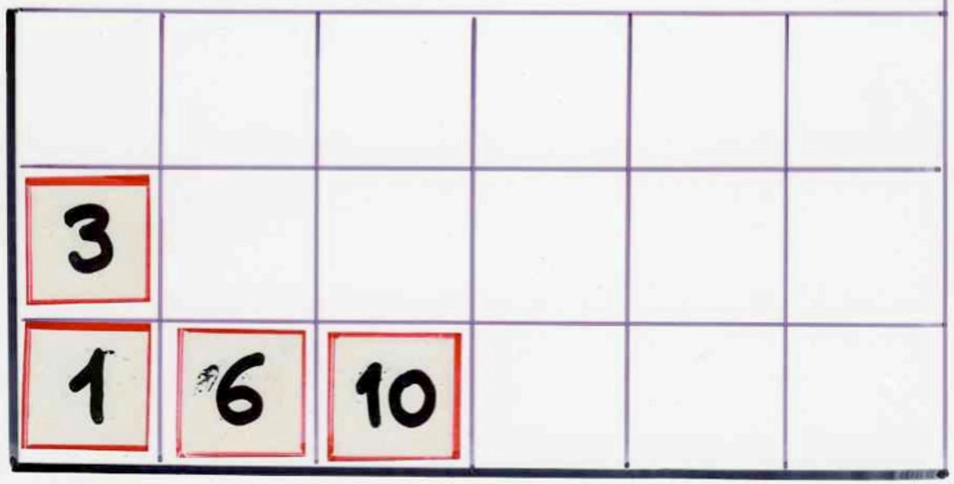
3					
1					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

