Combinatorial operators and quadratic algebras

part I: normal ordering, PASEP algebra and alternative tableaux

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"normal ordering"

Heisenberg
operators
U, D

$$UD = DU + 1$$

$$UD = DU + I$$

$$UD \to DU \qquad UD \to I$$

$$UD = DU + I$$

<u>Lemma</u>

Every word w with letters U and D can be written in a unique way

$$w = \sum_{i,j \geq 0} c_{ij}(w) D^i U^j$$

$$UD = DU + Id$$

$$U^n D^n = \; ?$$

$$UUUDDD = UU(DU + Id)DD$$

$$= UUDUDD + UUDD$$

$$= UDUUDD + 2\,UUDD$$

$$= DUUUDD + 3\,UUDD$$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2\,UD \\
&= DUDU + DU + 2\,(DU + \mathrm{Id}) \\
&= DDUU + 2\,DU \\
&= DDUU + 4\,DU + 2\,\mathrm{Id}
\end{aligned}
$$

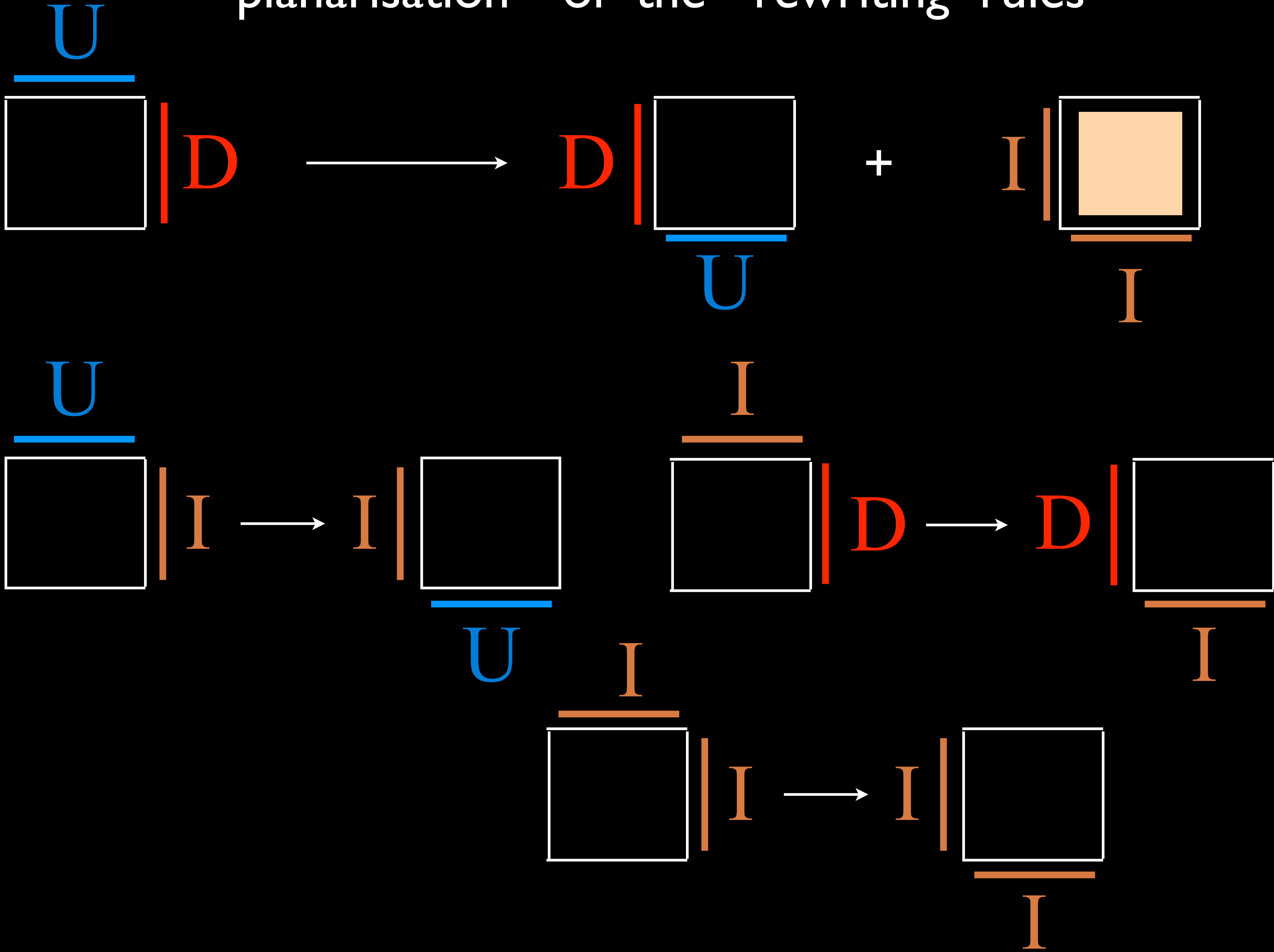
$$
\begin{aligned}
U^3D^3 &= DU\,(DDUU + 4DU + 2\,\mathrm{Id}) + \\
&\quad 3\,(DDUU + 4DU + 2\,\mathrm{Id}) \\
&= DDUDUU + DDUU \\
&\quad + 4\,(DDUU + DU) + 2\,DU \\
&\quad + 3\,DDUU + 12\,DU + 6\,\mathrm{Id} \\
&= D^3U^3 + 9\,D^2U^2 + 18\,DU + 6\,\mathrm{Id}
\end{aligned}
$$

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

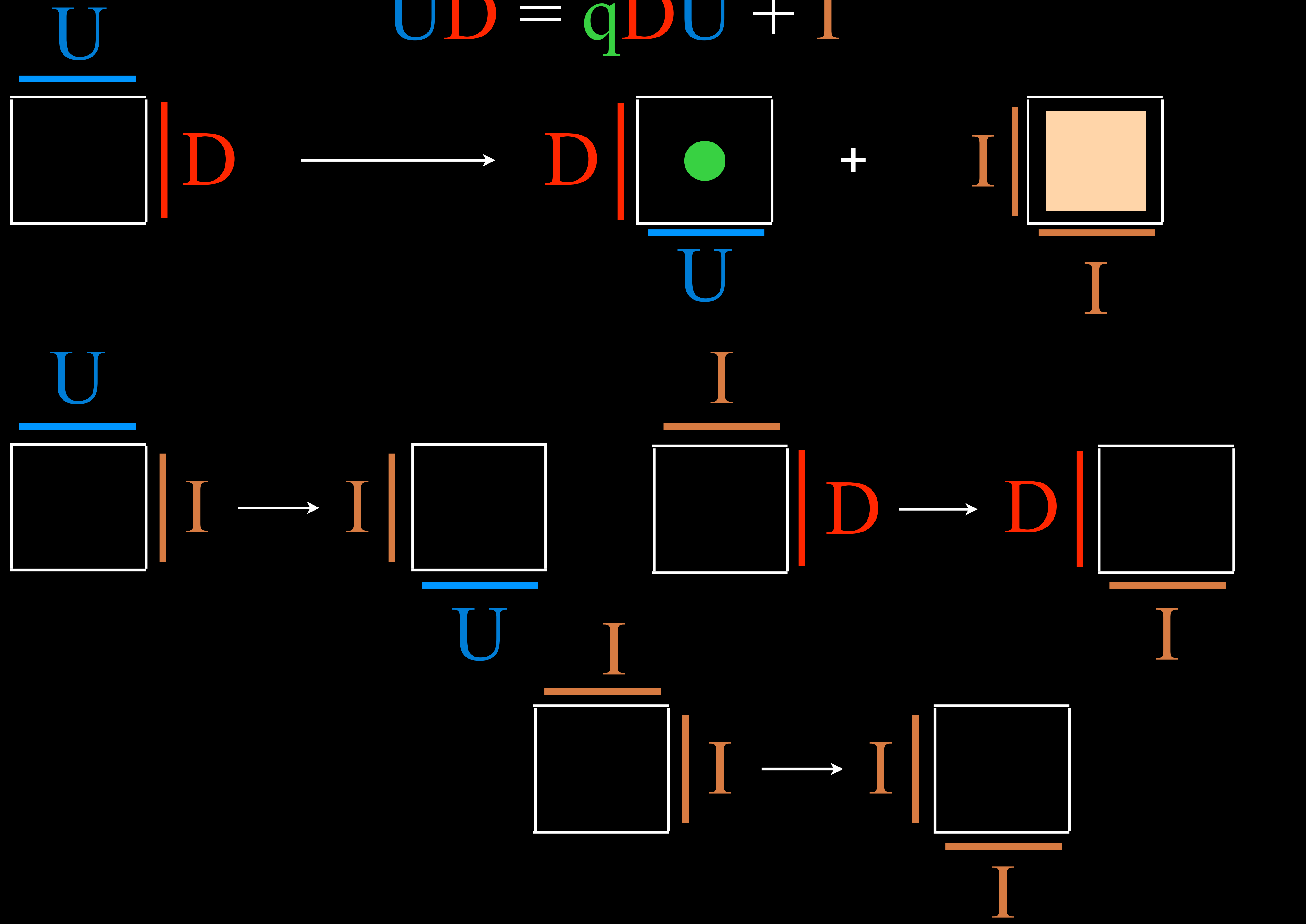
normal ordering

$$c_{n,0} = n!$$

“planarisation” of the “rewriting rules”



