Physique combinatoire
et
algèbres quadratiques

"L'Ansatz cellulaire"

Cours 1 (suite du colloquium)

7 Novembre 2011

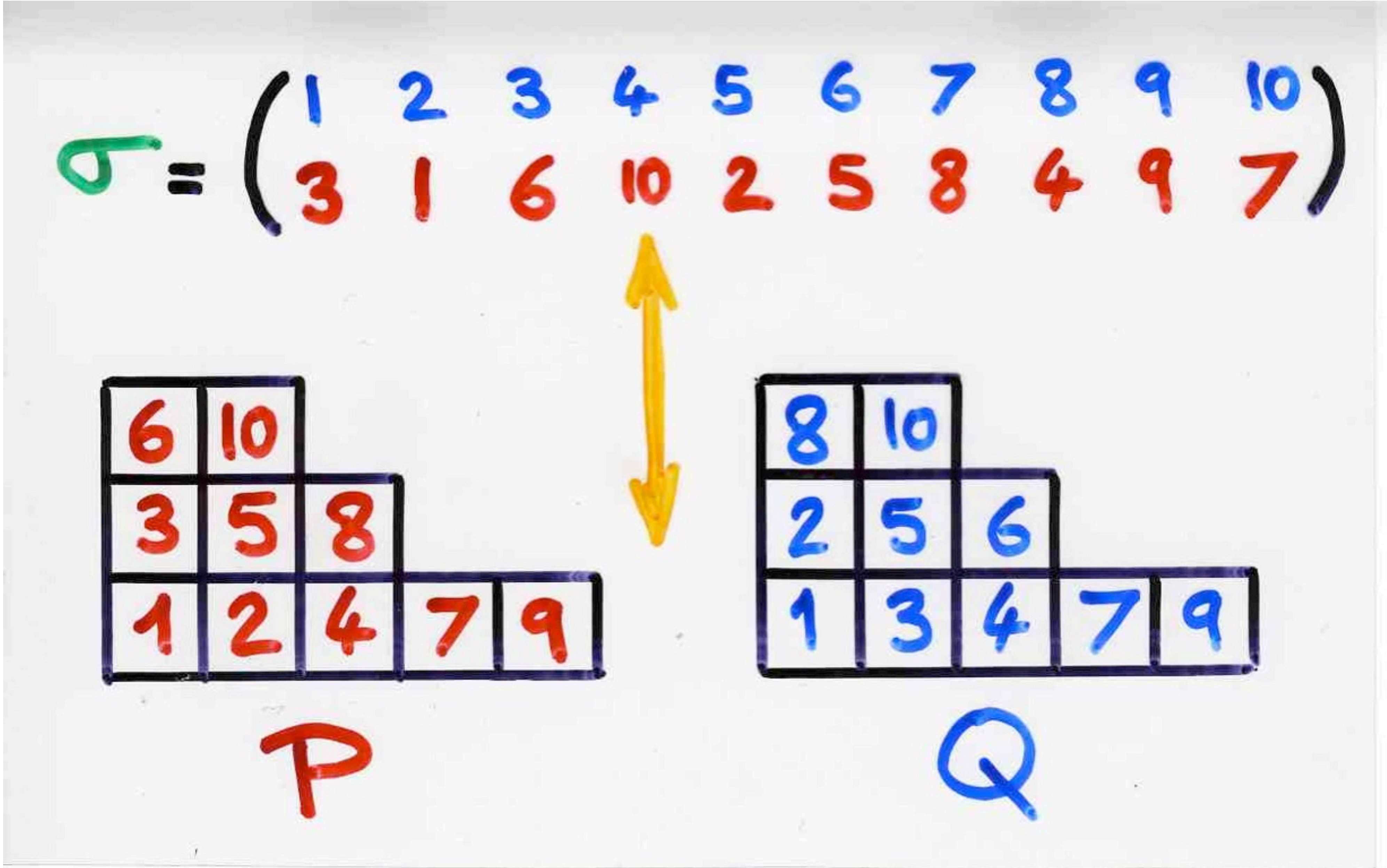
Nice

Xavier Viennot

LaBRI, CNRS, Bordeaux

An introduction to RSK

G. de B. Robinson, 1938
C. Schensted, 1961



The Robinson-Schensted correspondence (RSK) between permutations and pair of (standard) Young tableaux with the same shape

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

3					

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2					
1					

3					
1					

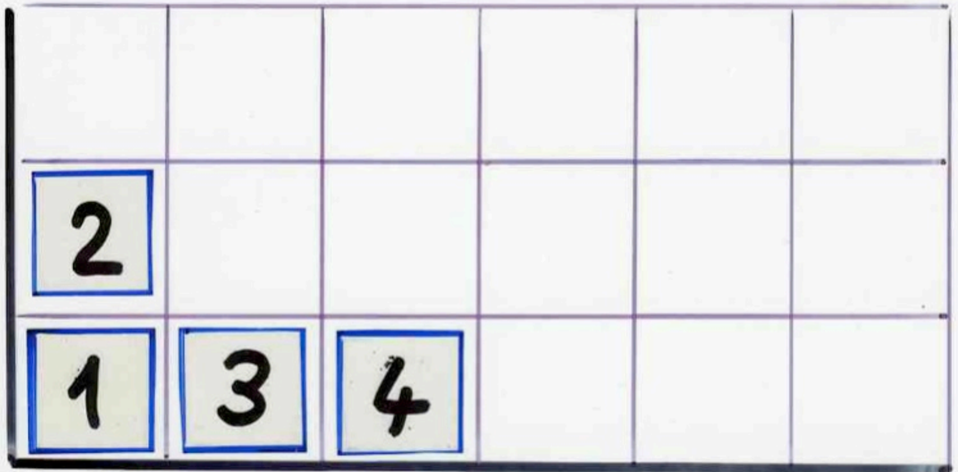
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

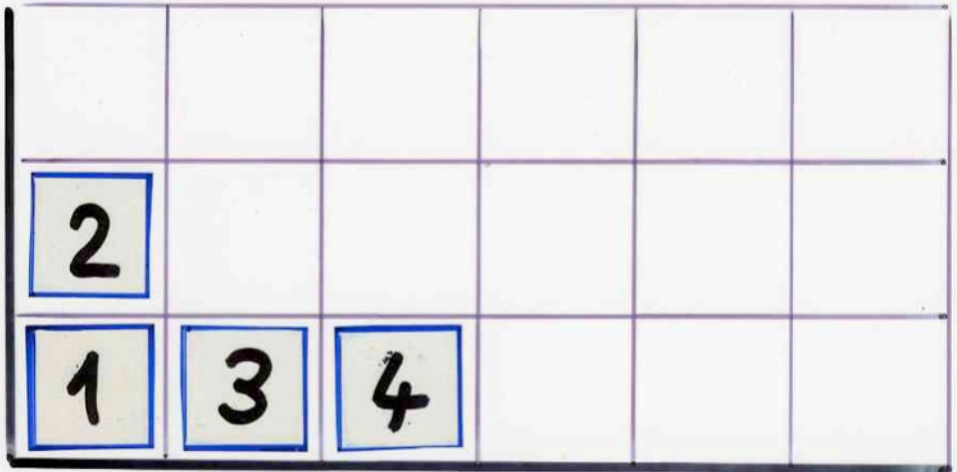
2					
1	3	4			

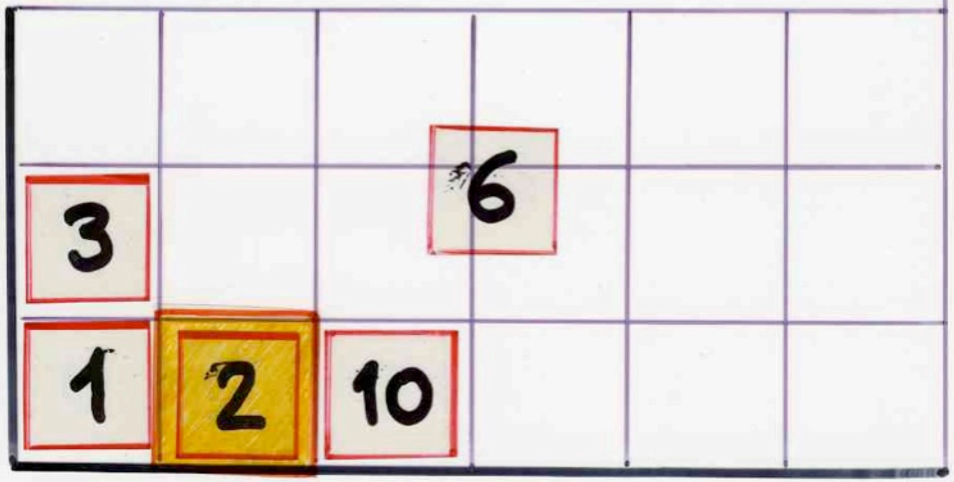
3					
1	6	10			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

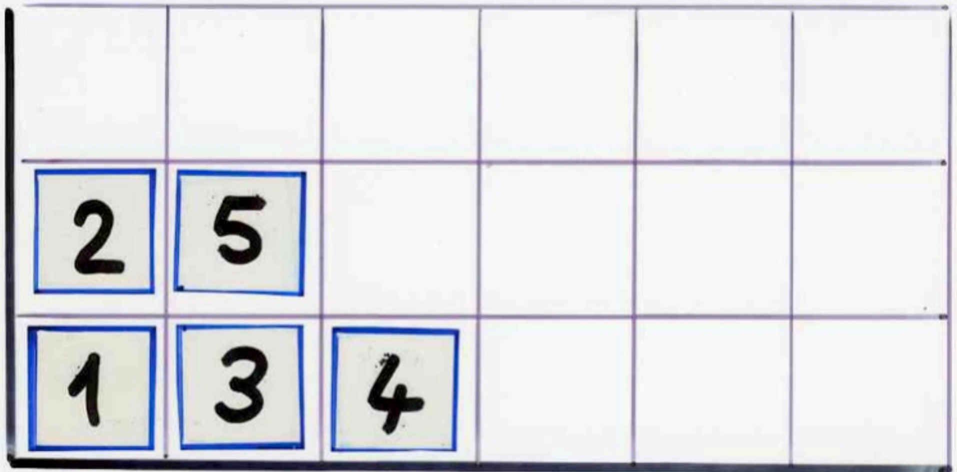


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

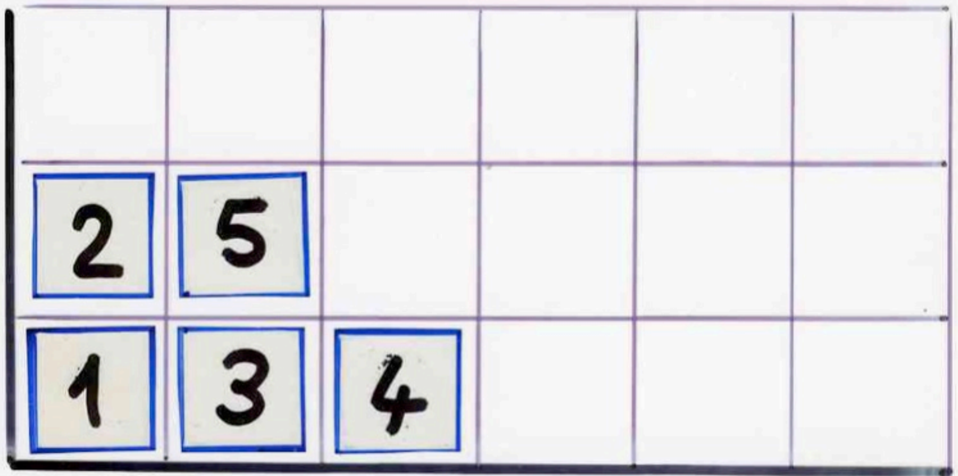


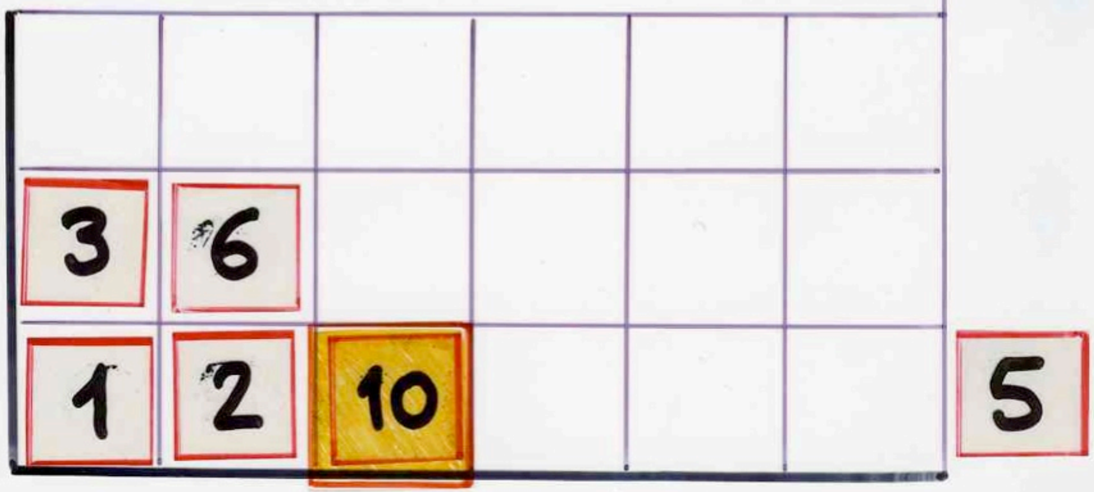


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

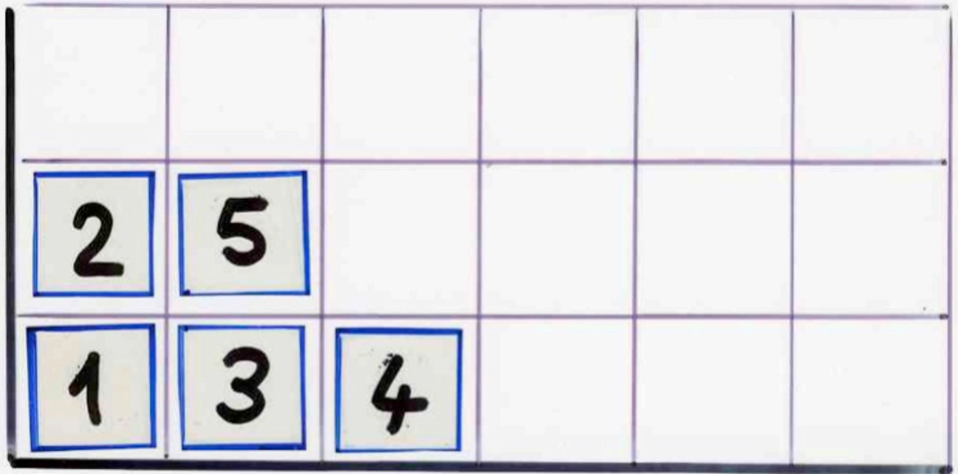


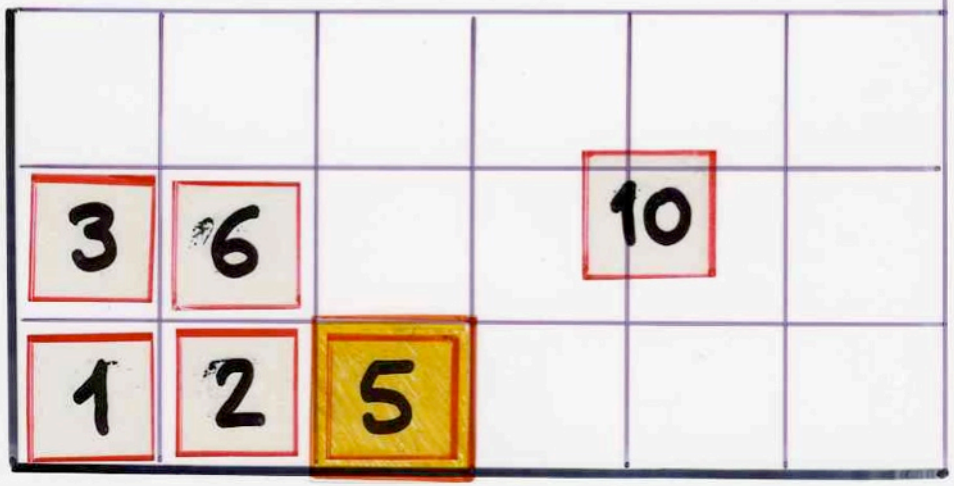
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

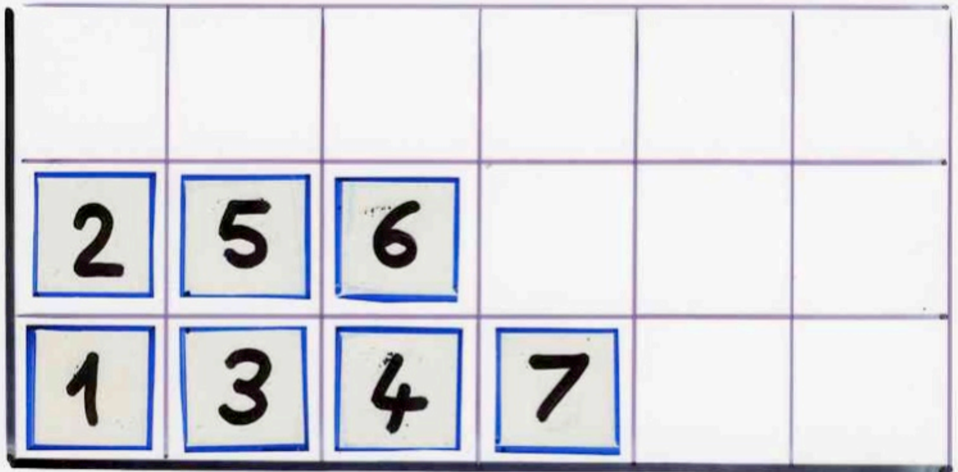
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

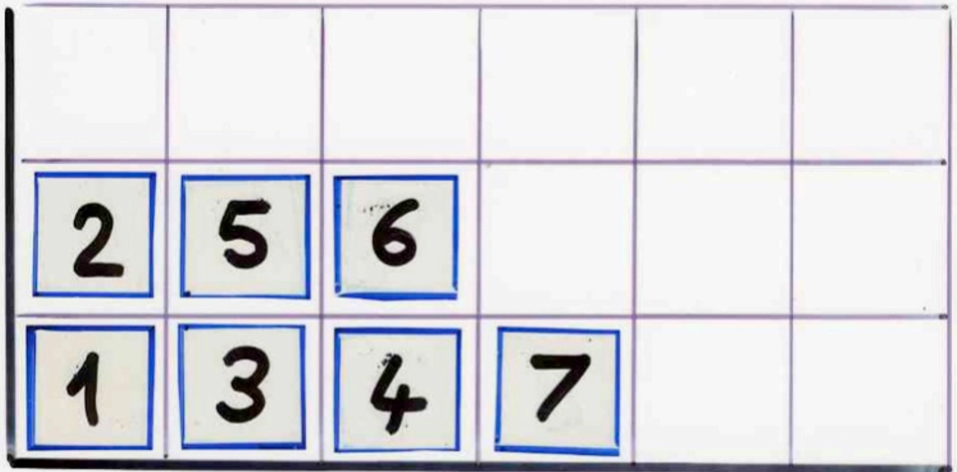
2	5	6			
1	3	4	7		

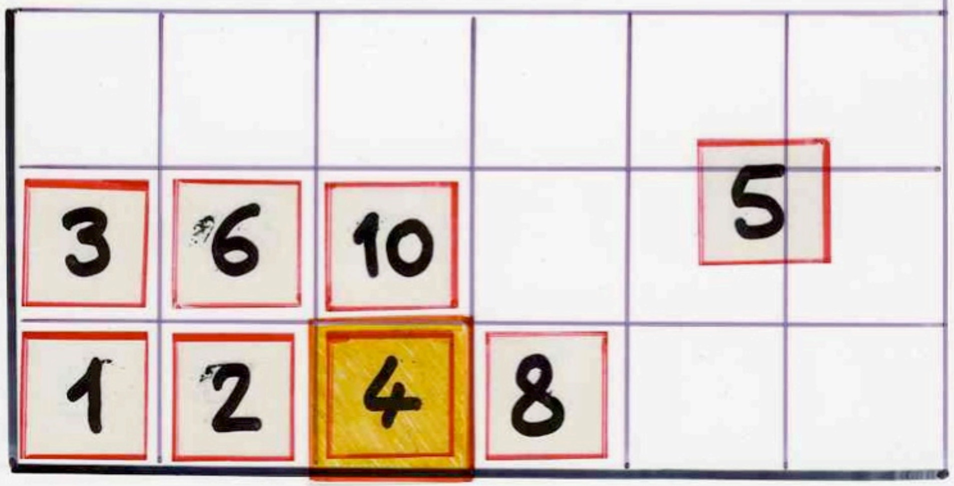
3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

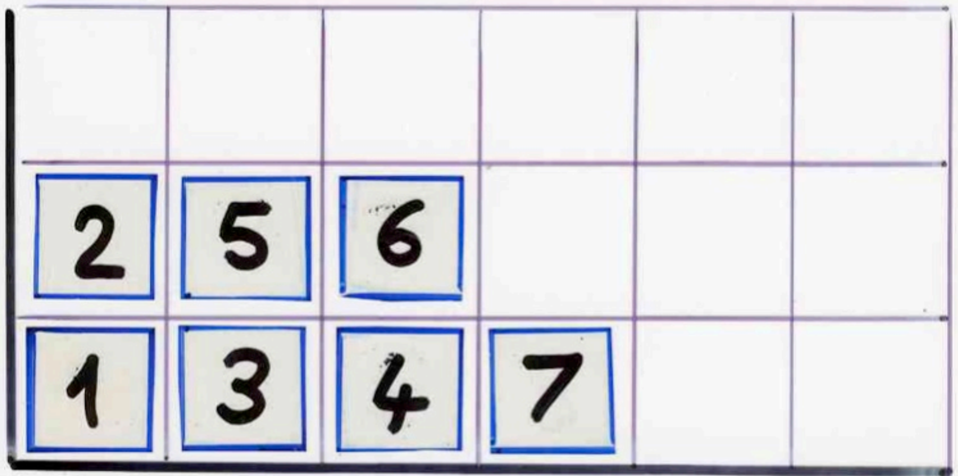


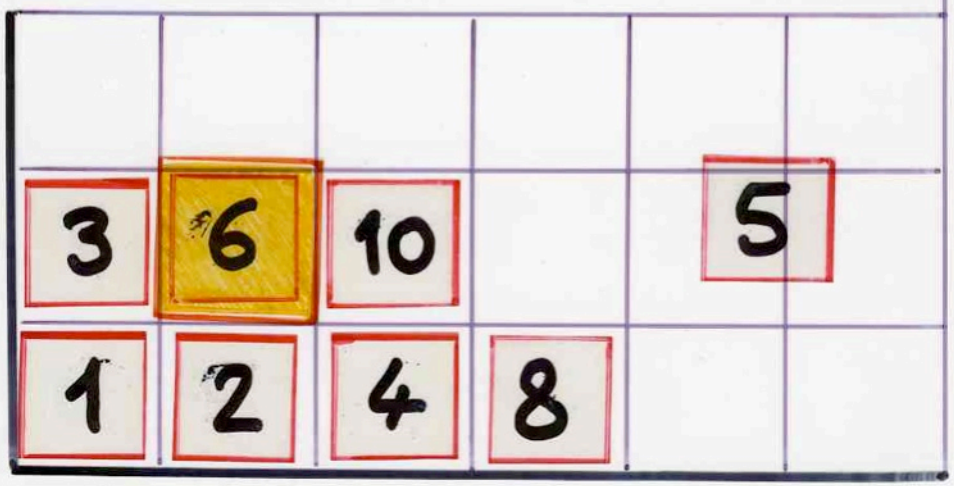
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



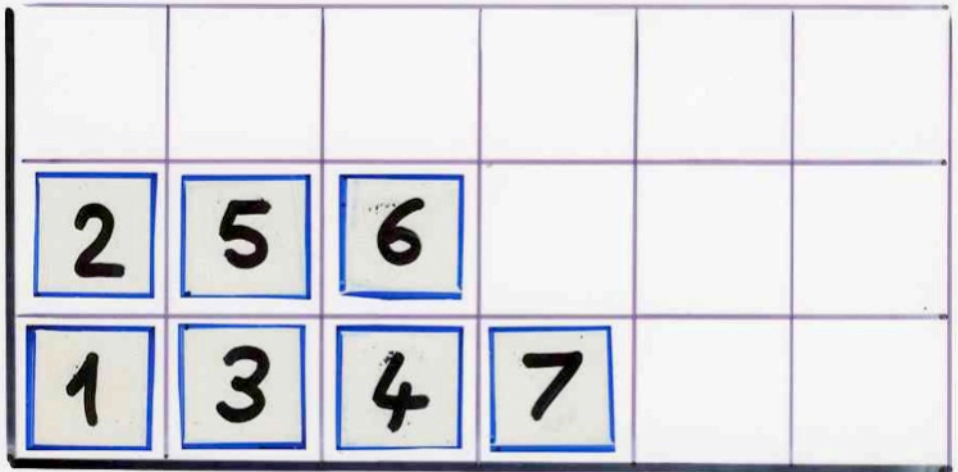


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

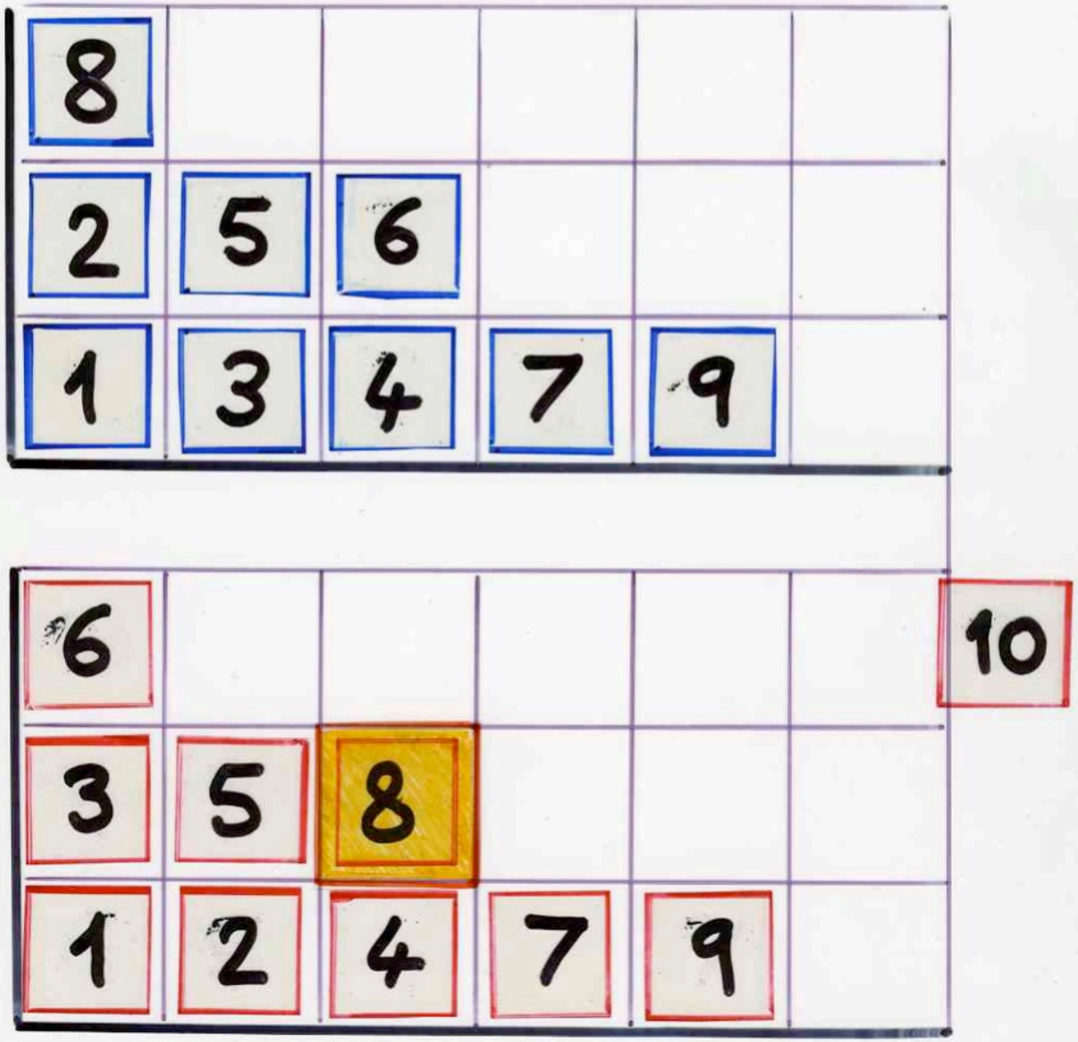
6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

6	10				
3	5	8			
1	2	4	7	9	

$$\sigma \longleftrightarrow (P, Q)$$

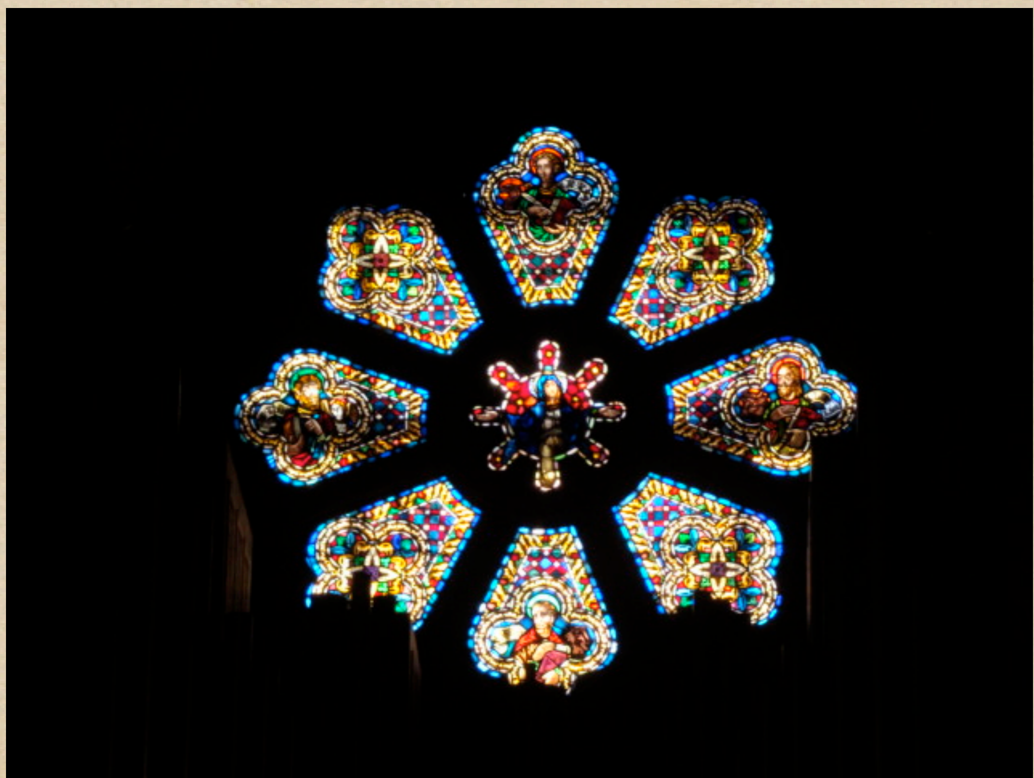
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

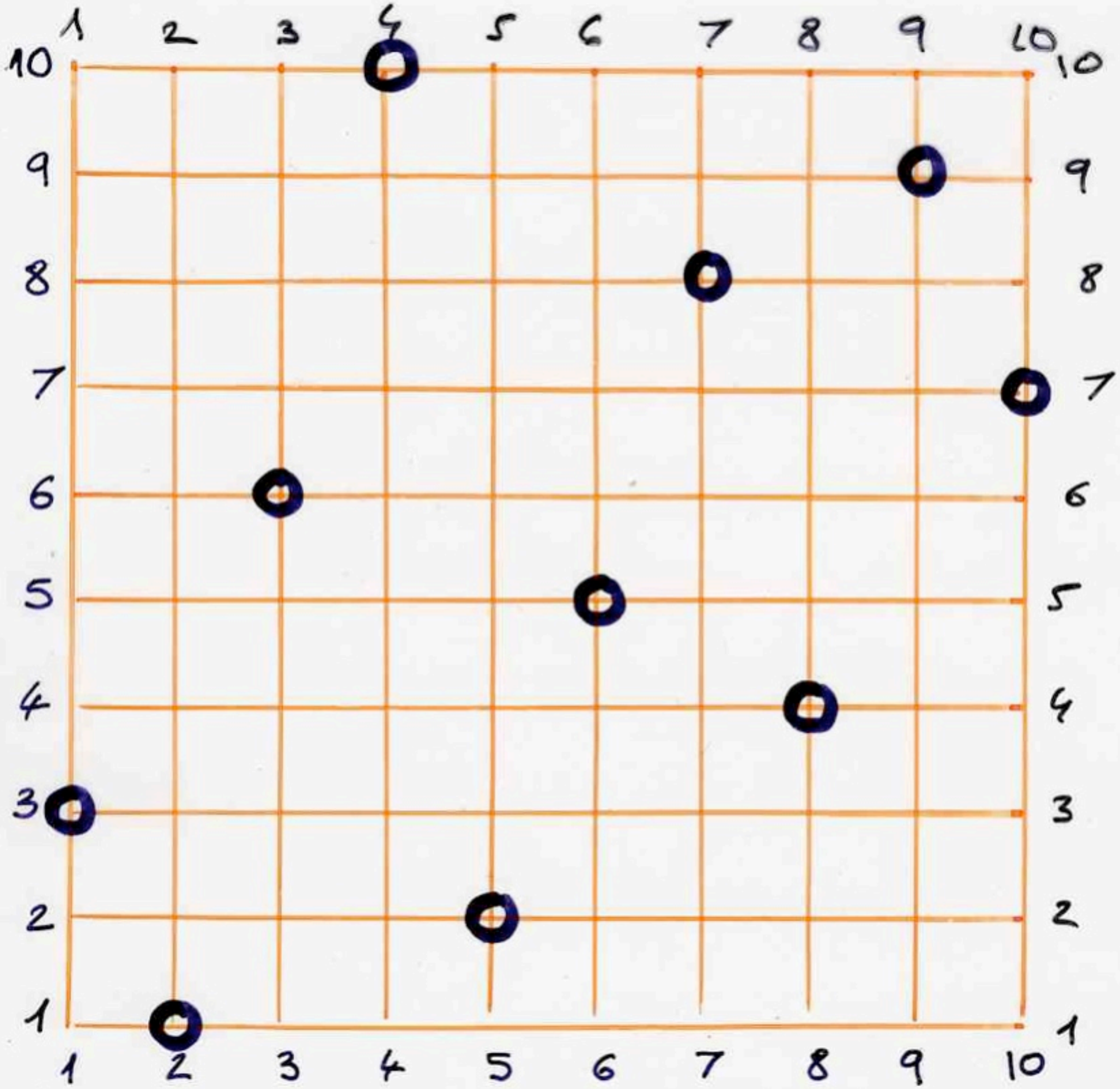
Donald Knuth (1972)