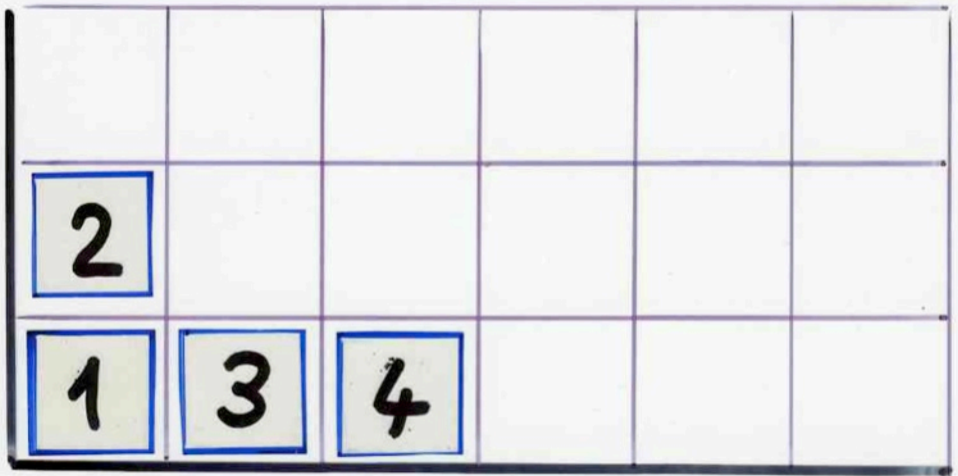


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

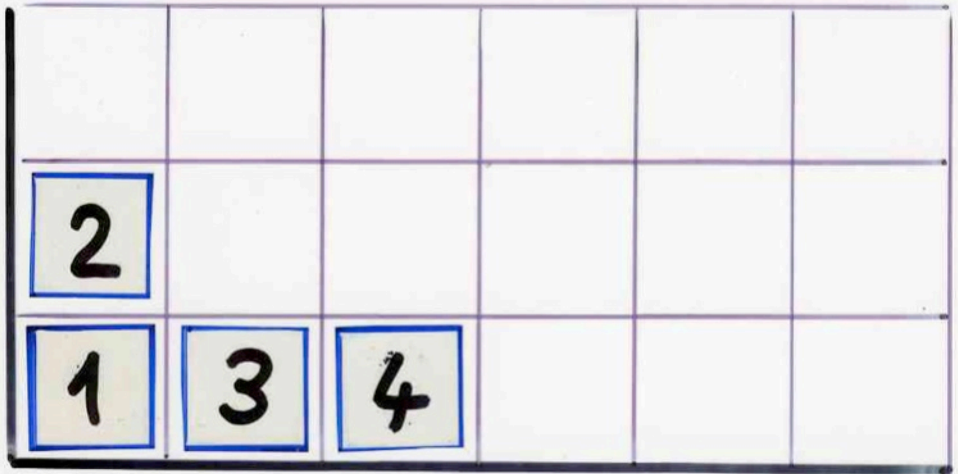


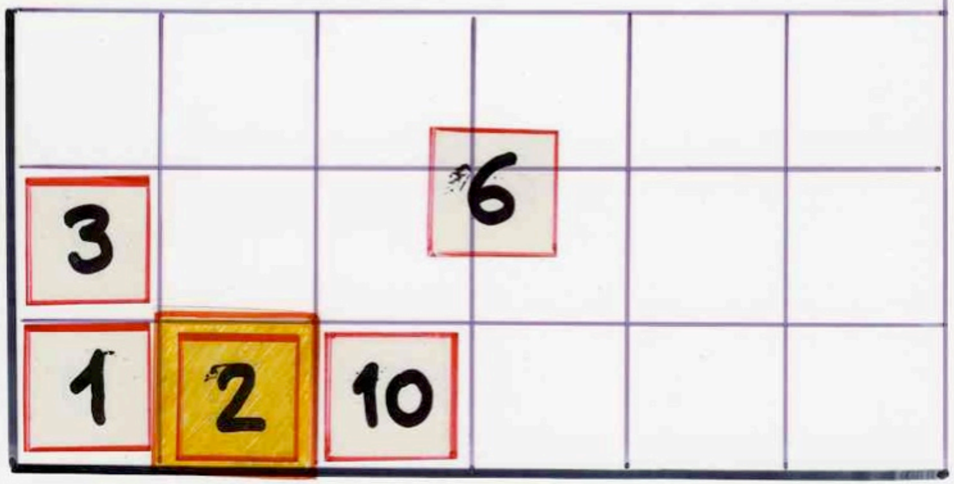


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



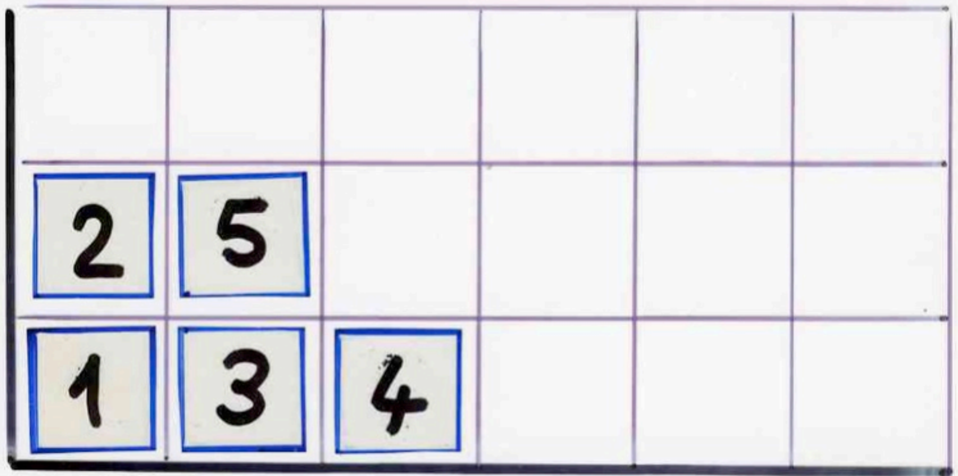


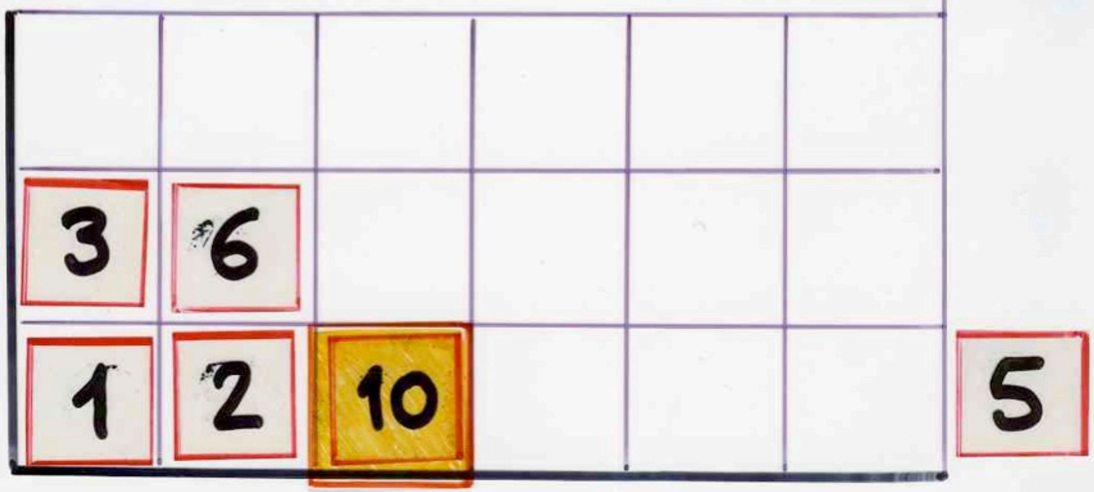
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5				
1	3	4			

3	6				
1	2	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
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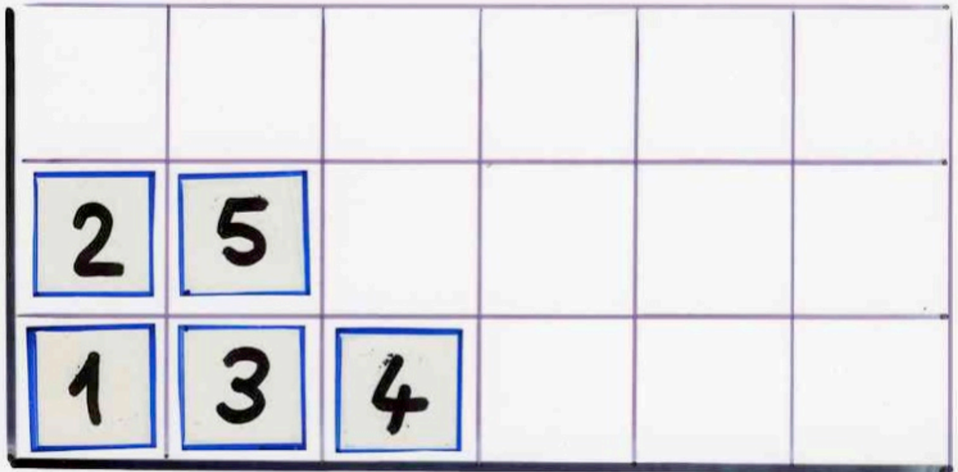
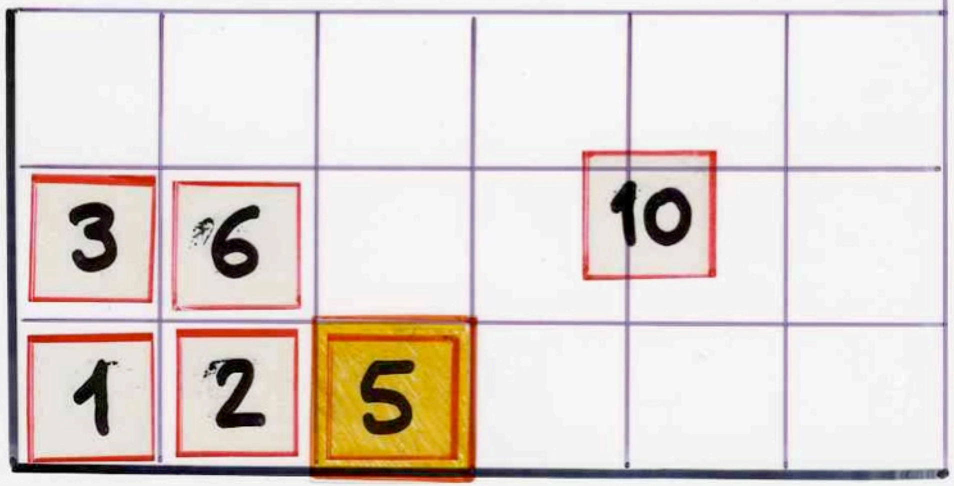




1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

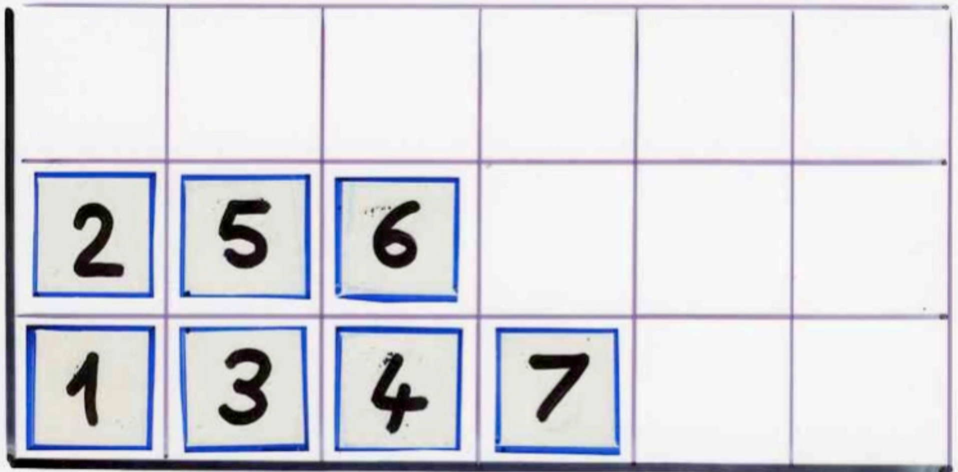
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