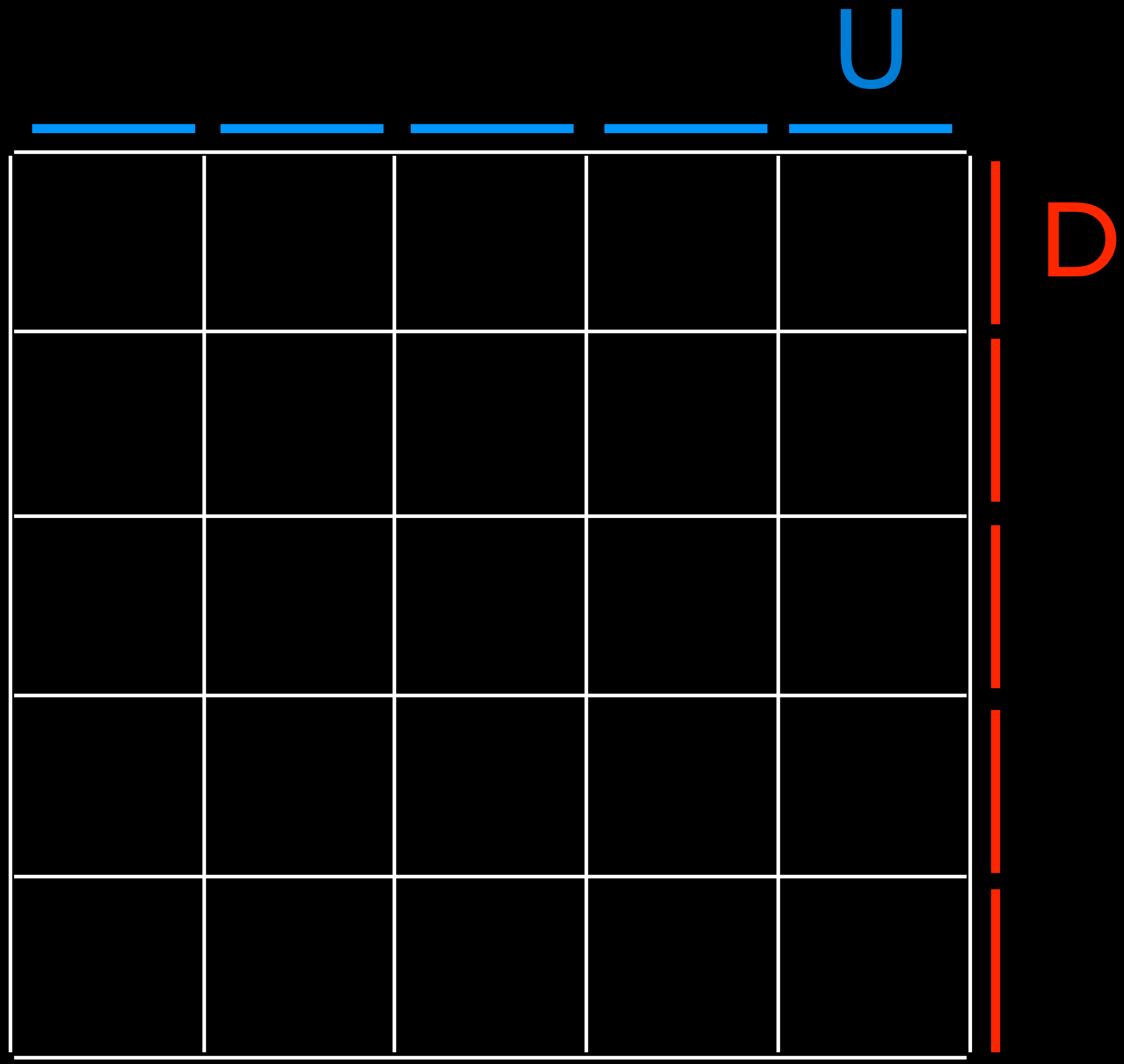
$UD = qDU + I$



$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

rewriting rules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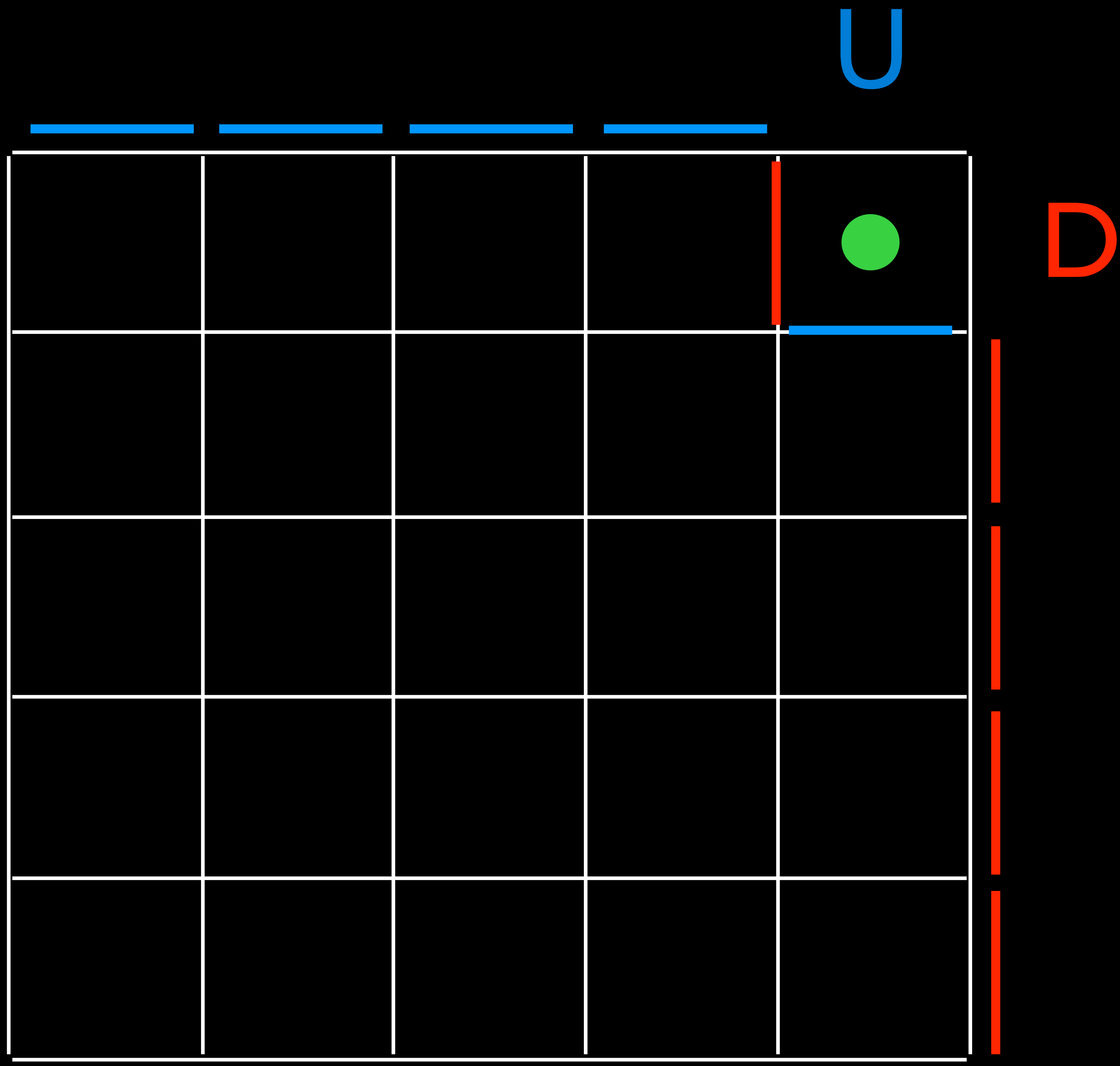
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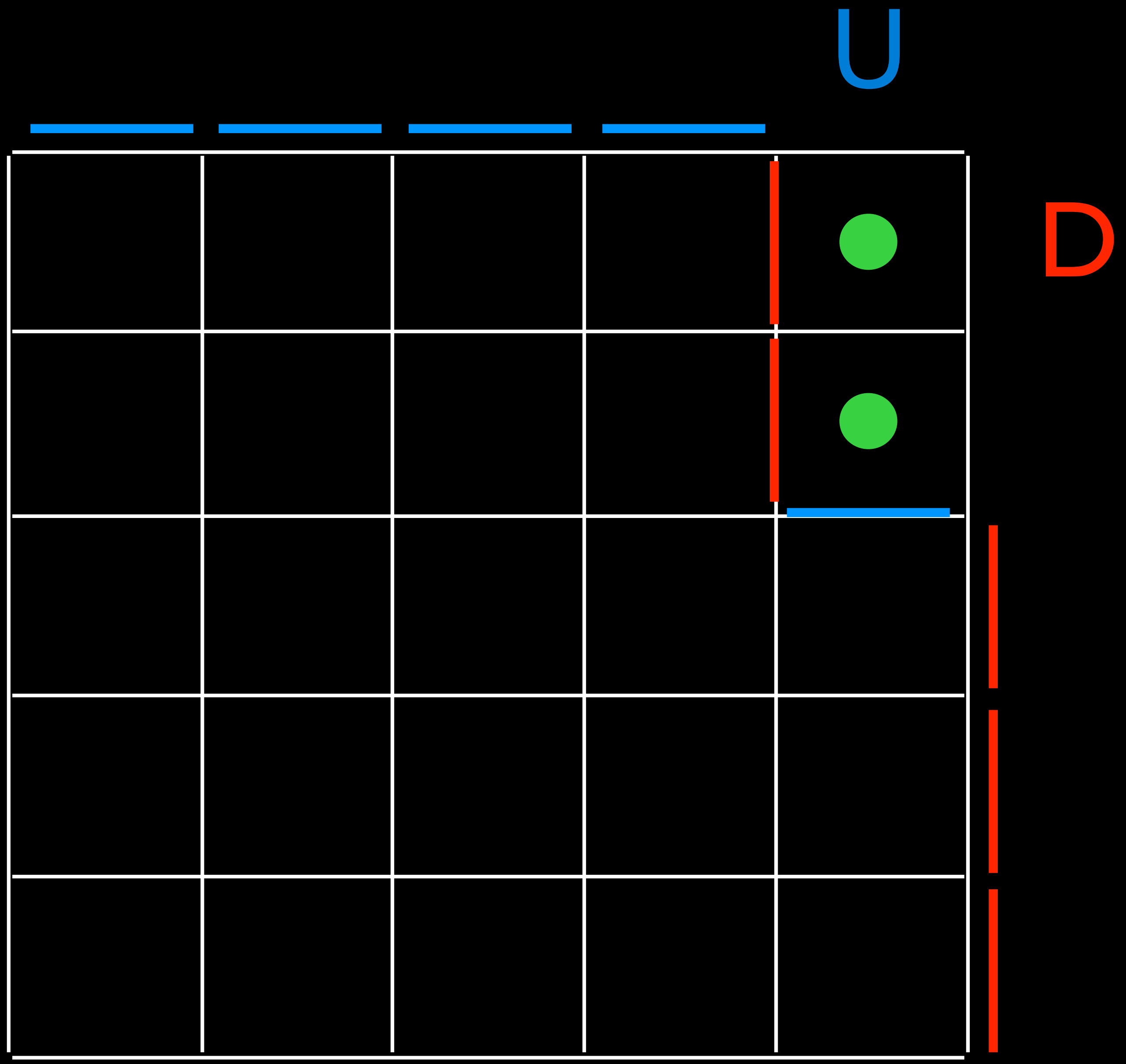
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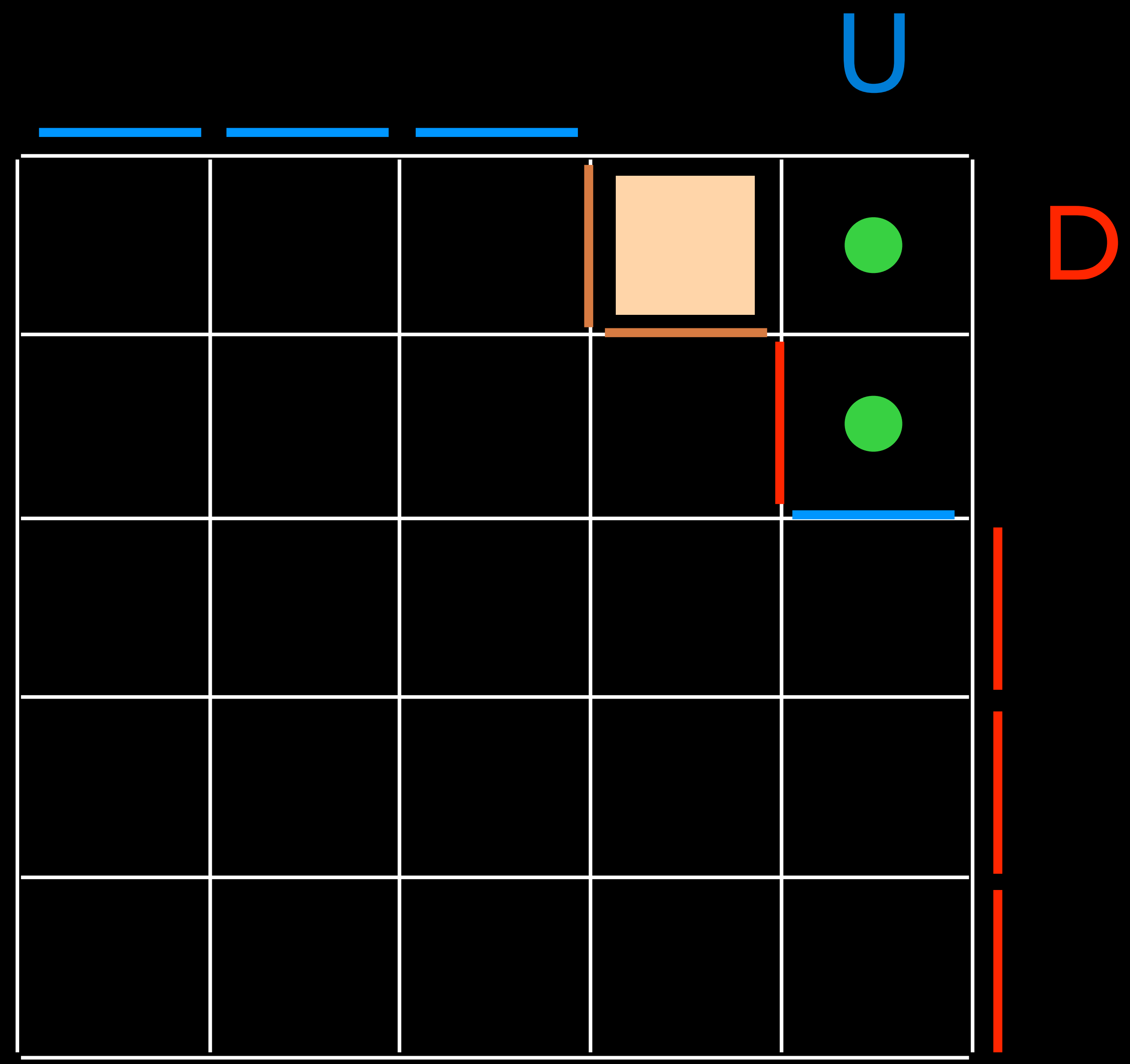
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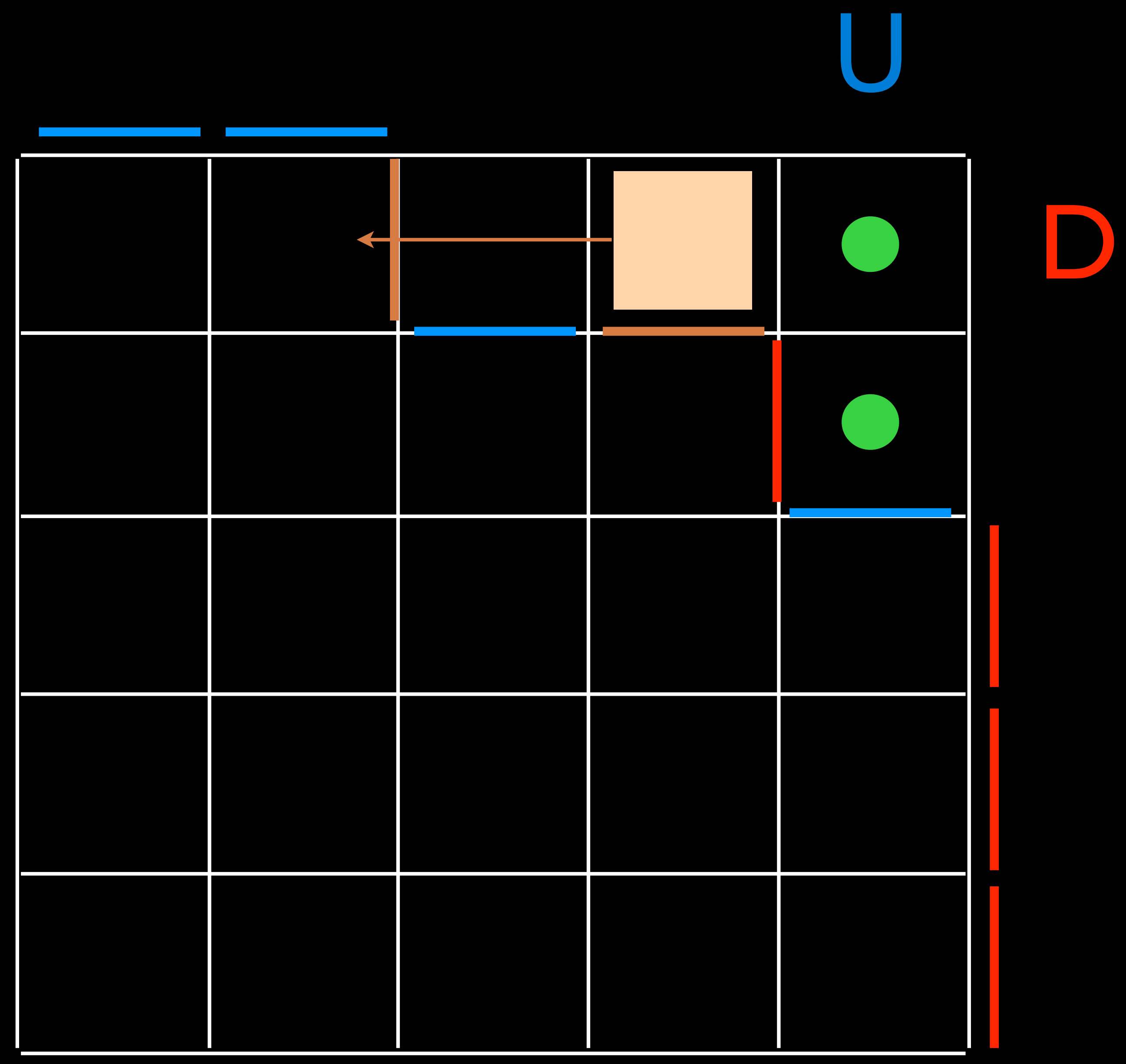
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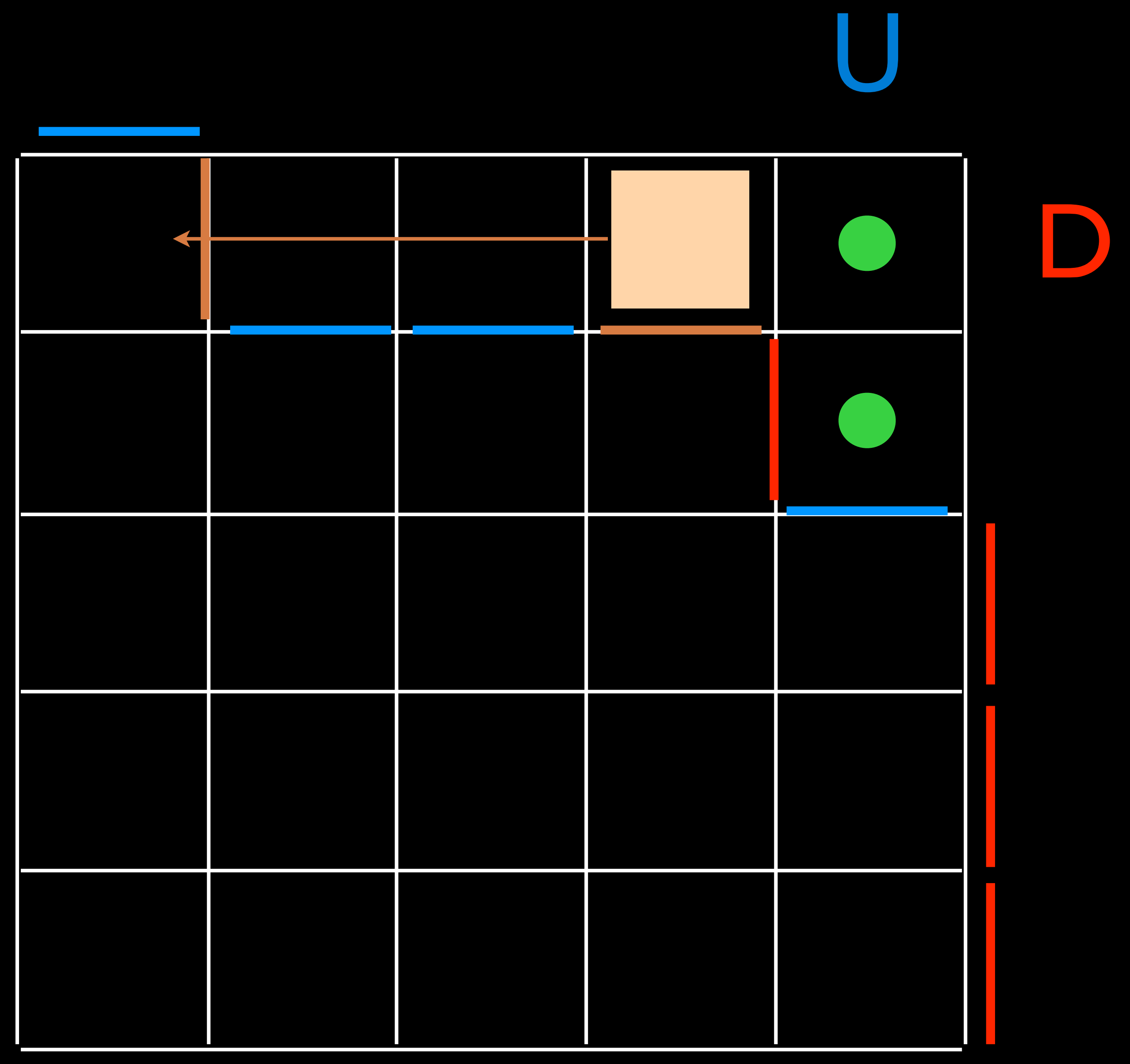
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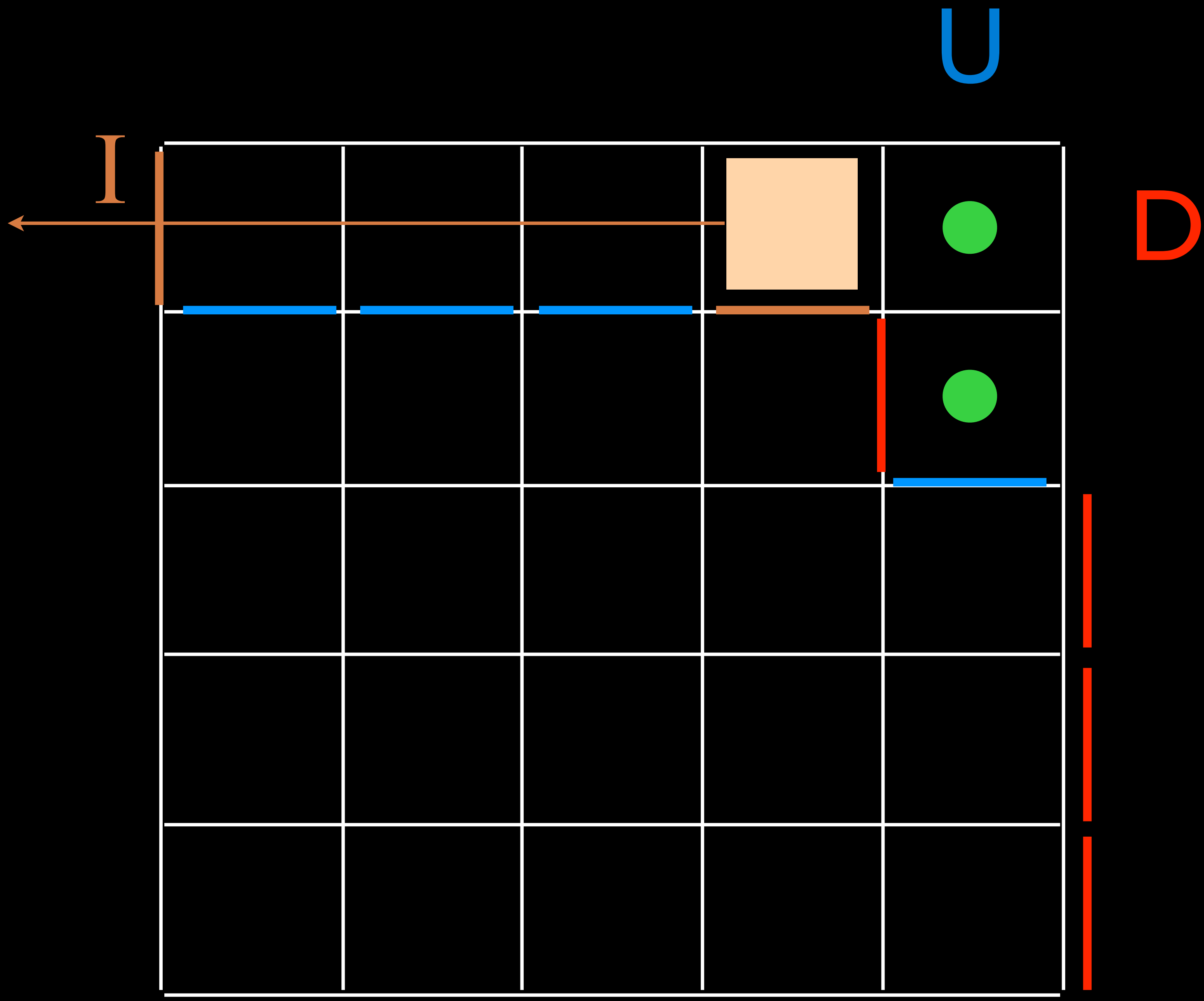
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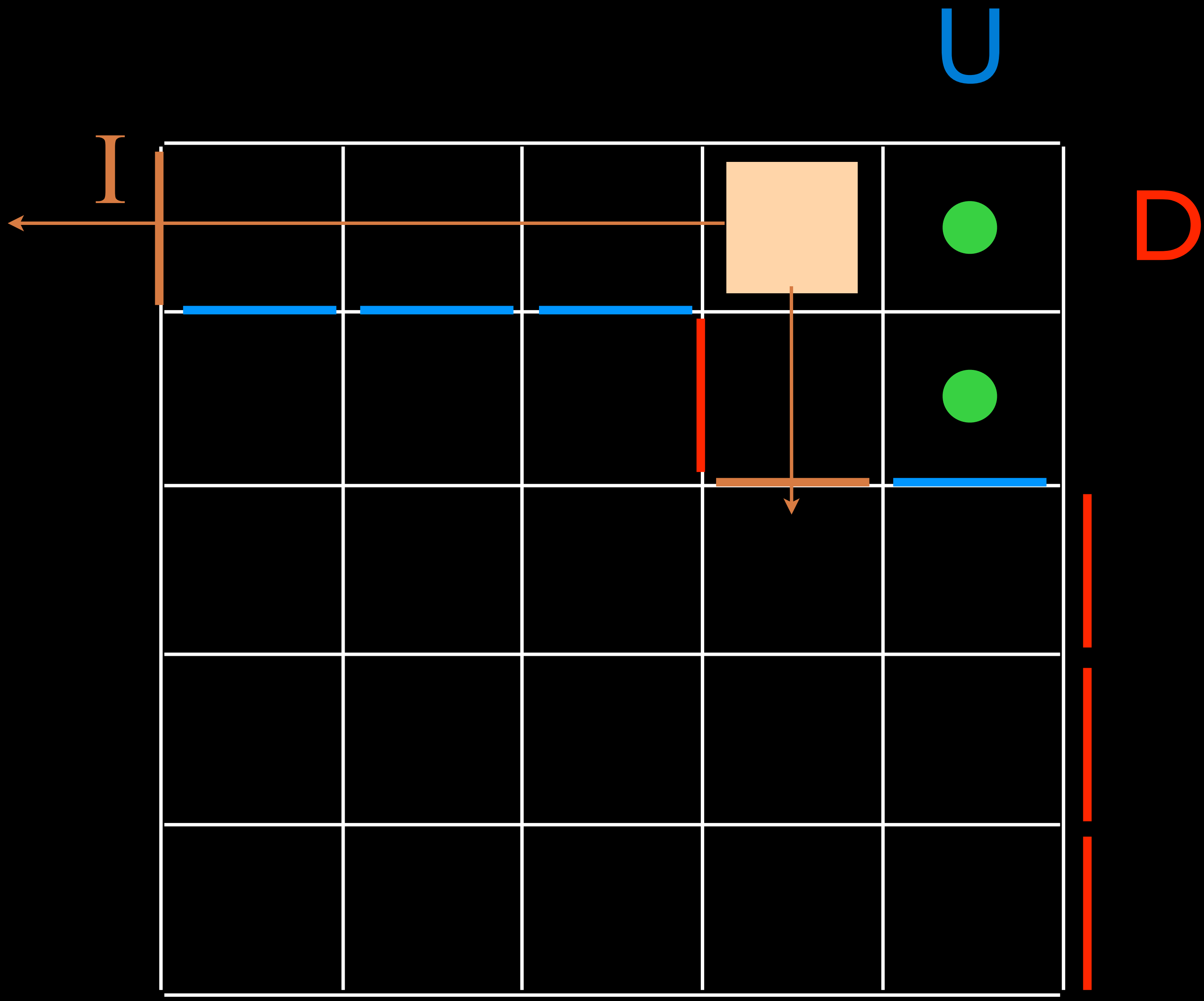
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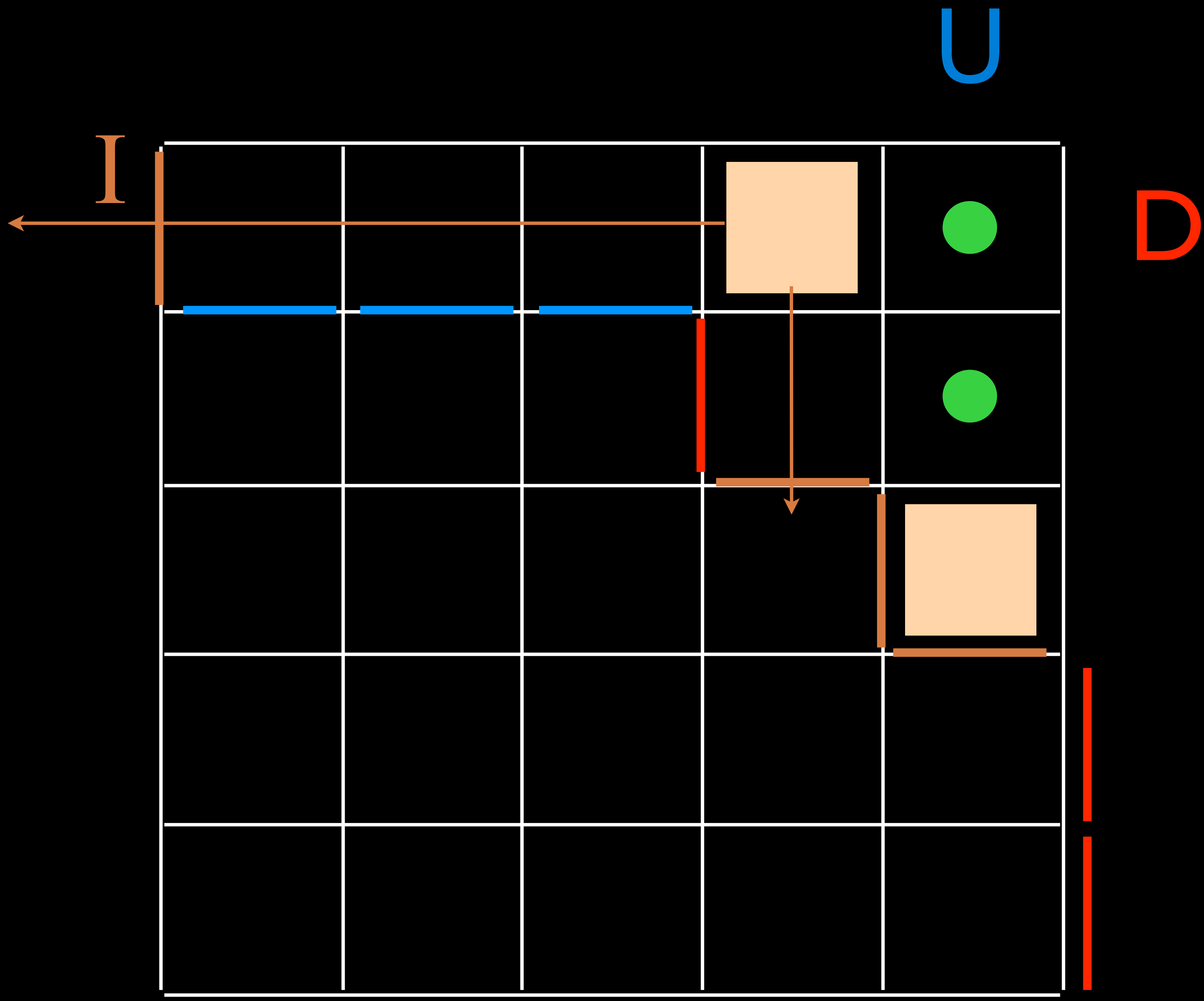
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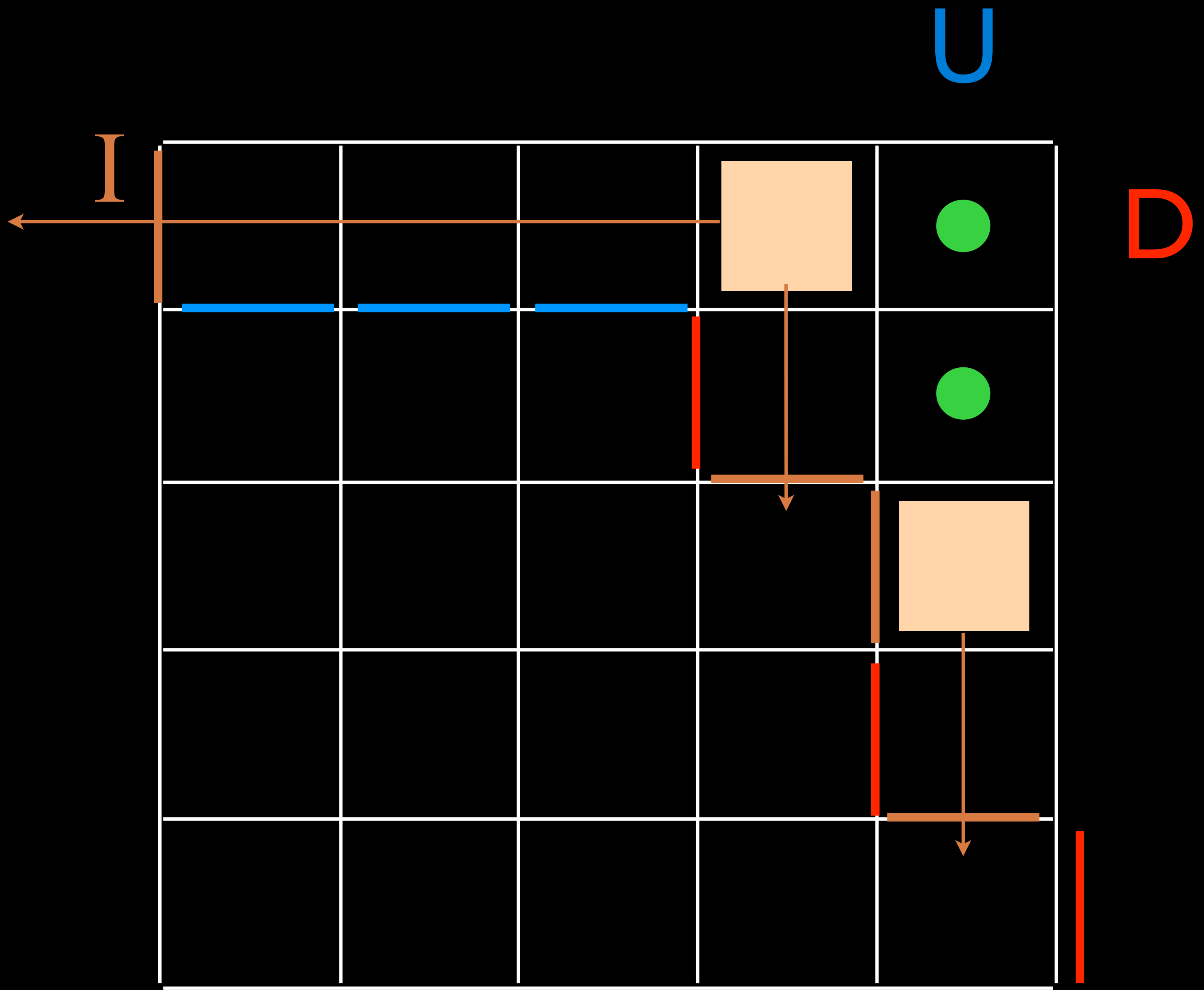
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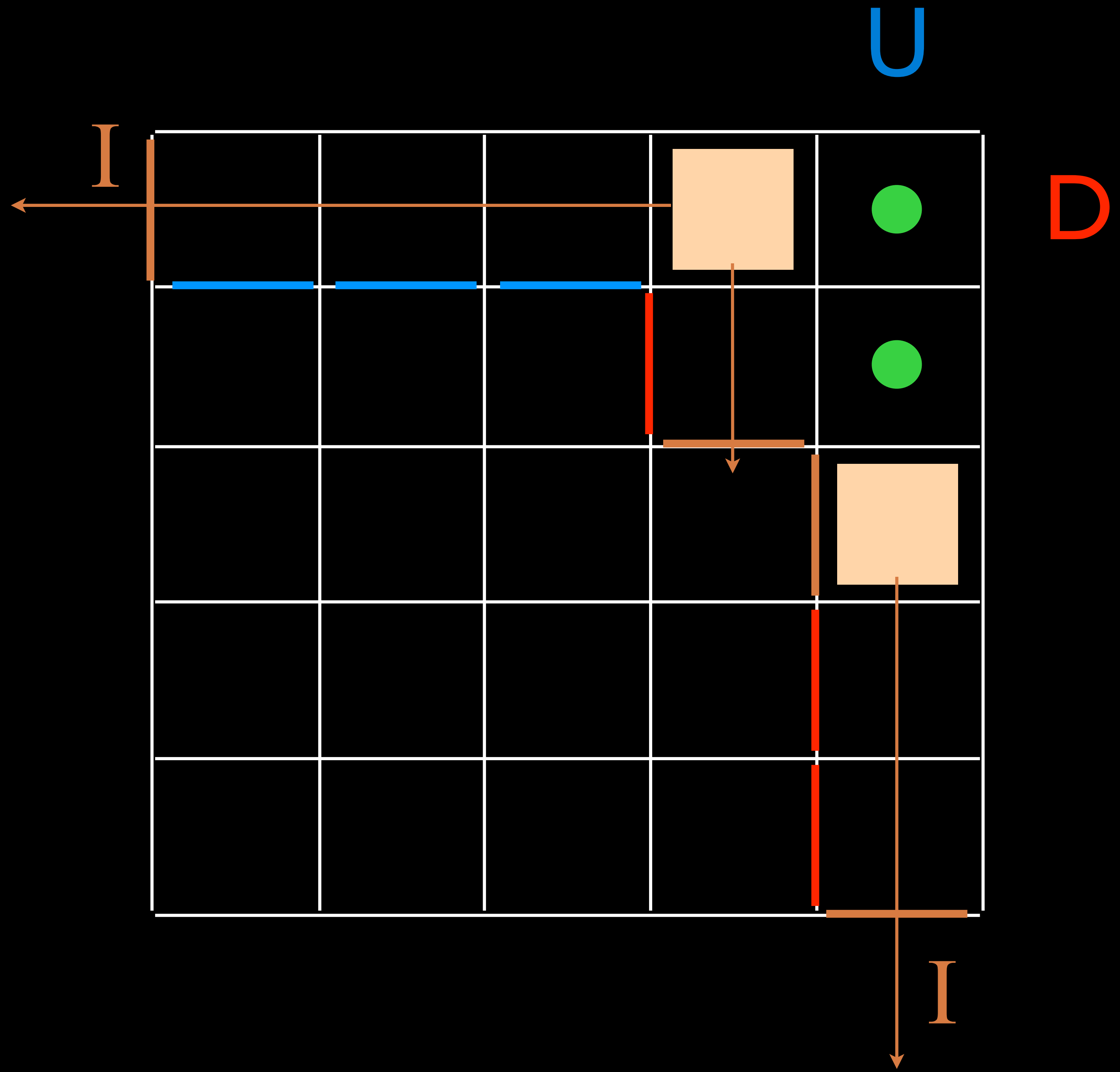
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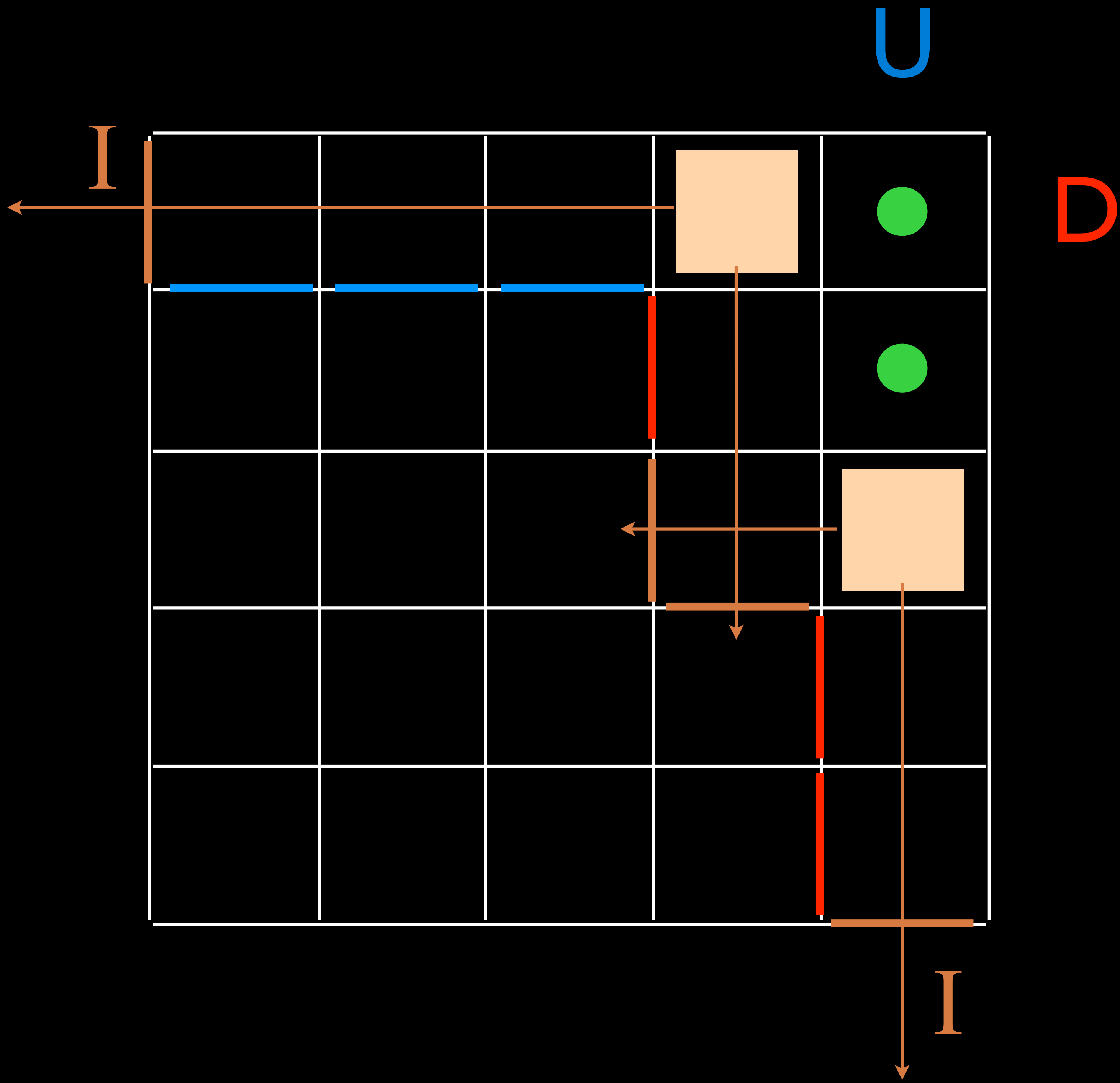
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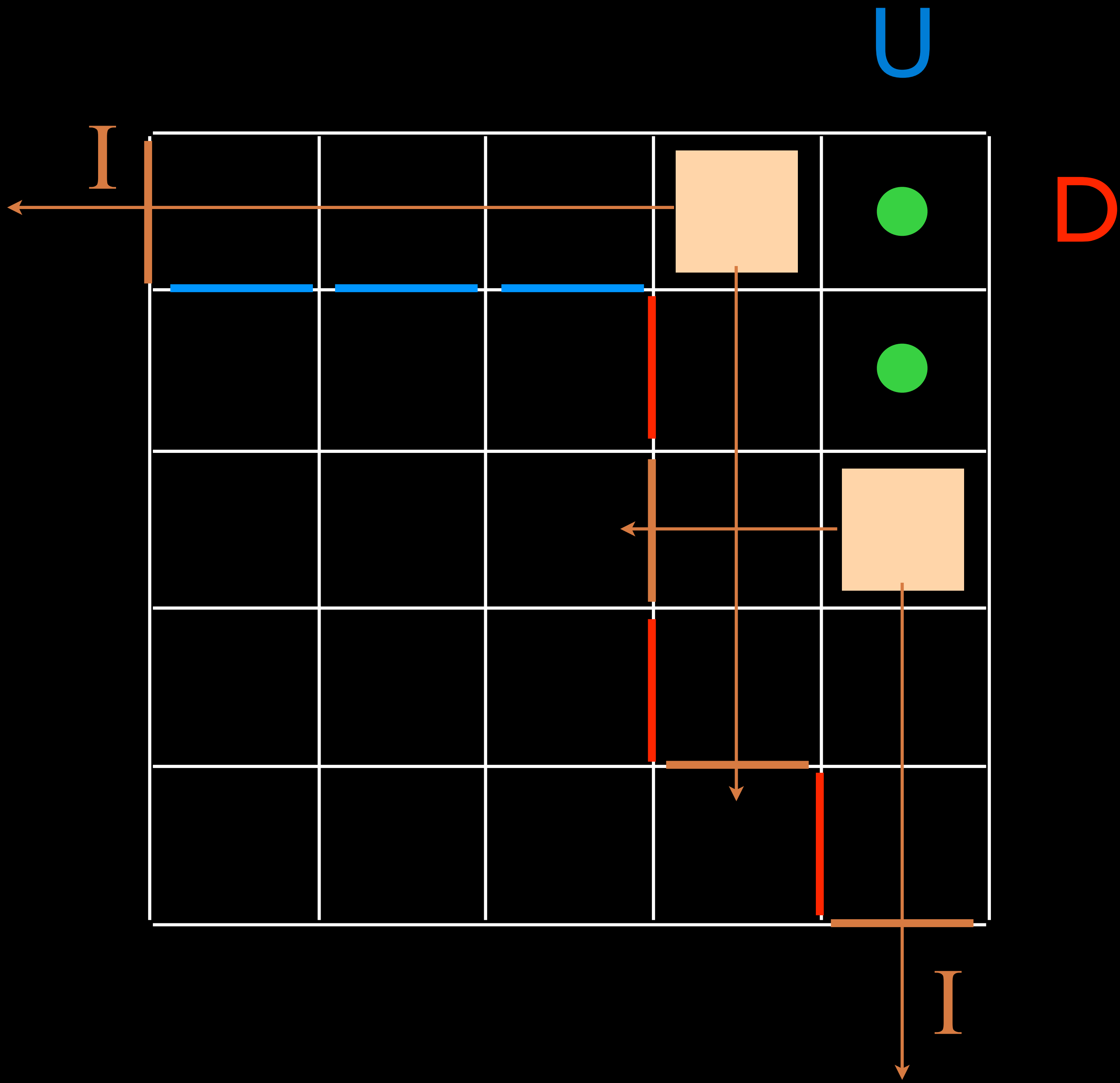