" The unusual nature of these coincidences might lead us to suspect that some sort of withcraft is operating behind the scenes "

A geometric version of RSK with "light" and "shadow lines"

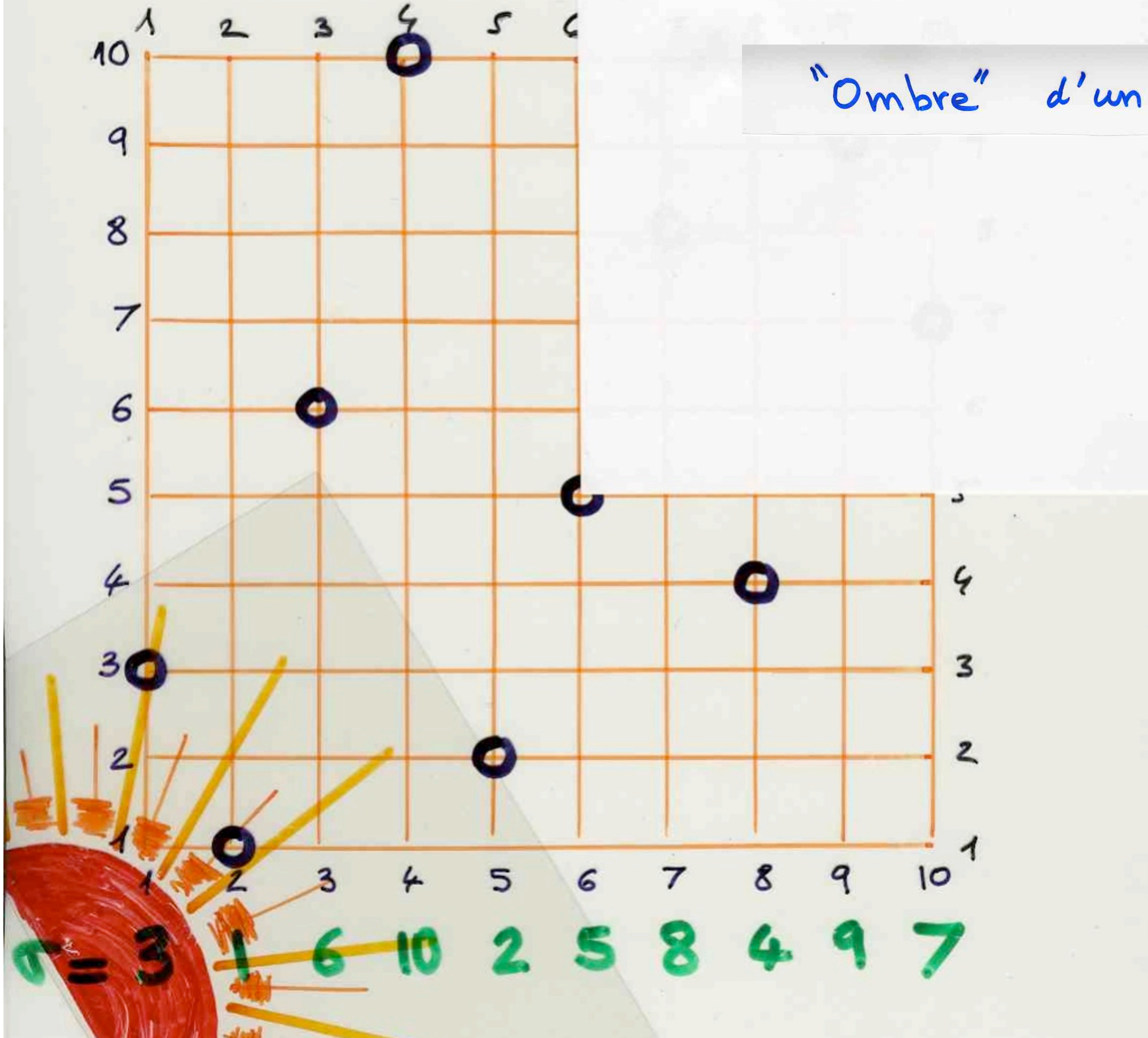
xgv, 1976





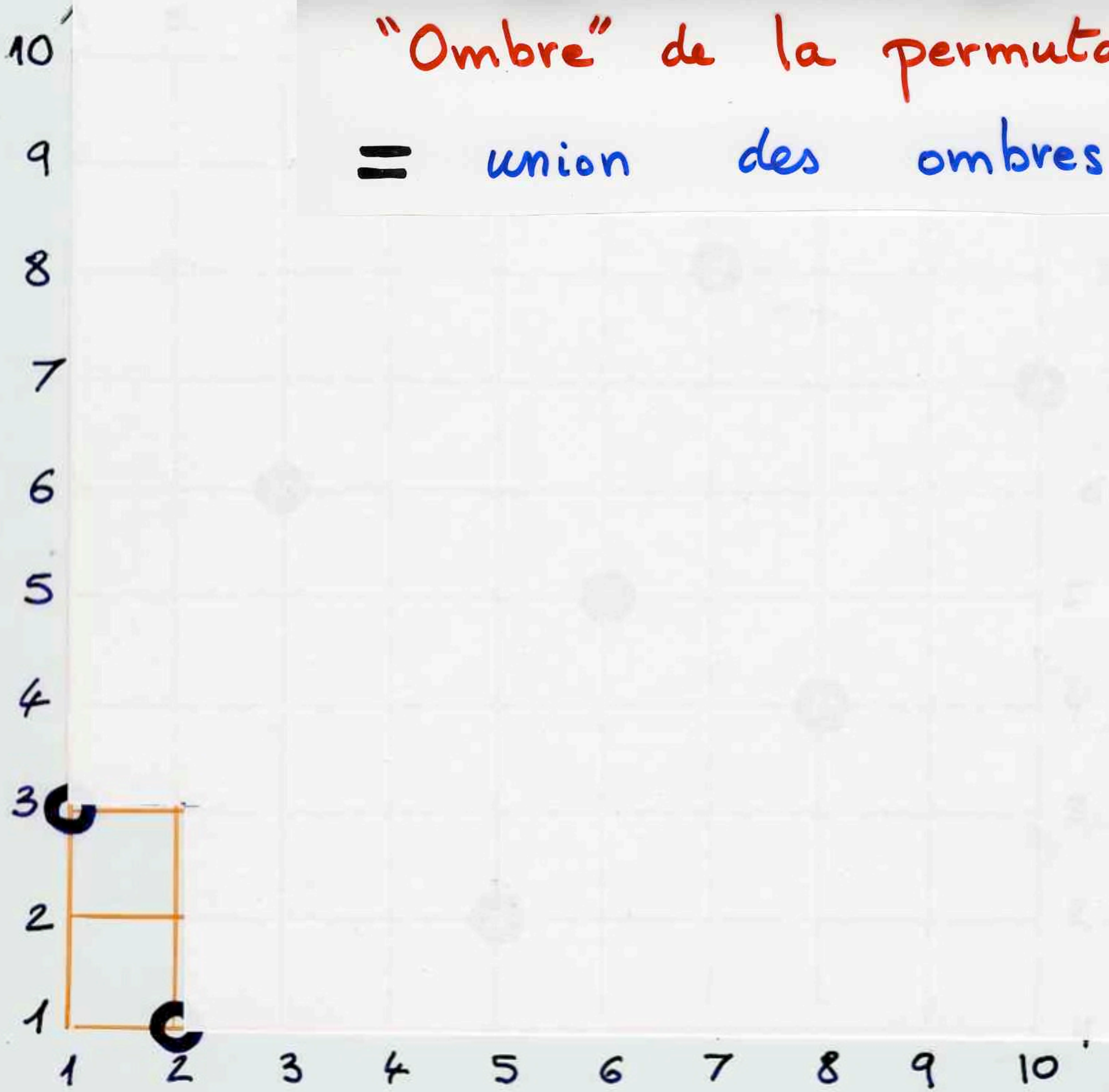
$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

"Ombre" d'un point

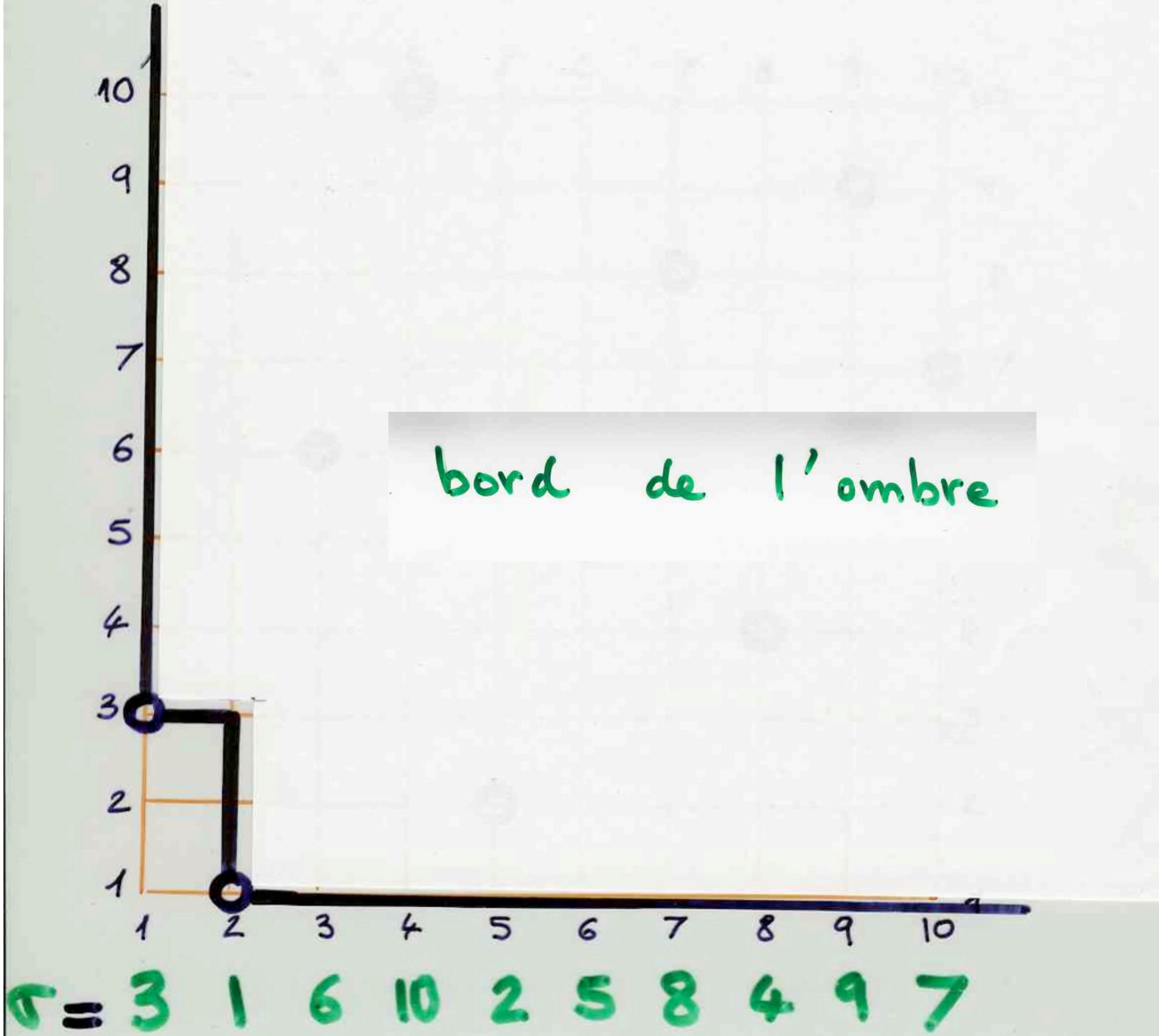


"Ombre" de la permutation

= union des ombres

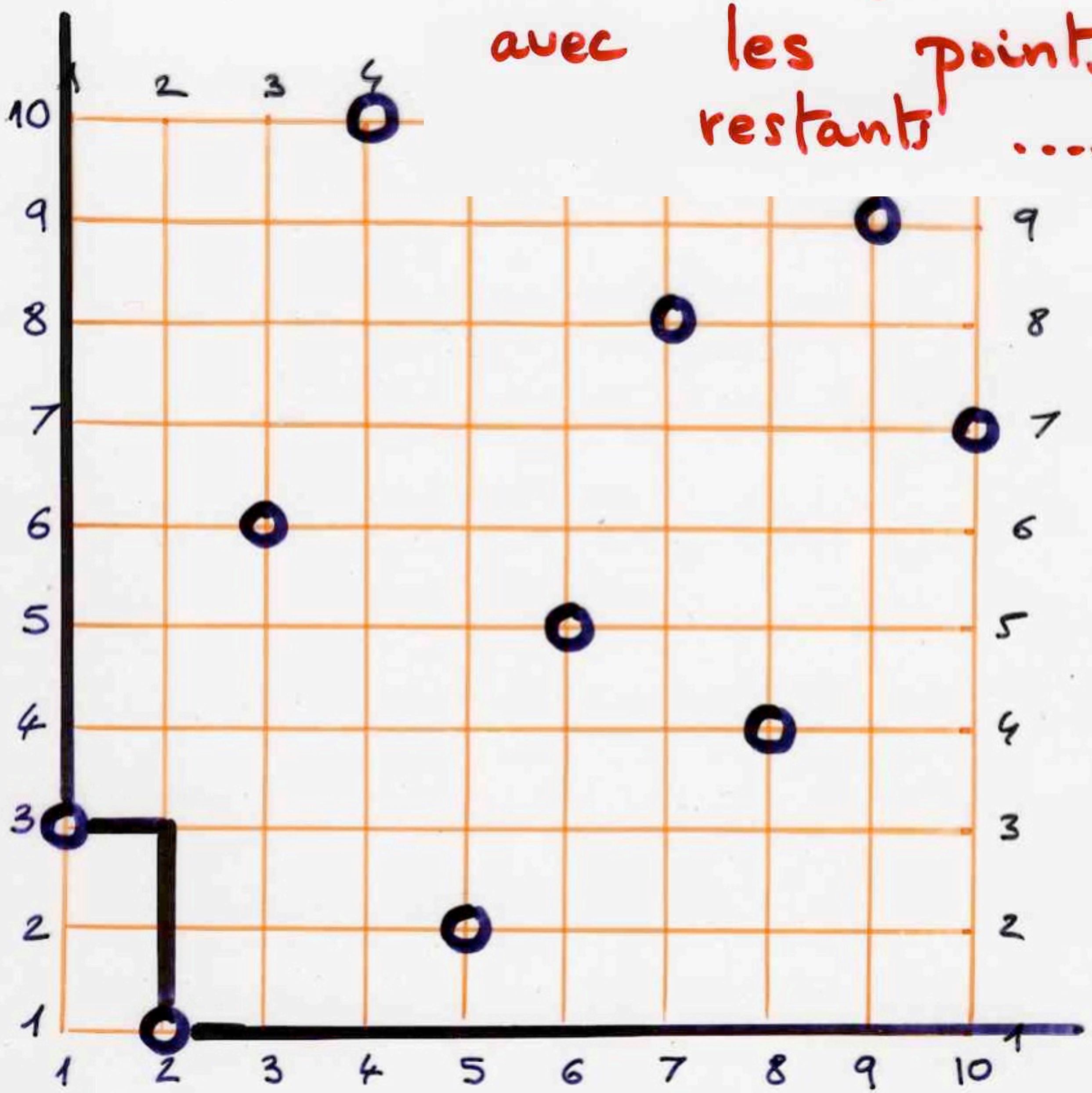


σ = 3 1 6 10 2 5 8 4 9 7

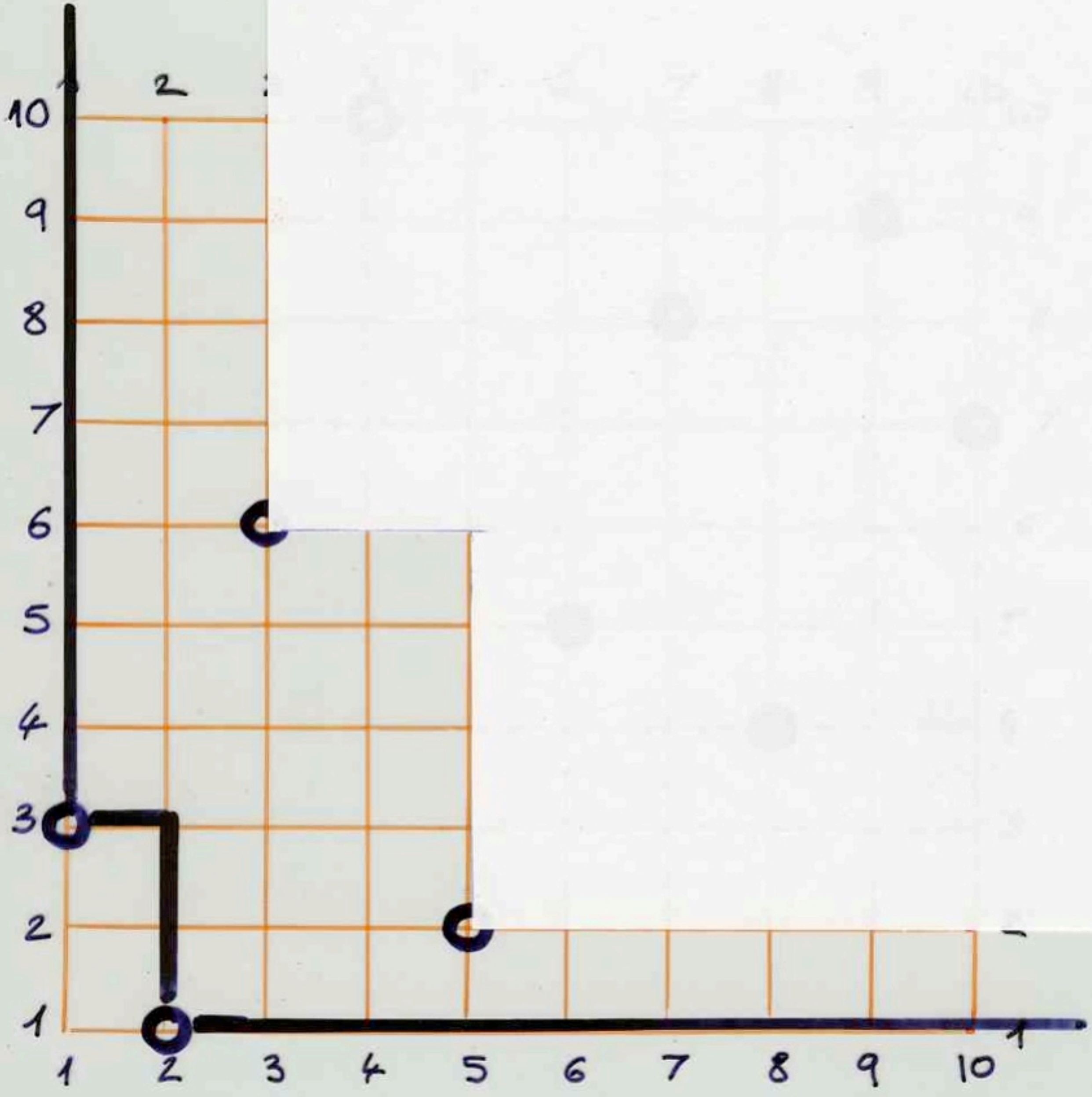


10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10
bord de l'ombre
σ = 3 1 6 10 2 5 8 4 9 7

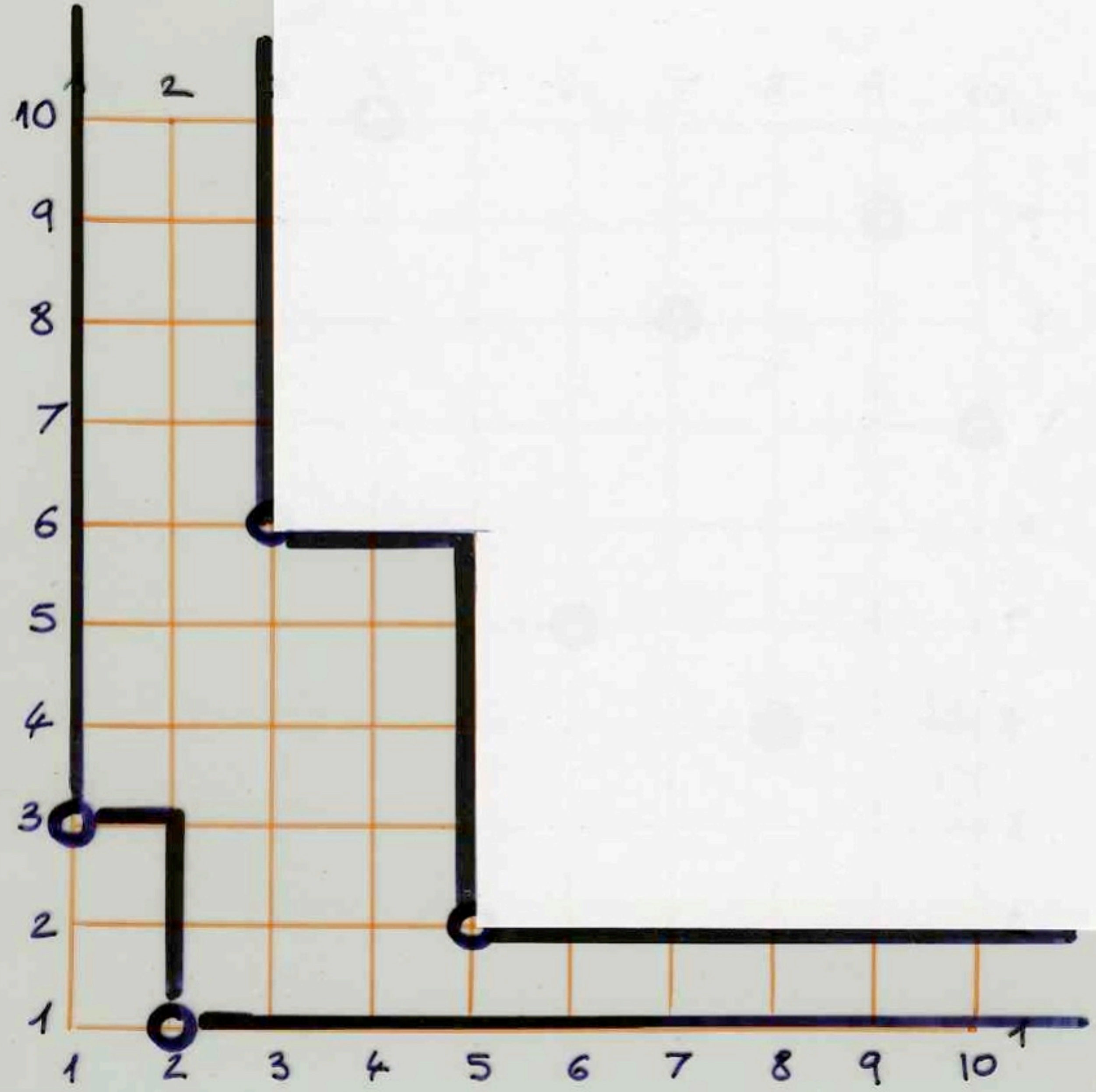
recommençons avec les points restants ....



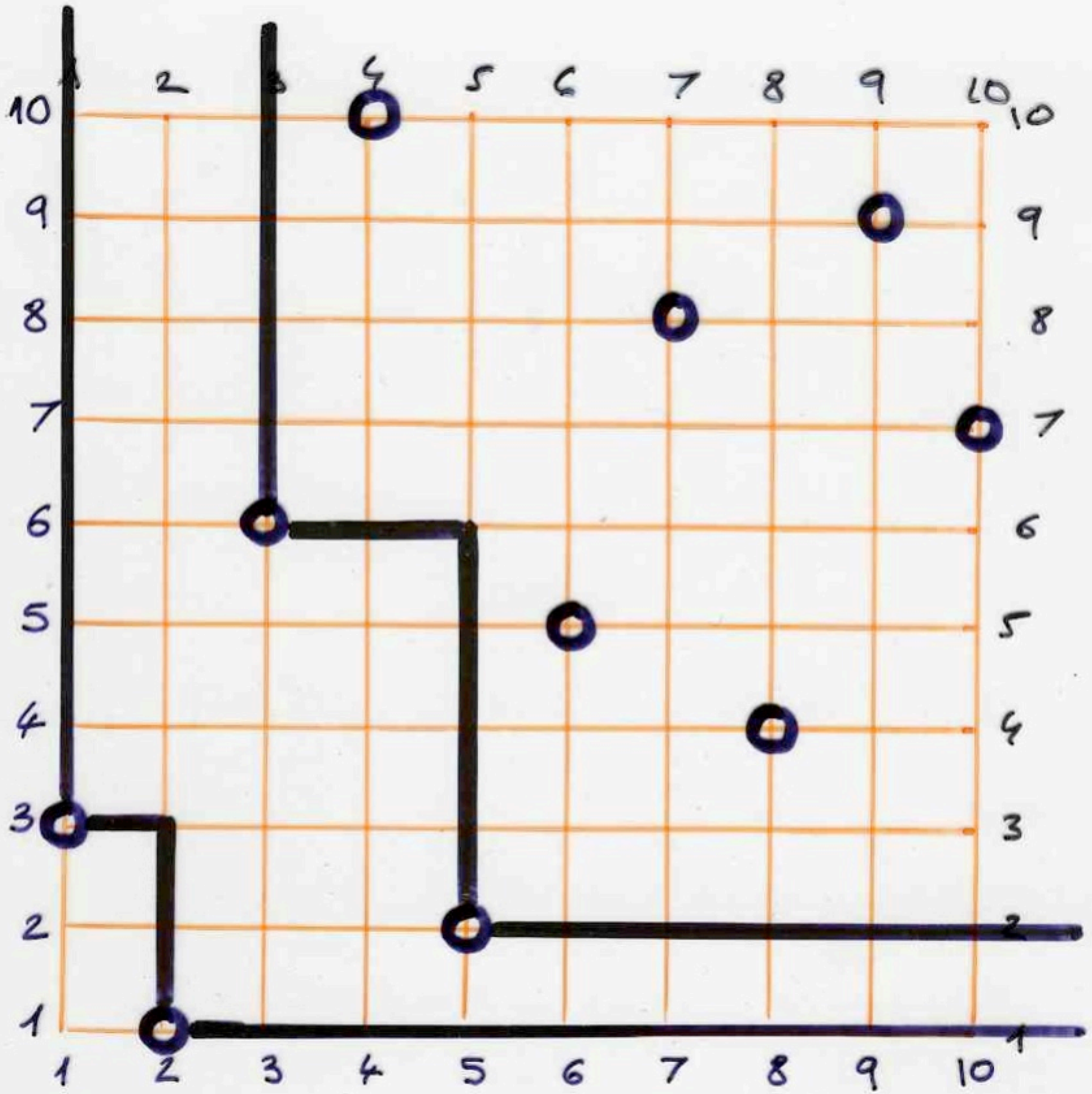
σ = 3 1 6 10 2 5 8 4 9 7



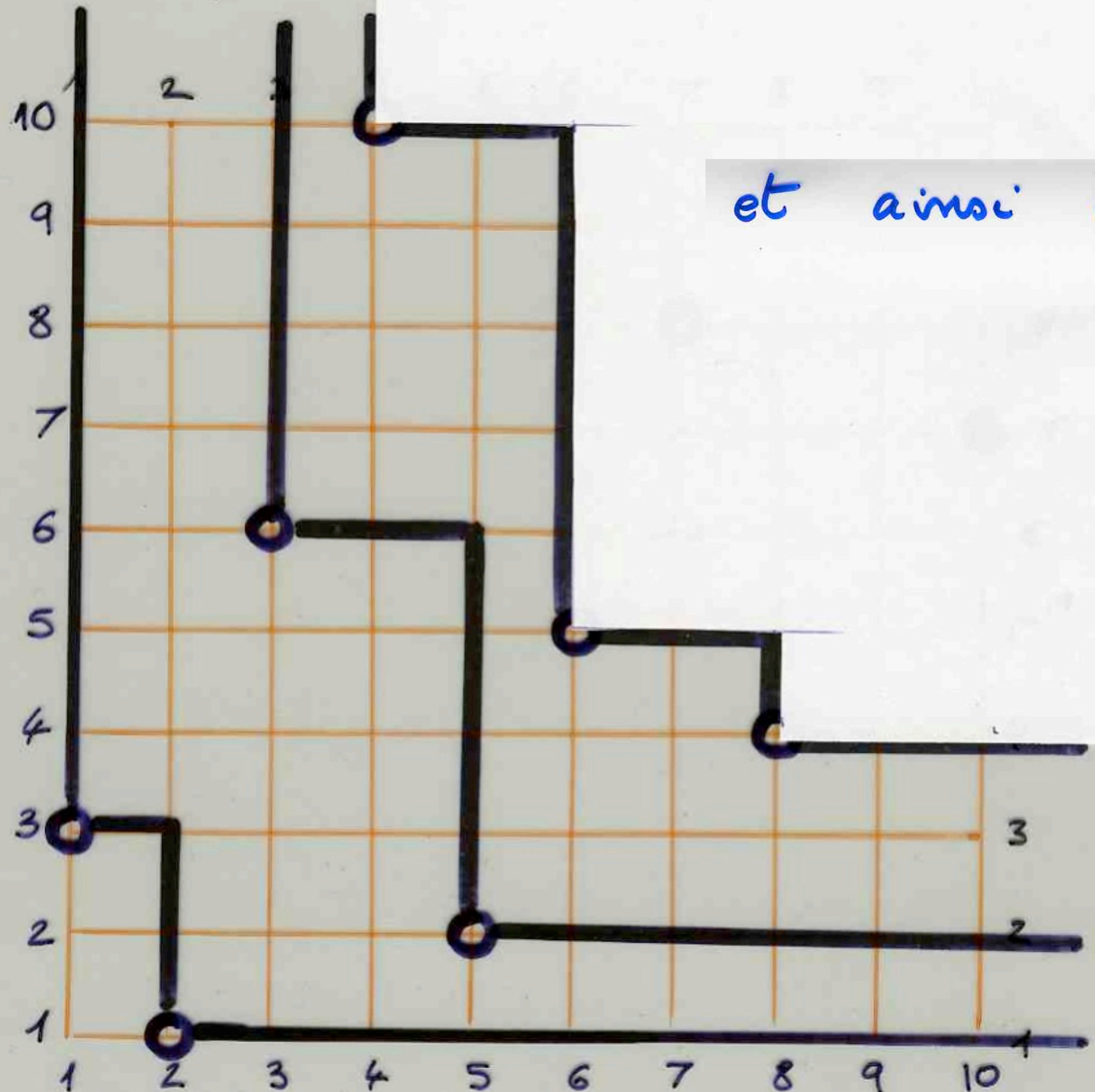
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