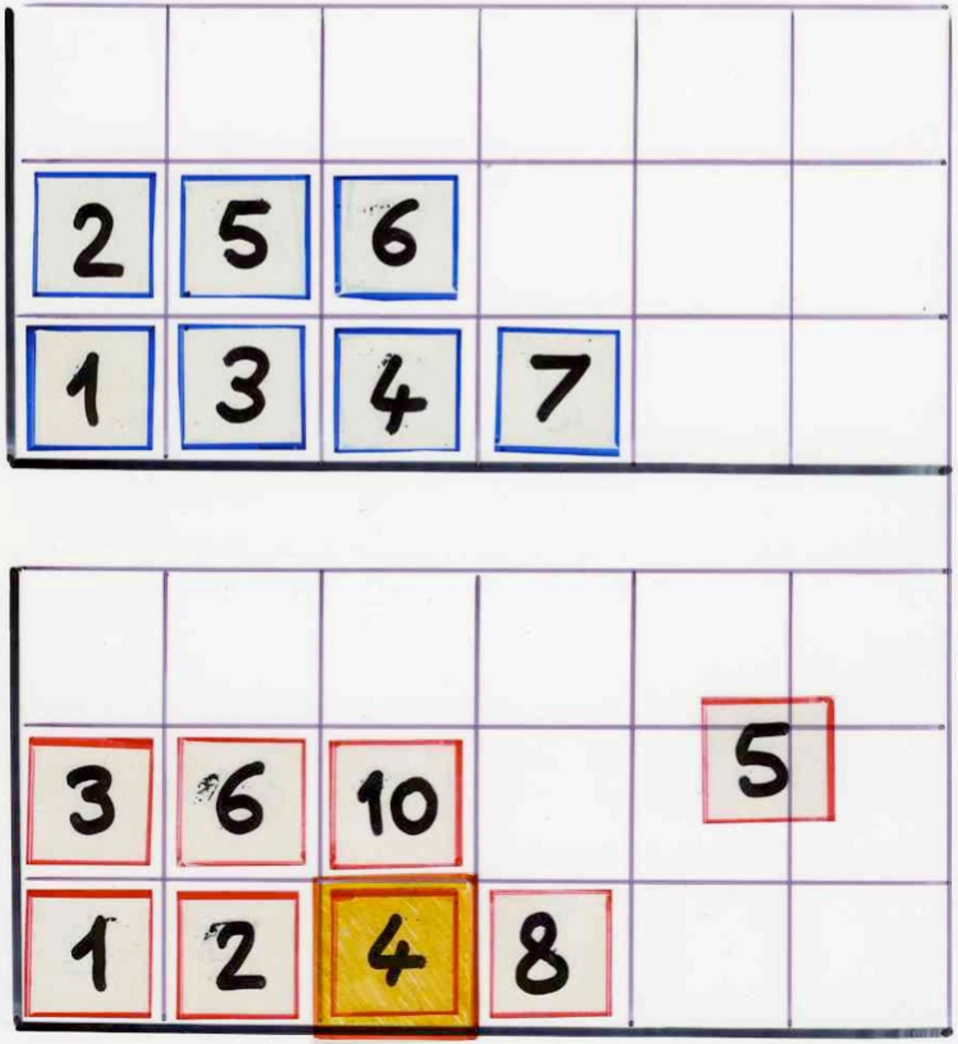
2	5	6			
1	3	4	7		

3	6	10			
1	2	5	8		

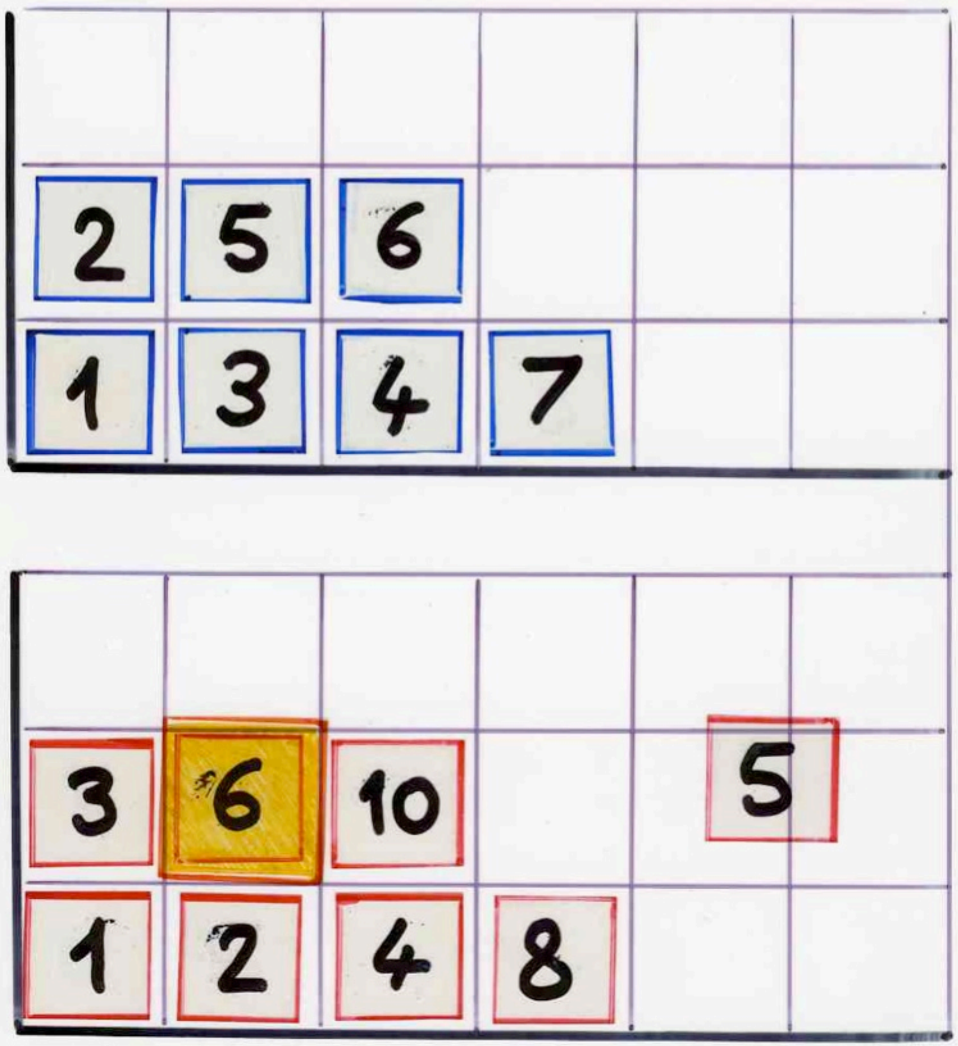
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



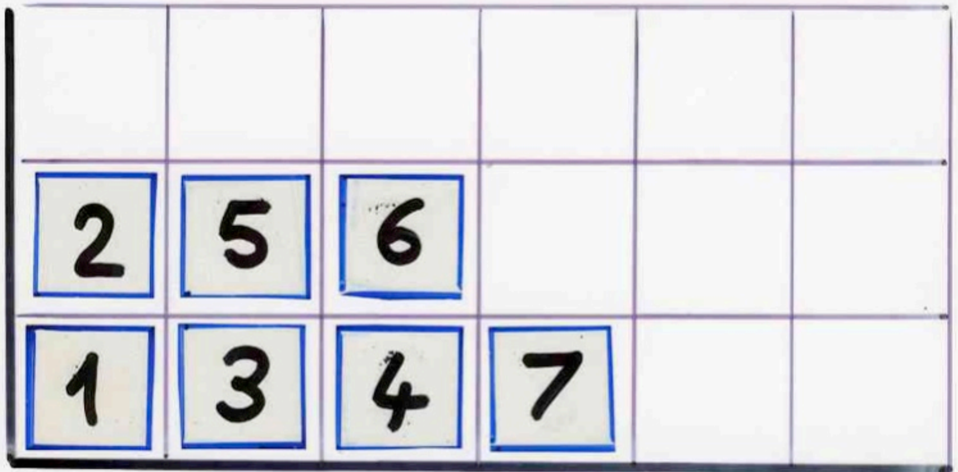
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

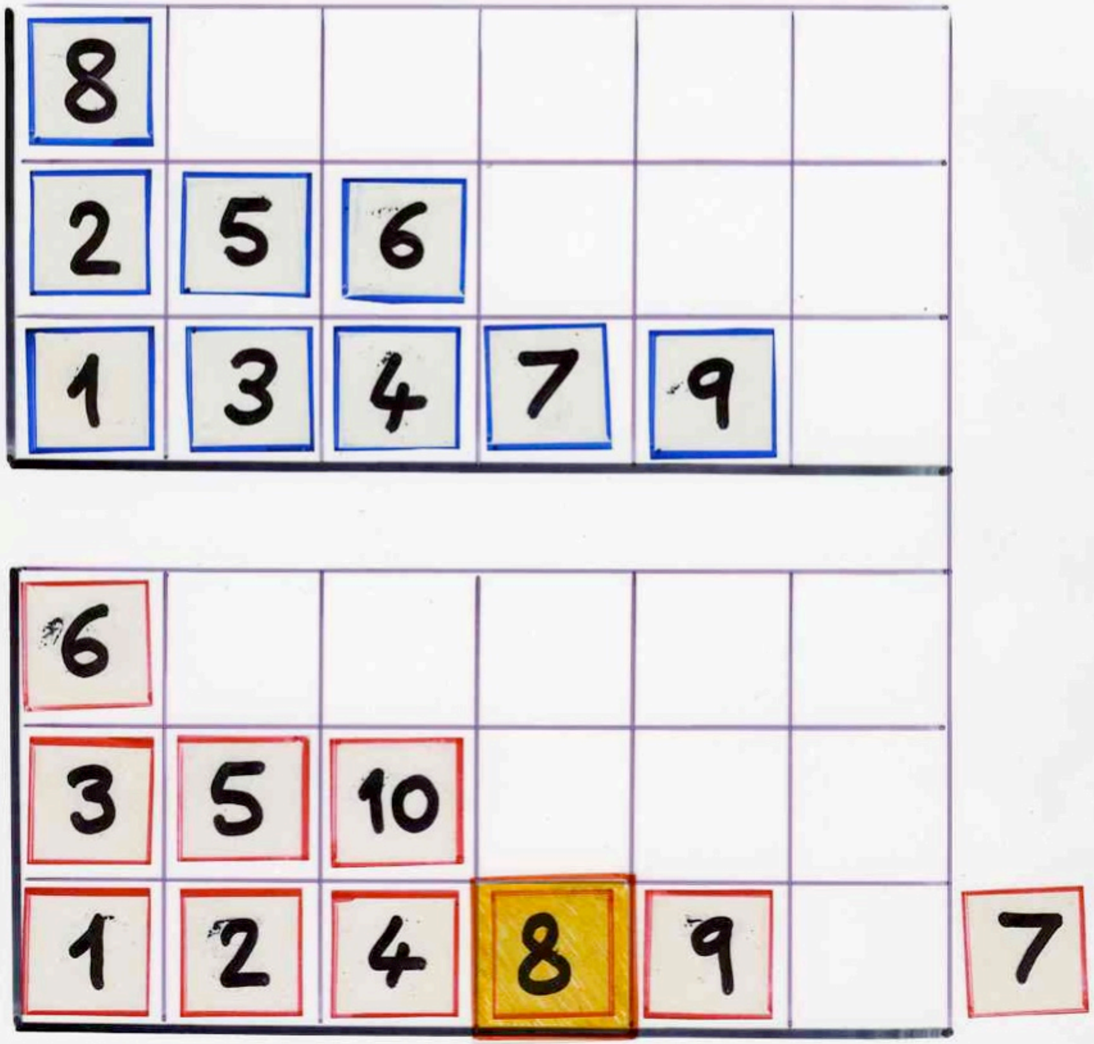
6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