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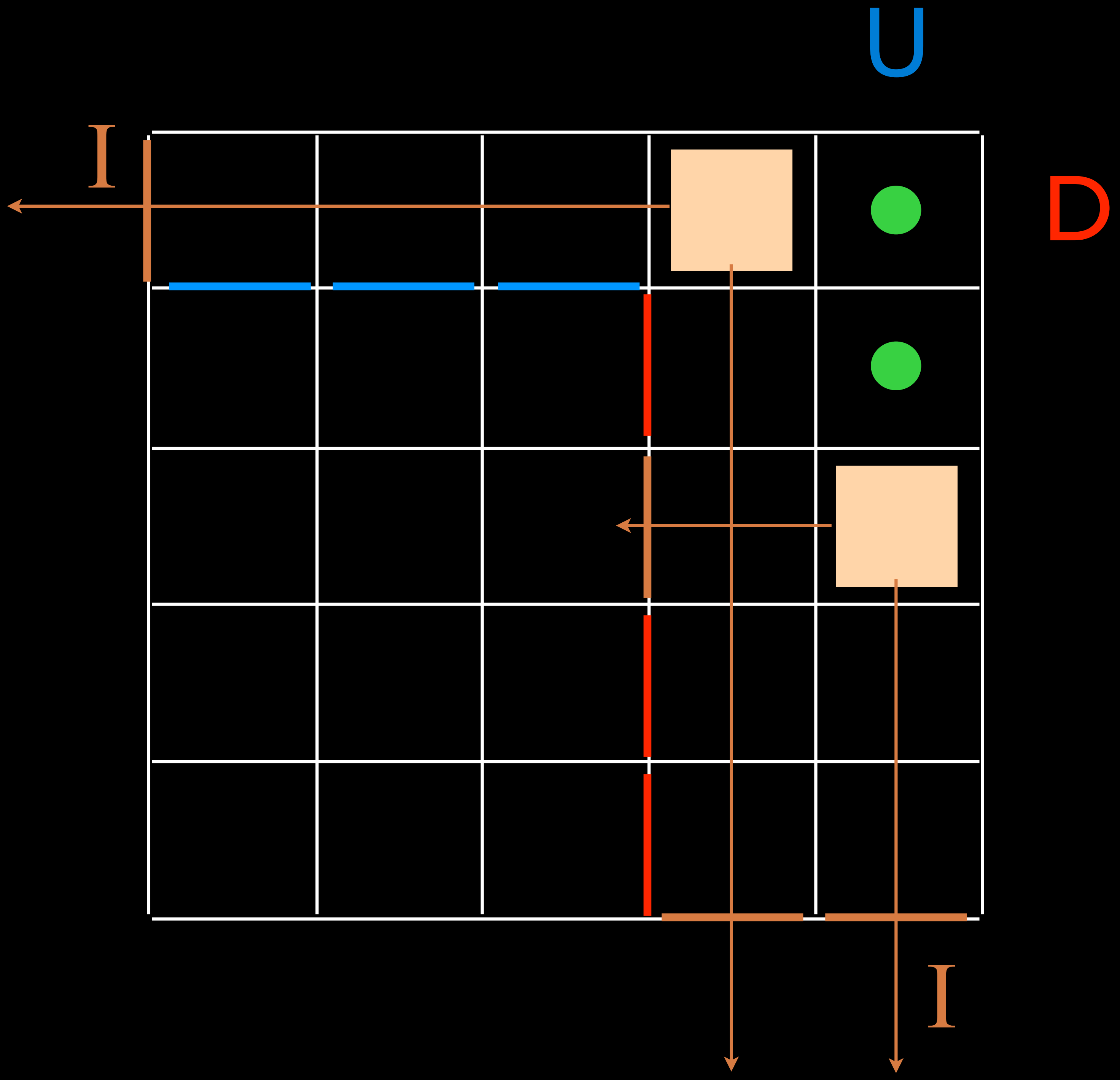