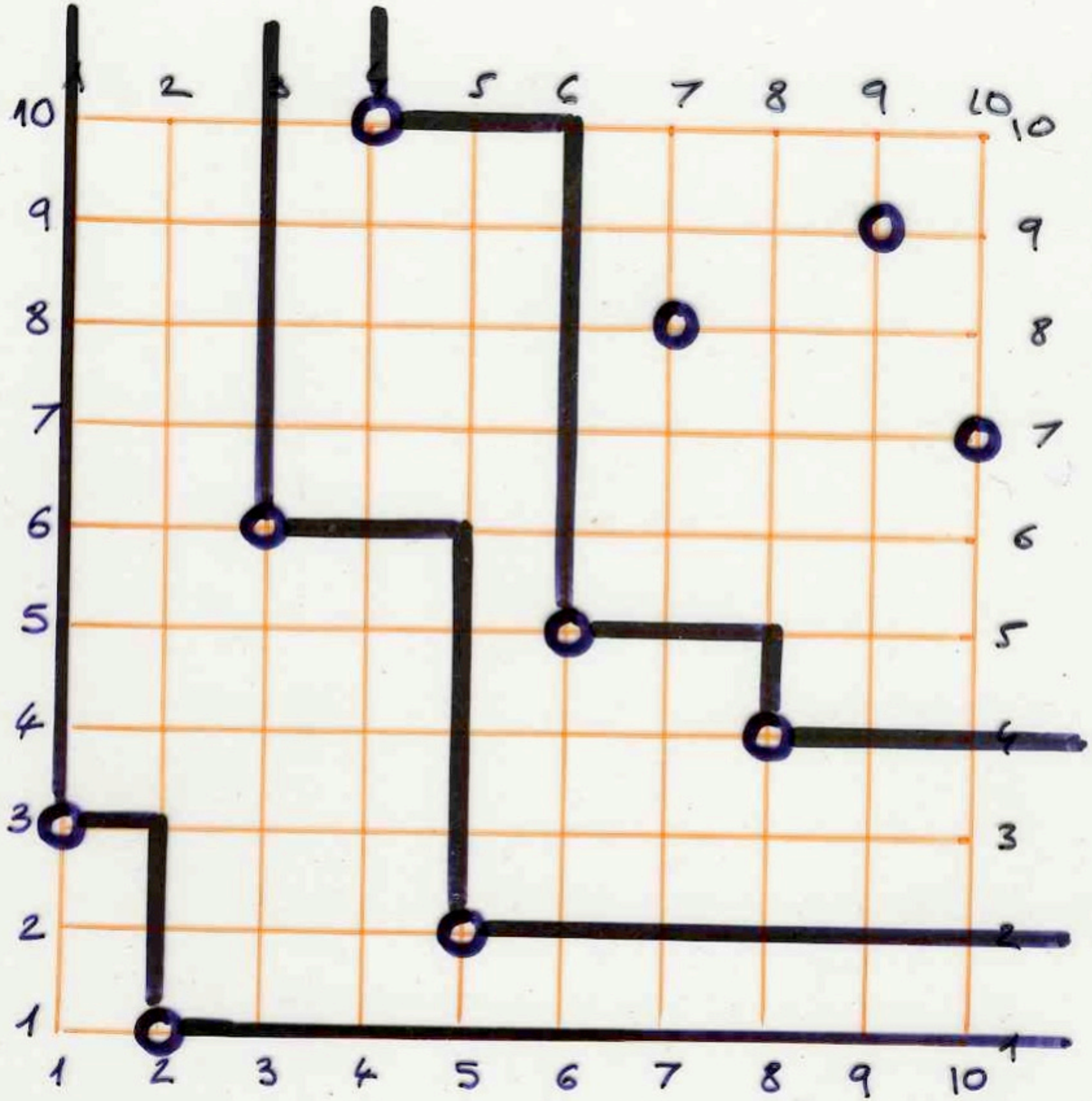


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

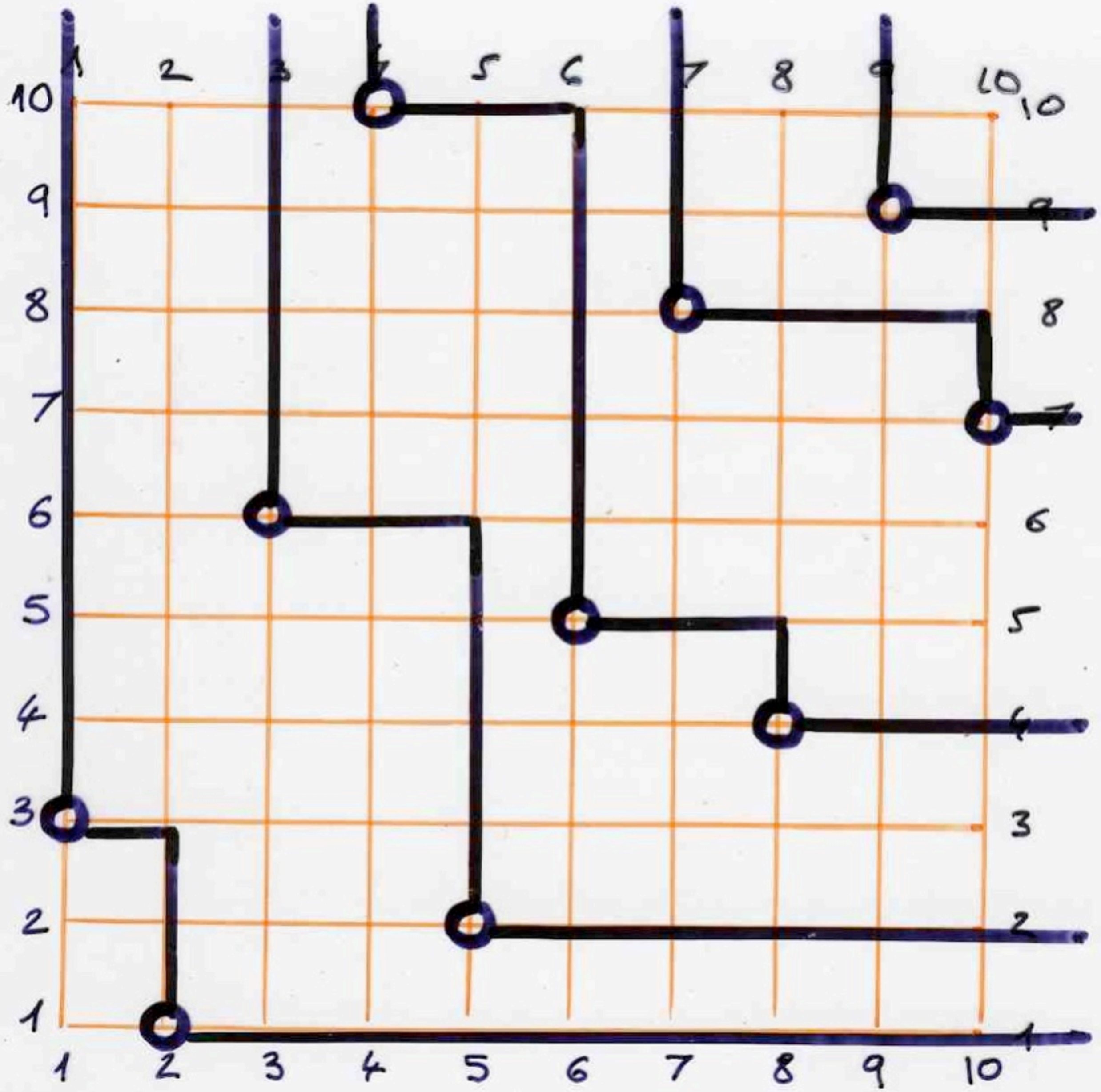


et ainsi de suite ...

$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

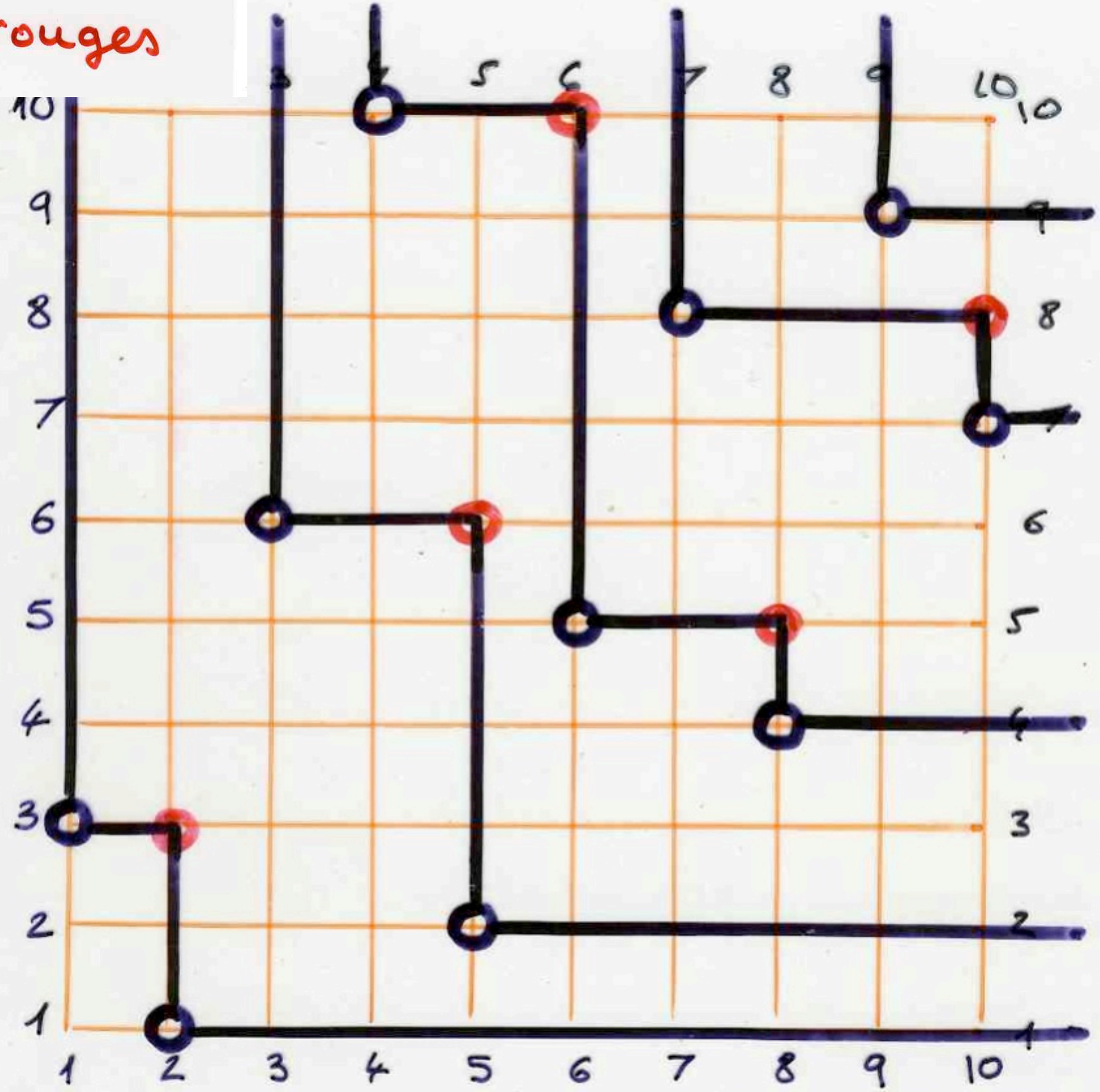


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

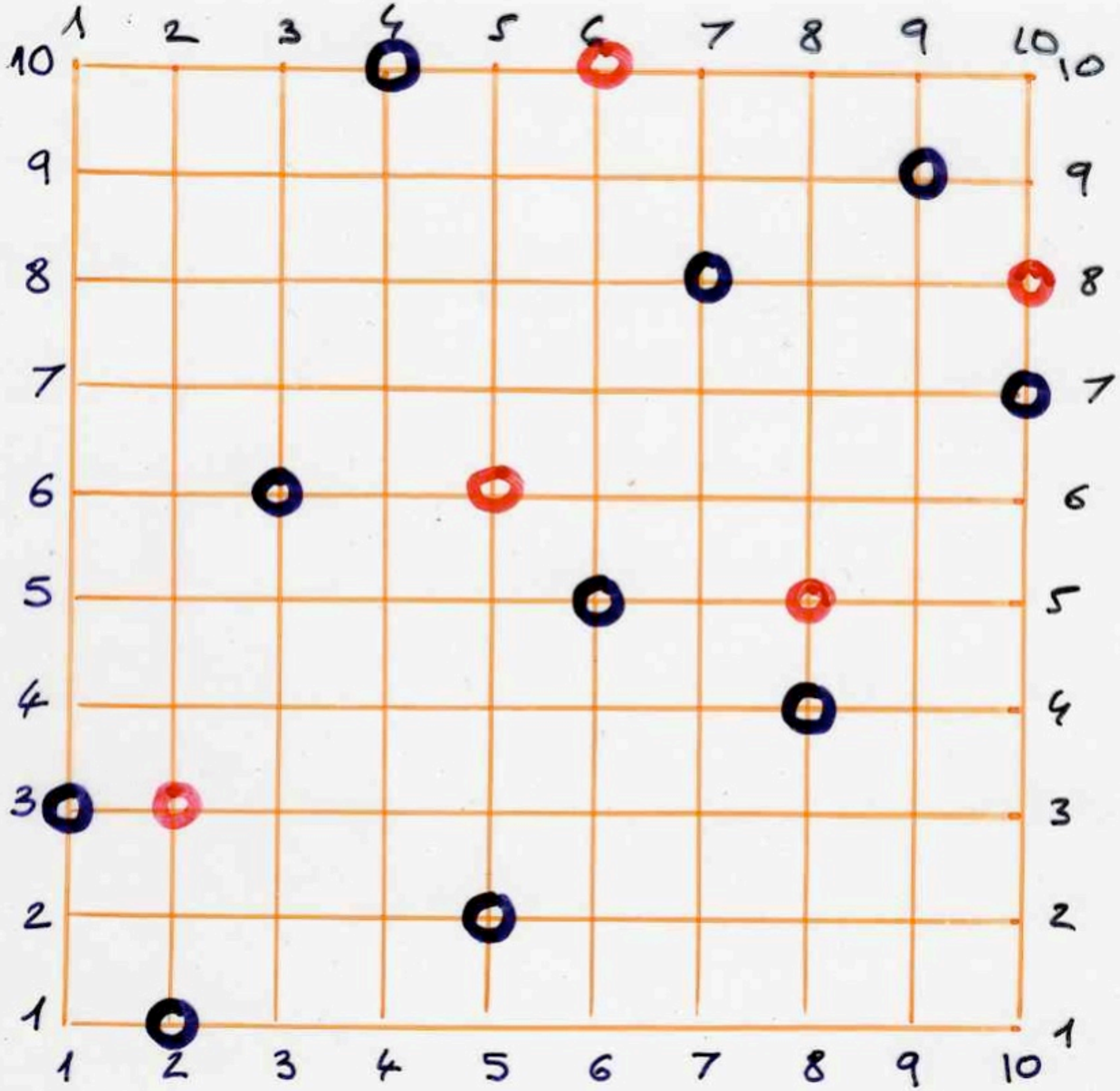


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

des nouveaux points
les rouges

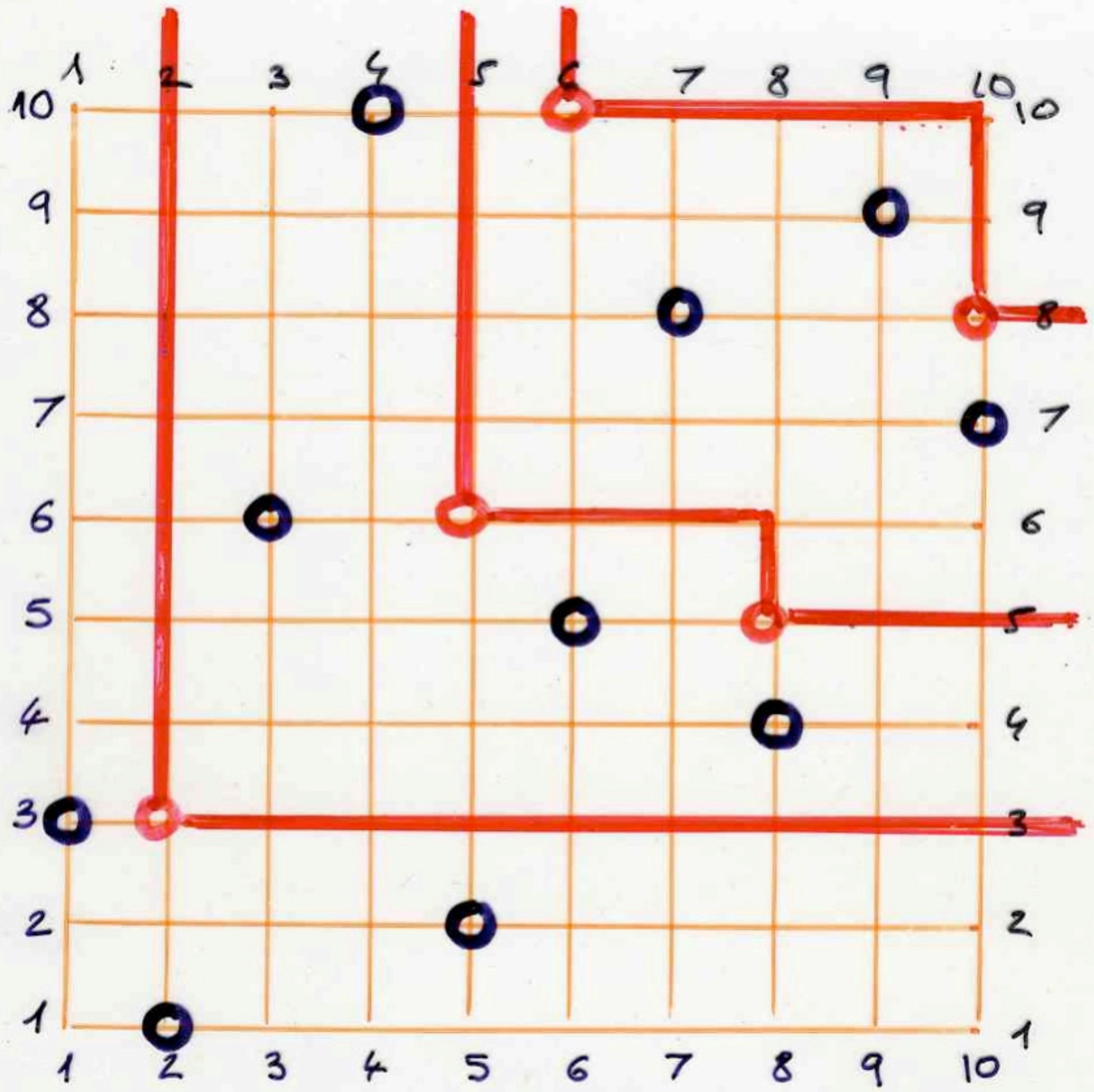


σ = 3 1 6 10 2 5 8 4 9 7

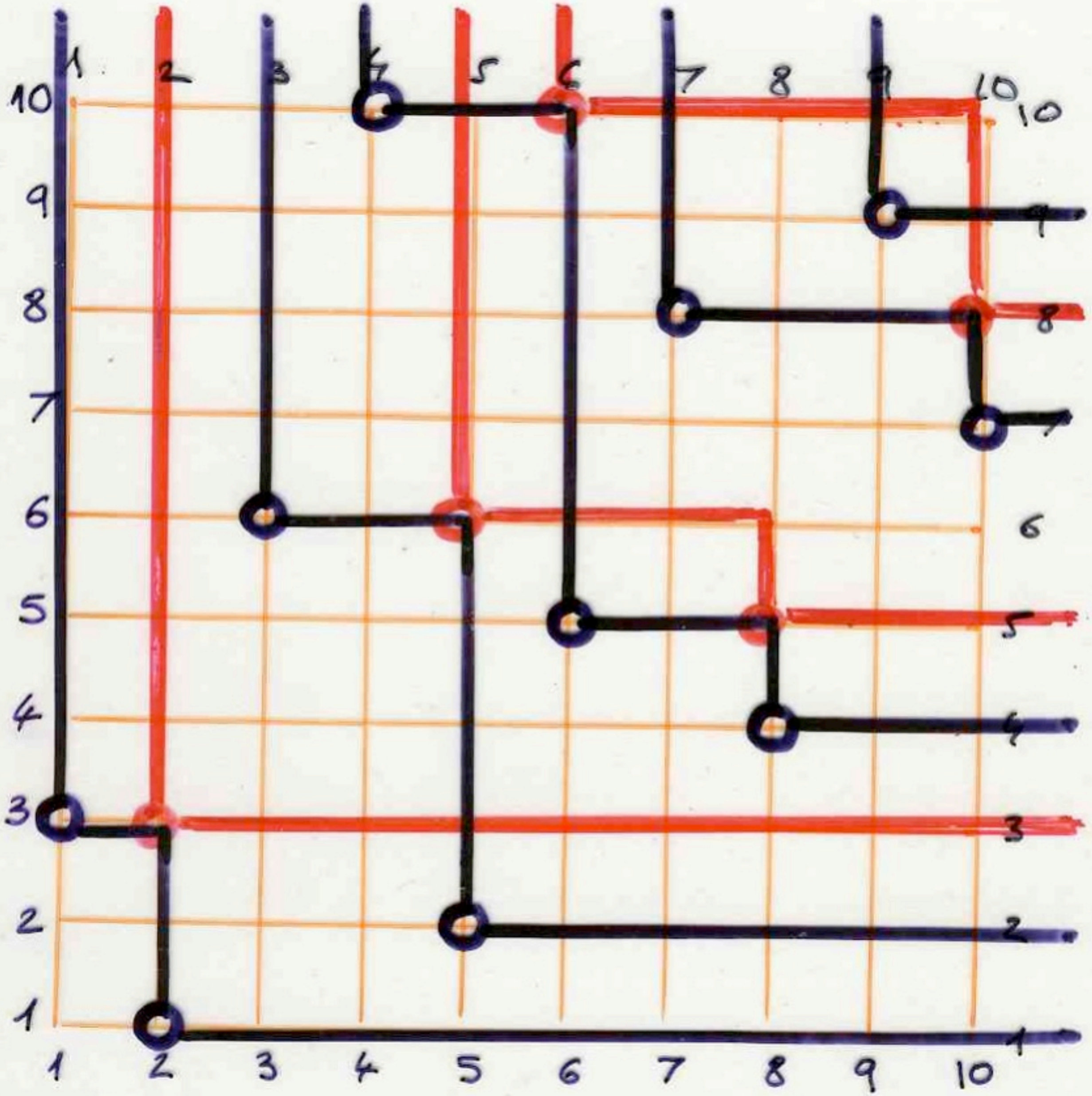


$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

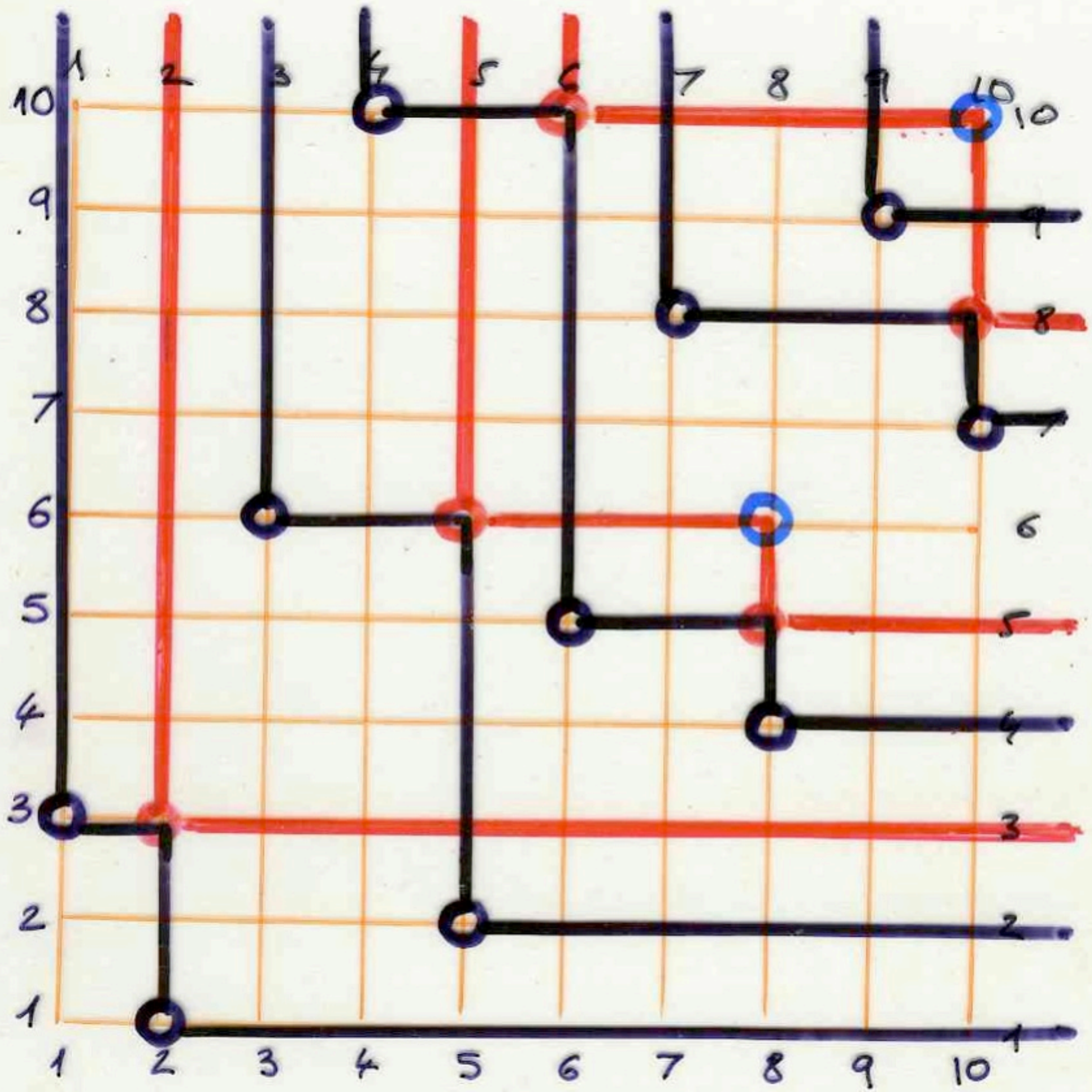
Répétons sur les points rouges la construction des bords d'ombres.



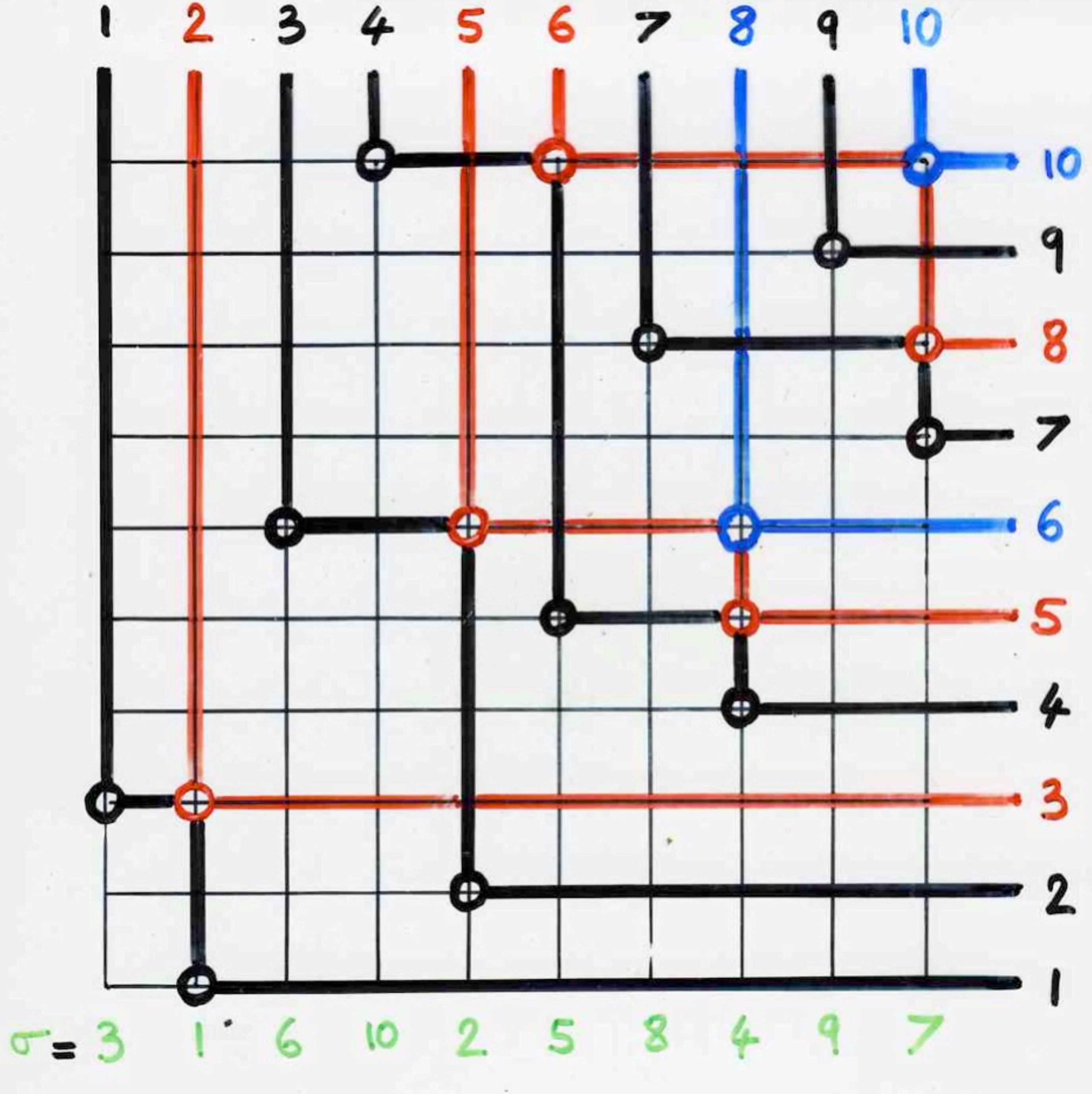
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