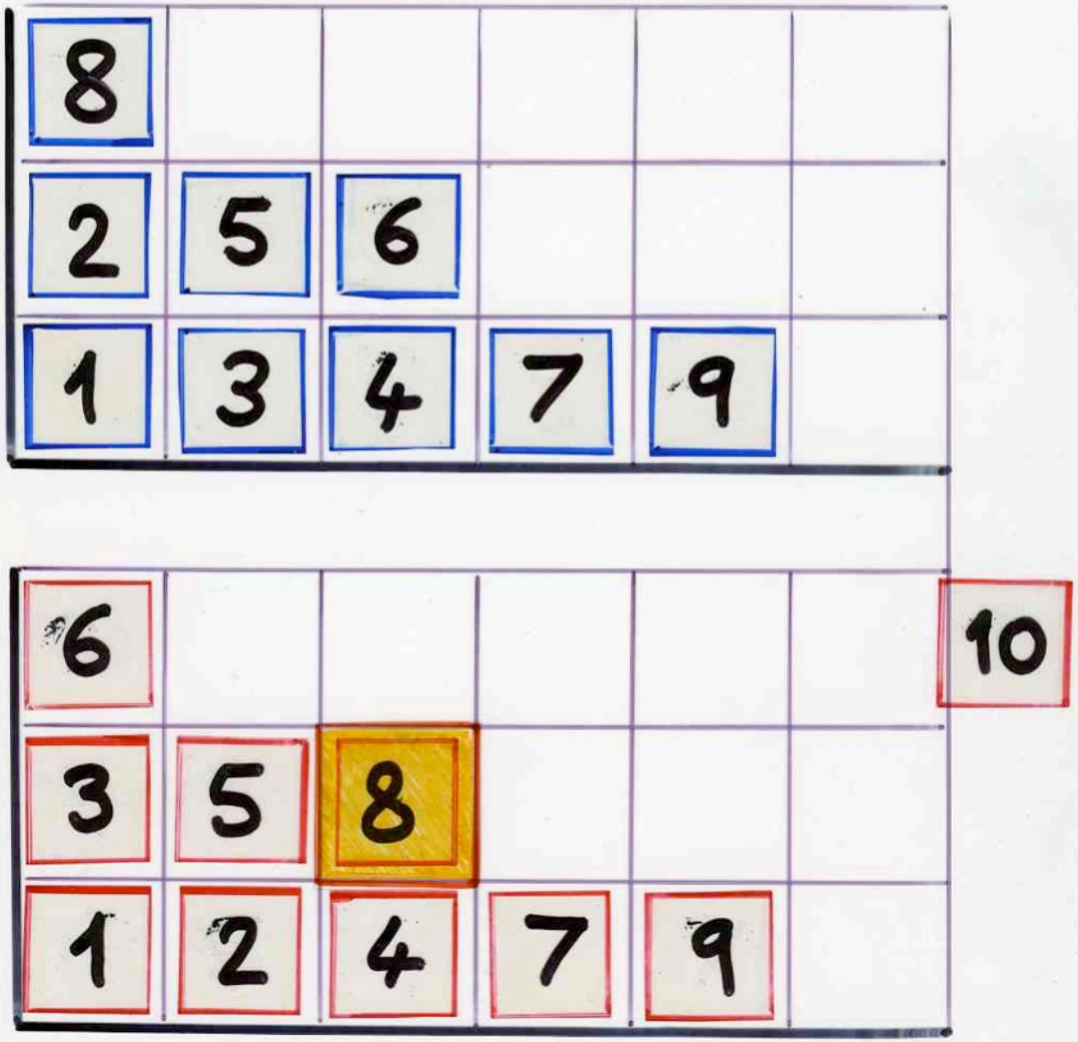
6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