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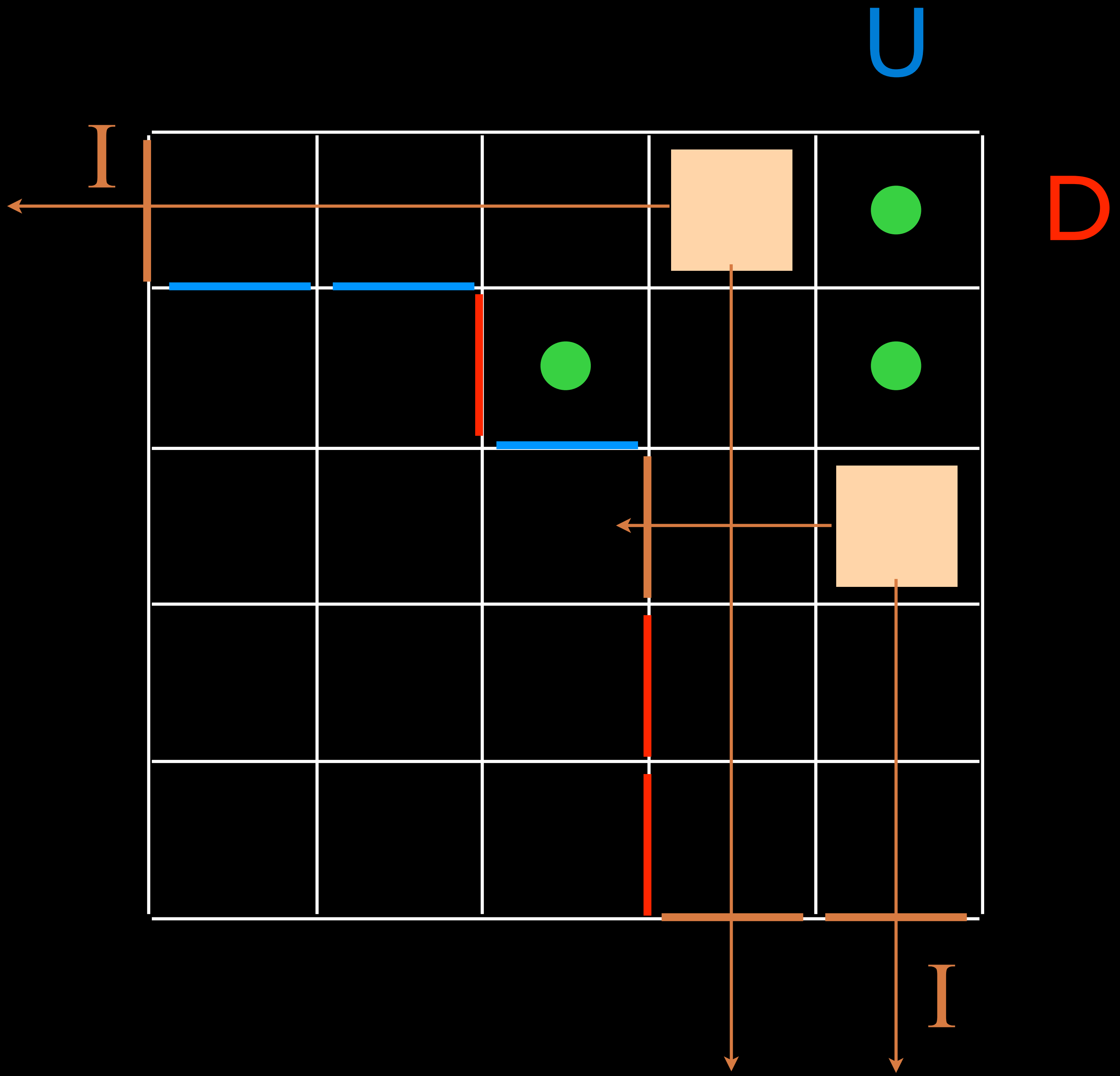
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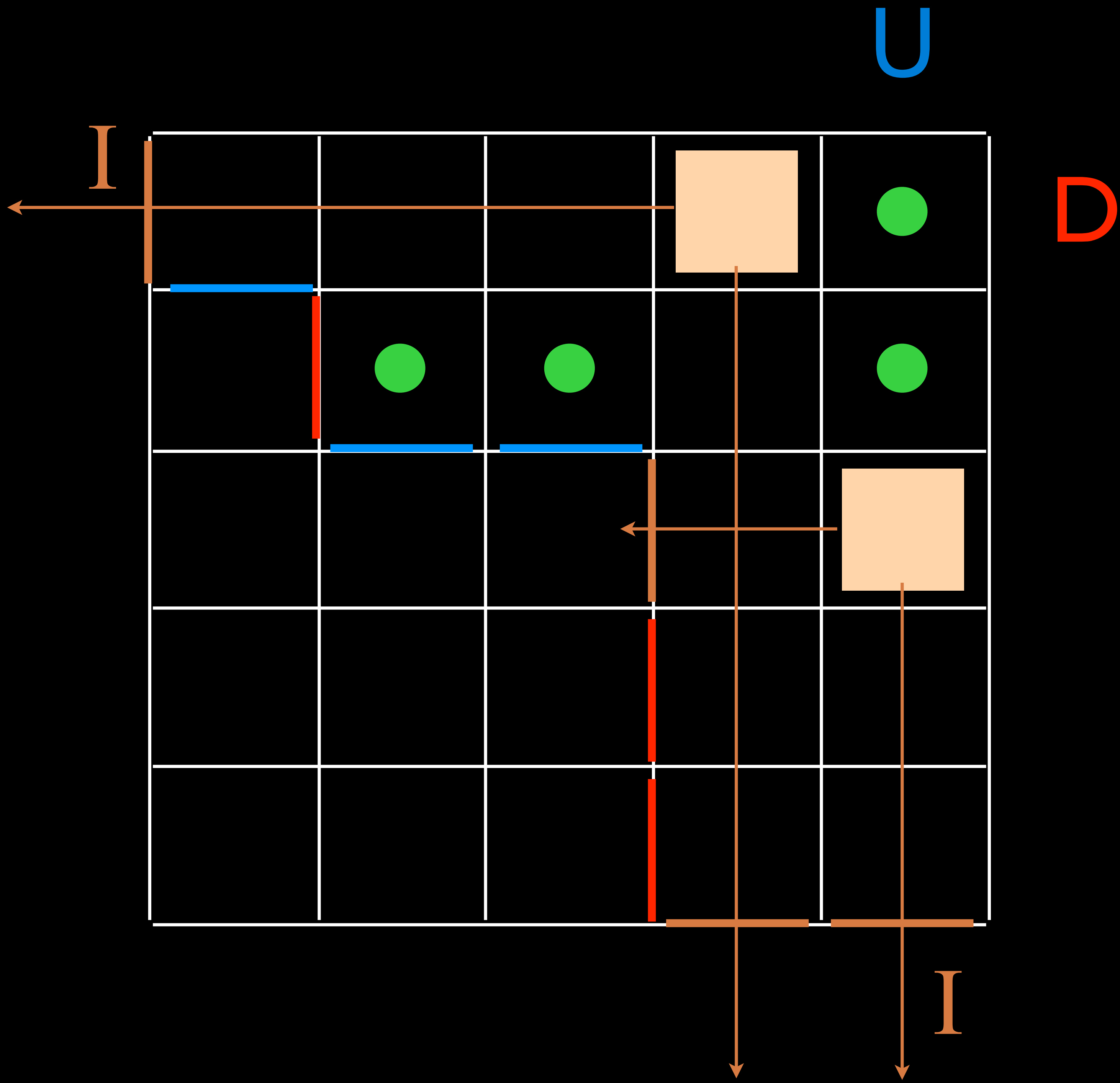
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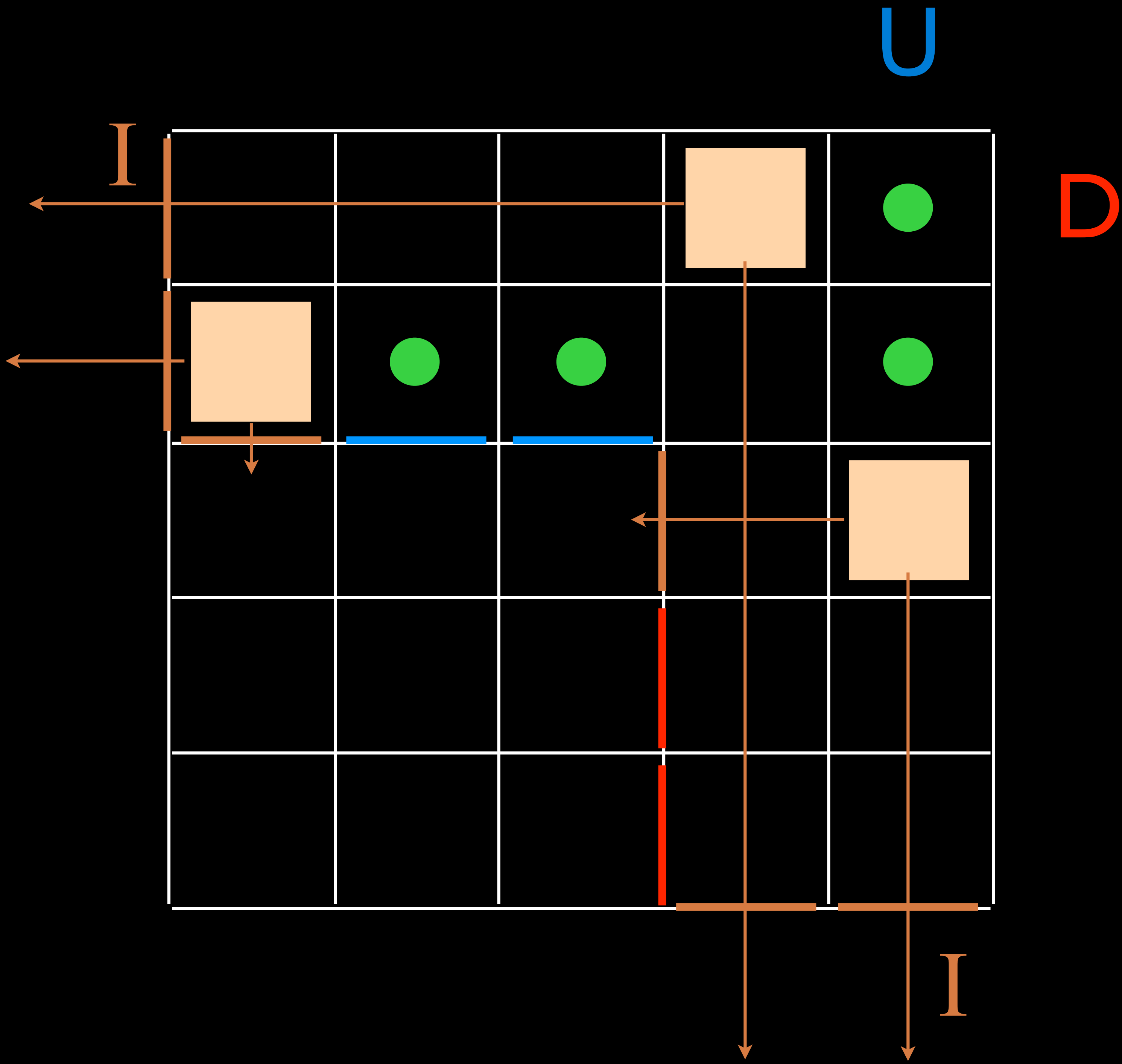
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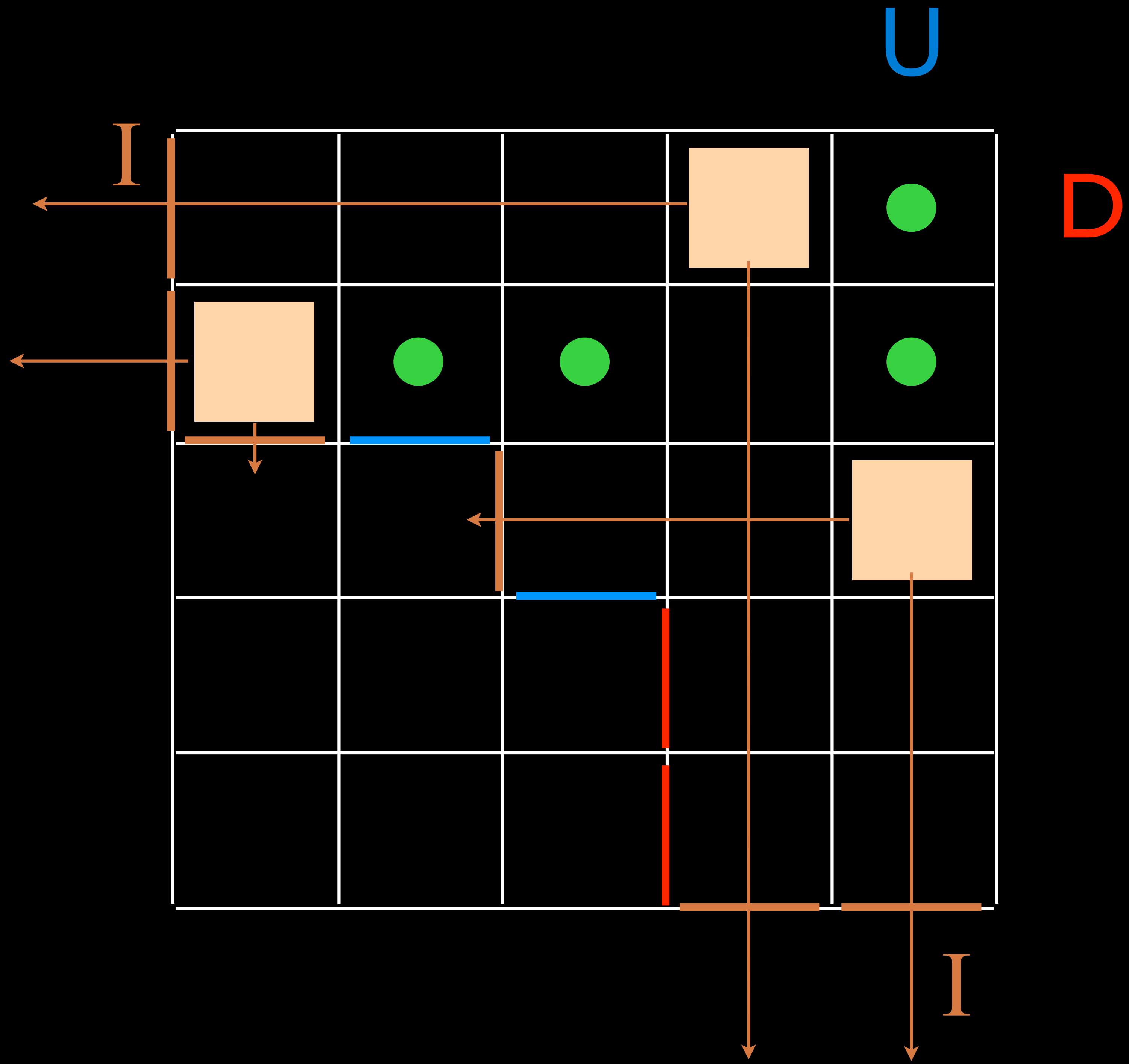