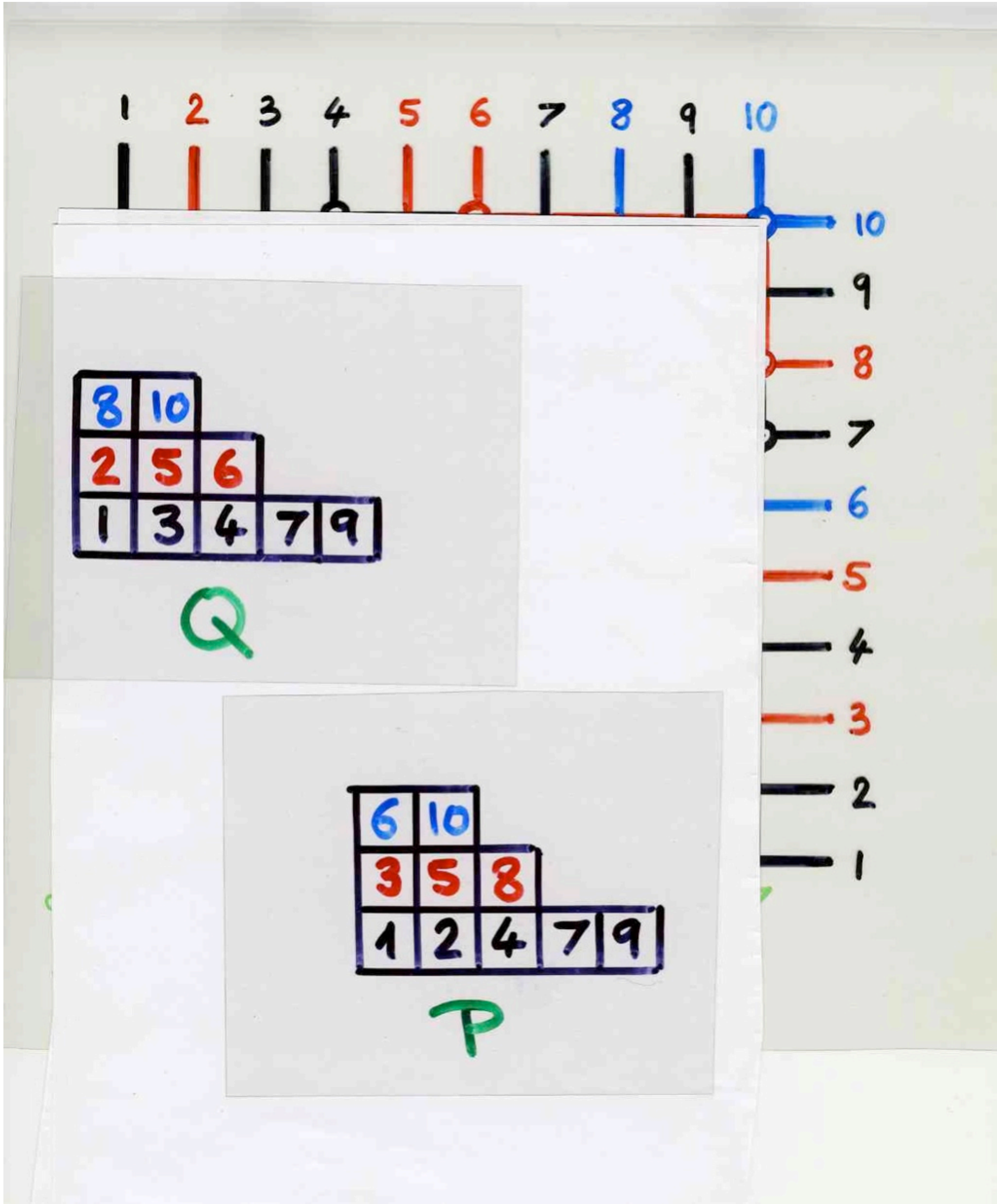
σ = 3 1 6 10 2 5 8 4 9 7

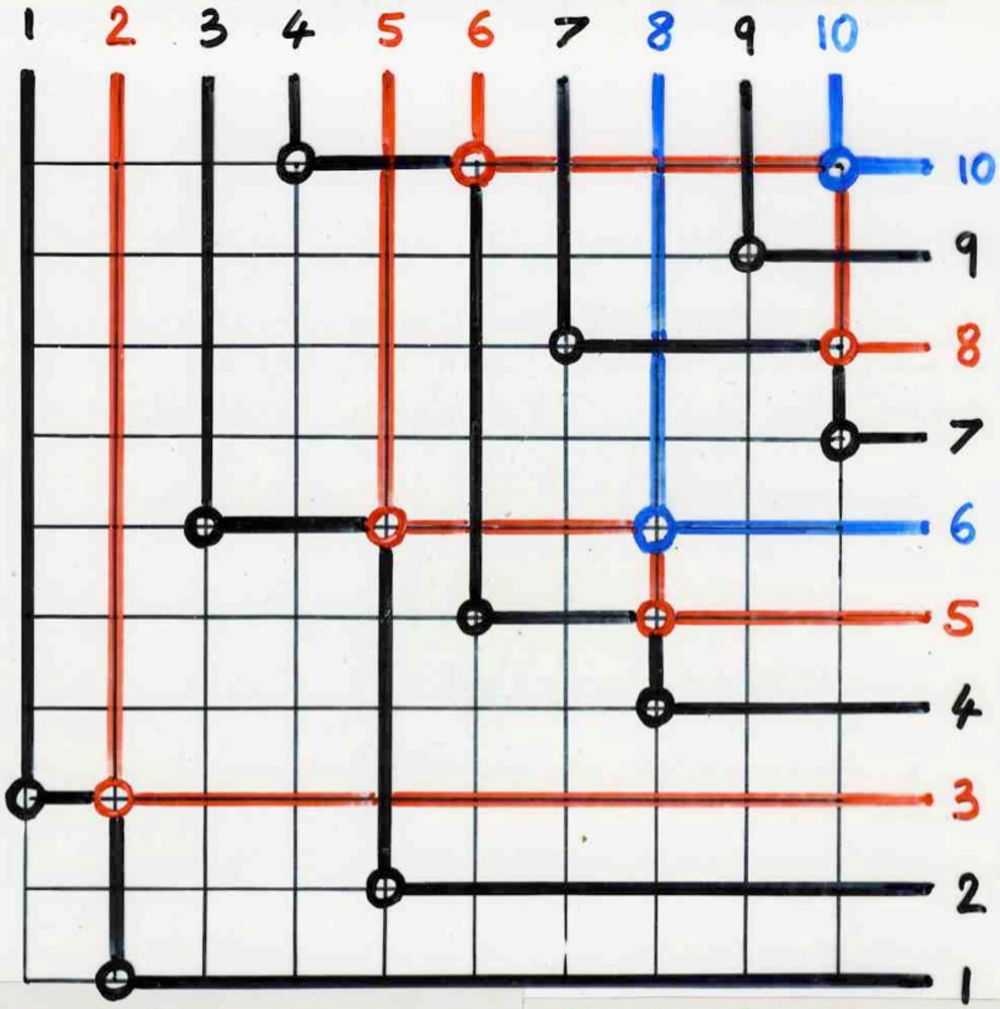


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7

1 2 3 4 5 6 7 8 9 10

10
9
8
7
6
5
4
3
2
1


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P



σ = 3 1 6 10 2 5 8 4 9 7

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

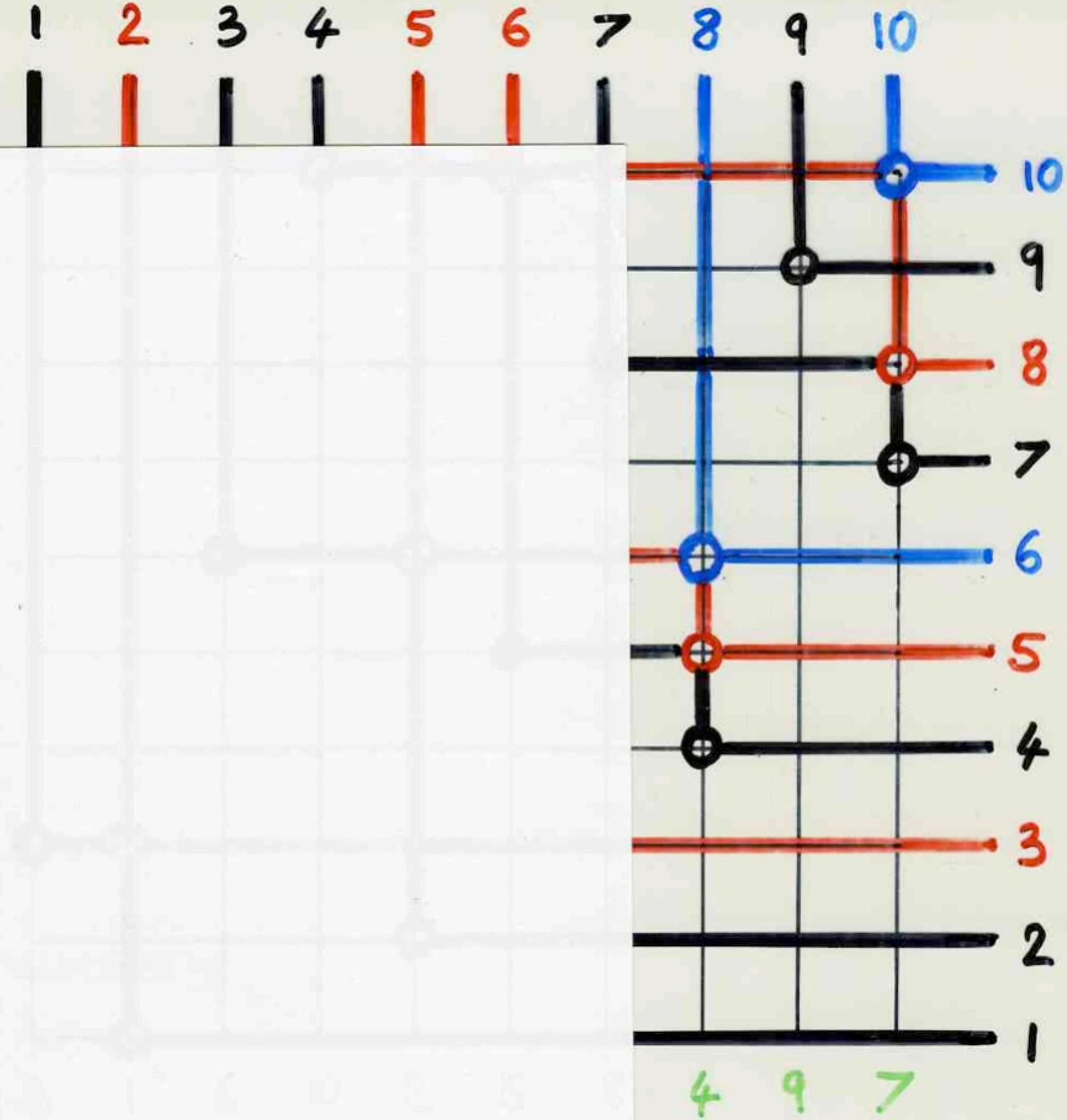
Q

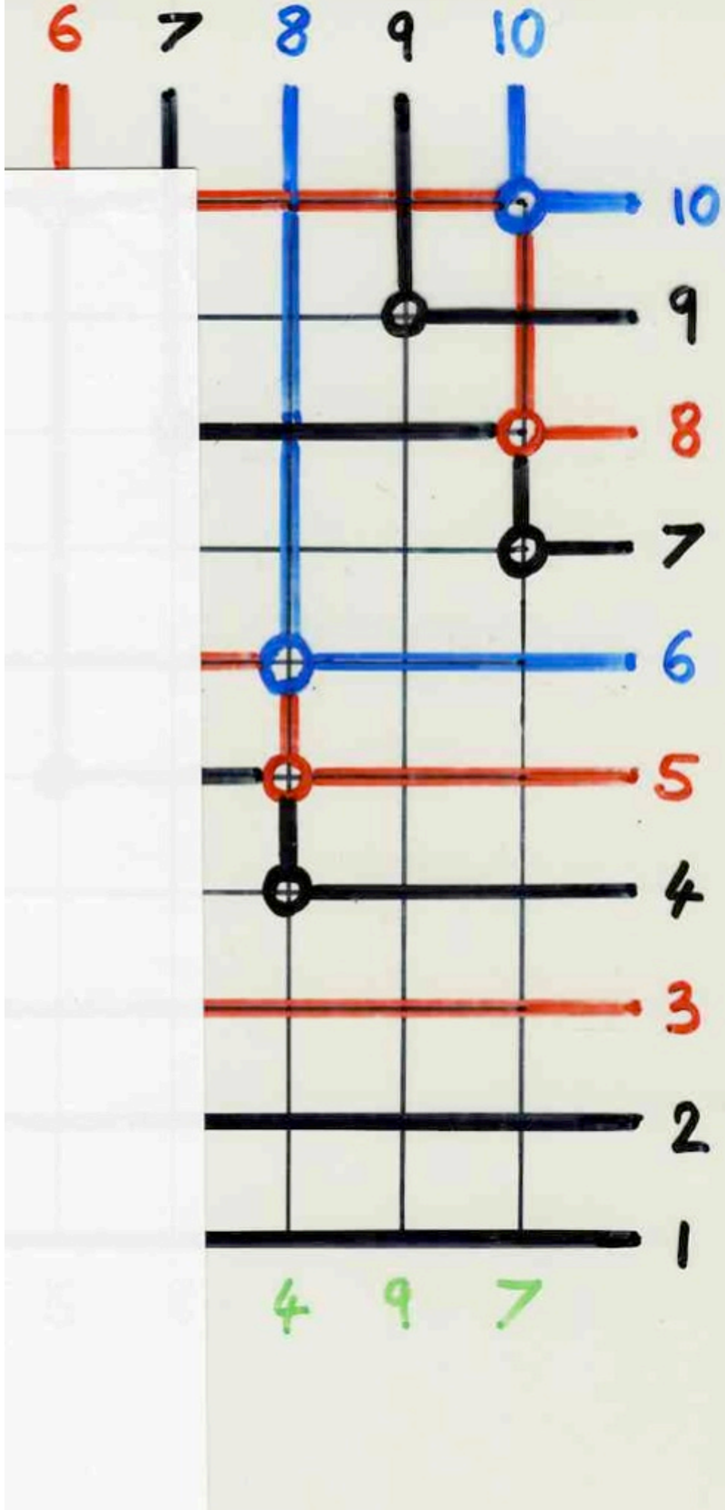
$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$

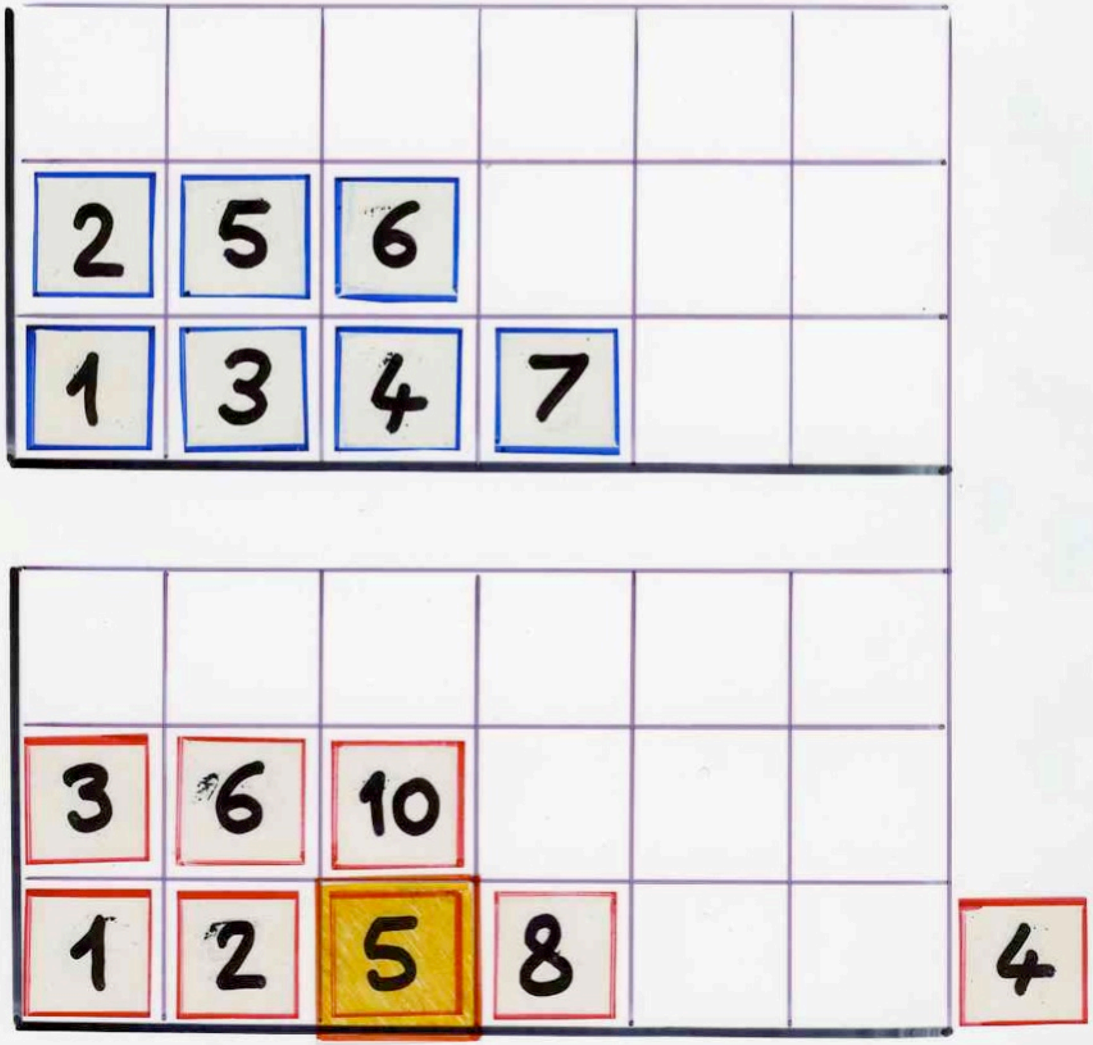
proof of the equivalence

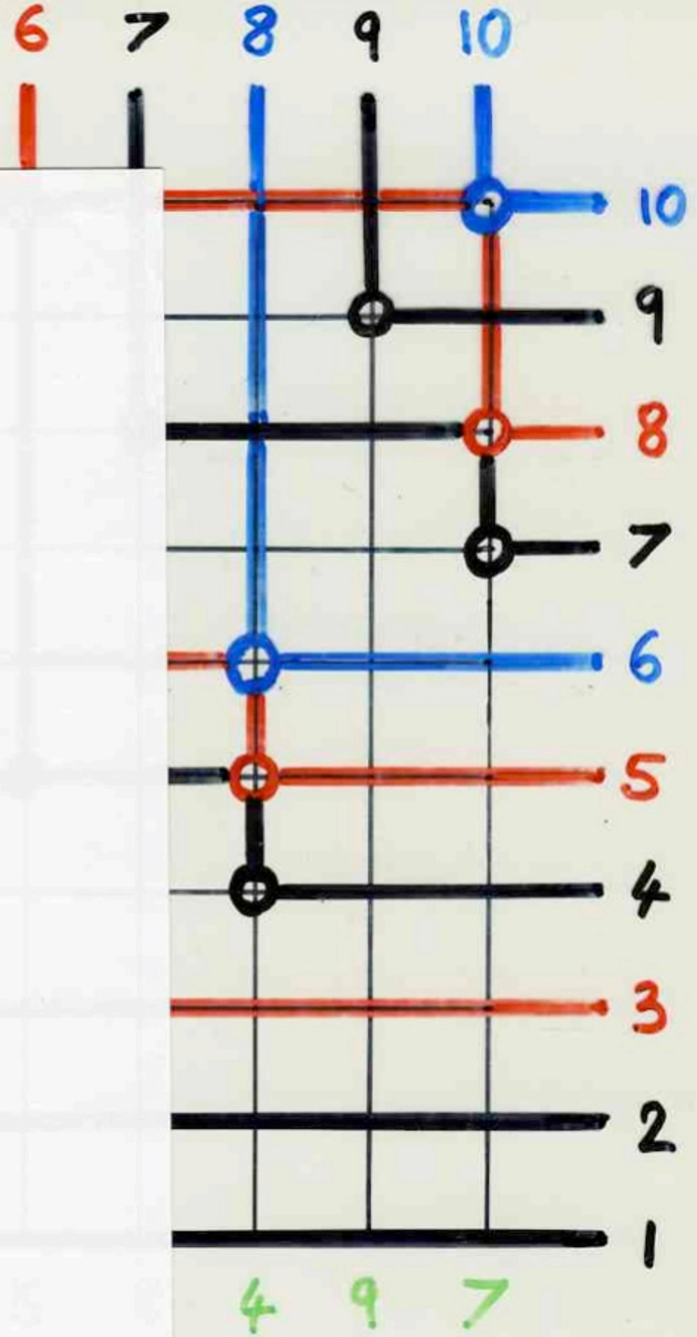
insertions --- geometric constrution


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
4 9 7

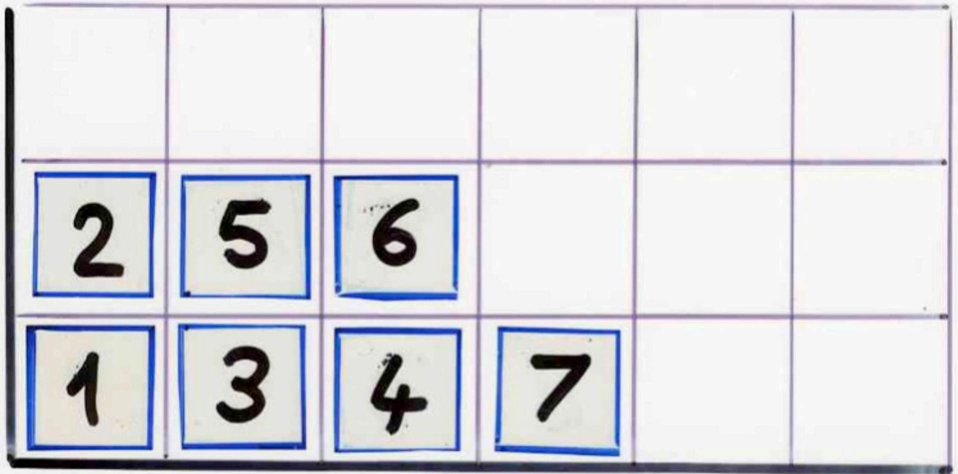


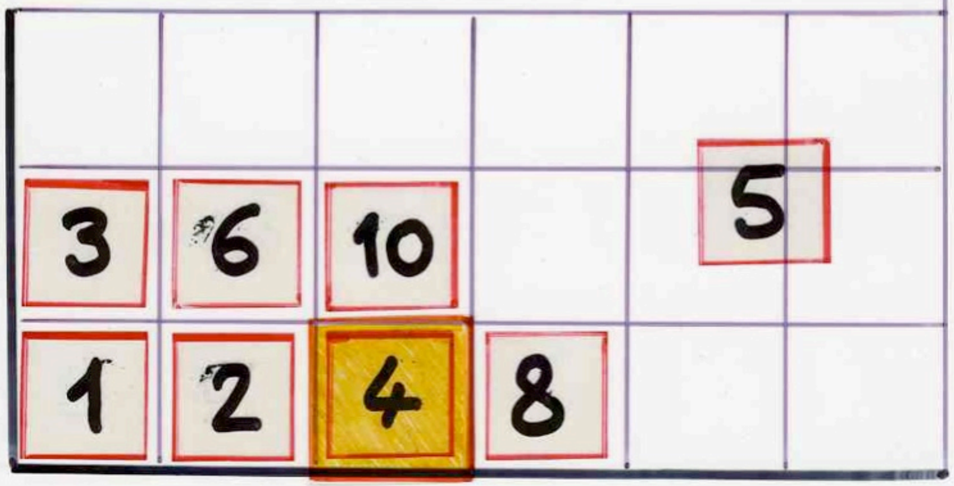
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

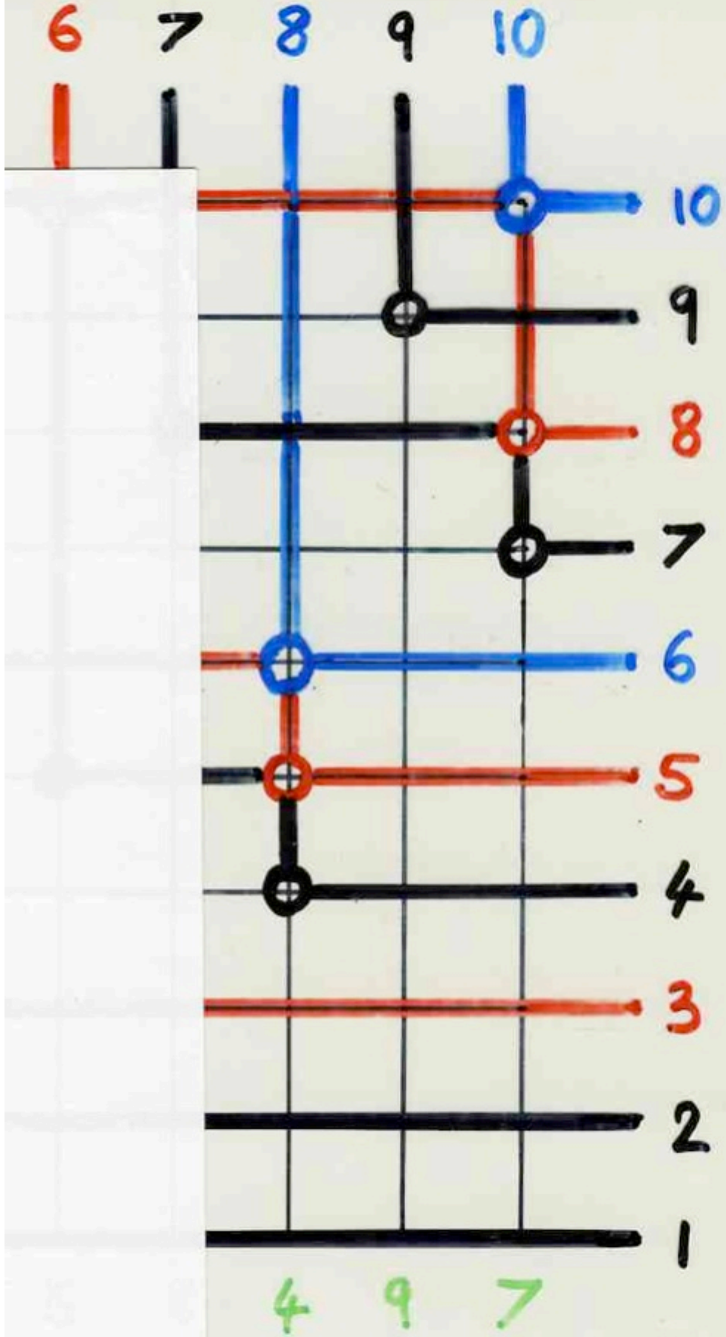




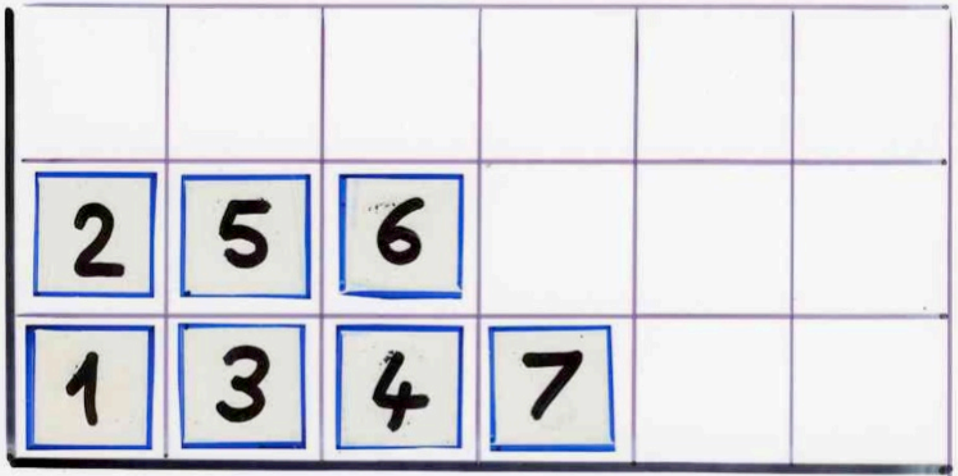
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

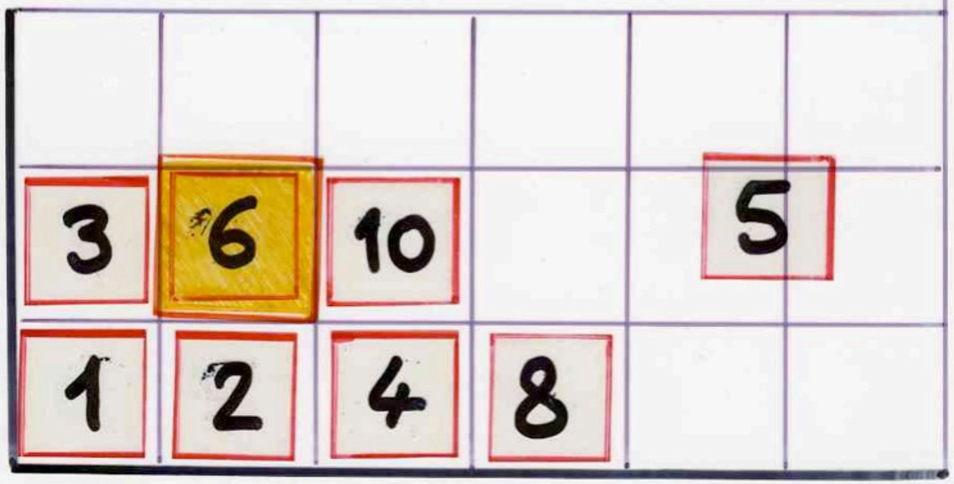


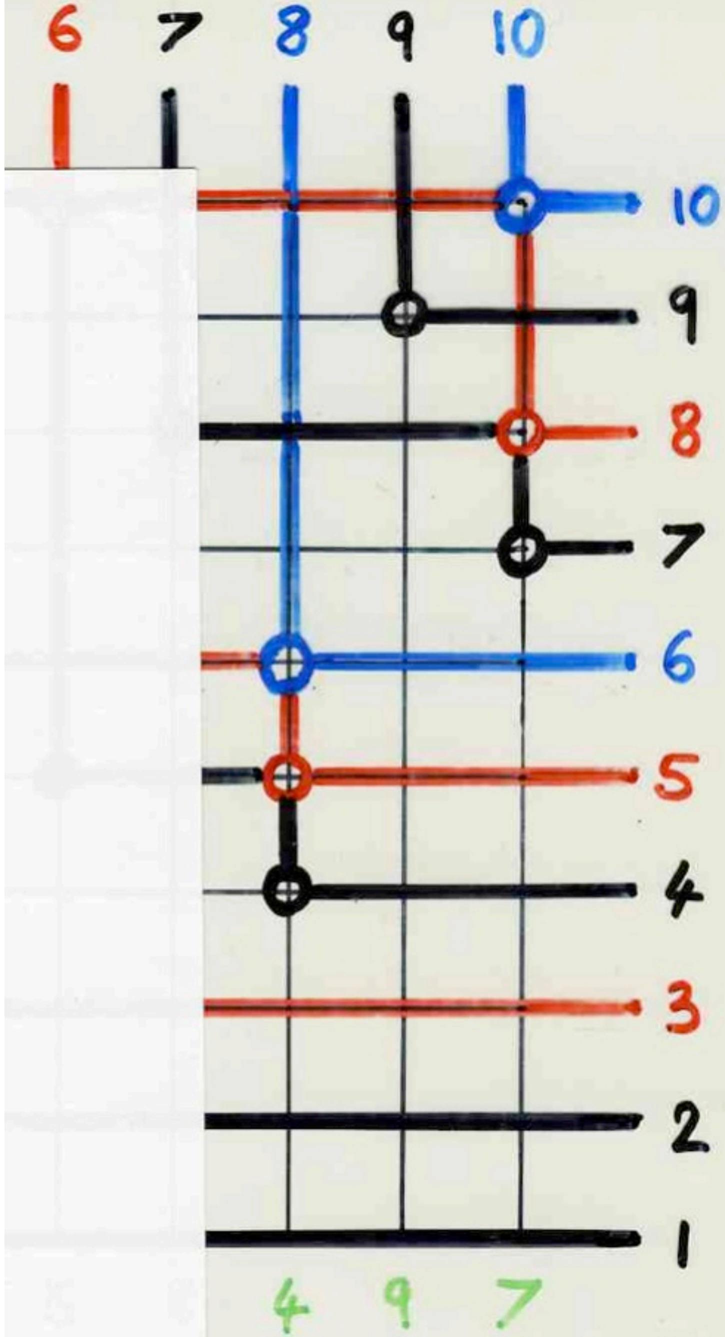




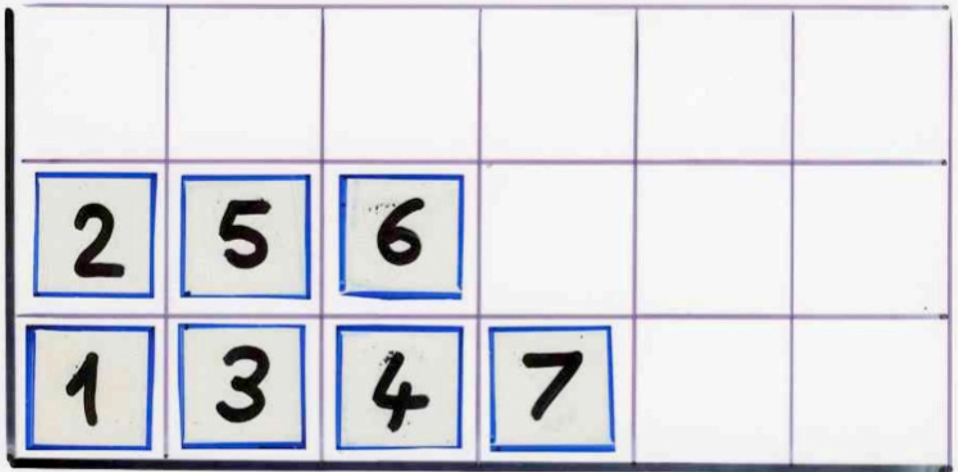
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

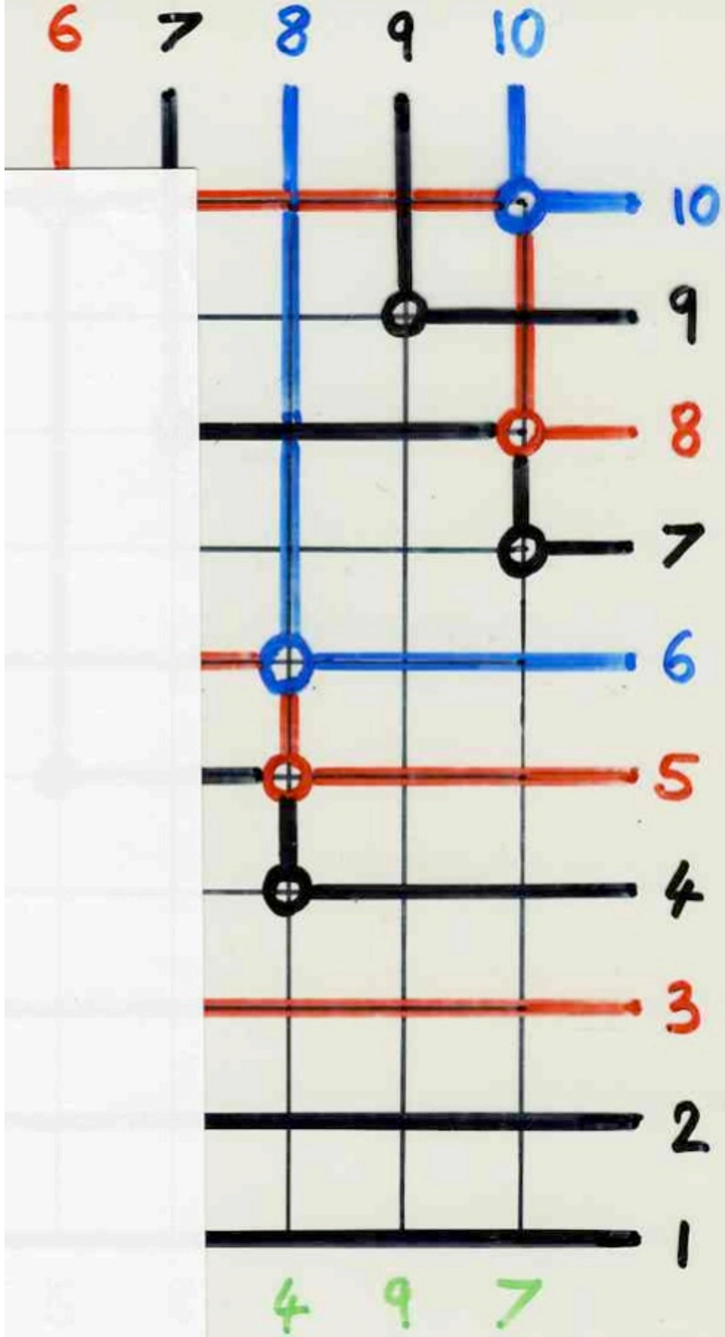






1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

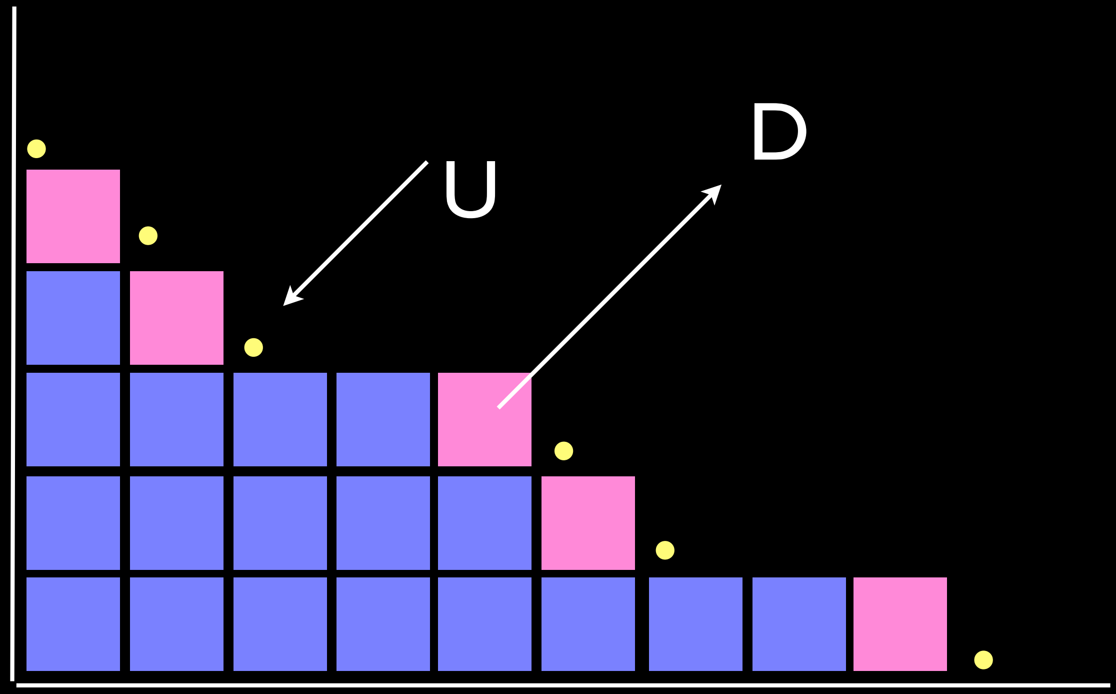
6					
3	5	10			
1	2	4	8		

representation of the operators U, D

Sergey Fomin

(with C. K.)