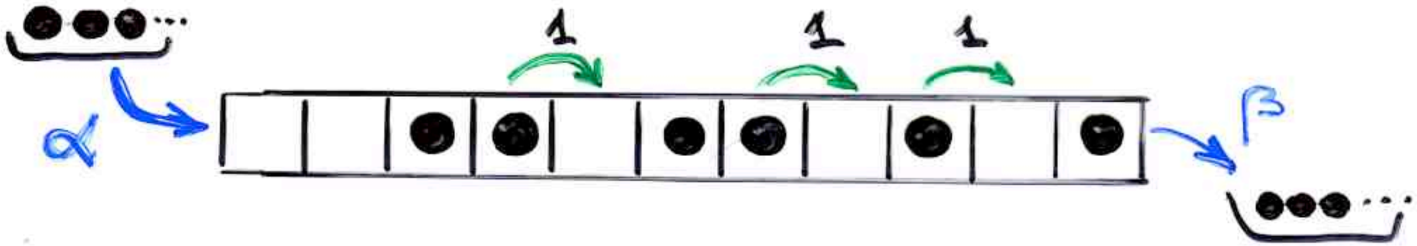
8	10				
2	5	6			
1	3	4	7	9	

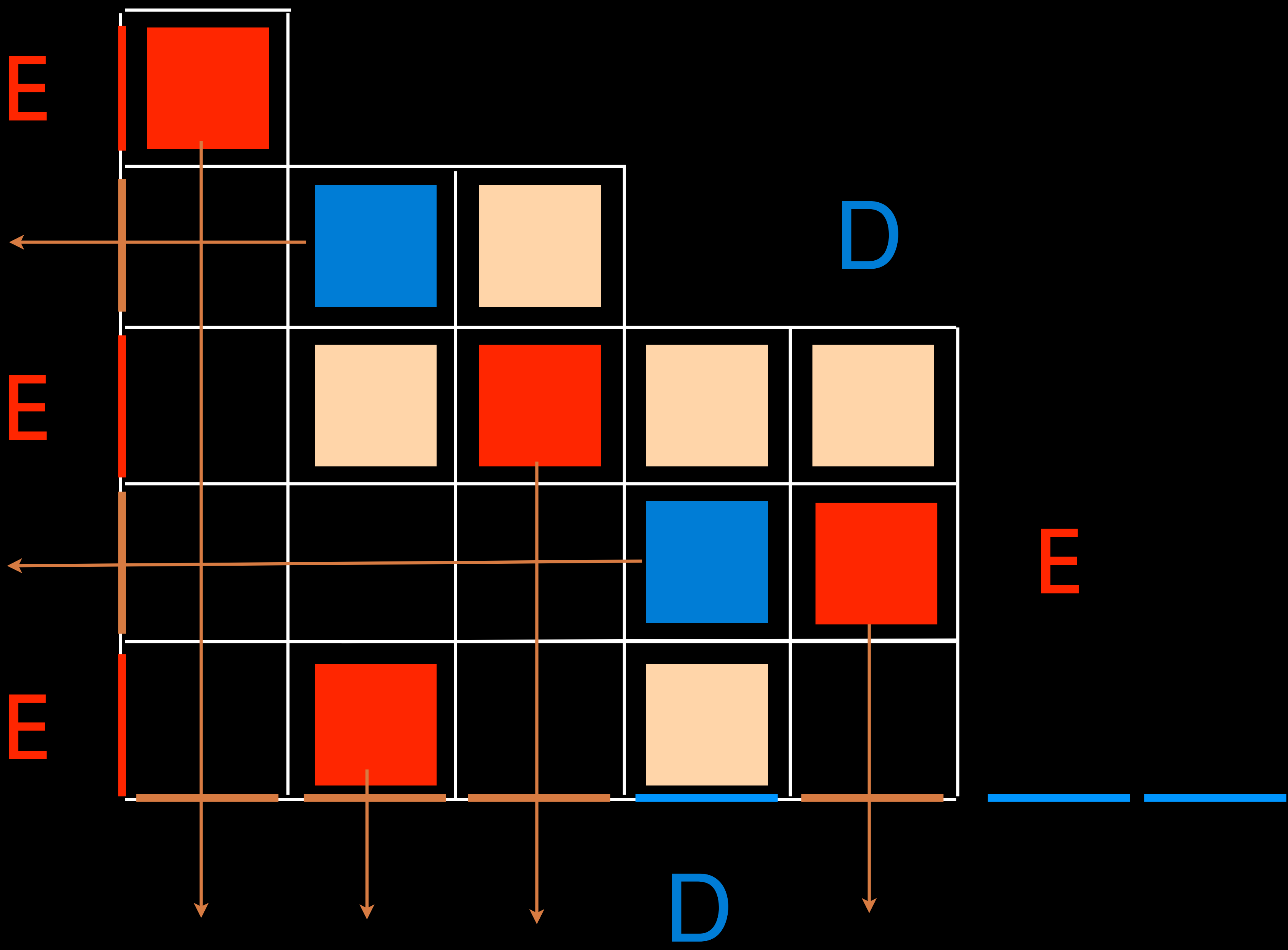
6	10				
3	5	8			
1	2	4	7	9	

TASEP
and
Catalan alternative tableaux

TASEP

"totally asymmetric exclusion process"