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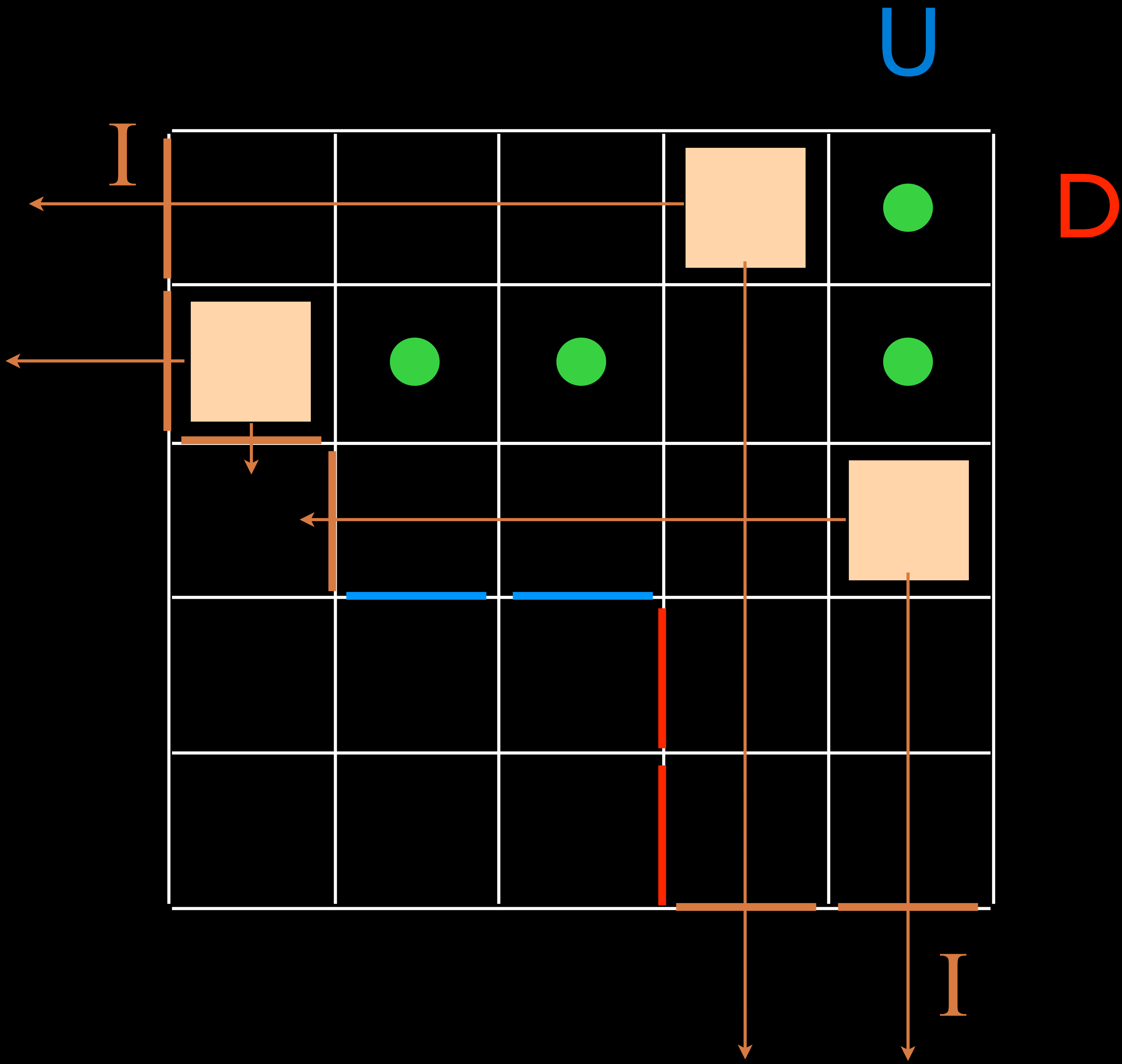

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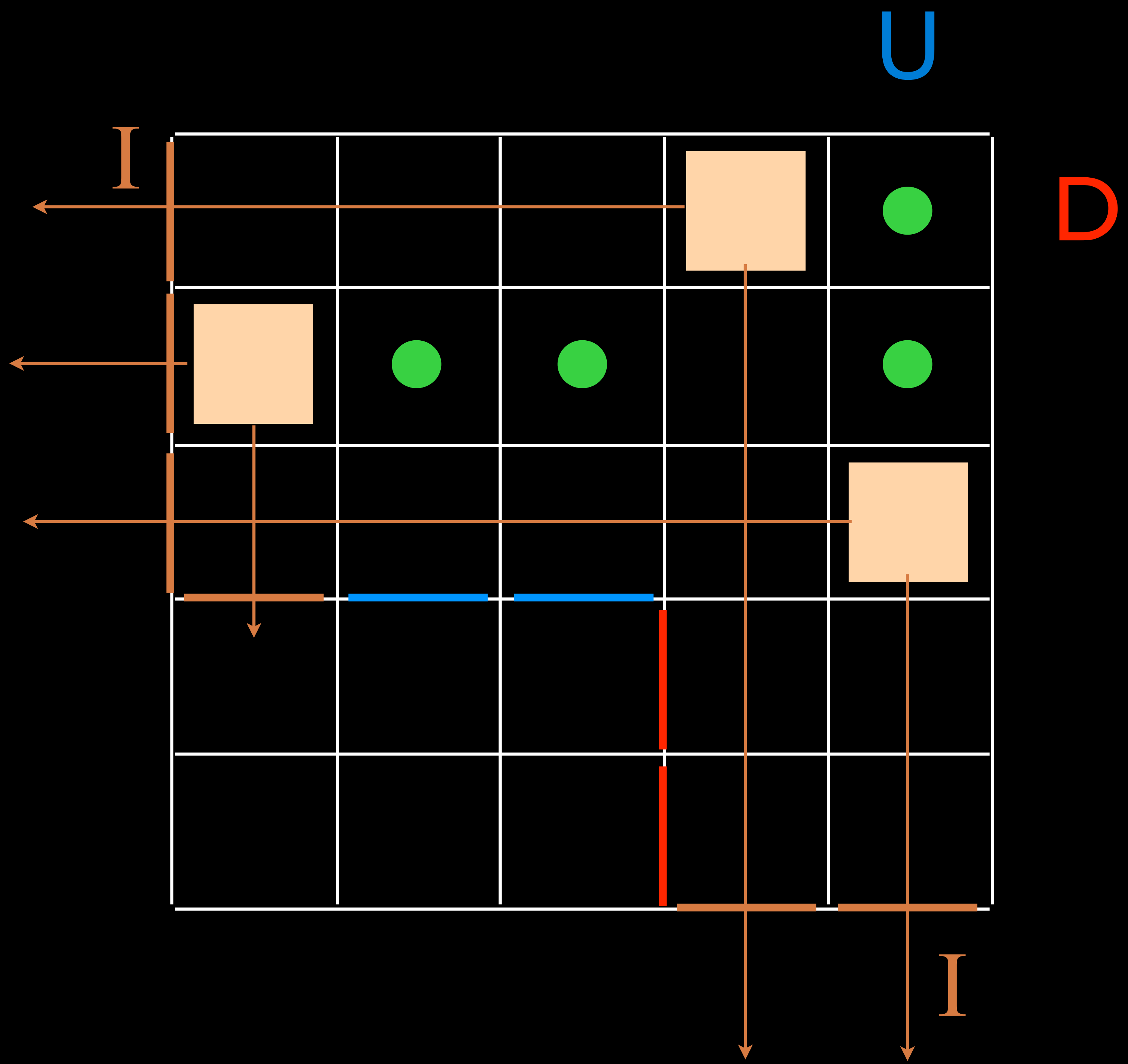
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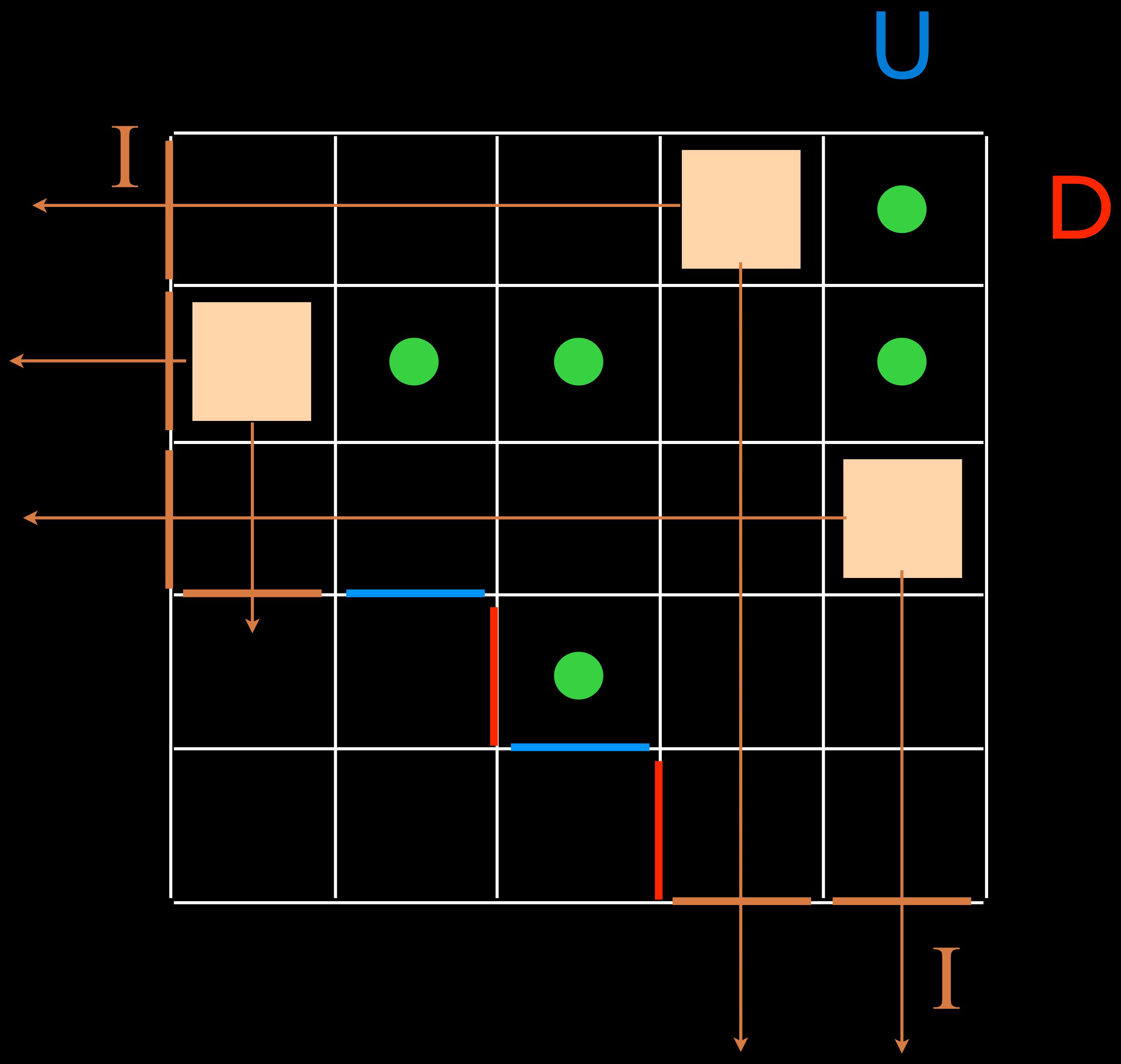
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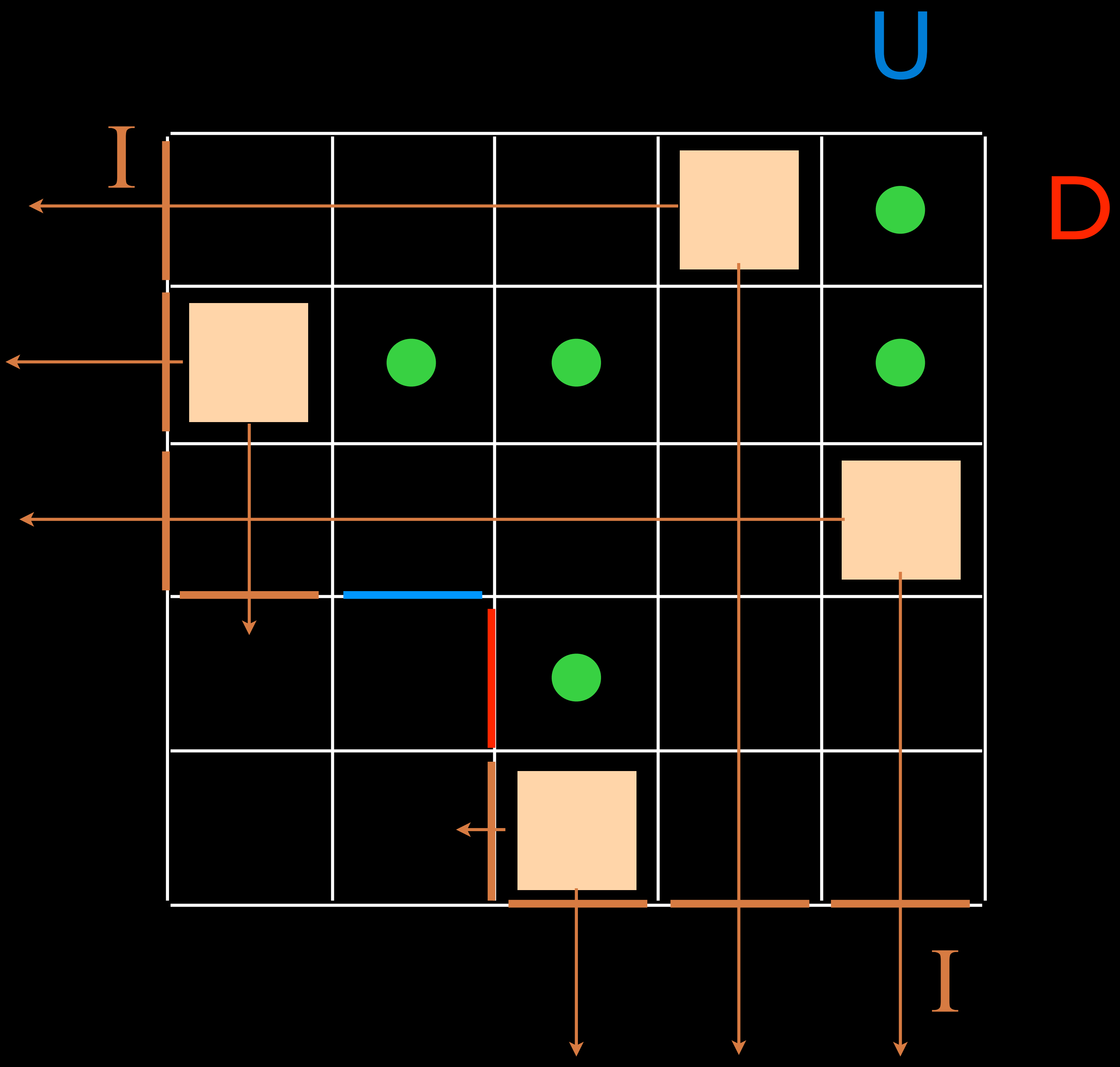