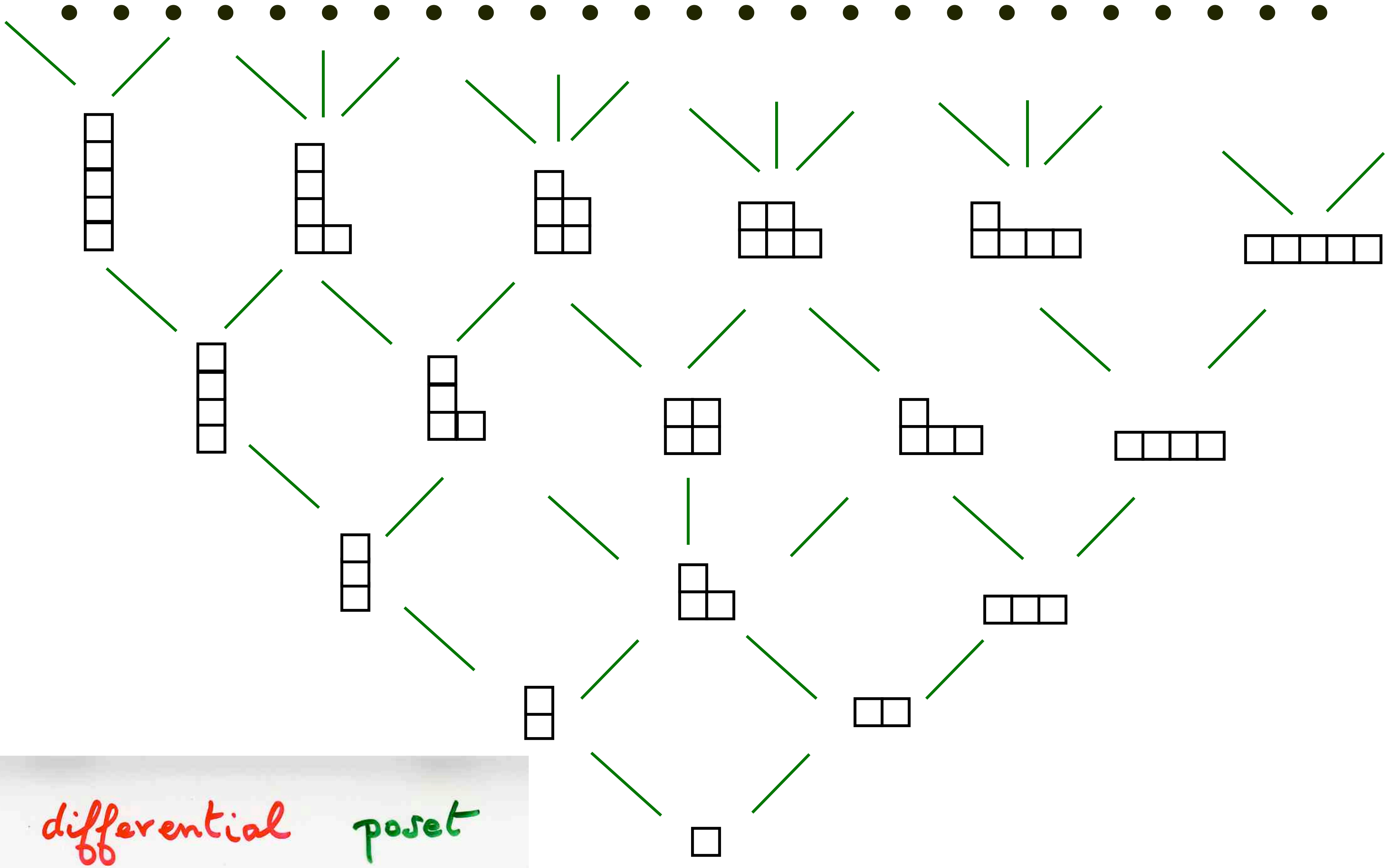
Operators U and D



adding
or deleting
a cell in
a Ferrers
diagram

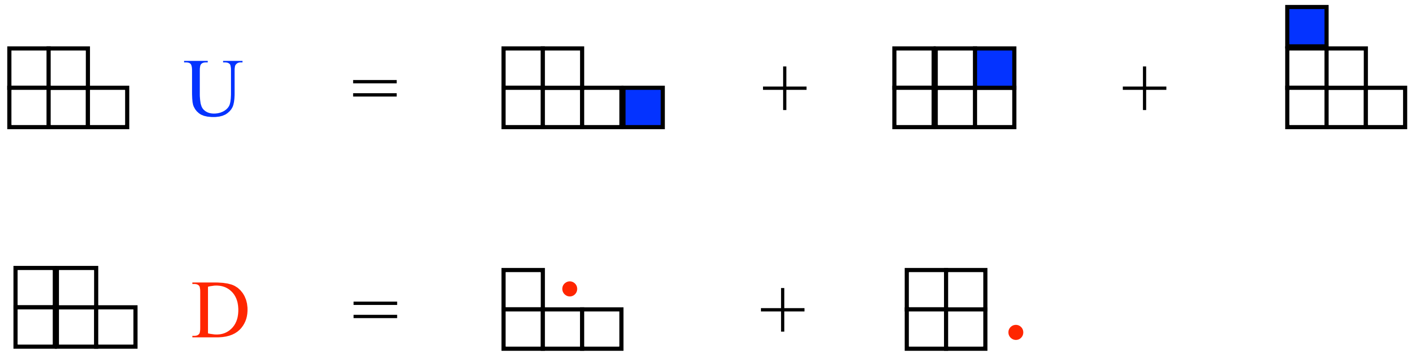
Young lattice

Young lattice



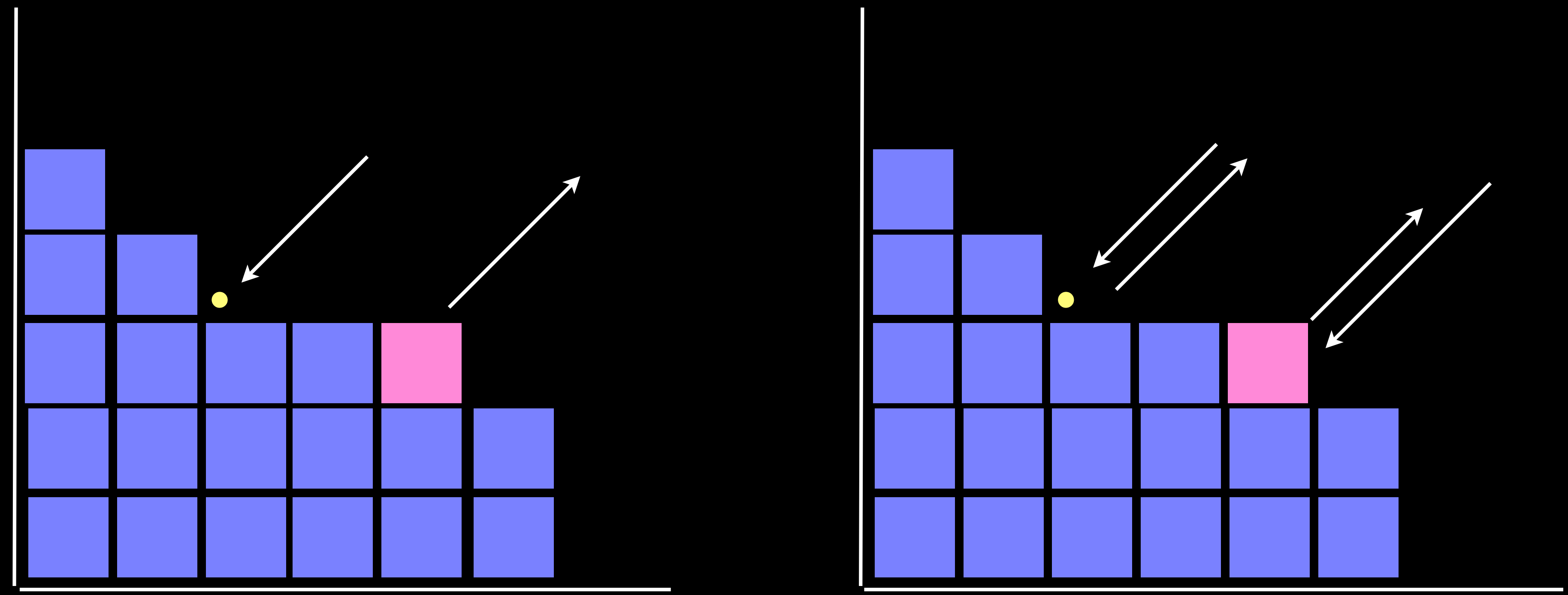
differential poset

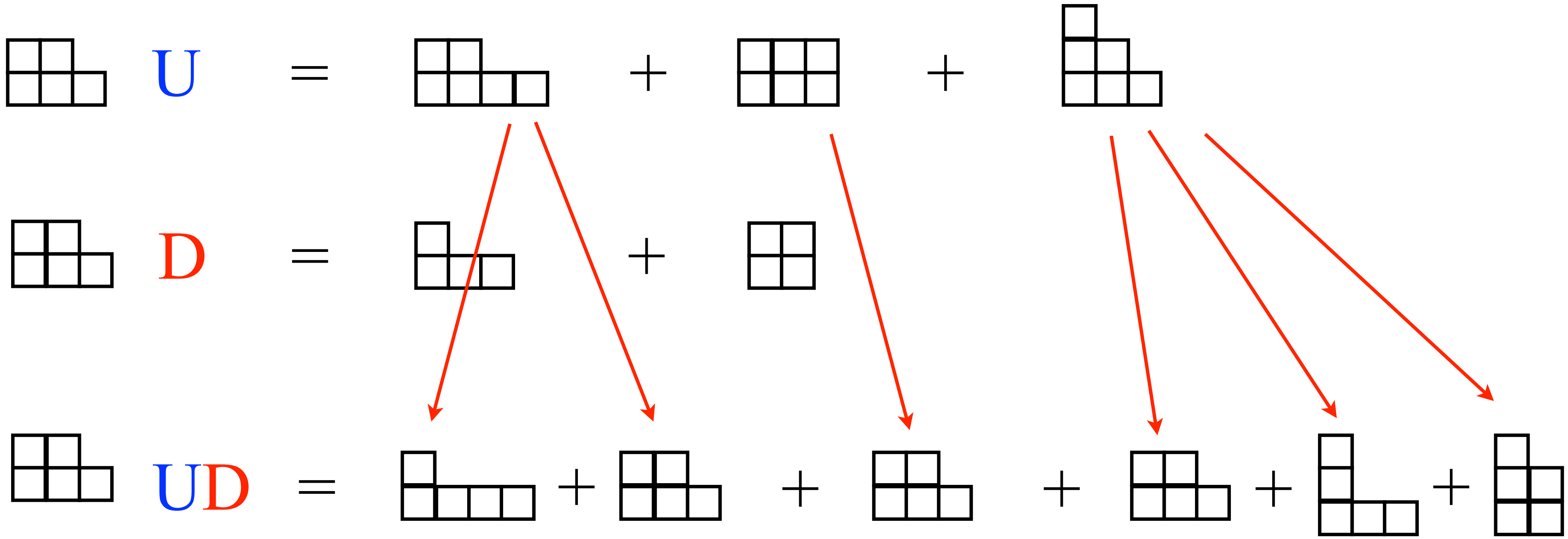
Fomin, Stanley

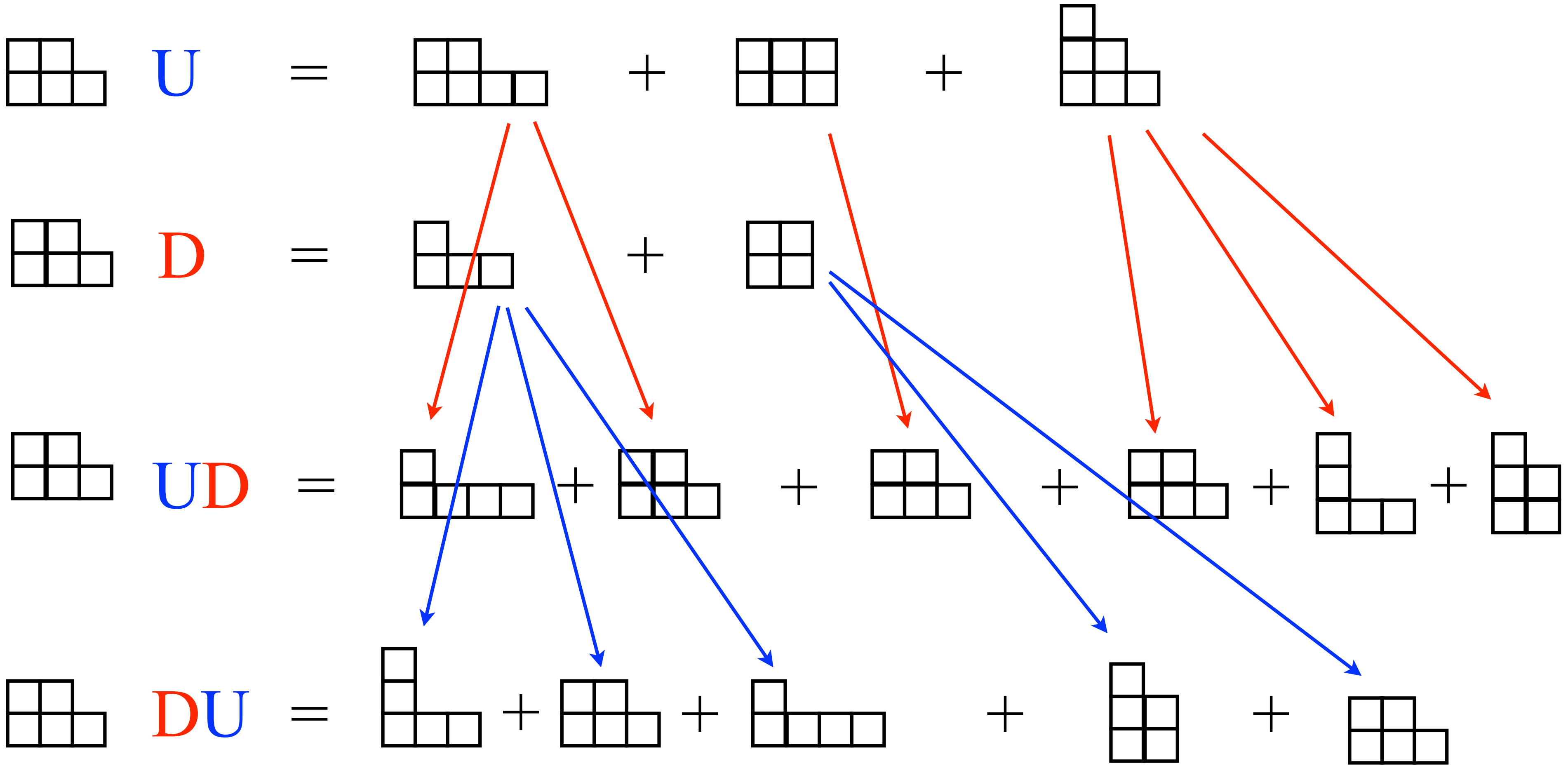

U
D

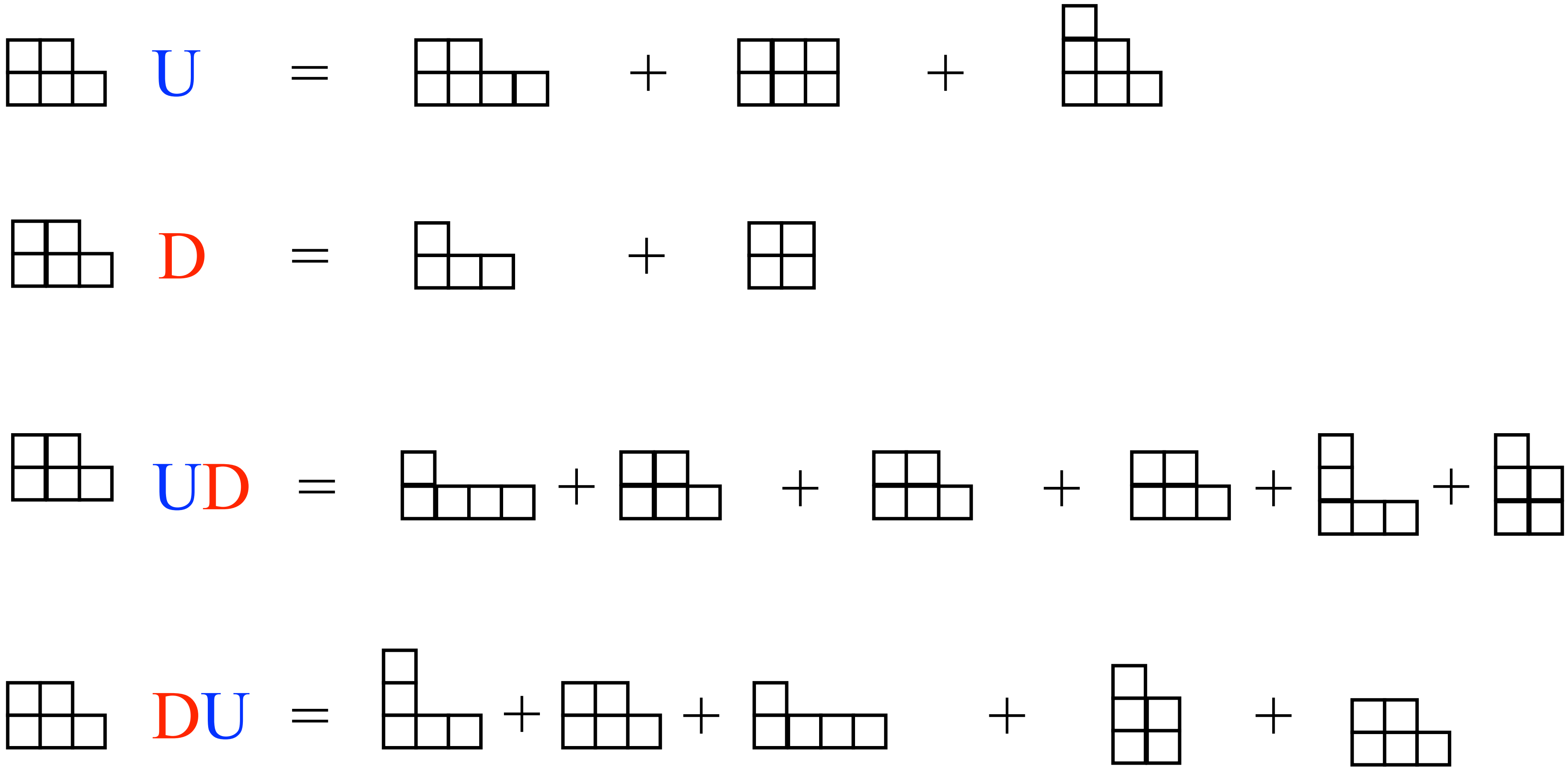
Heisenberg commutation relation

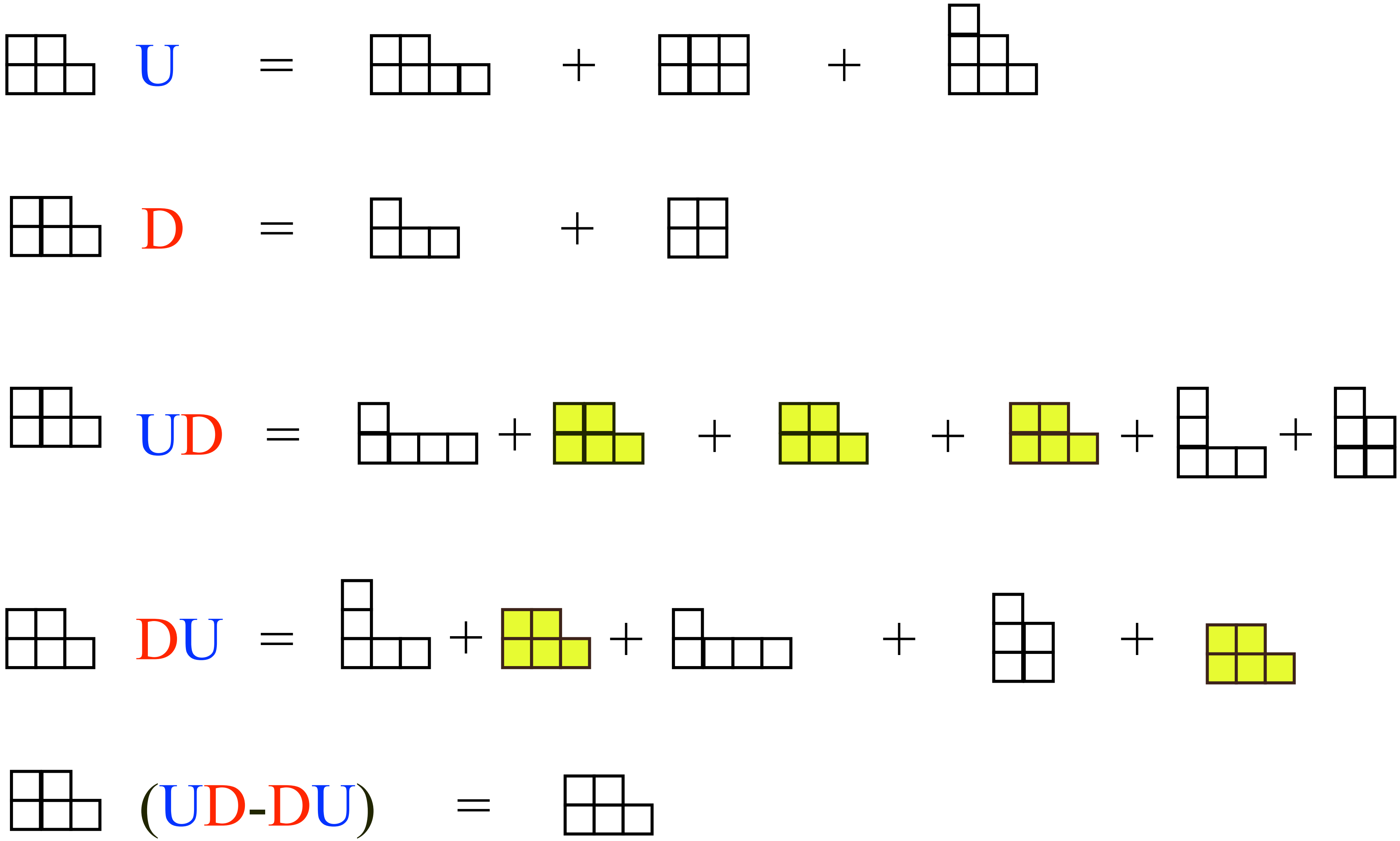
$$UD = DU + I$$



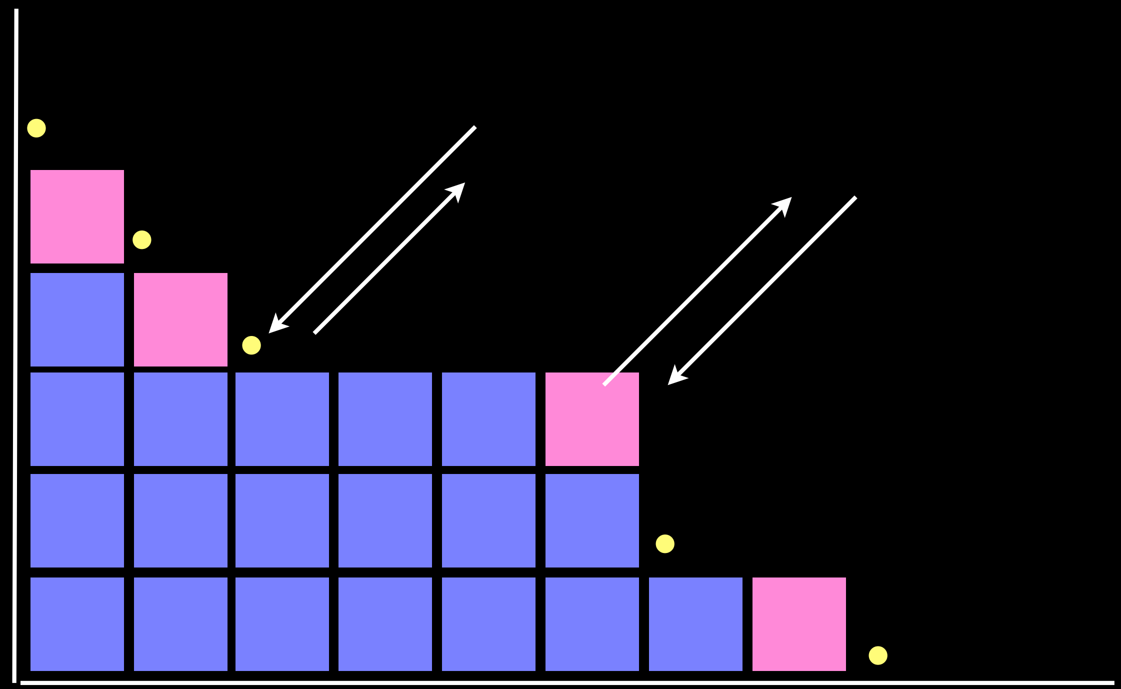

U
D
UD


U
D
UD
DU


U =
D =
UD =
DU =


U =
D =
UD =
DU =
(UD-DU) =

$$UD = DU + I$$

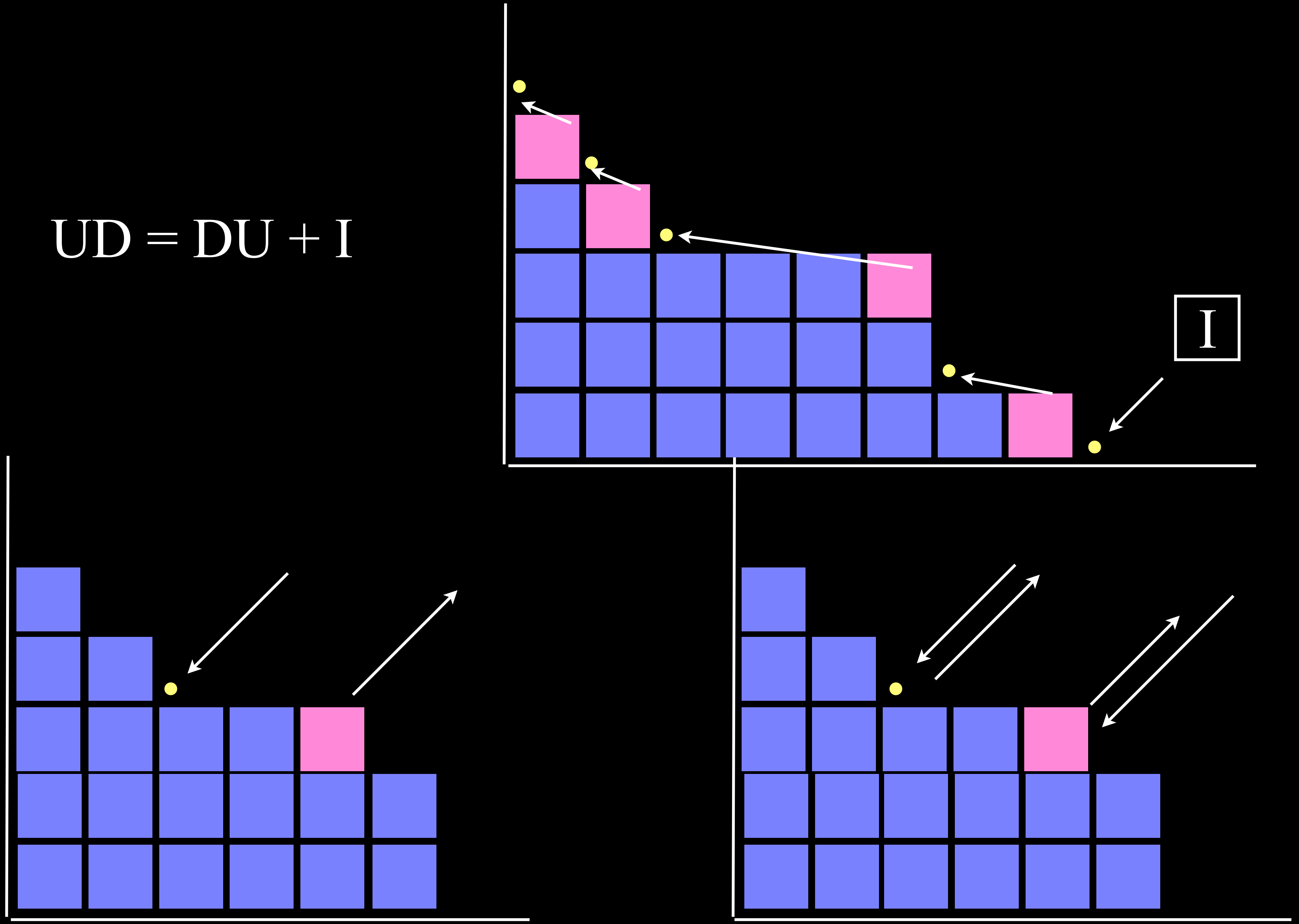


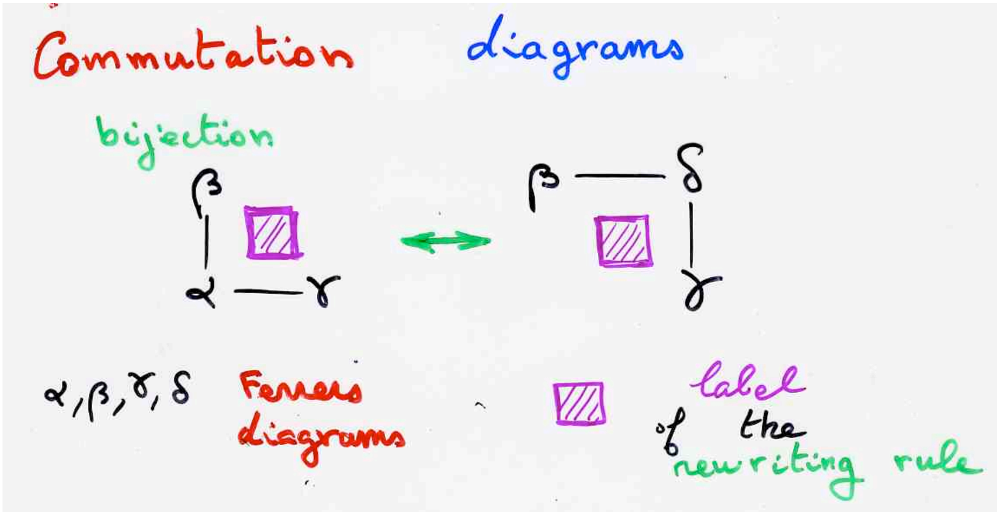
The cellular Ansatz
second part:

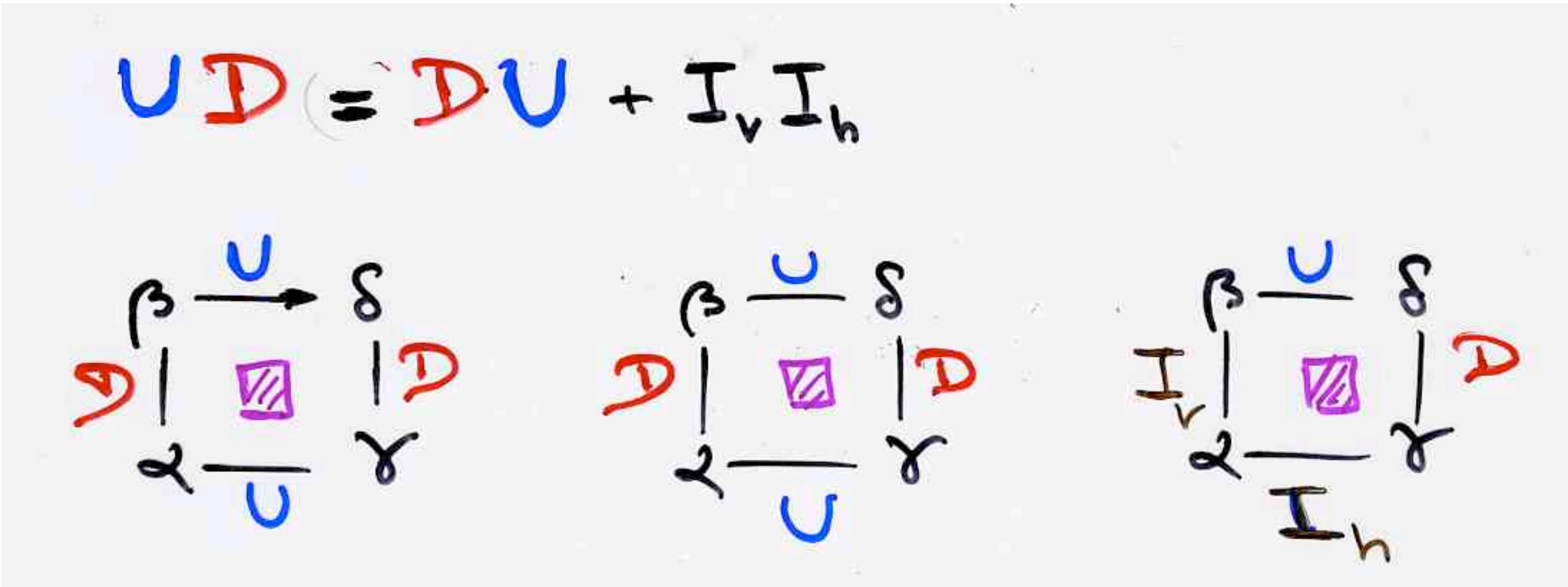
guided construction
of a bijection
(from the representation of U and D)