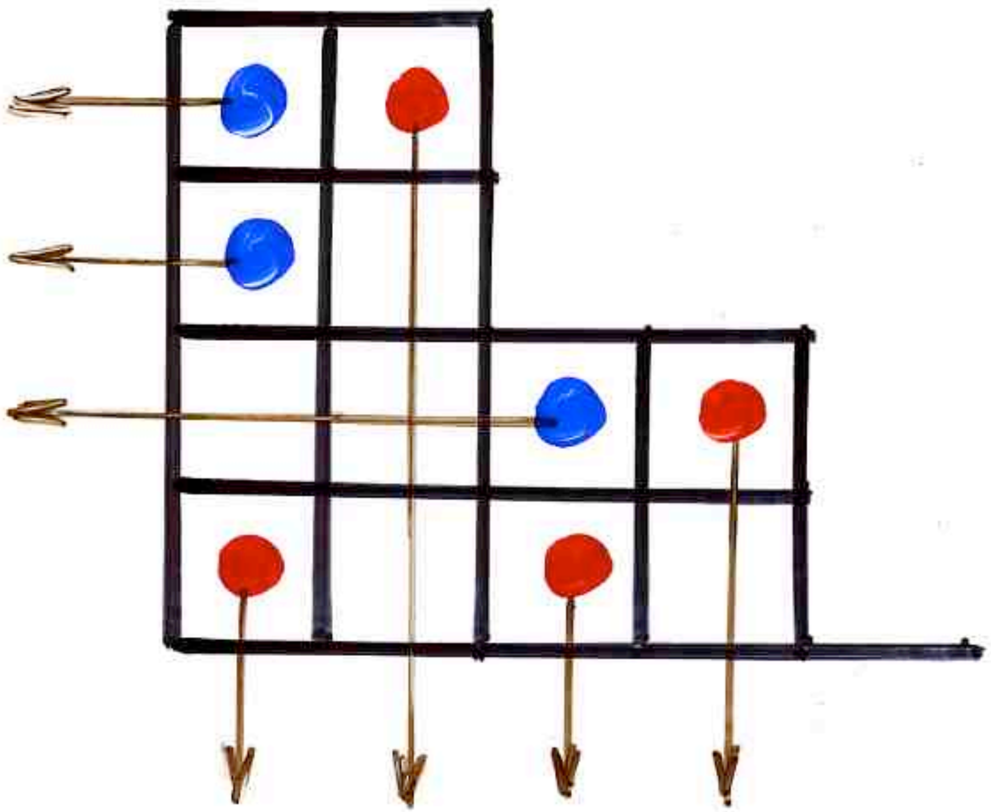


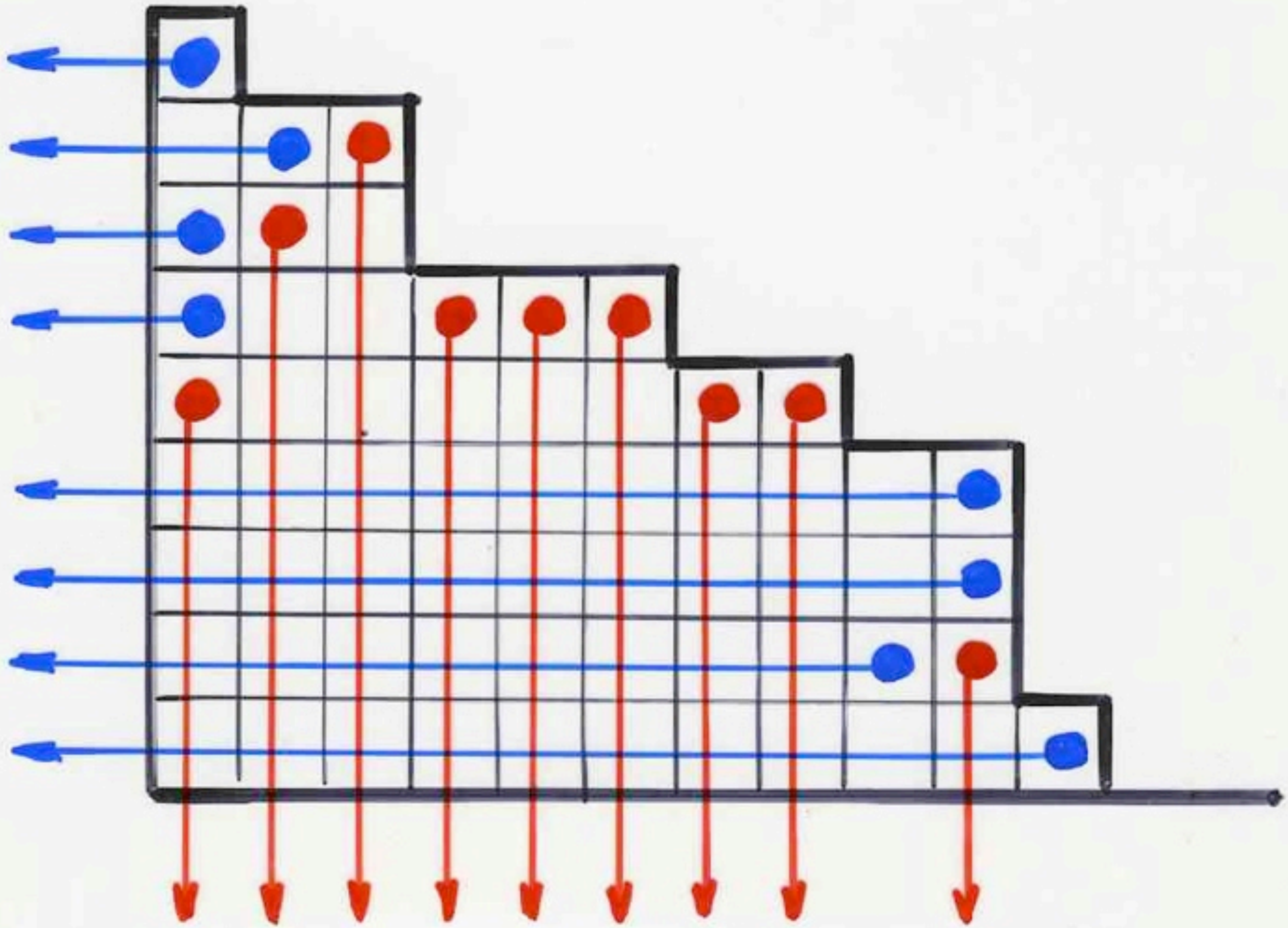

E
D
E
E
E
D

Def Catalan alternative tableau T

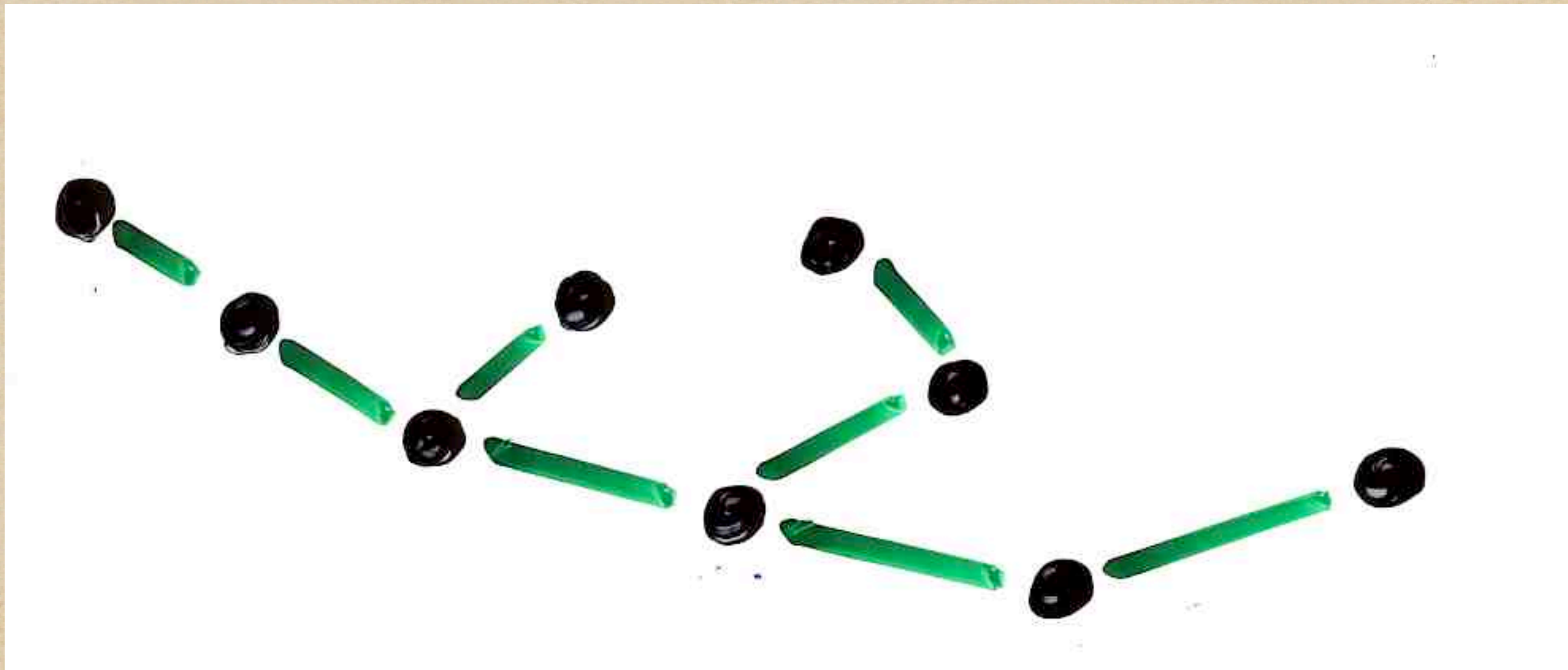
alt. tab. without cells ☒

i.e: every empty cell is below a red cell or on the left of a blue cell

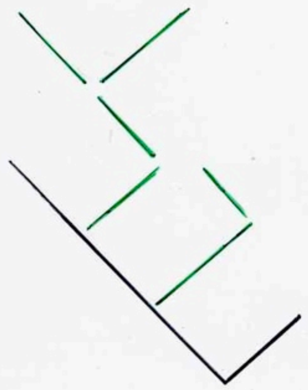


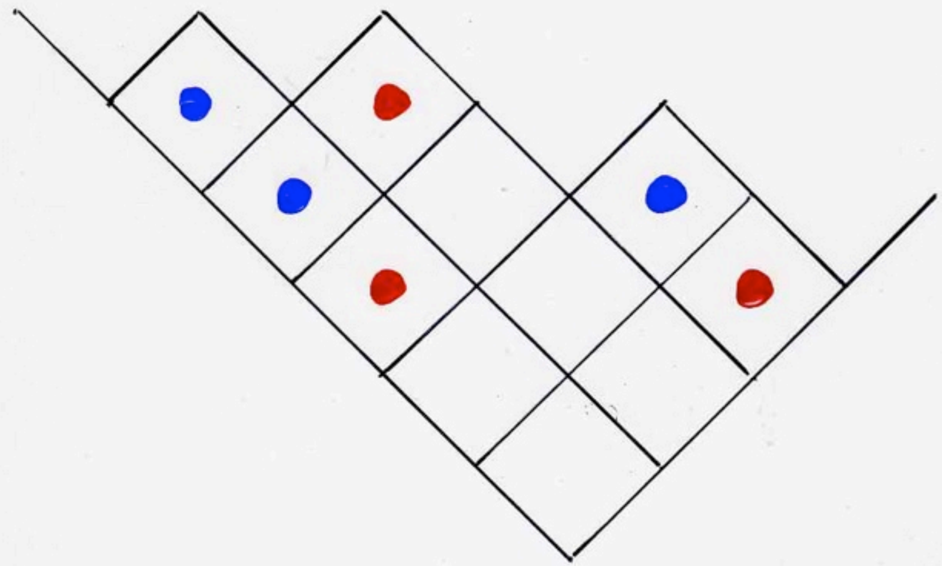


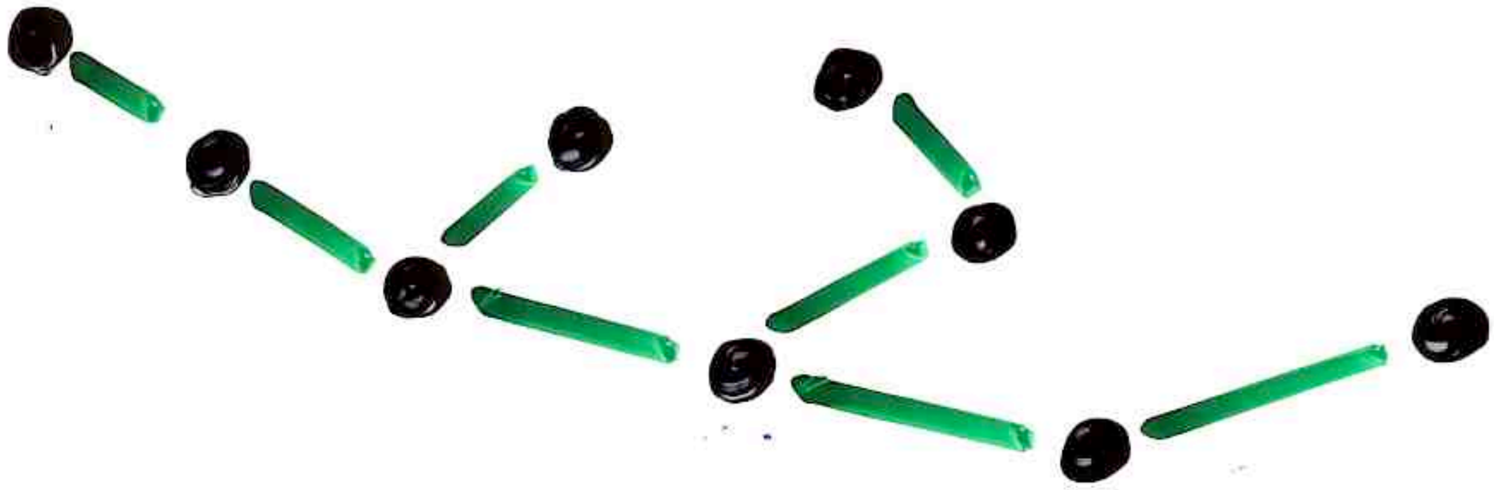
Bijection
alternative Catalan tableaux
binary trees