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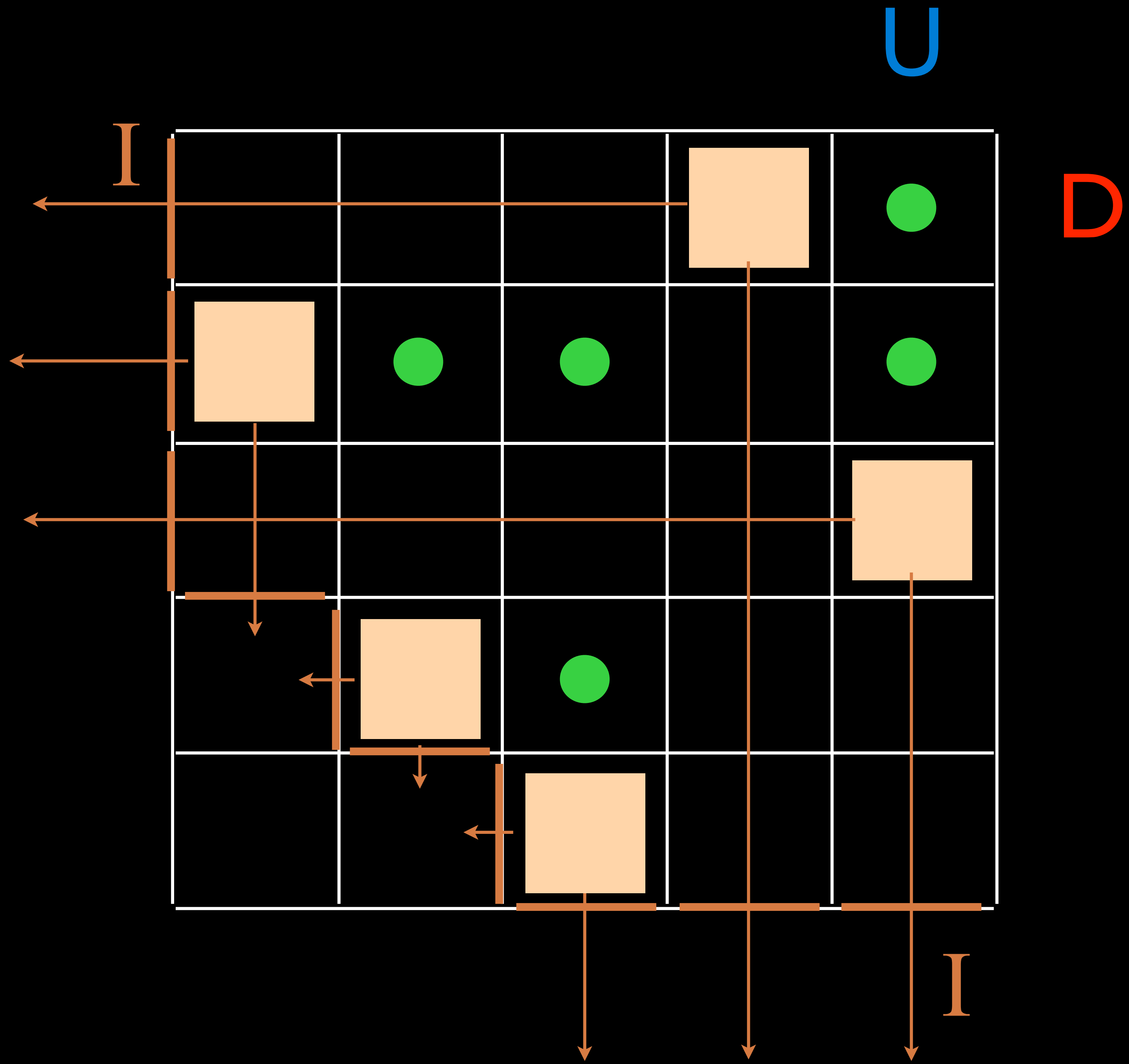

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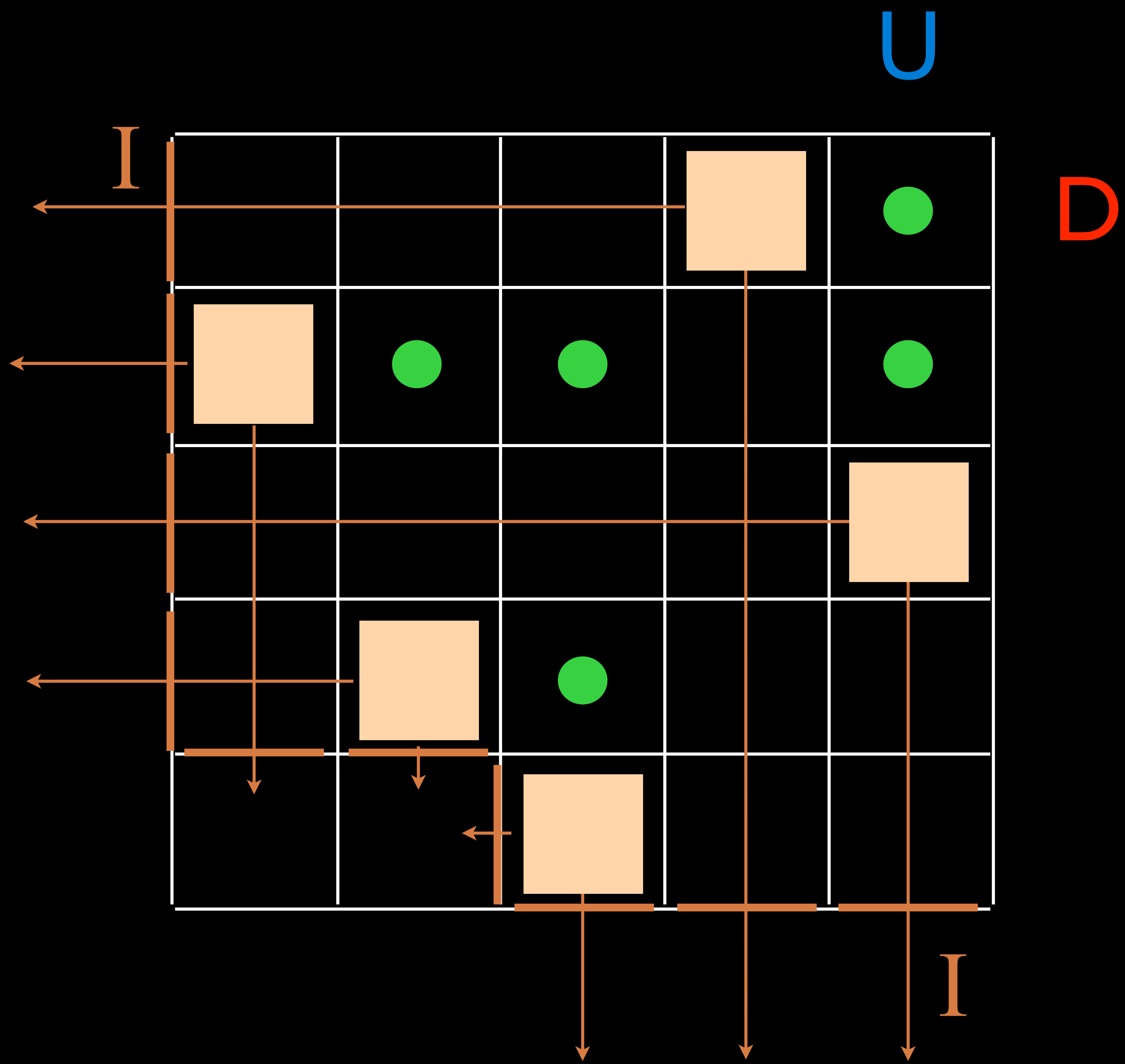
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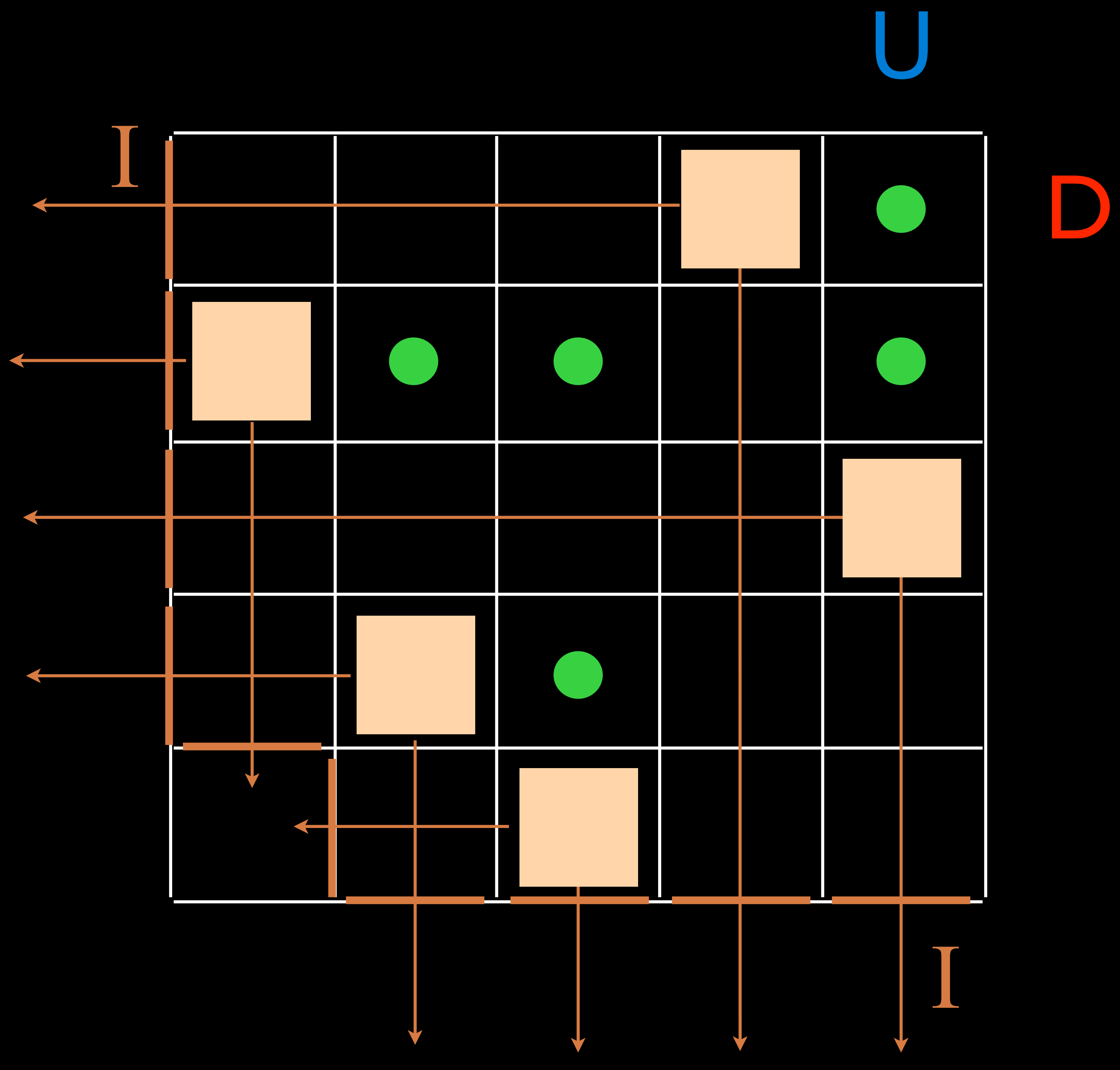

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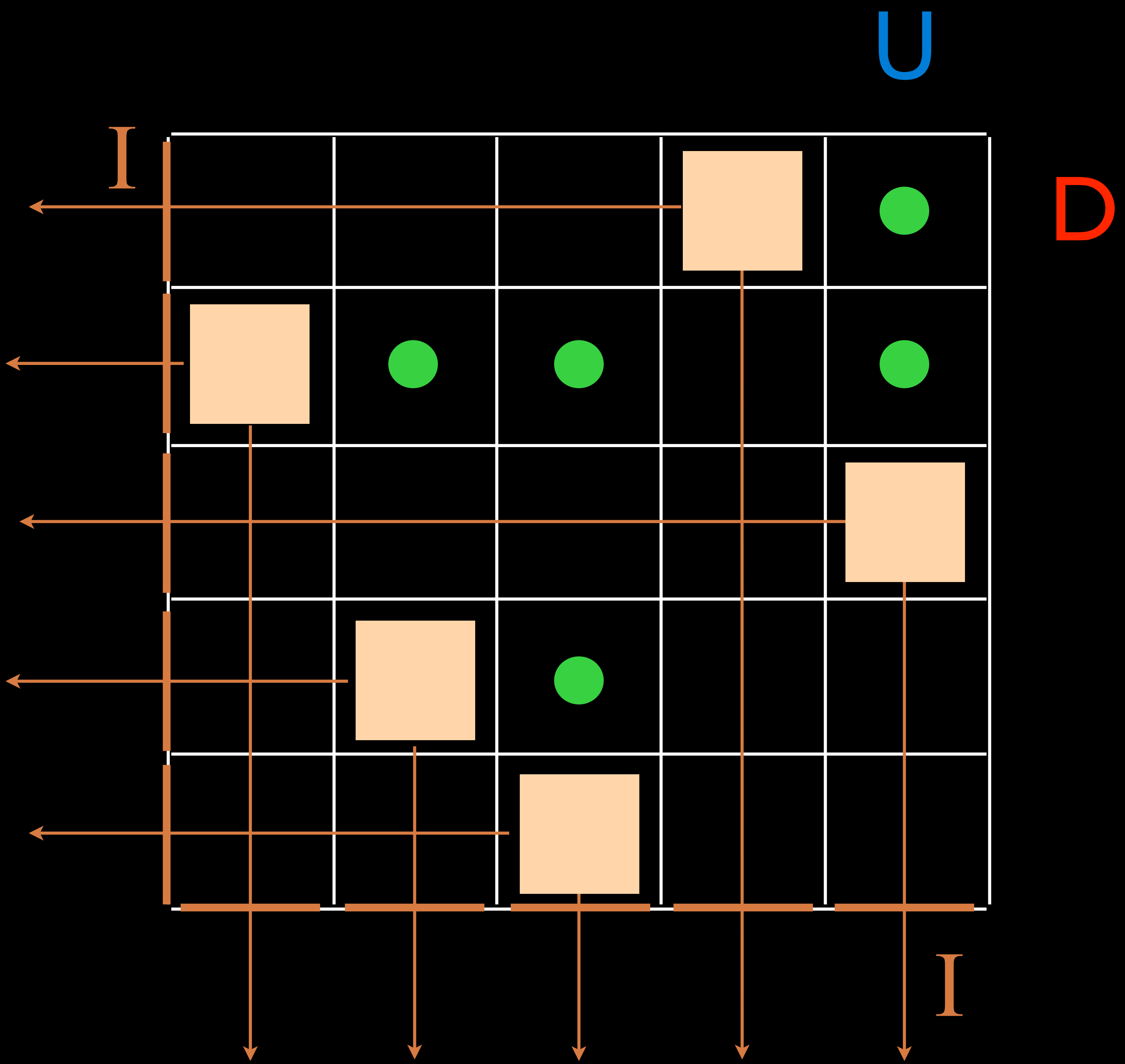