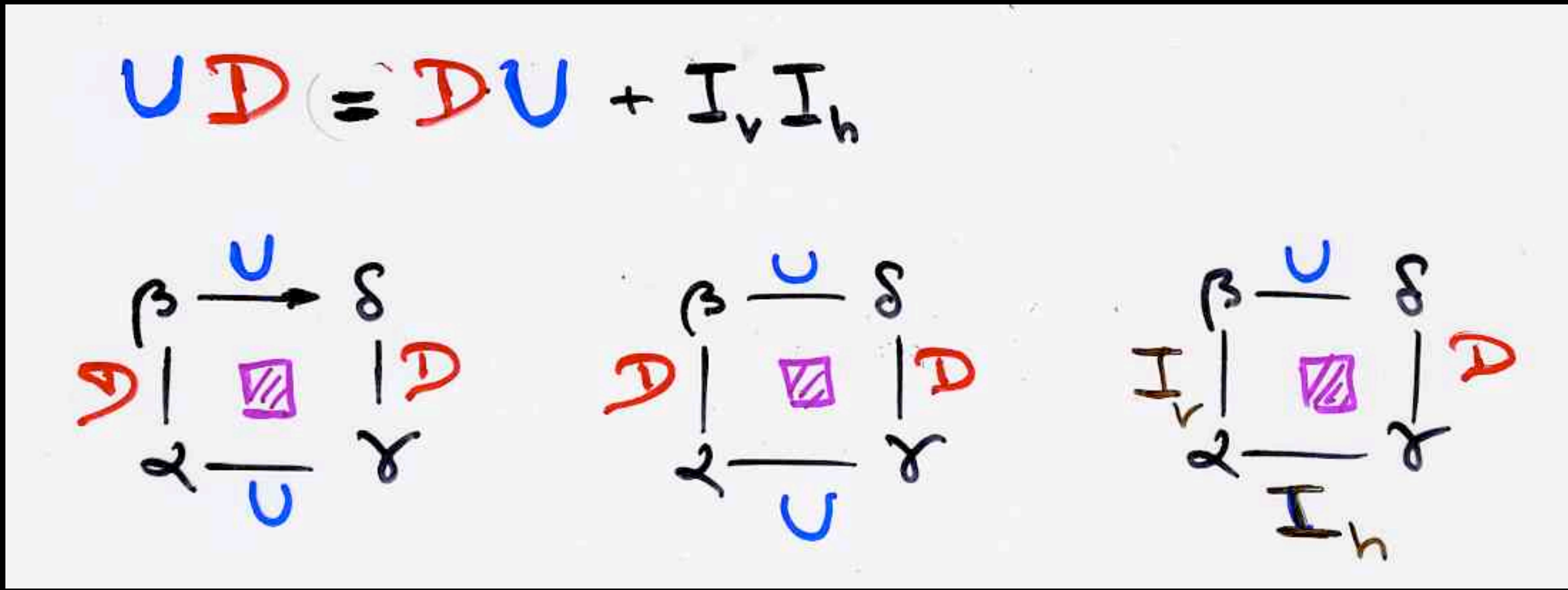
combinatorial “representation” of the commutation relation $UD = DU + I$

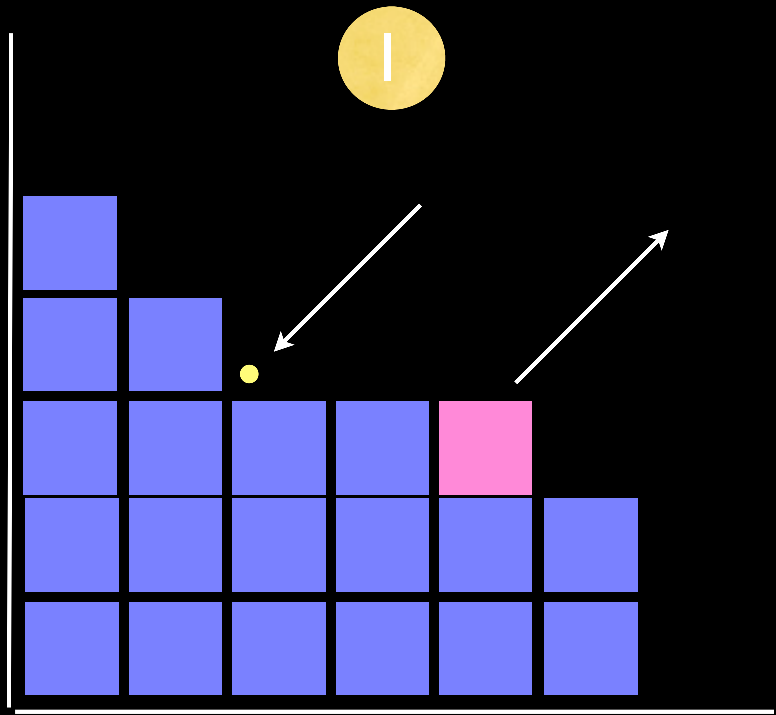
$$UD = DU + I$$




Commutation diagrams
bijection
β
δ
α
γ
α, β, γ, δ
Ferrers diagrams
label
of the
rewriting rule

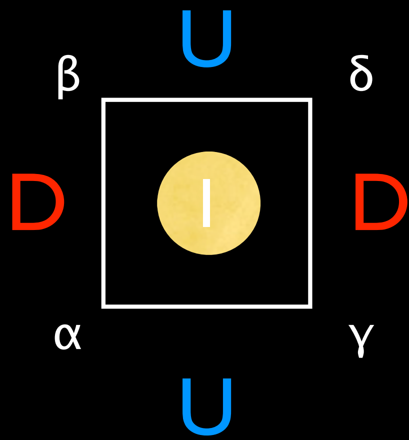

UD = DU + I_v I_h
U
D
β
δ
α
γ
I_v
I_h



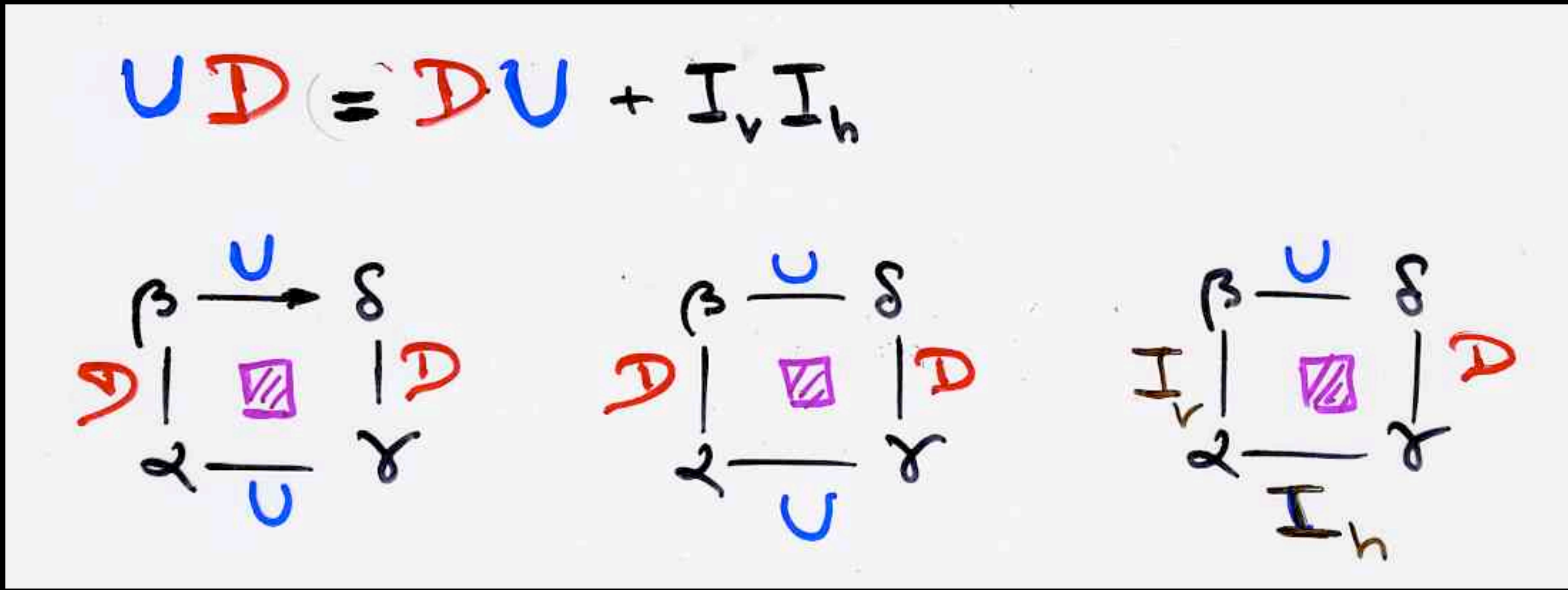


β

β ≠ γ

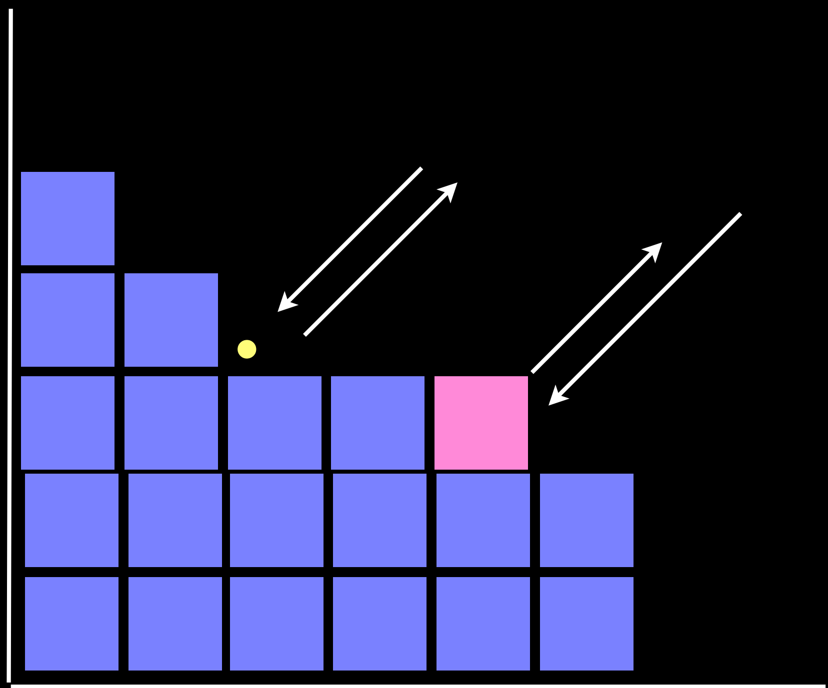


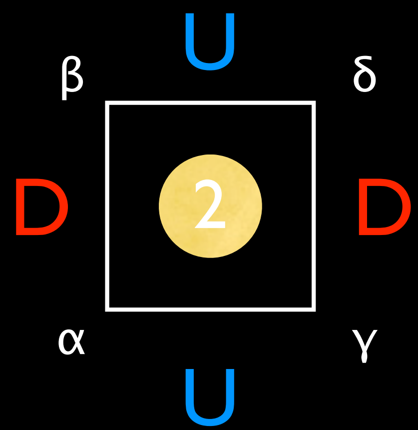
$\delta = \beta \cup \gamma$



2

β = γ

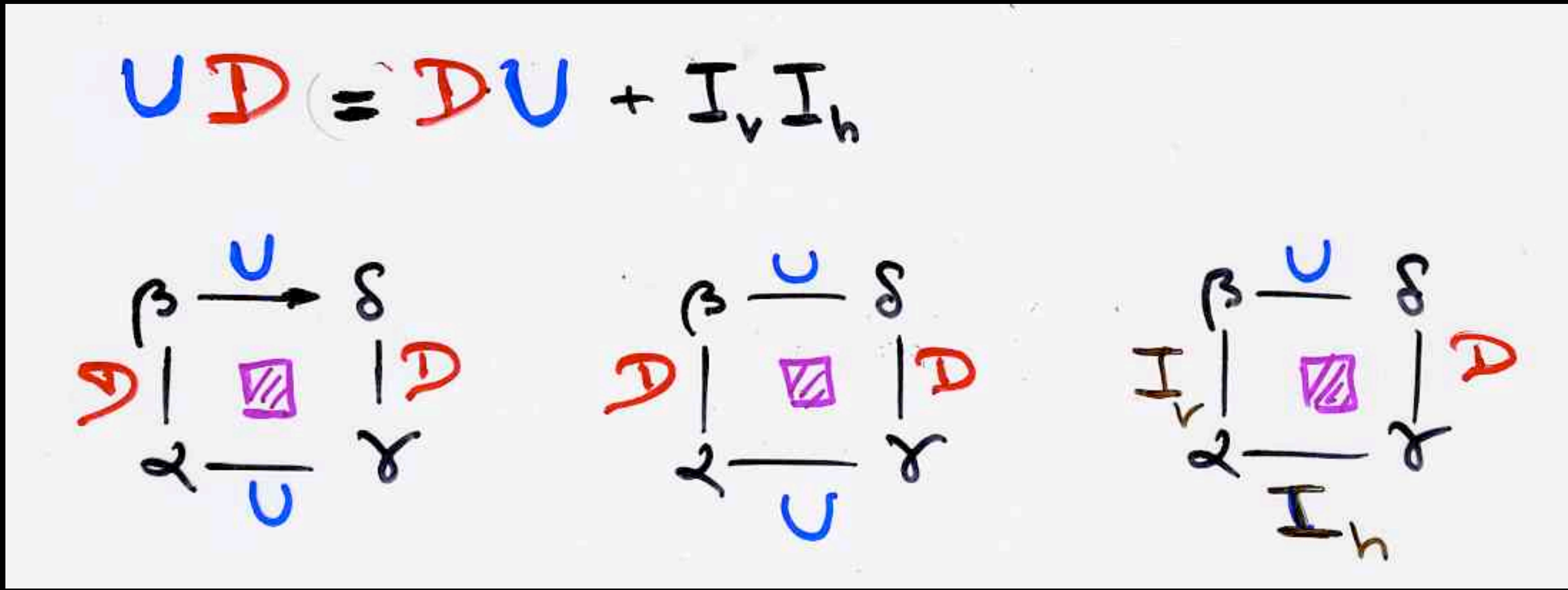


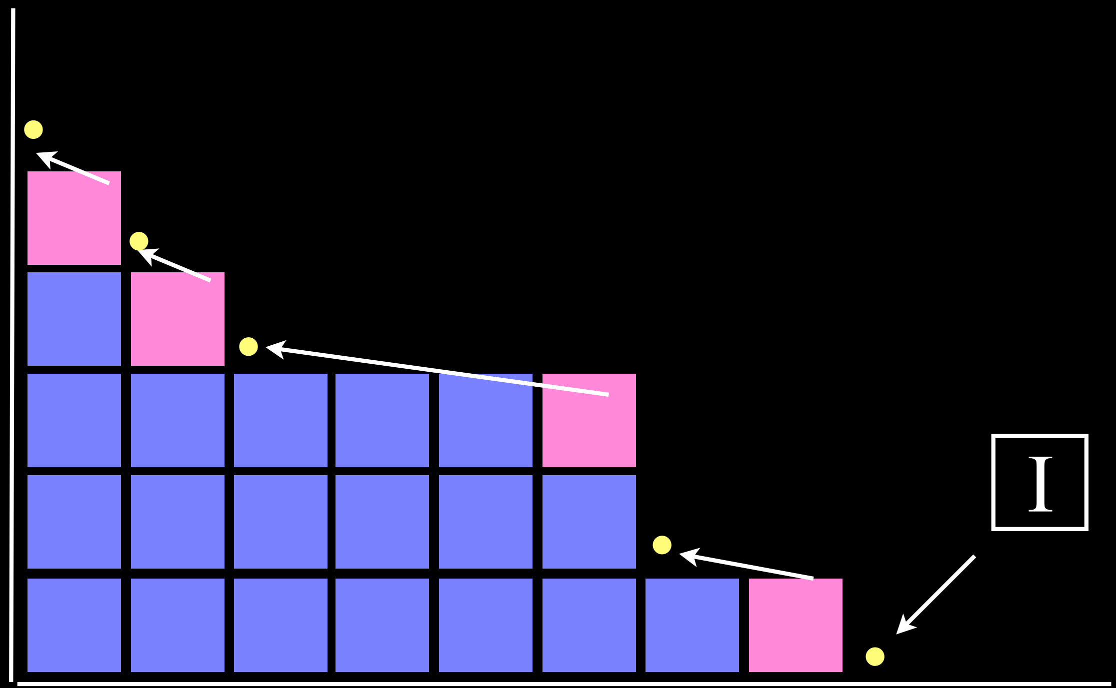


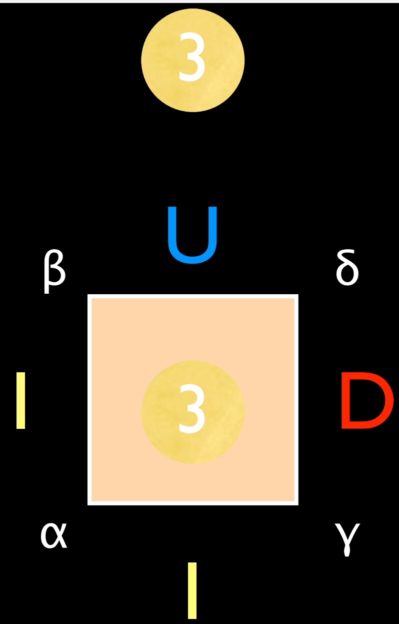
$\beta = \gamma = \alpha + (i)$

$\delta = \beta + (i+1)$

β = γ



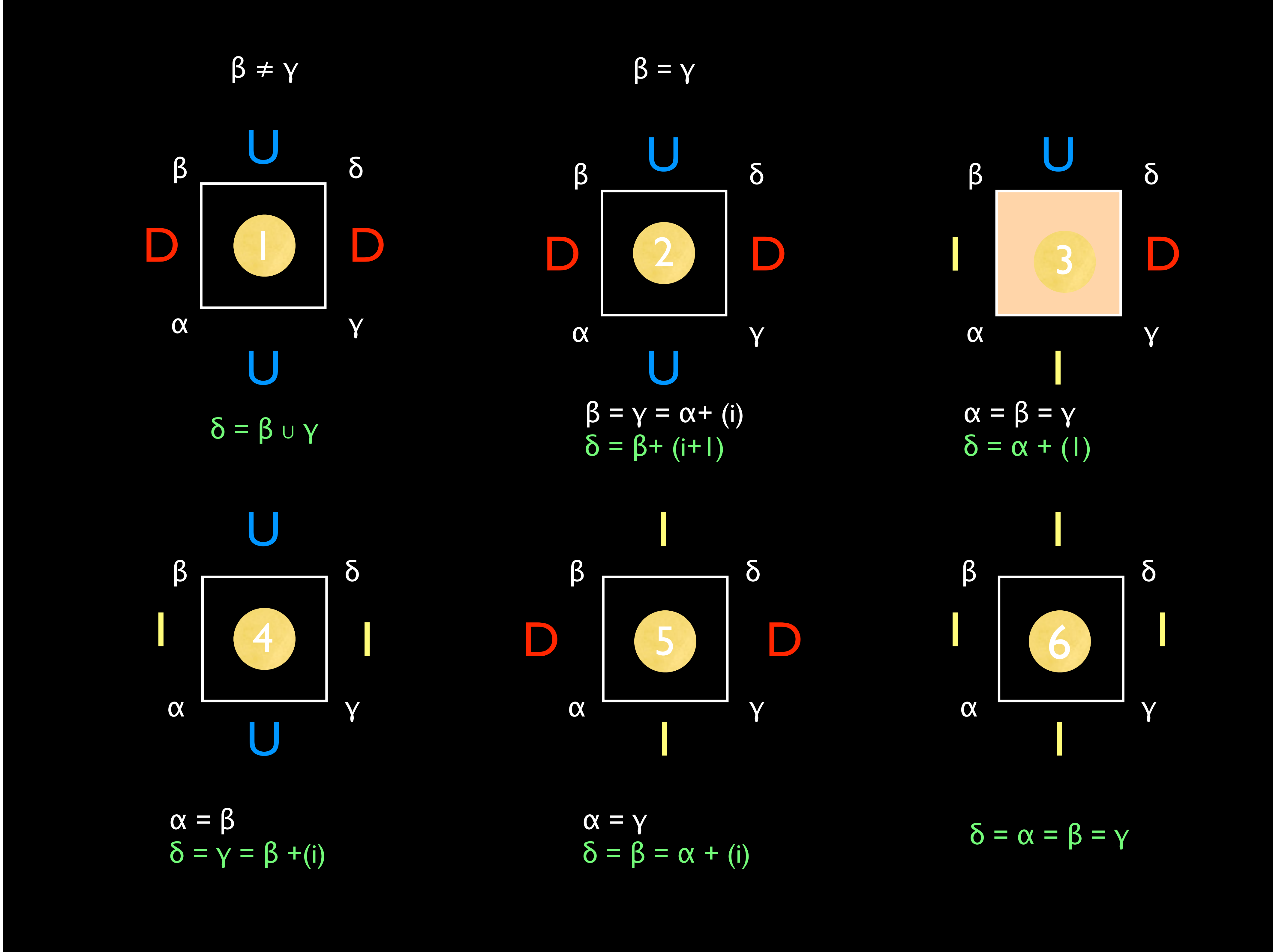




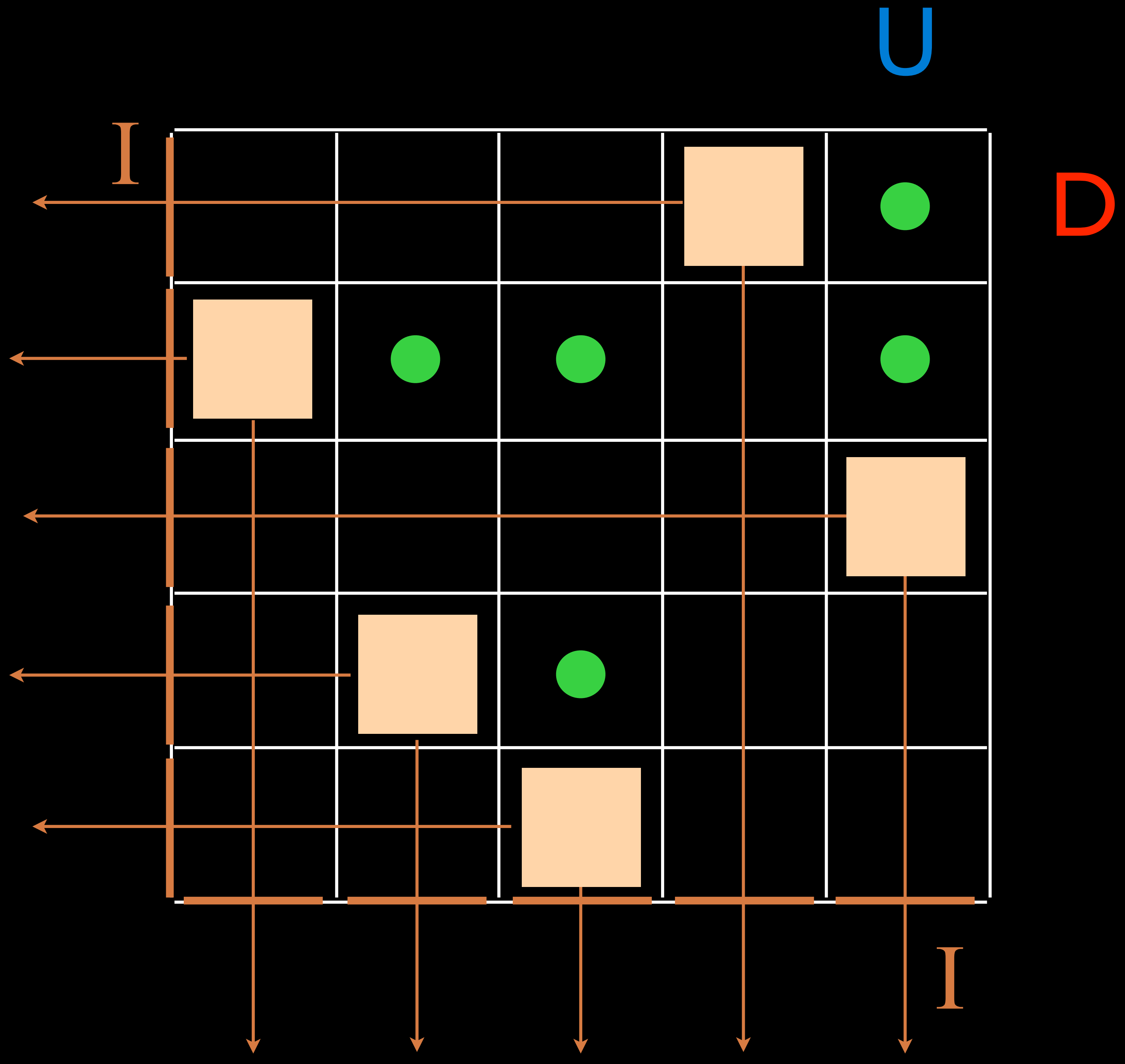
$\alpha = \beta = \gamma$

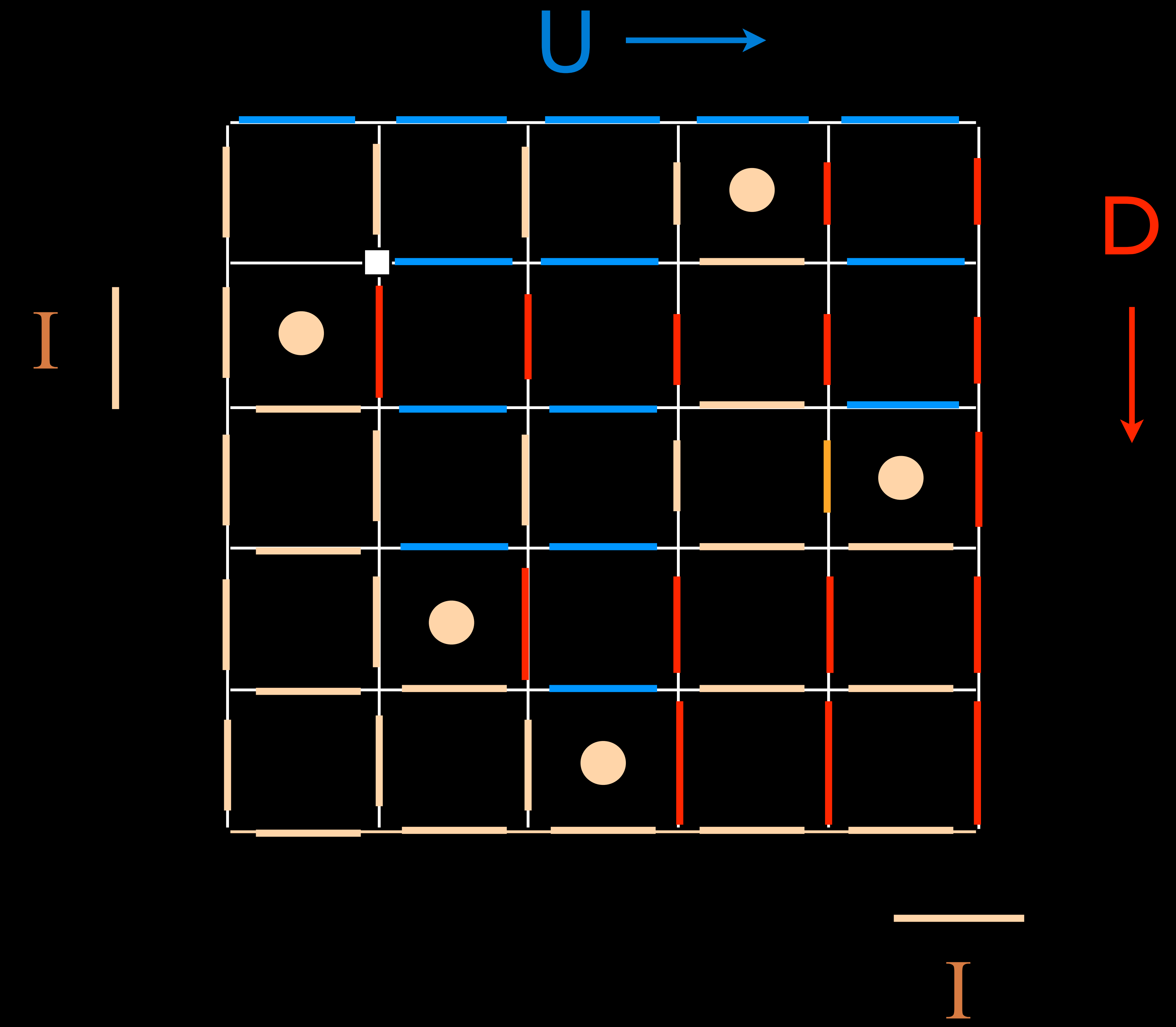
$\delta = \alpha + (I)$

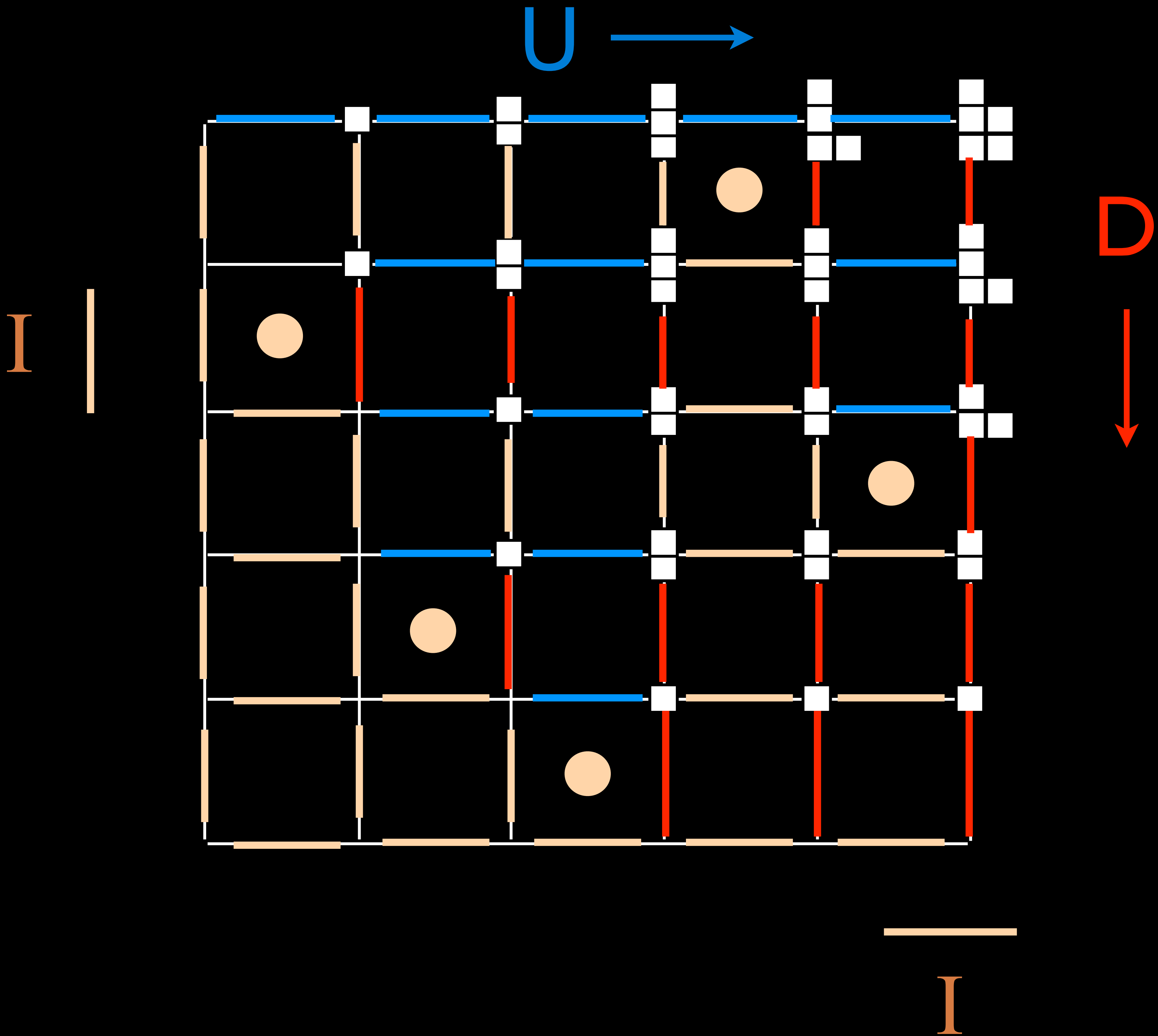
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