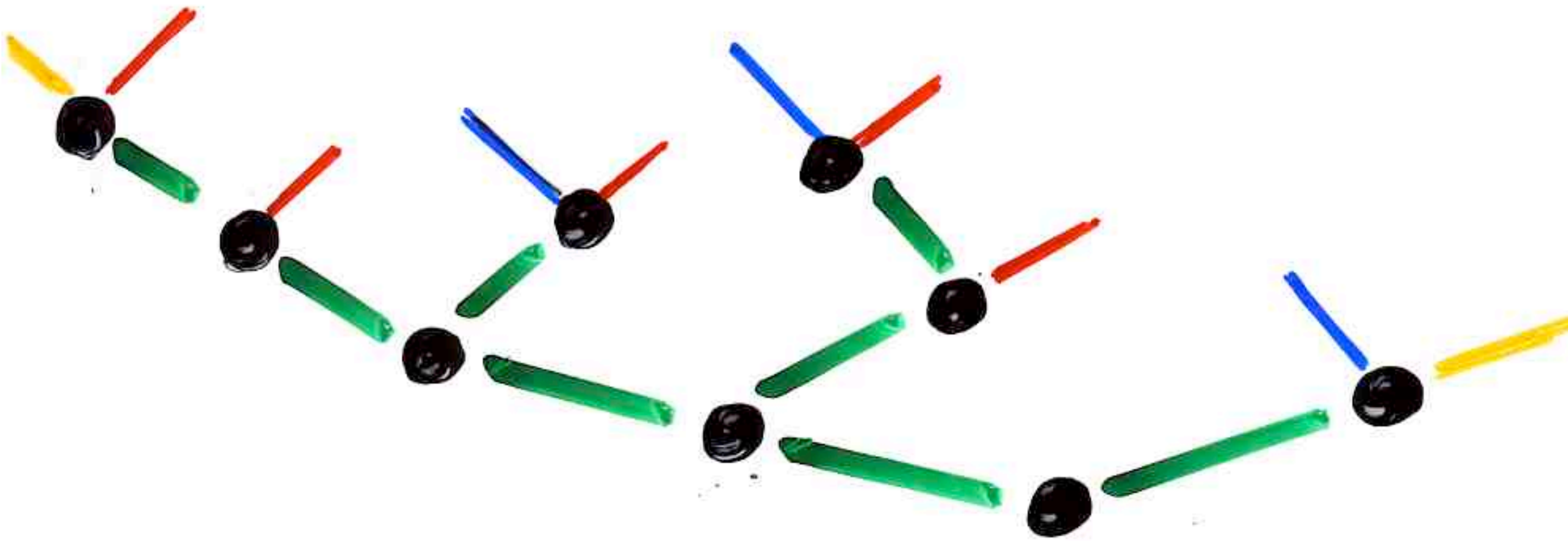


Bijection tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes

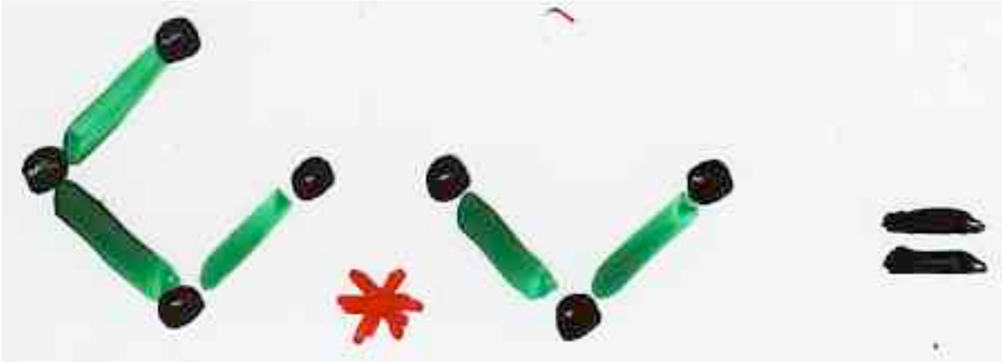


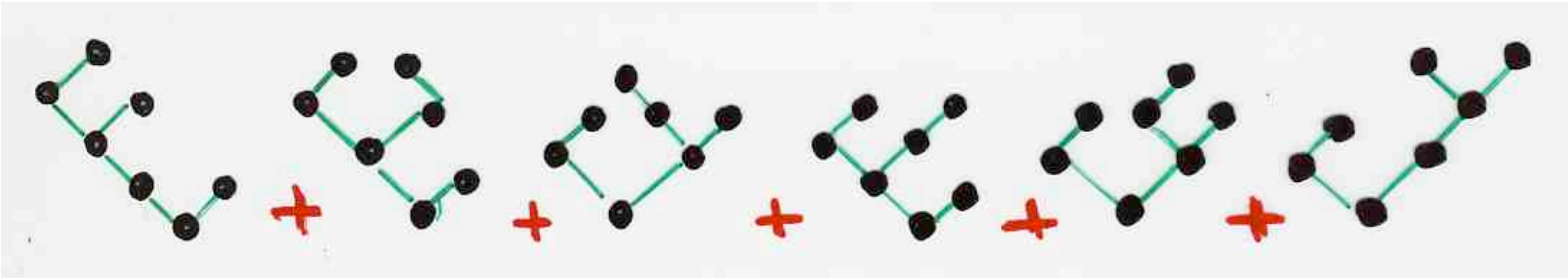






alternative Catalan tableaux
and
Loday - Ronco algebra



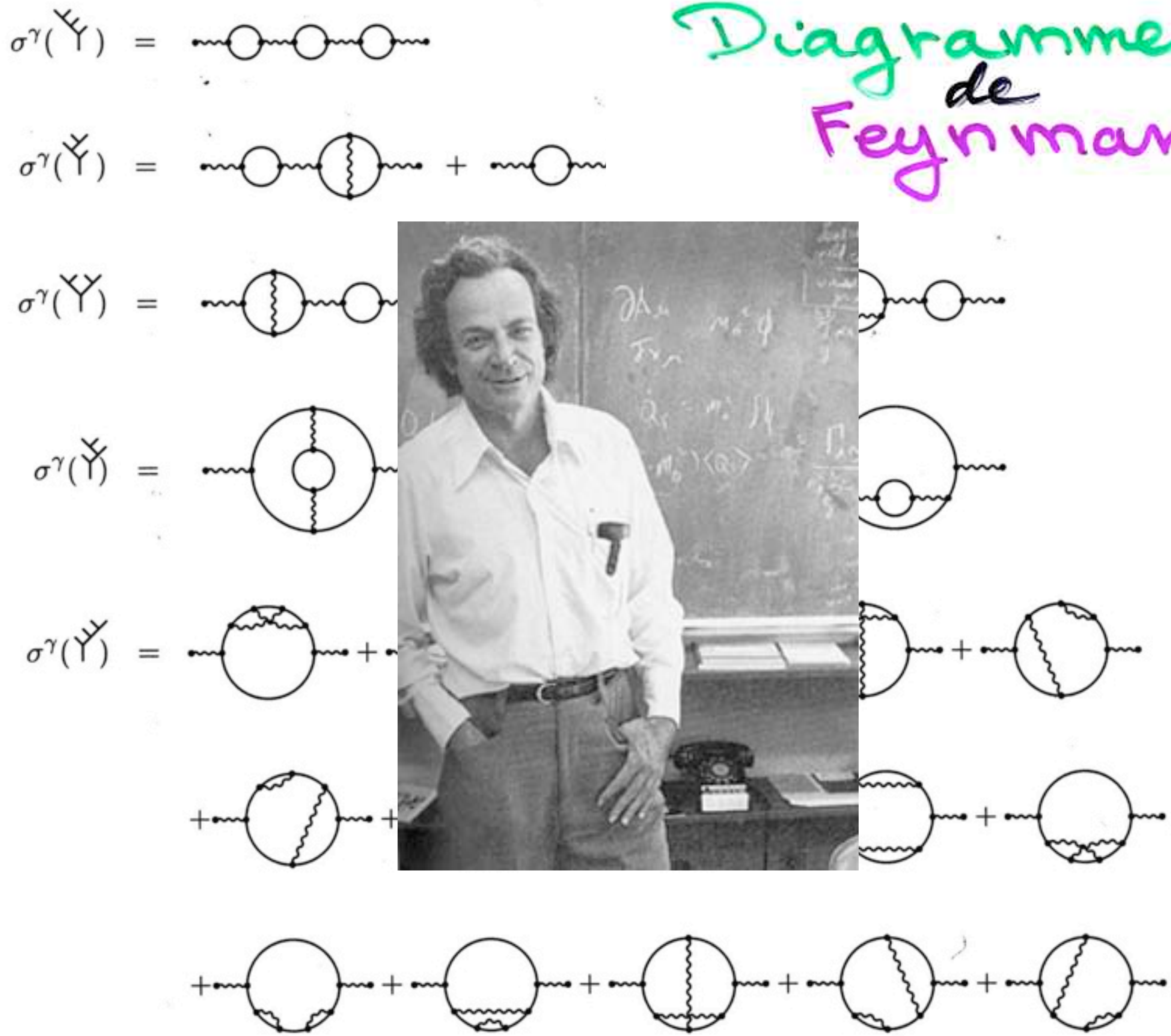


algèbre de Loday-Ronco

algèbre de Hopf
combinatoire


Renormalisation
quantique

Diagrammes de Feynman



"L'Ansatz cellulaire"

objets combinatoires planarisés

représentation par opérateurs

Physique

histoires de fichiers

bijections

polynômes orthogonaux

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

placements de tours

permutations ⟷ paires Tableaux Young

RSK

tableaux alternatifs ⟷ permutations

histoires de Laguerre

algèbre quadratique Q

Q-tableaux

ex: ASM, FPL

pavages, 8-vertex

?

commutations

réécritures

planarisation

automates planaires

- 1. Lundi, après le colloquium
  de l'algèbre UD=DU+Id à la correspondance RSK

- 2 Mercredi après-midi, 14h
  le PASEP, tableaux alternatifs,
  l'Ansatz matriciel et le calcul des probabilités stationnaires,
  de l'algèbre DE=qED+E+D à la bijection
  permutations -- tableaux alternatifs

- 3 Jeudi matin, 9h
  rappels: théorie combinatoire des polynômes orthogonaux,
  liens avec le PASEP
  le TASEP, arbres binaires, algèbre de Loday-Ronco