U
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I

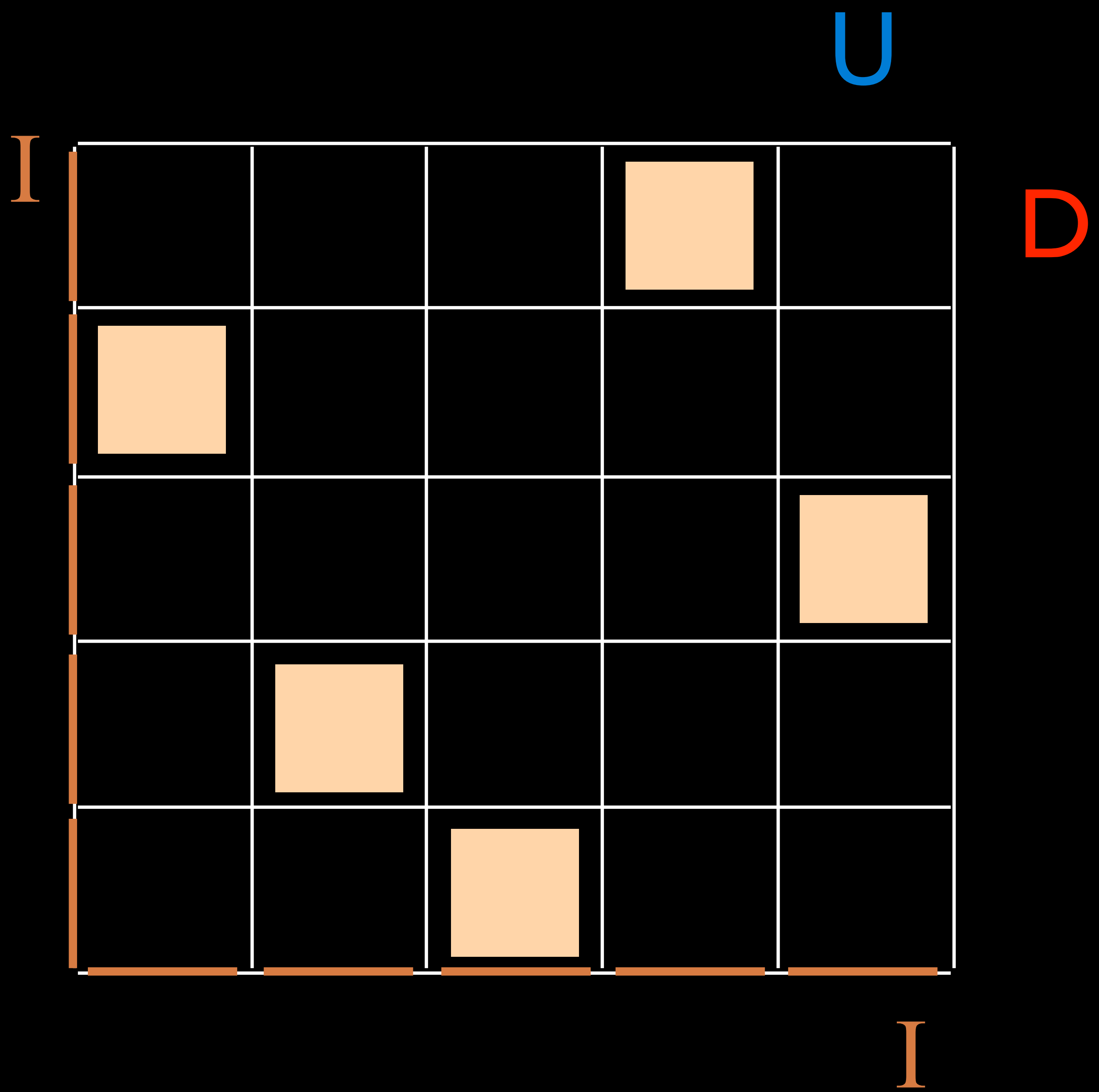
$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

Quadratic algebra $\mathbf{Q}$

5 rewriting rules

"complete" Q-tableau (5 labels)

↓ Q-tableau (2 labels)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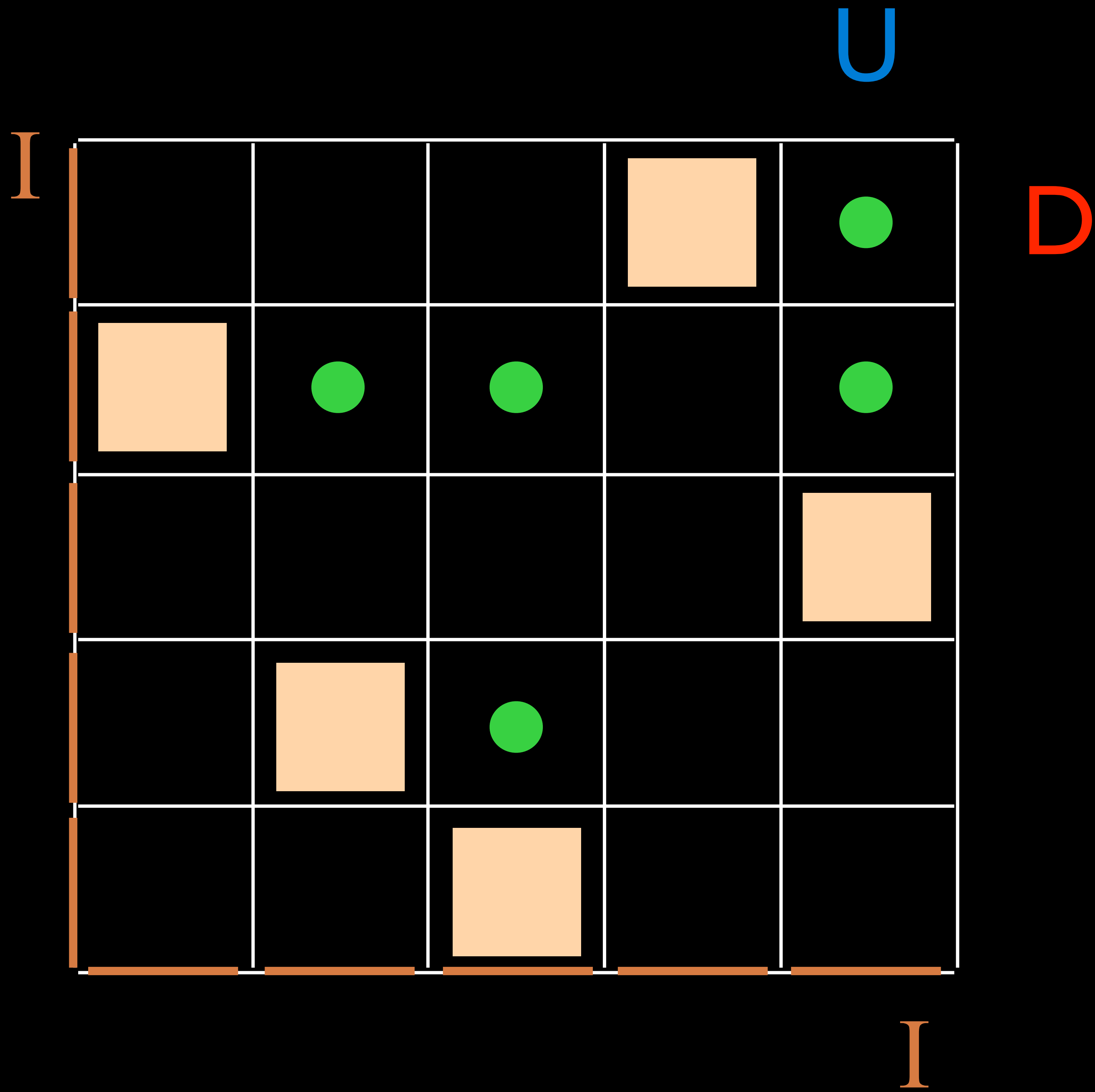
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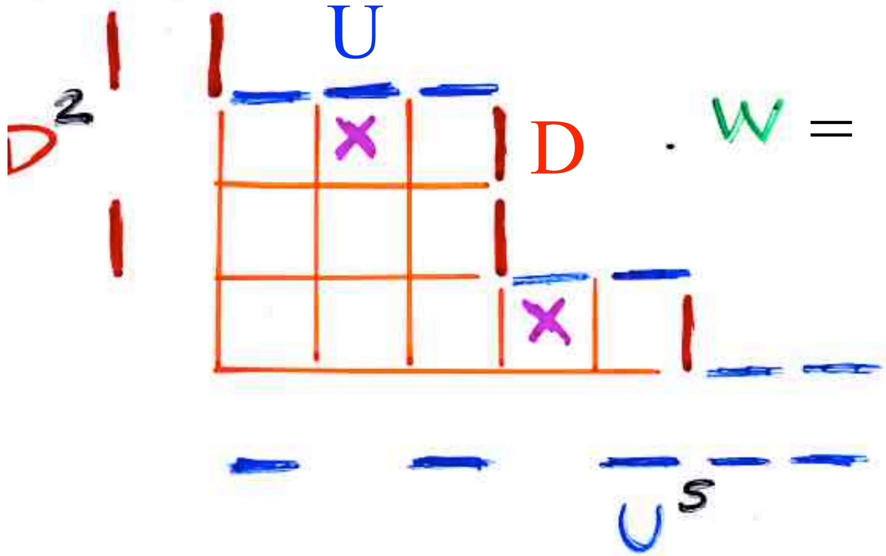
$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

$$c_{n,i} = \binom{n}{i}^2 (n-i)!$$

notation $w \rightarrow F_w$ Ferrers diagram



w = D U U U D D U U D U U

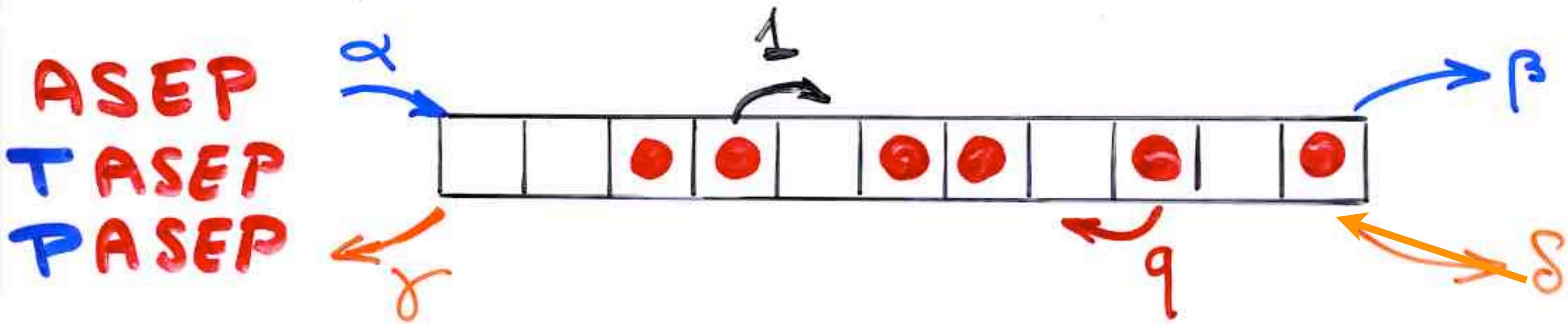
Towers
placements on a
Ferrers diagram

Prop. $c_{i,j}(w)$ = nb de "placement" de k tours sur F

avec $i = |w|_D - k$

$j = |w|_U - k$

The PASEP algebra



ASEP
TASEP
PASEP
α
1
β
γ
q
δ

The Matrix Ansatz

Derrida, Evans, Hakim, Pasquier (1993)

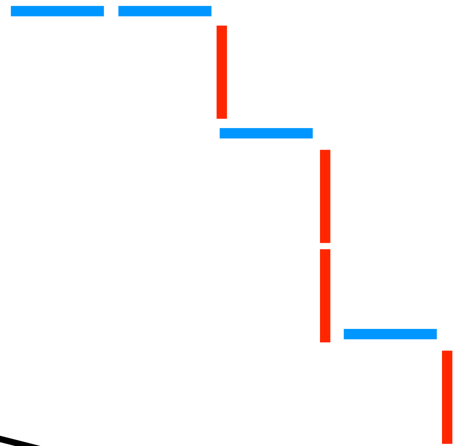
$$DE = qED + E + D$$

The PASEP algebra

$$DE = qED + E + D$$

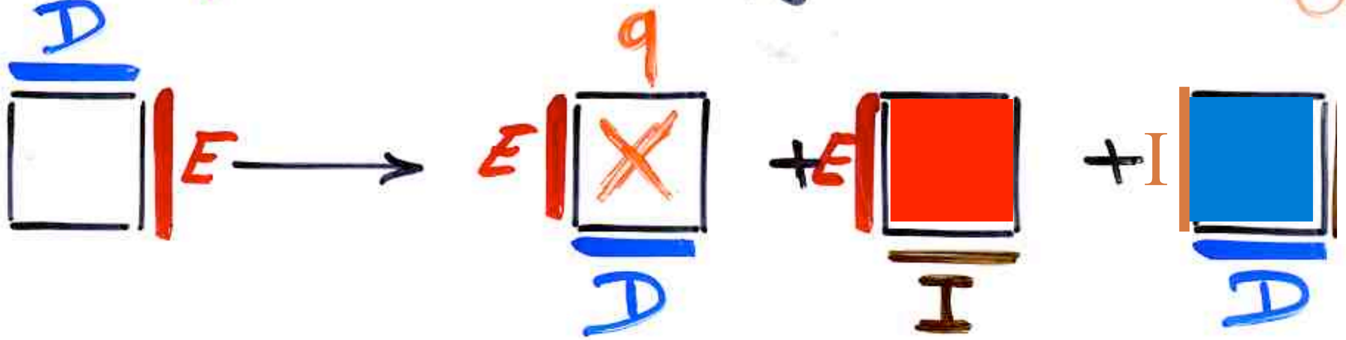
D D E D E E D E

D DE (D E) E D E



DDE(E)EDE + DDE(ED)EDE + DDE(D)EDE

Proof: "planarization" of the rewriting rules



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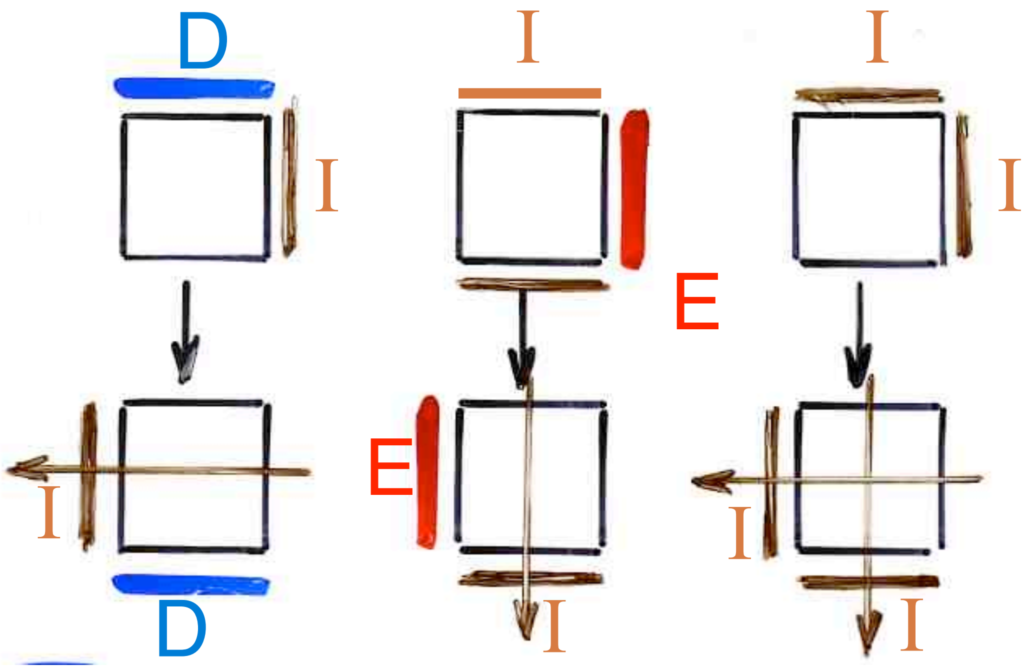
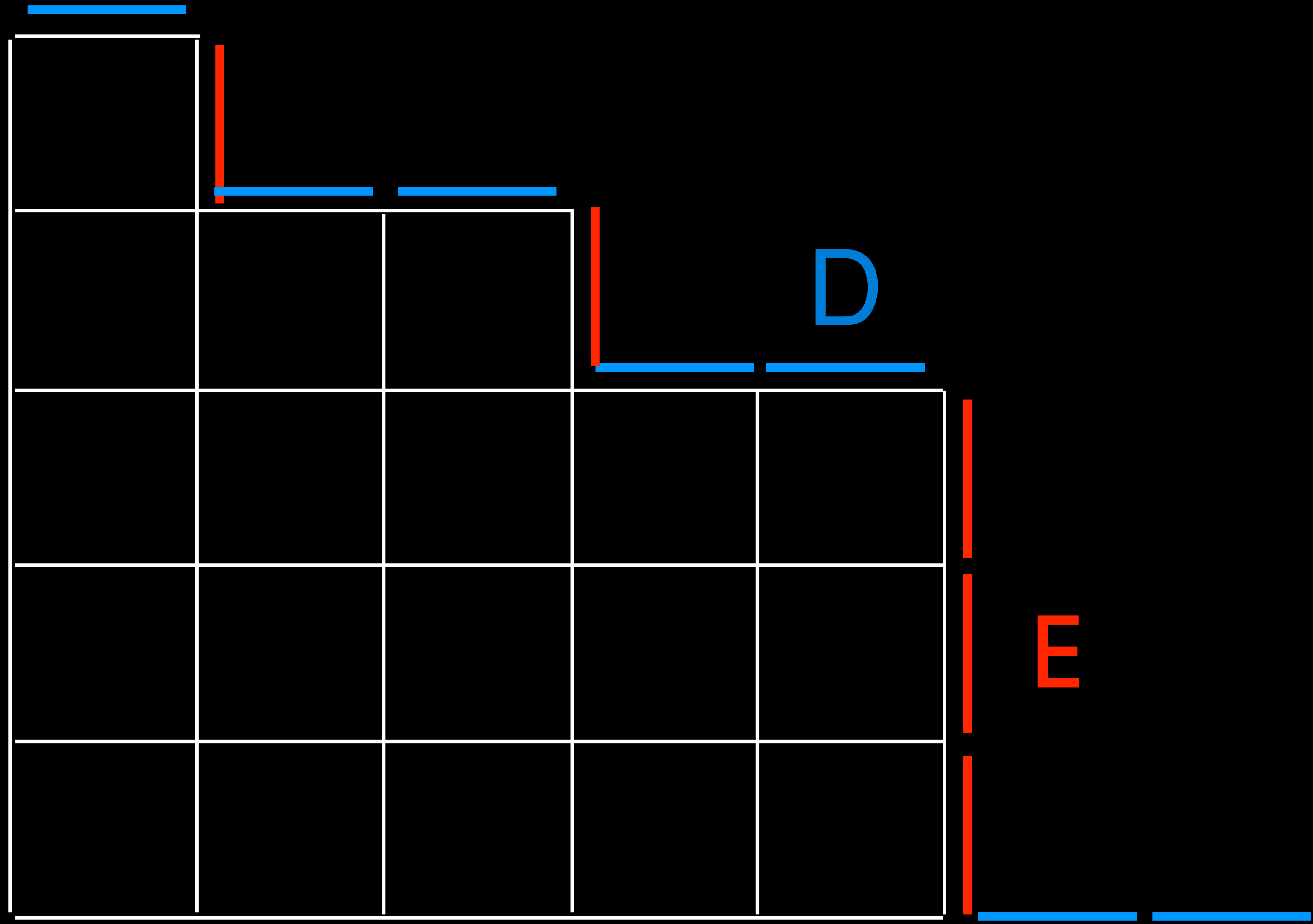