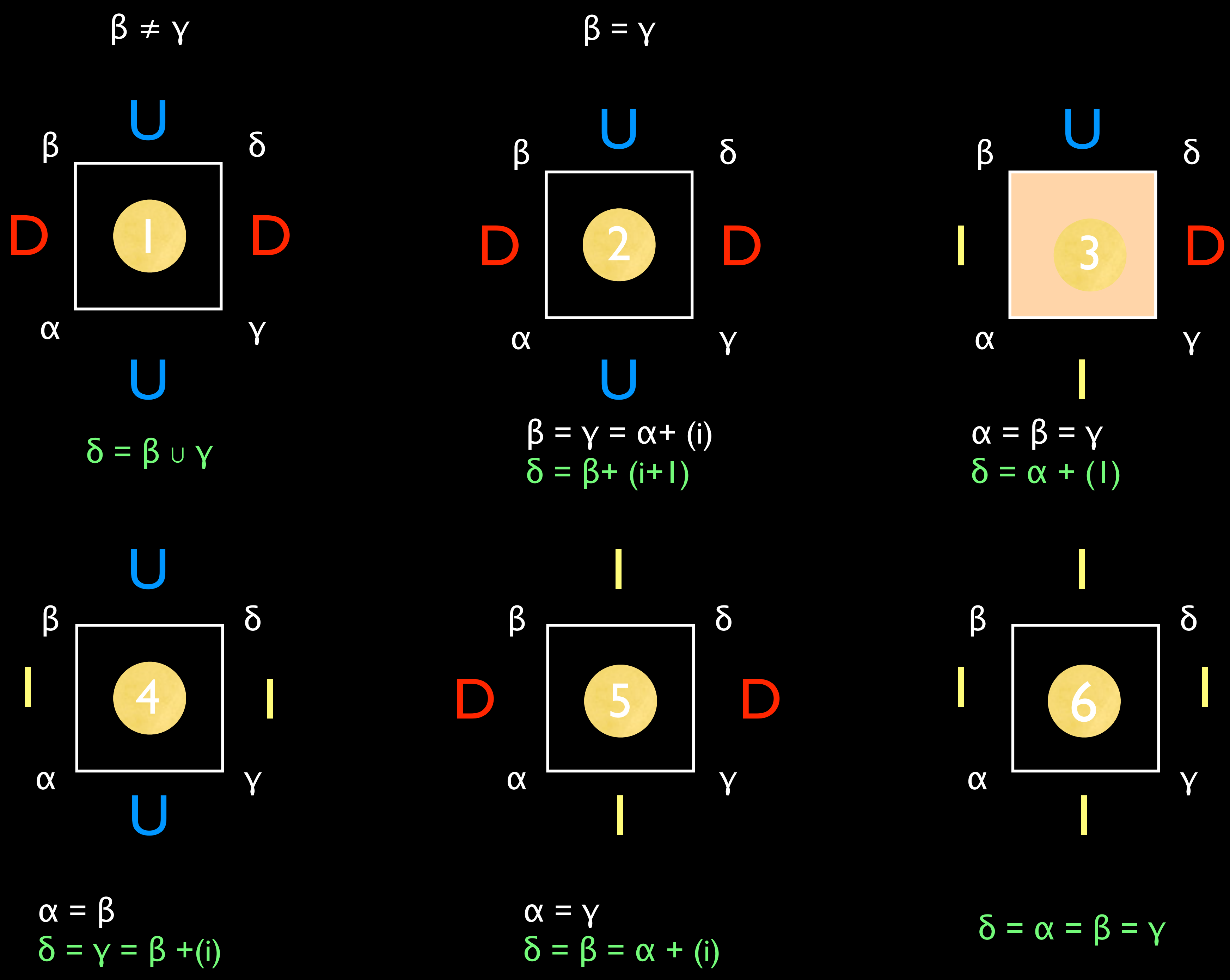


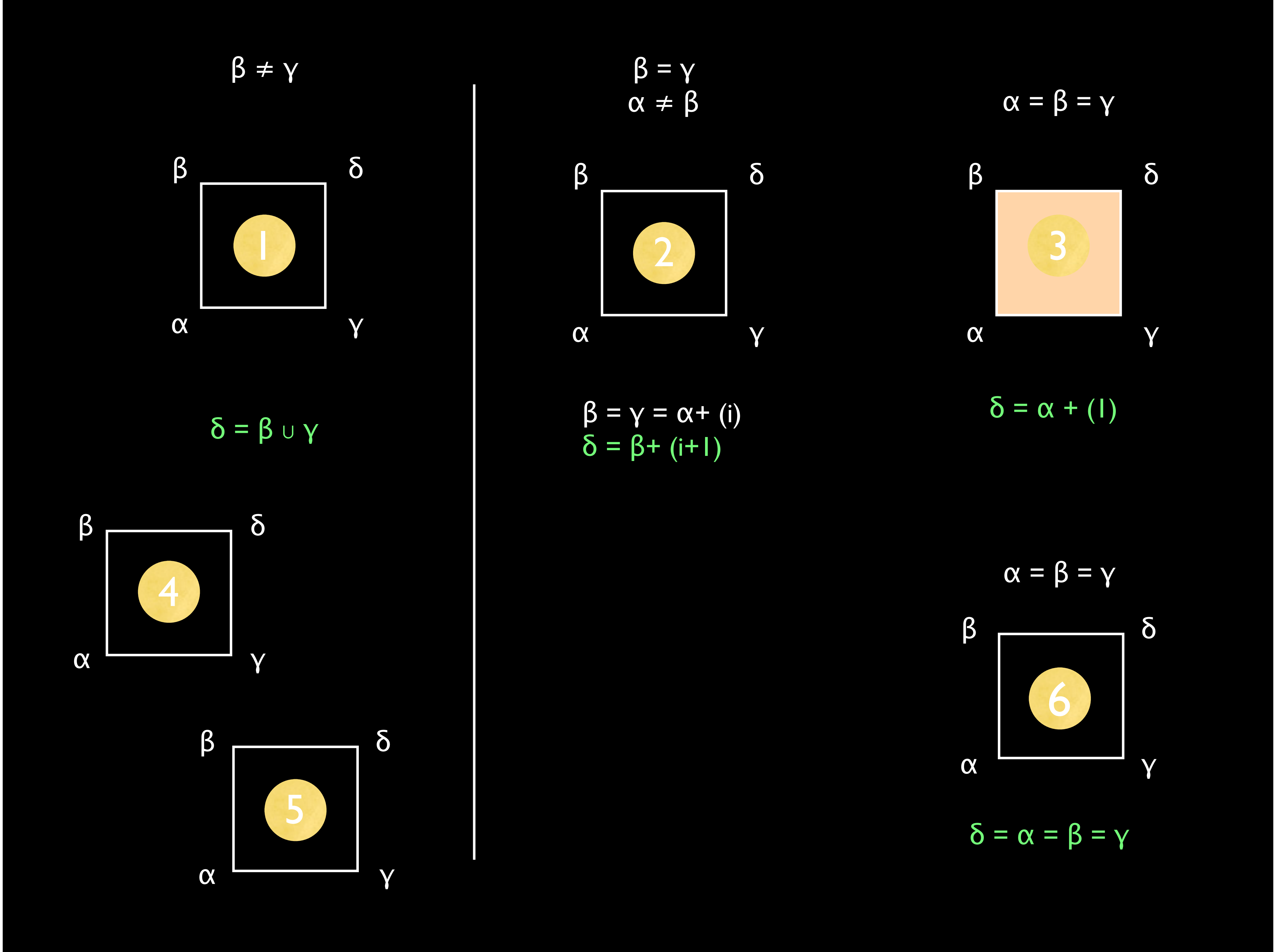
β ≠ γ
β = γ
U
β
δ
D
1
D
α
γ
U
δ = β ∪ γ
U
β
δ
D
2
D
α
γ
U
β = γ = α+ (i)
δ = β+ (i+1)
U
β
δ
I
3
D
α
γ
I
α = β = γ
δ = α + (1)
U
β
δ
I
4
I
α
γ
U
α = β
δ = γ = β +(i)
I
β
δ
D
5
D
α
γ
I
α = γ
δ = β = α + (i)
I
β
δ
I
6
I
α
γ
I
δ = α = β = γ

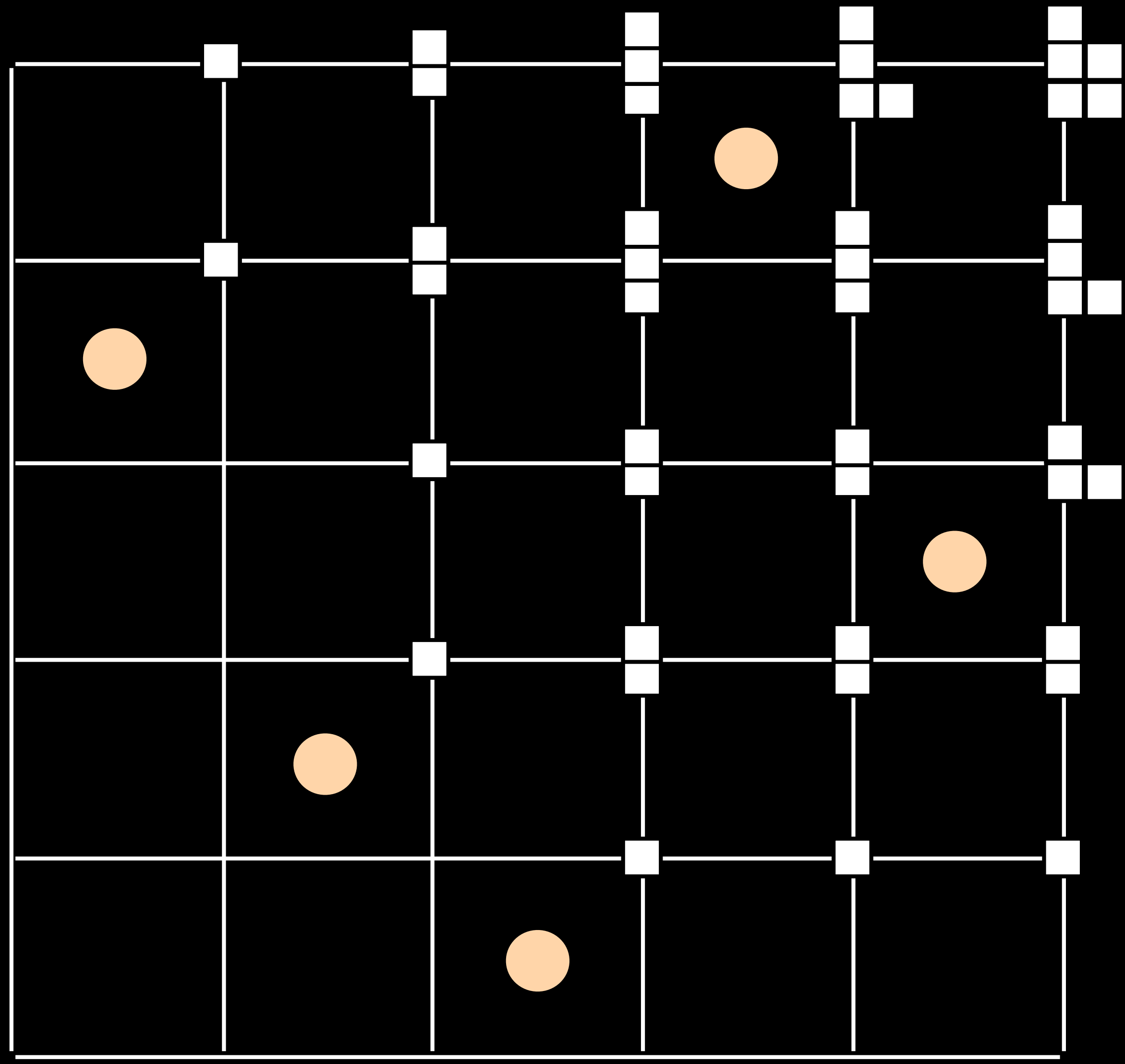

U
D
I
I


U
D
I
I


U
D
I
I


β ≠ γ
β = γ
U
β
δ
D
1
D
α
γ
U
δ = β ∪ γ
U
β
δ
D
2
D
α
γ
U
β = γ = α+ (i)
δ = β+ (i+1)
U
β
δ
I
3
D
α
γ
I
α = β = γ
δ = α + (1)
U
β
δ
I
4
I
α
γ
U
α = β
δ = γ = β +(i)
I
β
δ
D
5
D
α
γ
I
α = γ
δ = β = α + (i)
I
β
δ
I
6
I
α
γ
I
δ = α = β = γ


β ≠ γ
β δ α γ 1
δ = β ∪ γ
β δ α γ 4
β δ α γ 5
β = γ
α ≠ β
β δ α γ 2
β = γ = α+ (i)
δ = β+ (i+1)
α = β = γ
β δ α γ 3
δ = α + (1)
α = β = γ
β δ α γ 6
δ = α = β = γ



RSK with

Fomin's

"local rules"

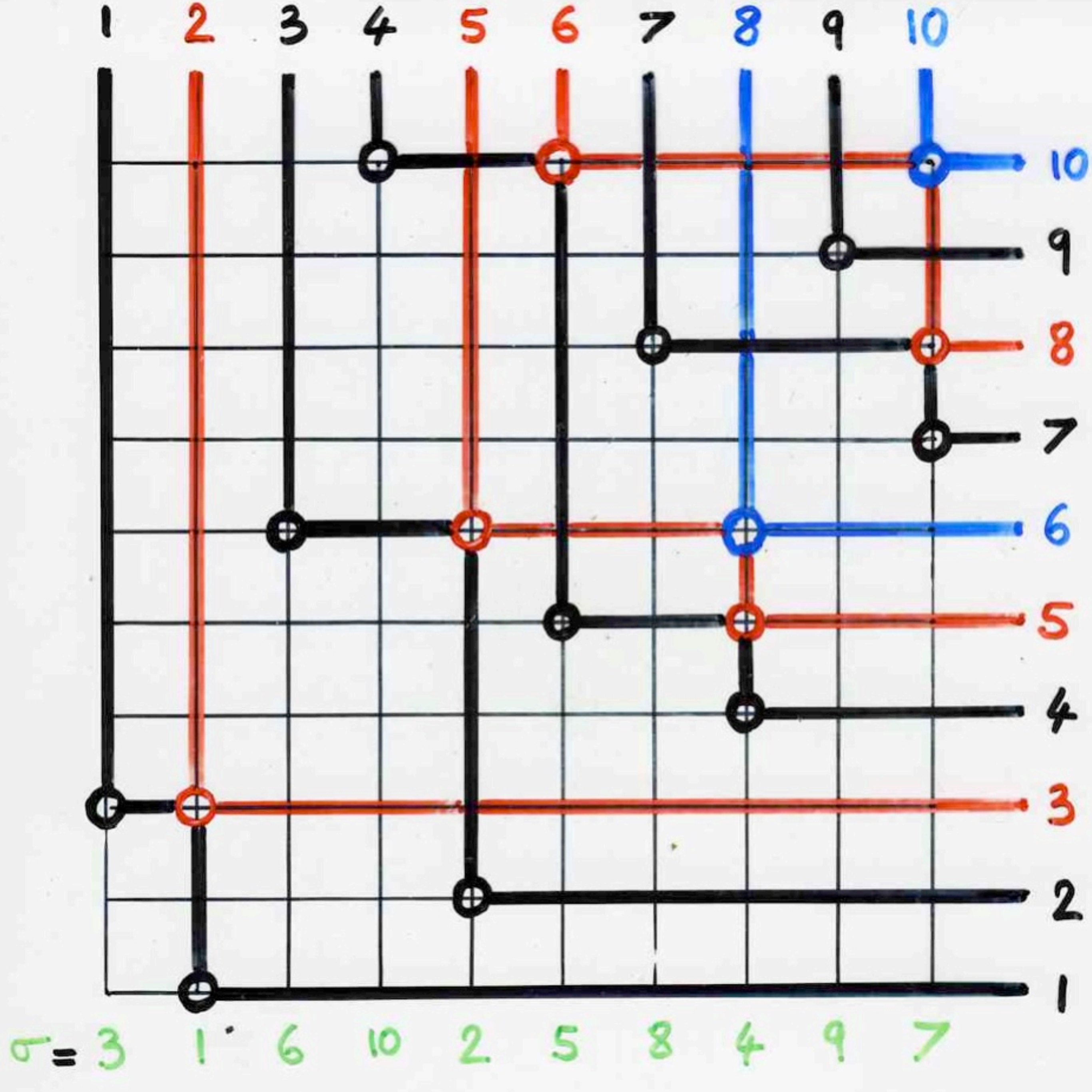
$$UD = q\,DU + 1$$

Sergey Fomin

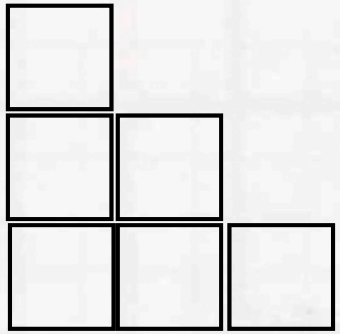
(with C. K.)

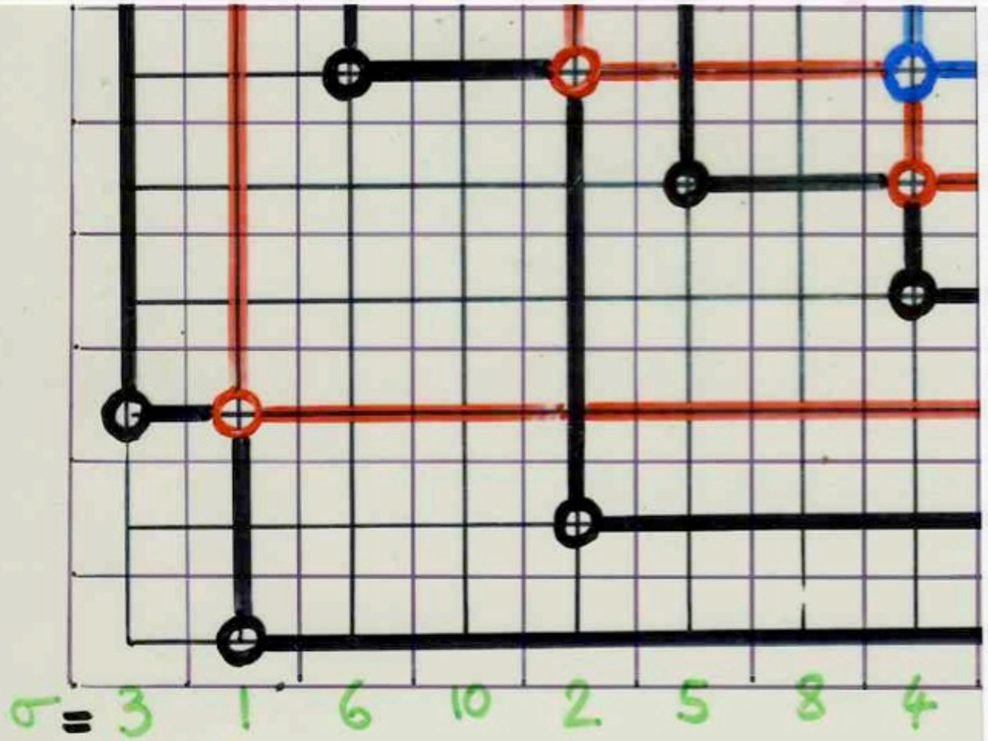
local RSK and geometric RSK

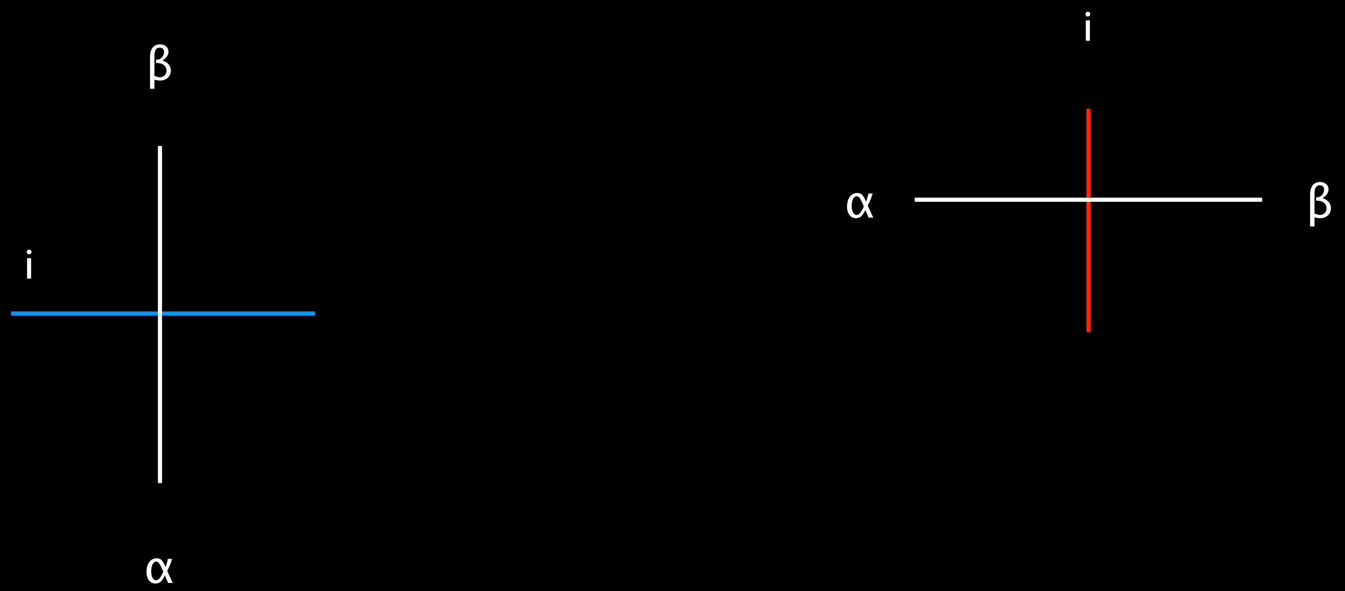
(the geometric construction with "light" and "shadow" for RSK leads to a simple proof of the fact that RSK and the "local rules" give the same bijection)

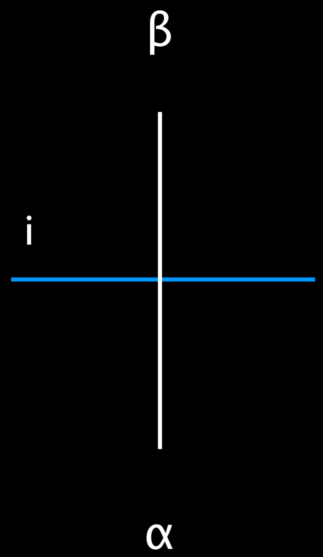


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7

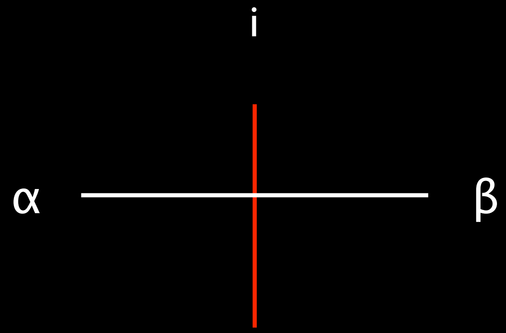


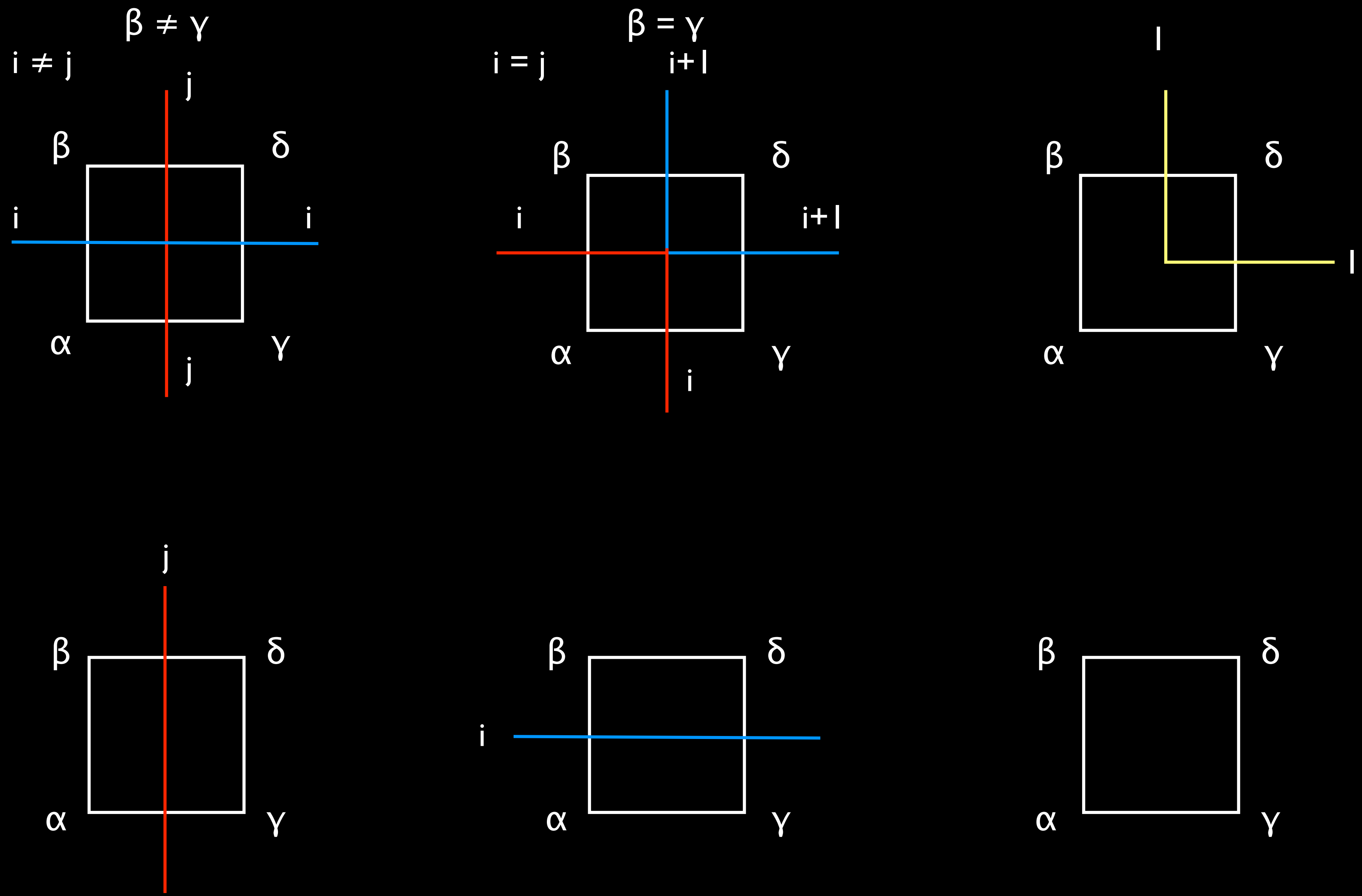

σ = 3 1 6 10 2 5 8 4


β
i
α
i
α
β

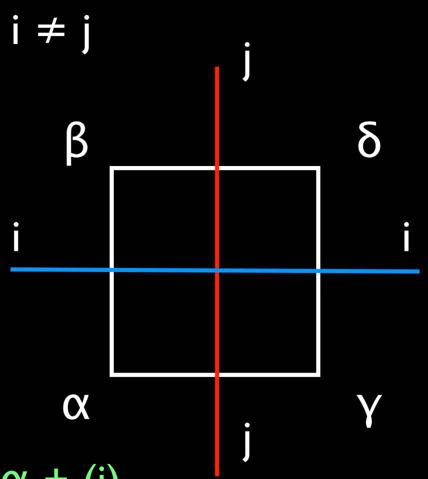


$\beta = \alpha + (i)$

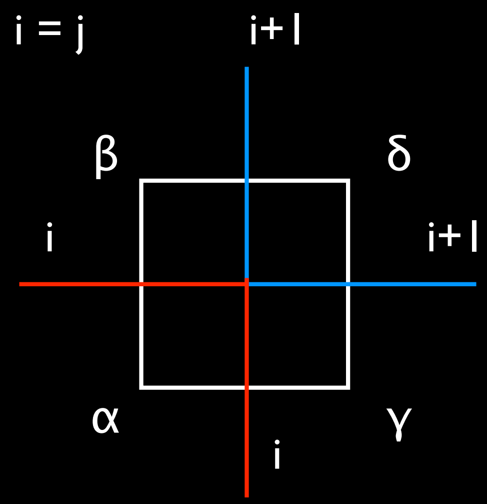




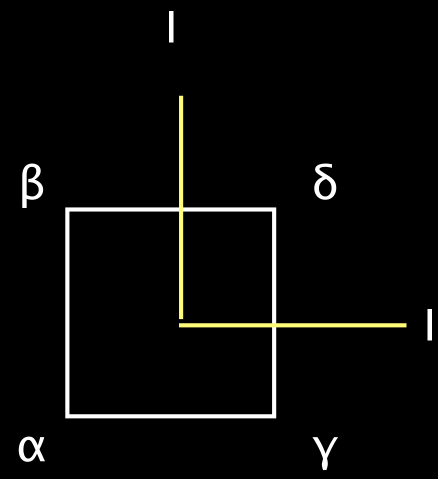
β ≠ γ
i ≠ j
j
β
δ
i
i
α
j
γ
β = γ
i = j
i+1
β
δ
i
i+1
α
i
γ
1
β
δ
1
α
γ
j
β
δ
α
γ
β
δ
i
α
γ
β
δ
α
γ



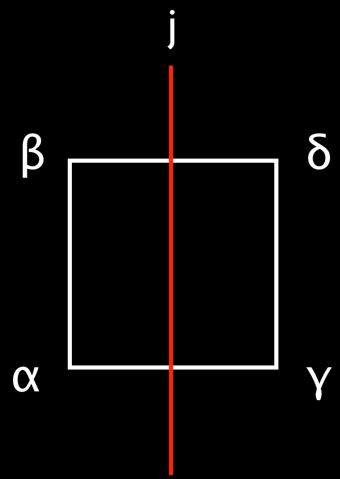
$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$



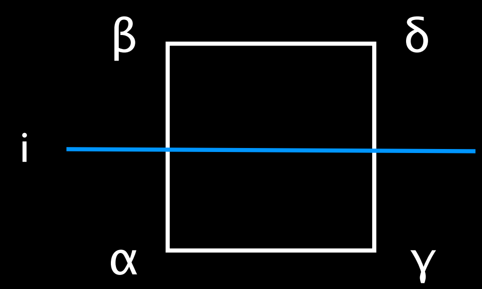
$\beta = \gamma = \alpha + (i)$
$\delta = \alpha + (i) + (i+1)$



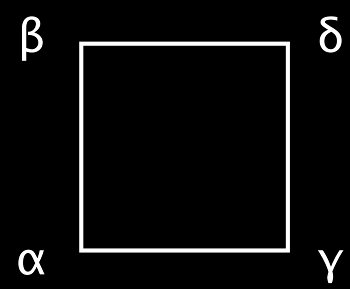
$\beta = \gamma = \alpha$
$\delta = \alpha + (1)$



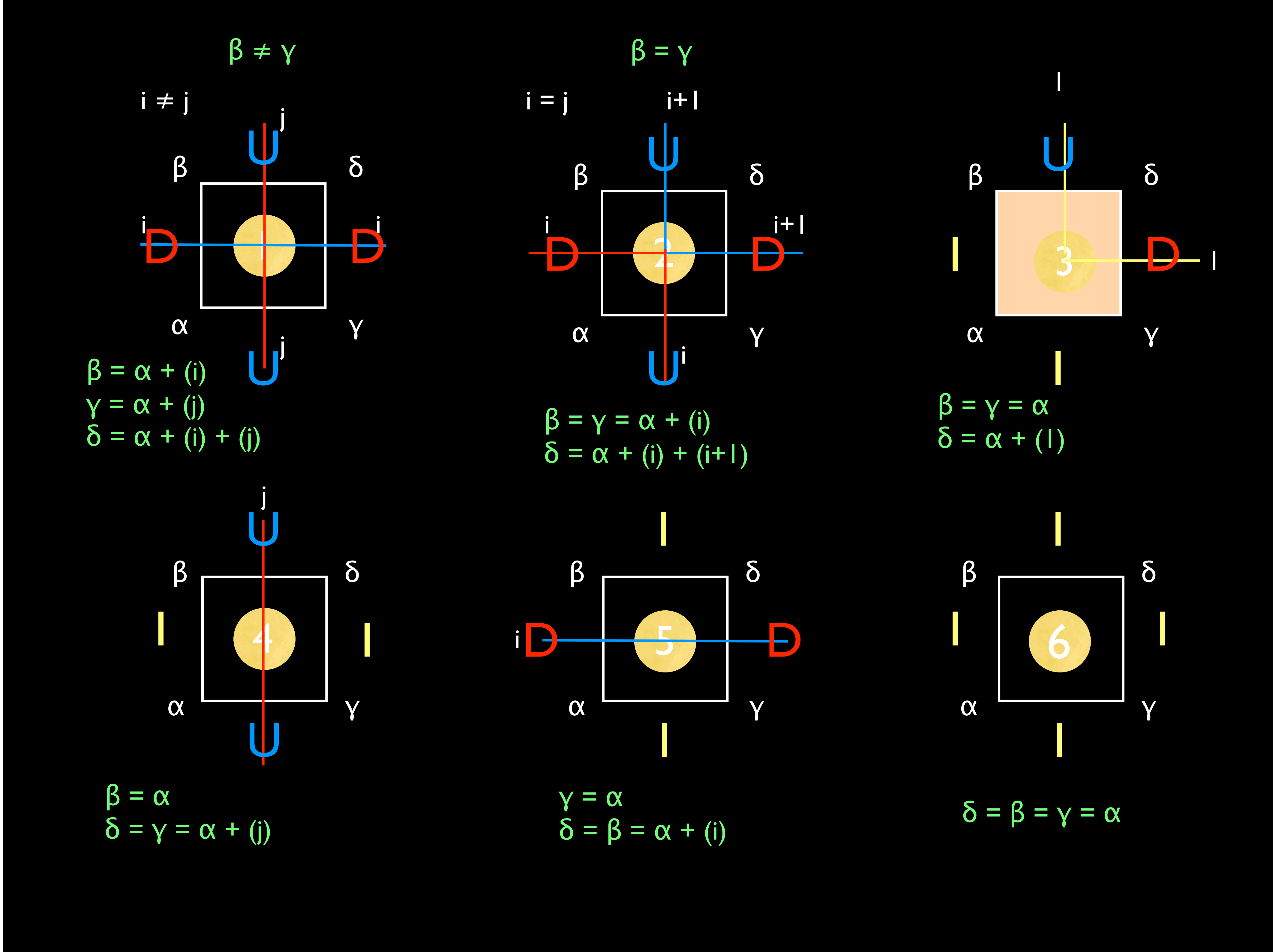
$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$



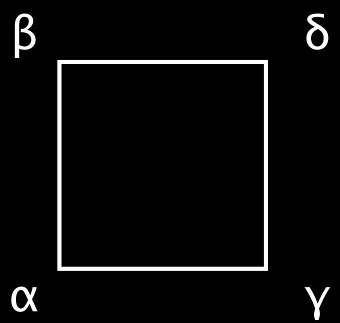
$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$