-4 Jeudi après-midi, 14h
  l'Ansatz cellulaire: théorie générale
  Q-tableaux, automates planaires, algèbre du 8-vertex

pour plus de détails
voir les diaporamas du cours donné à Talca:

Cours XGV, Universidad de Talca

(December 2010 - January 2011)

Combinatorics and interactions (with physics) (24h)

«The Cellular Ansatz»

accessible sur les sites:

http://www.labri.fr/perso/viennot/

Recherche, cv, publications, exposés, diaporamas, livres, petite école, photos: voir ma page personnelle ici

Vulgarisation scientifique voir la page de l'association Cont'Science

http://web.me.com/xgviennot/Xavier_Viennot/

http://web.me.com/xgviennot/Xavier_Viennot2/

"Petite école de combinatoire"

LaBRI, 2011-2012

Cours XGV
Universidad de Talca

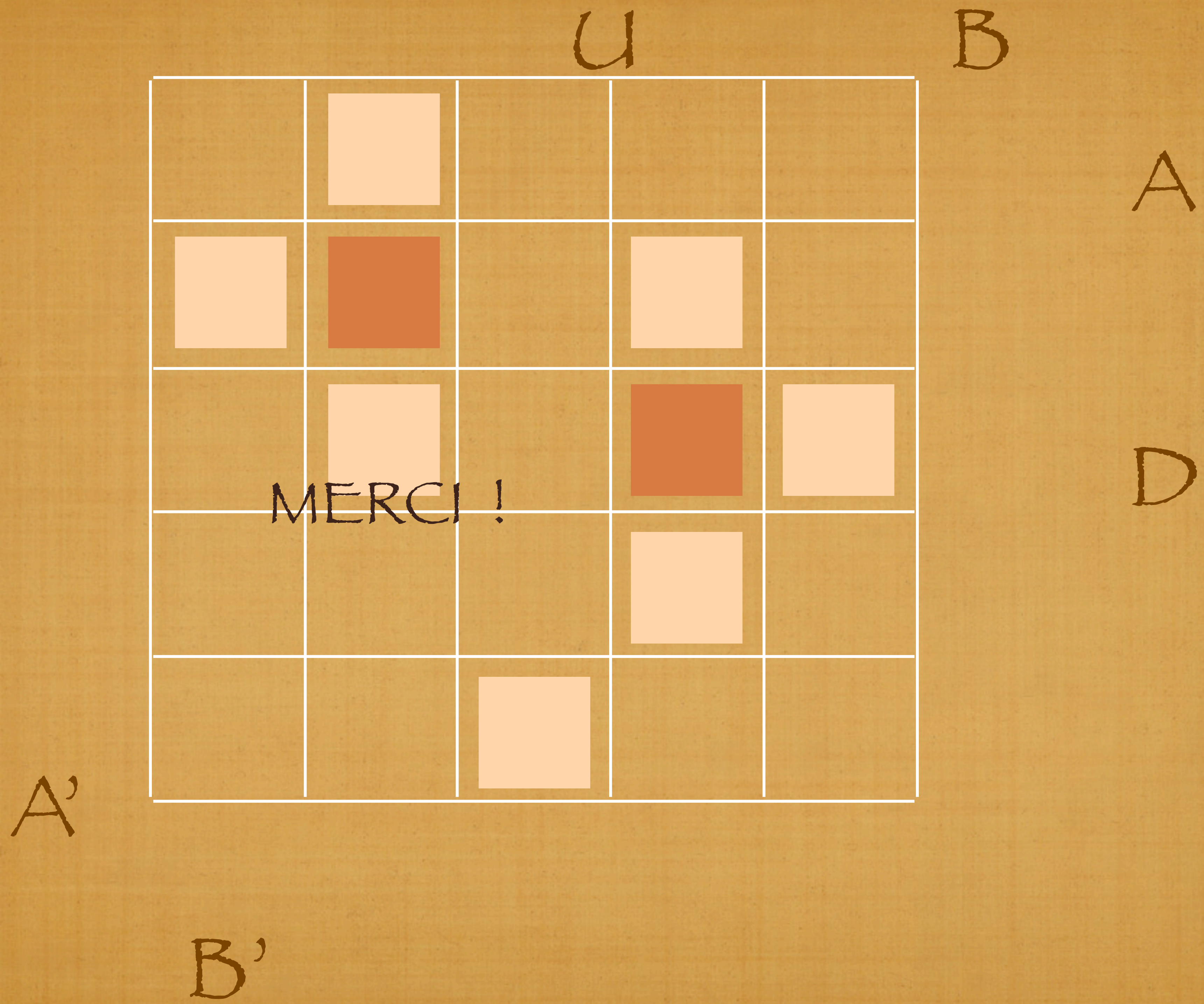
(December 2010 - January 2011)
24 h

Combinatorics and interactions
(with physics)
«The Cellular Ansatz»

Universidad de Talca

La Universidad de Los mejores

www.utalca.cl


U
B
A
MERCI !
D
A'
B'

Young tableaux　　hook-length

alg combin: repr Sn　　bij combin: RSK

comb phys: spins chain　　ASM　　enum ASM

TSSCPP　　6-vertex　　R-S conj

UD=DU+I　　C.A. (1) for UD=DU+I

Q for ASM

PASEP　　C.A. (1) for DE=ED+E+D

C.A. (2)　　representation U,D　　RSK: insertions

TASEP　　binary trees　　Loday-Ronco

plan du cours　　Cours 1　　complements