$\delta = \beta = \gamma = \alpha$


β ≠ γ
i ≠ j
j
β
δ
i
1
i
α
γ
j
β = α + (i)
γ = α + (j)
δ = α + (i) + (j)
β = γ
i = j
i+1
β
δ
i
2
i+1
α
γ
i
β = γ = α + (i)
δ = α + (i) + (i+1)
1
β
δ
3
1
α
γ
β = γ = α
δ = α + (1)
j
β
δ
4
α
γ
β = α
δ = γ = α + (j)
β
δ
i
5
α
γ
γ = α
δ = β = α + (i)
β
δ
6
α
γ
δ = β = γ = α

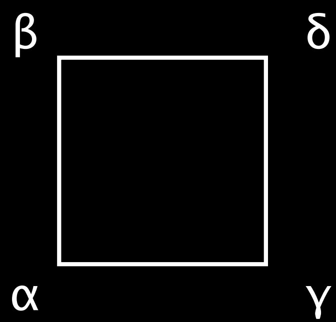
$\beta \neq \gamma$



$\delta = \beta \cup \gamma$

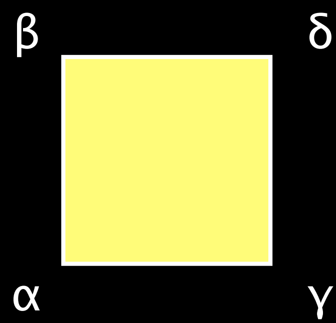
$\beta = \gamma$

$\beta = \gamma$
$\alpha \neq \beta$



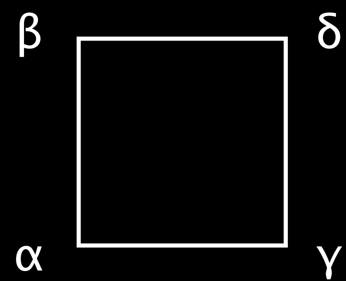
$\beta = \gamma = \alpha + (i)$
$\delta = \beta + (i+1)$

$\alpha = \beta = \gamma$

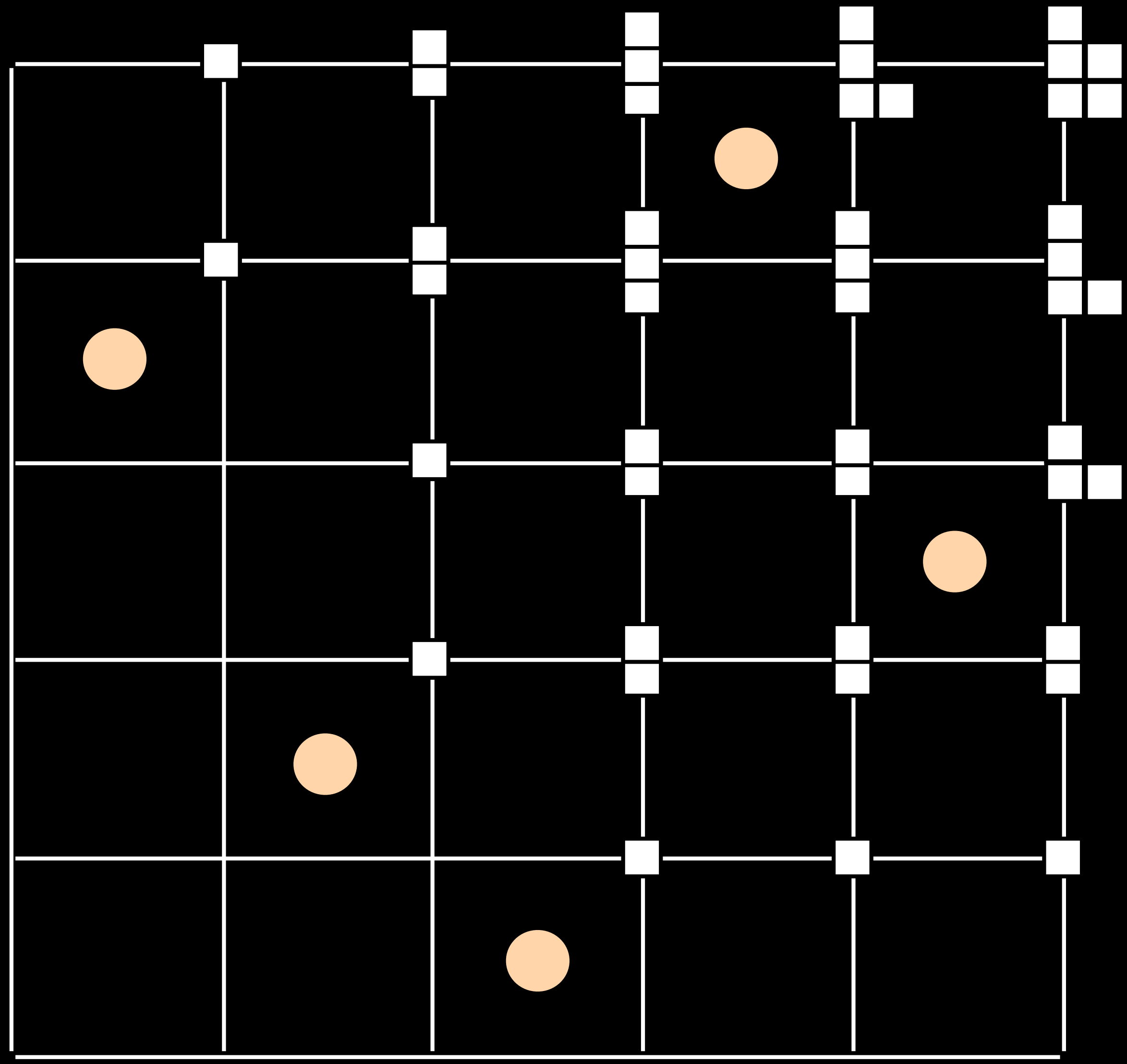


$\delta = \alpha + (1)$

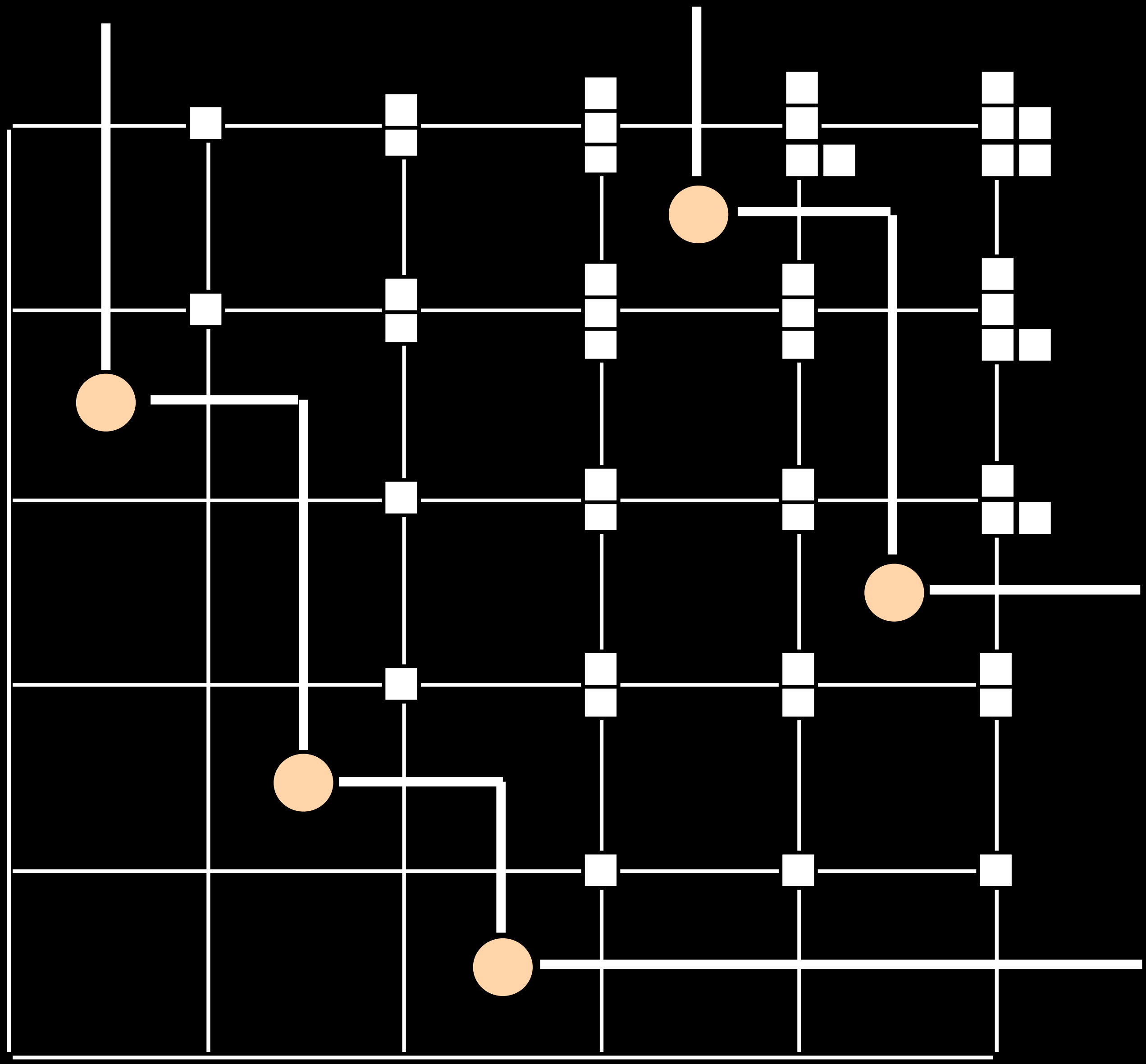
$\alpha = \beta = \gamma$

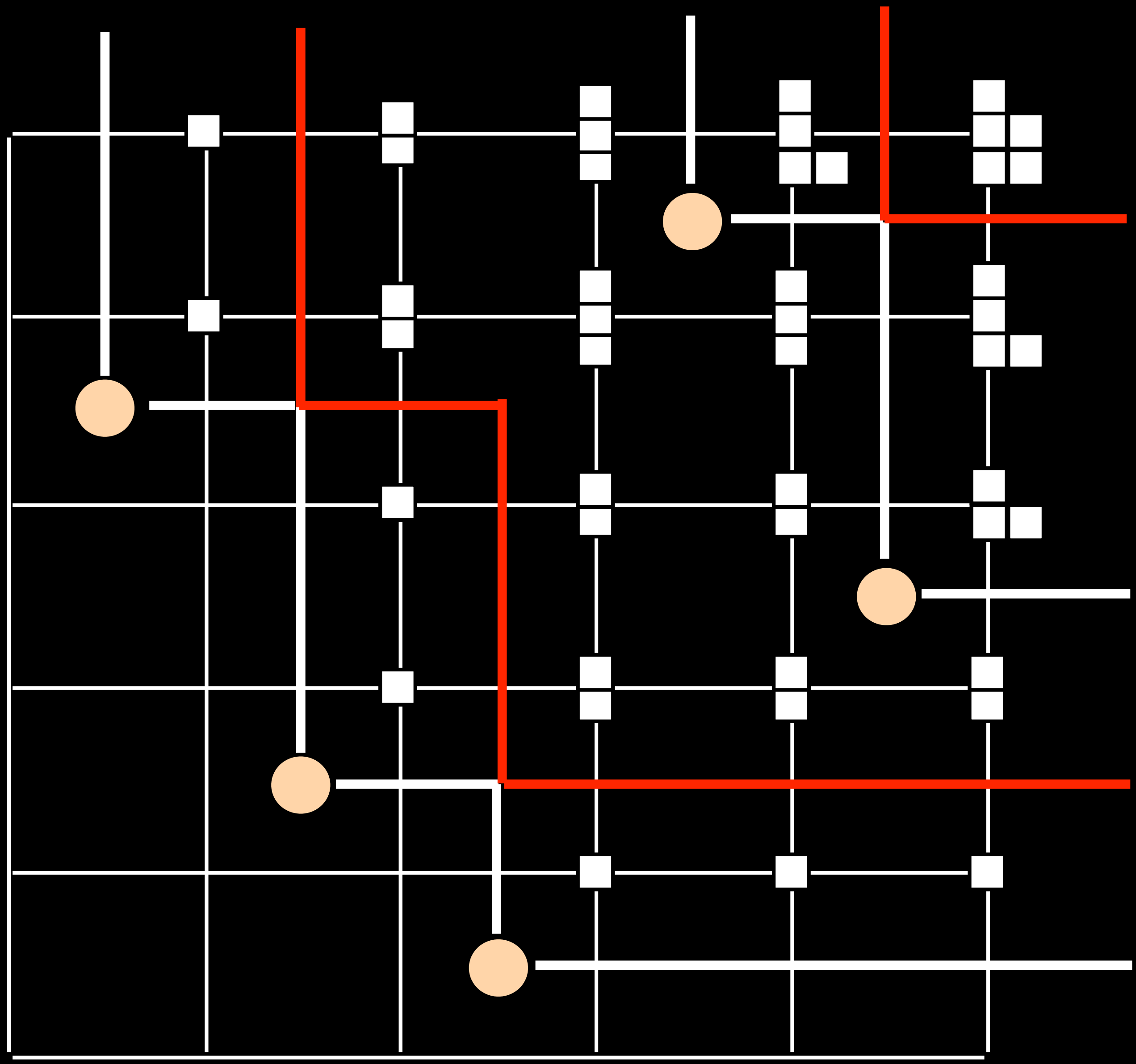


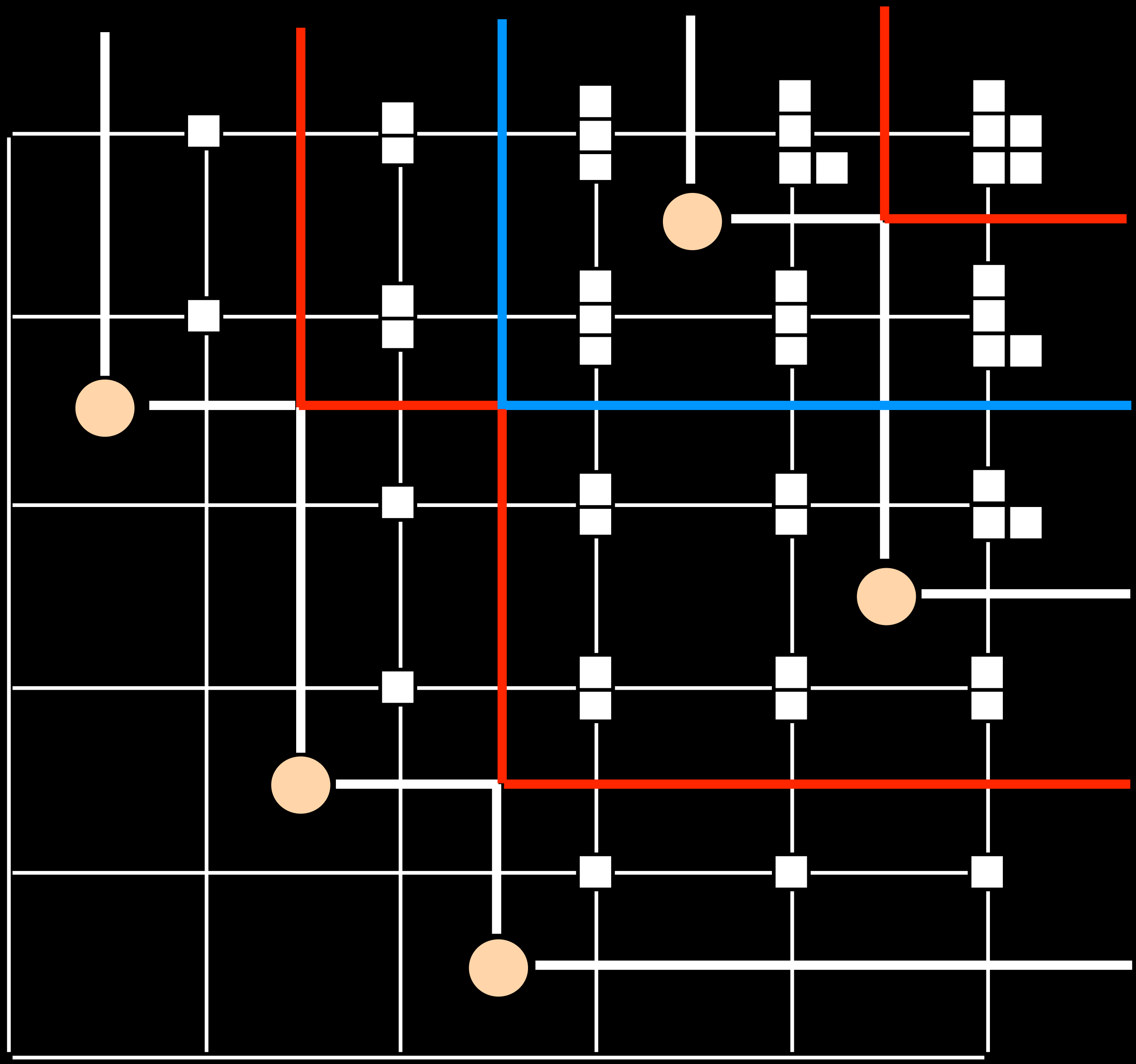
$\delta = \alpha = \beta = \gamma$

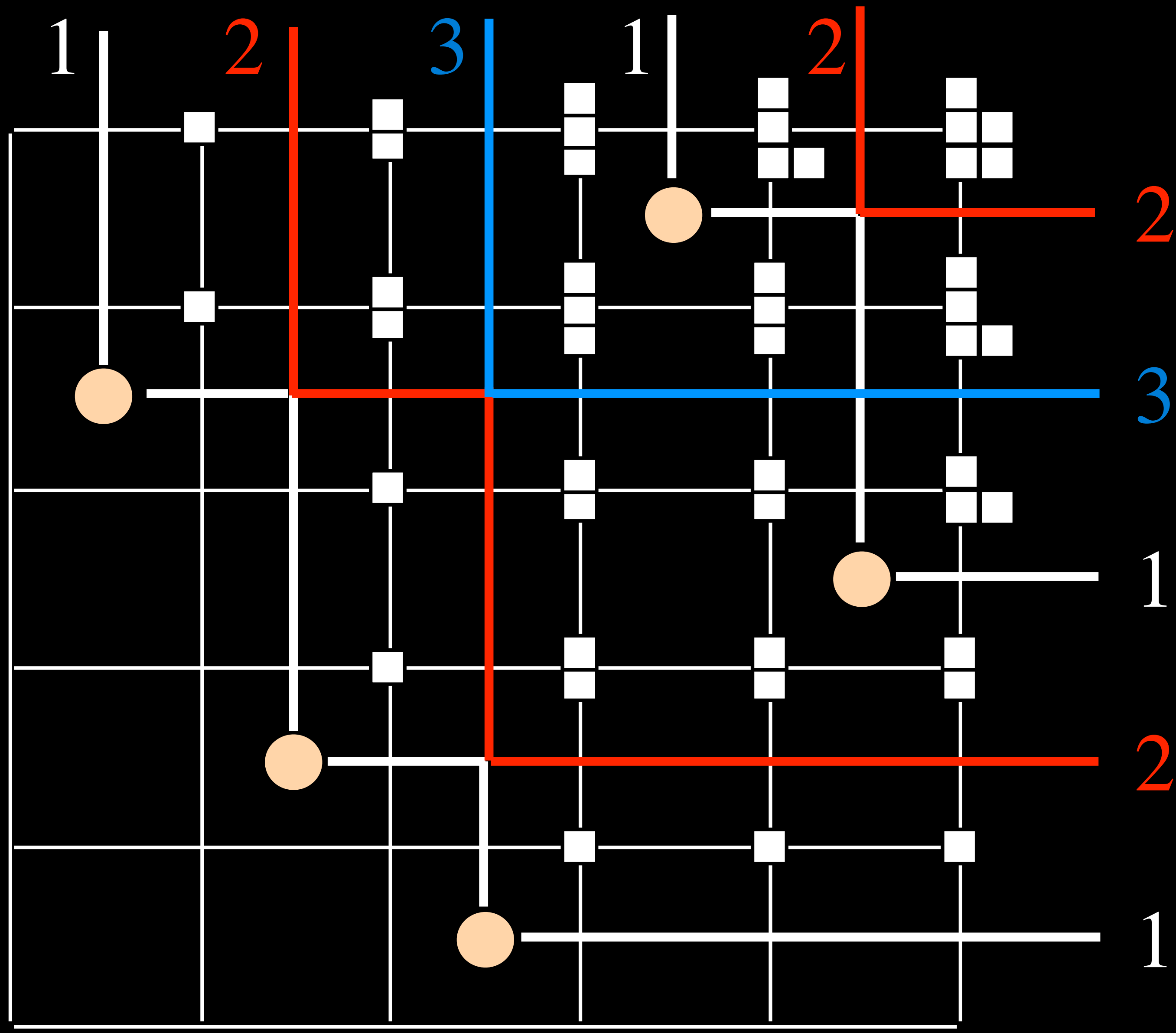


4 2 1 5 3

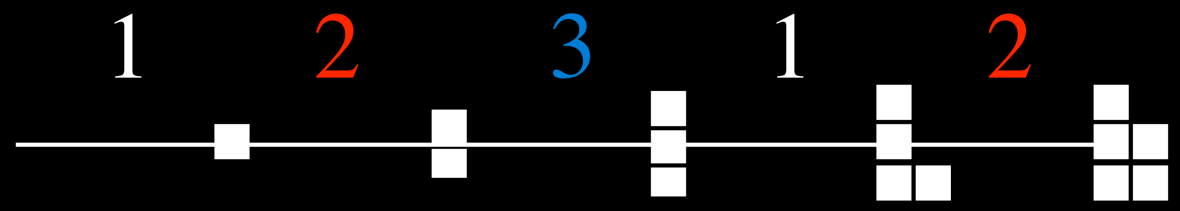


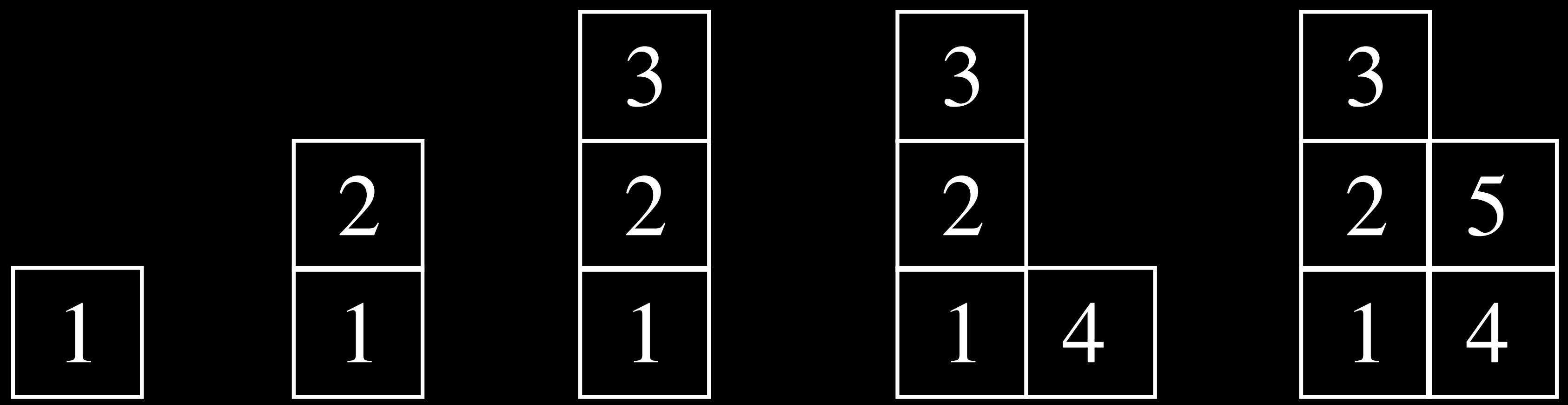






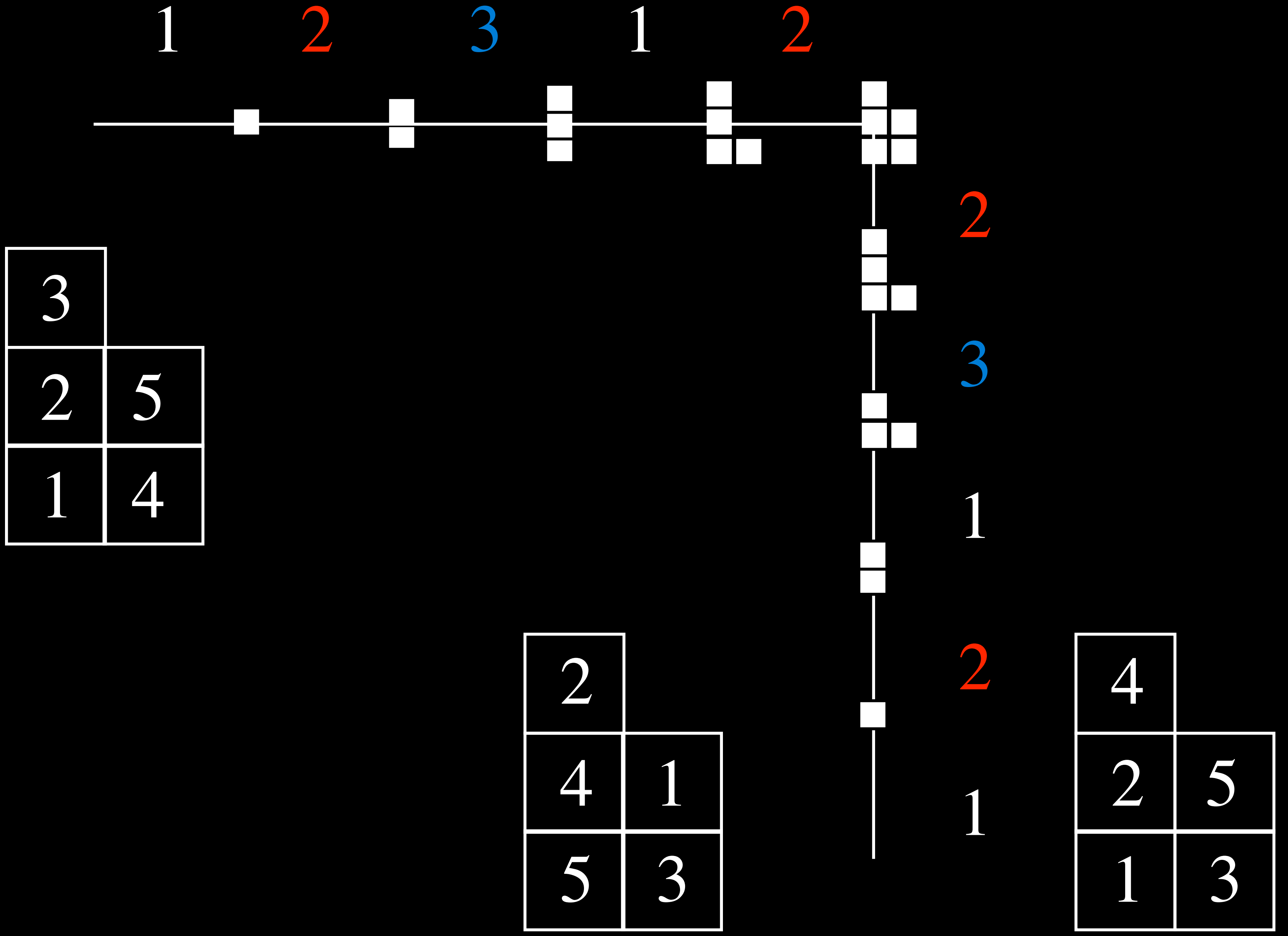
1
2
3
1
2
2
3
1
2
1



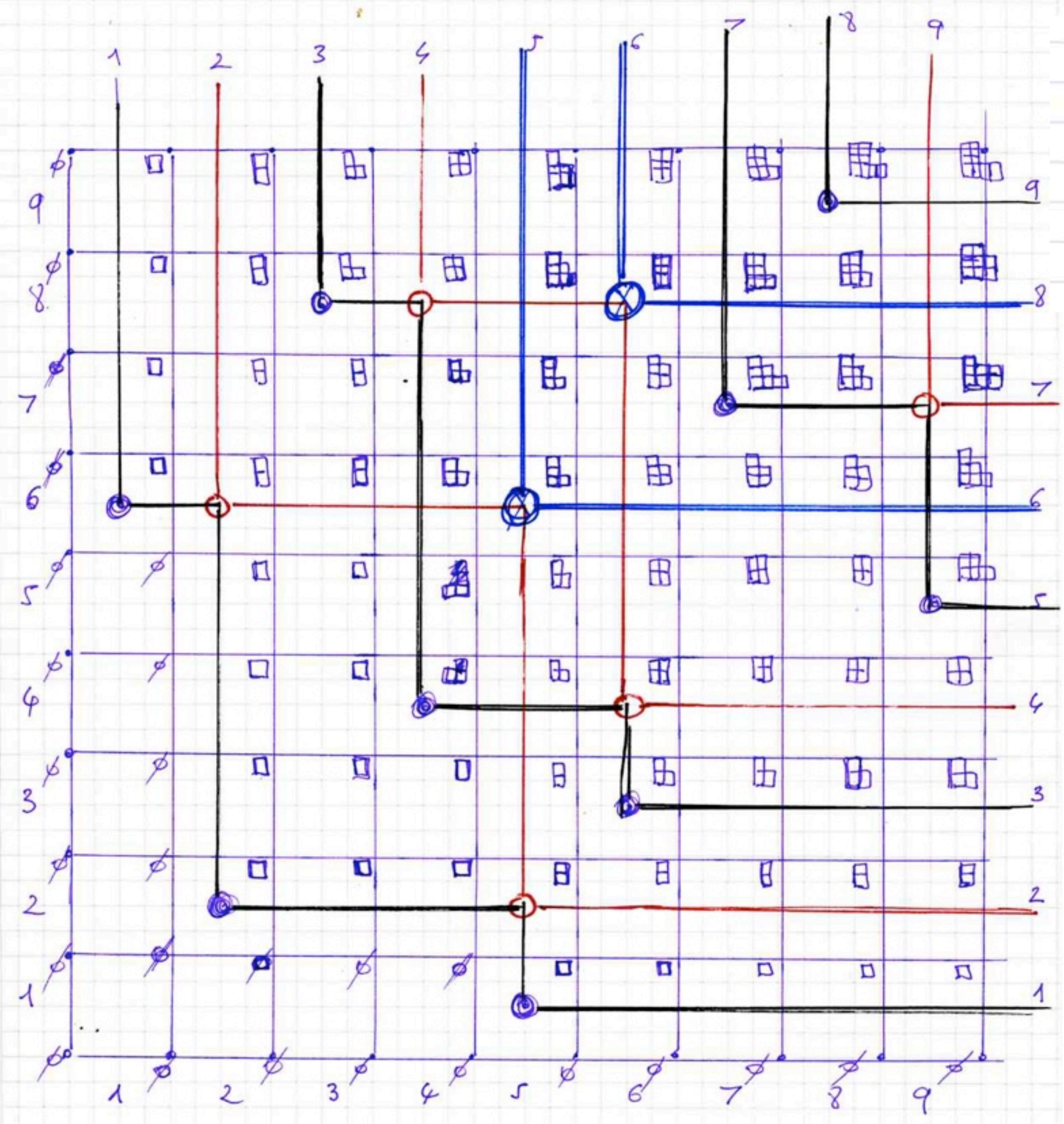


w = 1 2 3 1 2

Yamanuchi word



1 2 3 1 2
2 3 1 2 1
3
2 5
1 4
2
4 1
5 3
4
2 5
1 3



5	6		
2	4	9	
1	3	7	8

Q

another example
with
6 2 8 4 1 3 7 9 5

6	8		
2	4	7	
1	3	5	9

P

Sergey Fomin

- **Schur operators and Knuth correspondences**, *Journal of Combinatorial Theory, Ser.A* **72** (1995), 277-292.
- **Duality of graded graphs**, *Journal of Algebraic Combinatorics* **3** (1994), 357-404.
- **Schensted algorithms for dual graded graphs**, *Journal of Algebraic Combinatorics* **4** (1995), 5-45.
- **Dual graphs and Schensted correspondences**, *Series formelles et combinatoire algebrique*, P.Leroux and C.Reutenauer, Ed., Montreal, LACIM, UQAM, 1992, 221-236.

- **Finite posets and Ferrers shapes** (with T.Britz, 41 pages)

Advances in Mathematics **158** (2000), 86-127.
A survey on the Greene-Kleitman correspondence; many proofs are new.

- **Knuth equivalence, jeu de taquin, and the Littlewood-Richardson rule** (30 pages)

Appendix 1 to Chapter 7 in: R.P.Stanley, *Enumerative Combinatorics, vol.2*,
Cambridge University Press, 1999.

Richard P. Stanley

- **Differential posets, *J. Amer. Math. Soc.* 1 (1988), 919-961.**
- **Variations on differential posets, in *Invariant Theory and Tableaux* (D. Stanton, ed.),**

The IMA Volumes in Mathematics and Its Applications, vol. 19, Springer-Verlag, New York, 1990, pp. 145-165.

Xavier Gérard Viennot

- Une forme géométrique de la correspondance de Robinson-Schensted, in “Combinatoire et Représentation du groupe symétrique” (D. Foata ed.) Lecture Notes in Mathematics n° 579, pp 29-68, 1976

Marc van Leeuwen

- The Robinson-Schensted and Schützenberger algorithms, an elementary approach

(a 272 Kb dvi file) Electronic Journal of Combinatorics, Foata Festschrift, Vol 3(no.2), R15 (1996)

Guoniu Han

http://math.u-strasbg.fr/~guoniu/software/rsk/index.html
Autour de la correspondance de Robinson-Schensted
Exposé au SLC 52 et LascouxFest, 29/03/2004