$$DE = qED + EI_h + I_vD$$

$$DI_v = I_vD$$

$$I_hE = EI_h$$

$$I_hI_v = I_vI_h$$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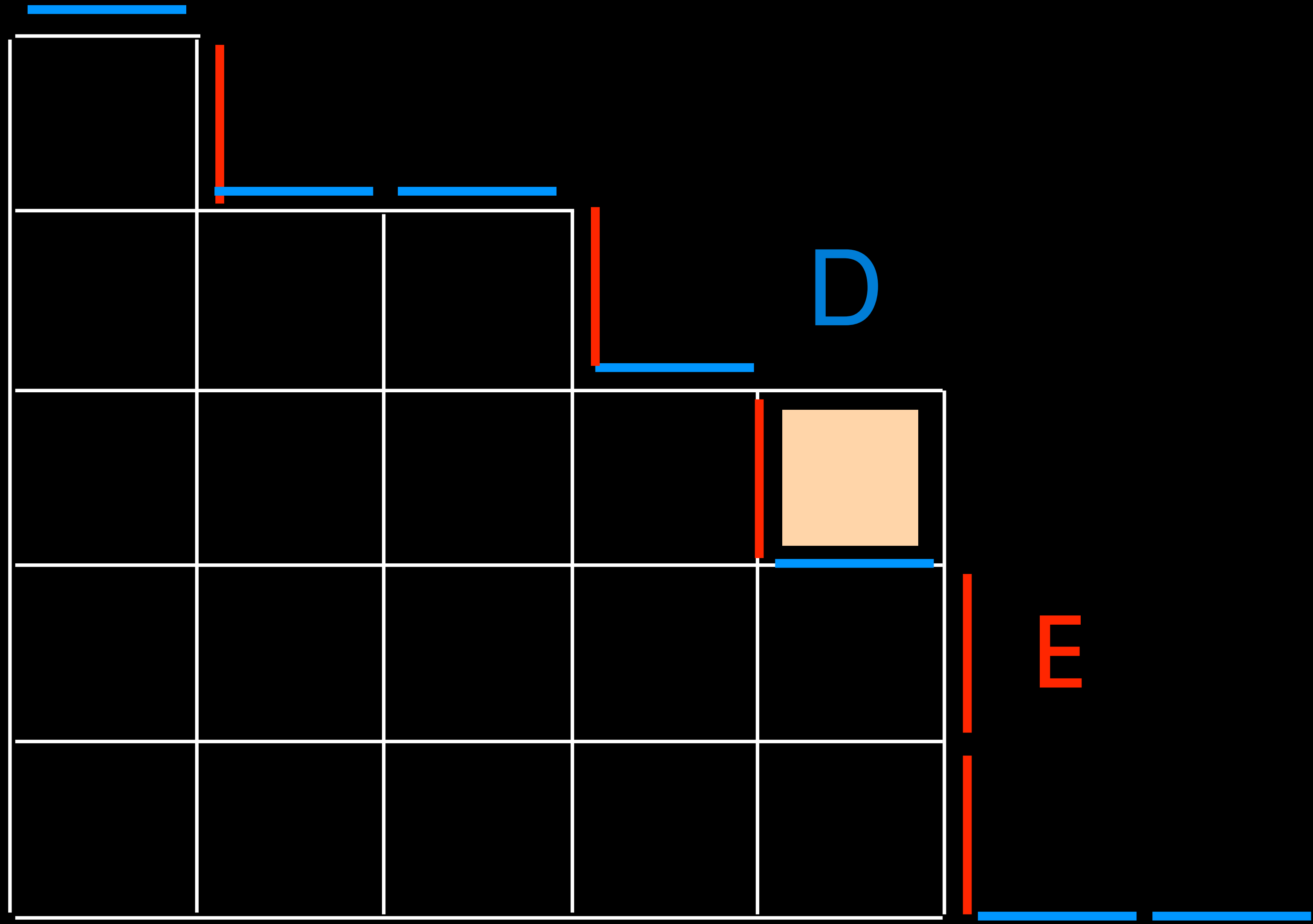
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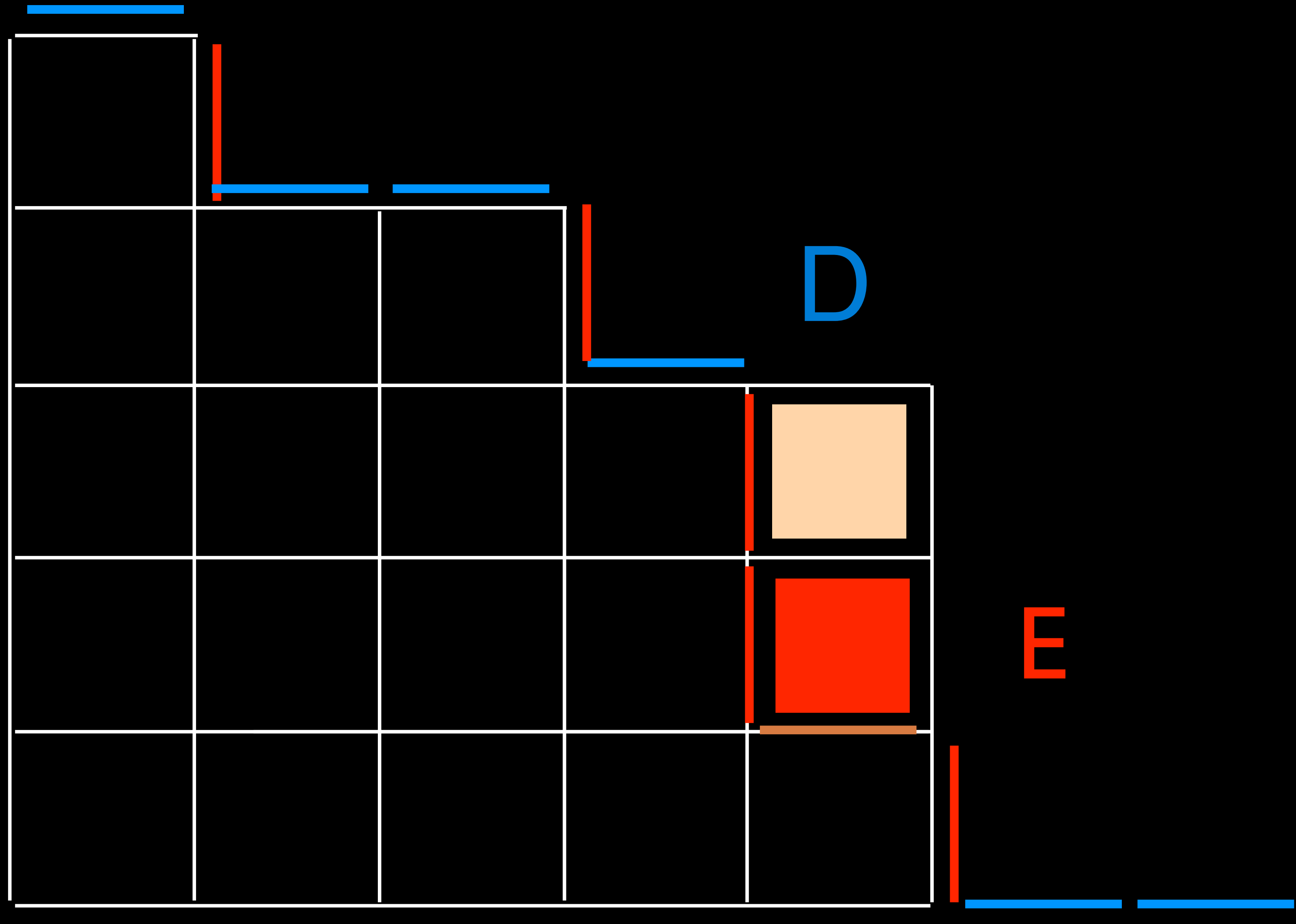
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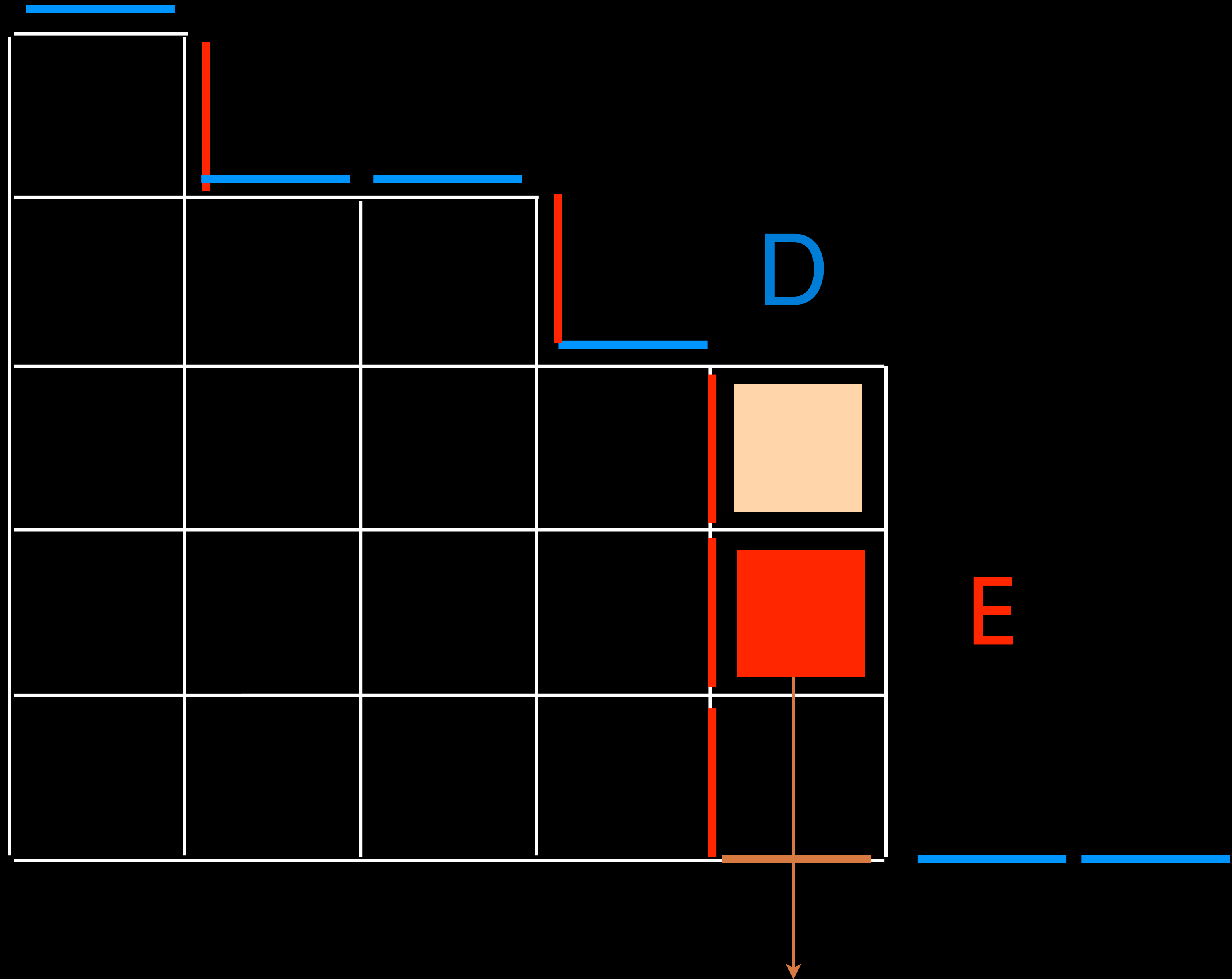
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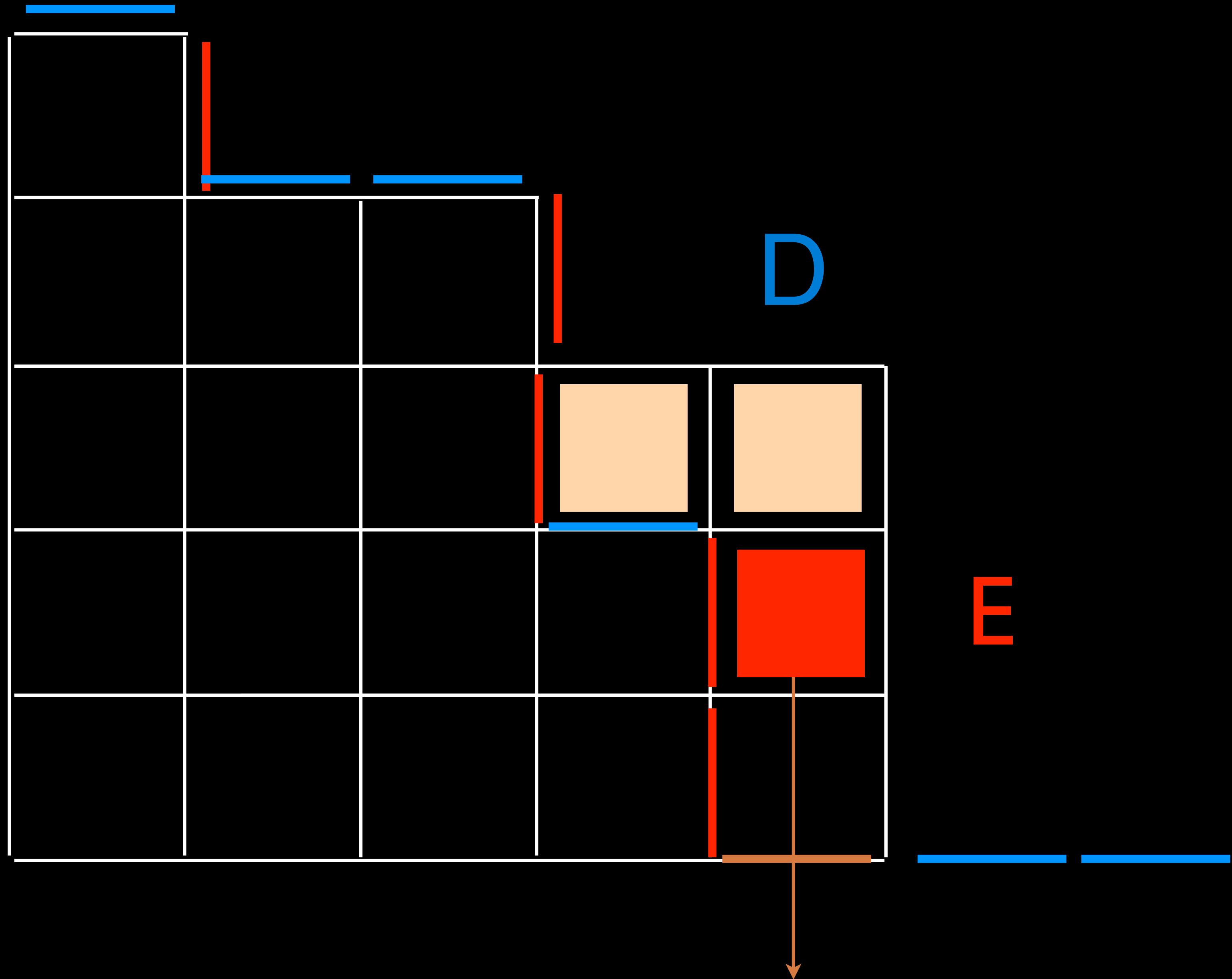

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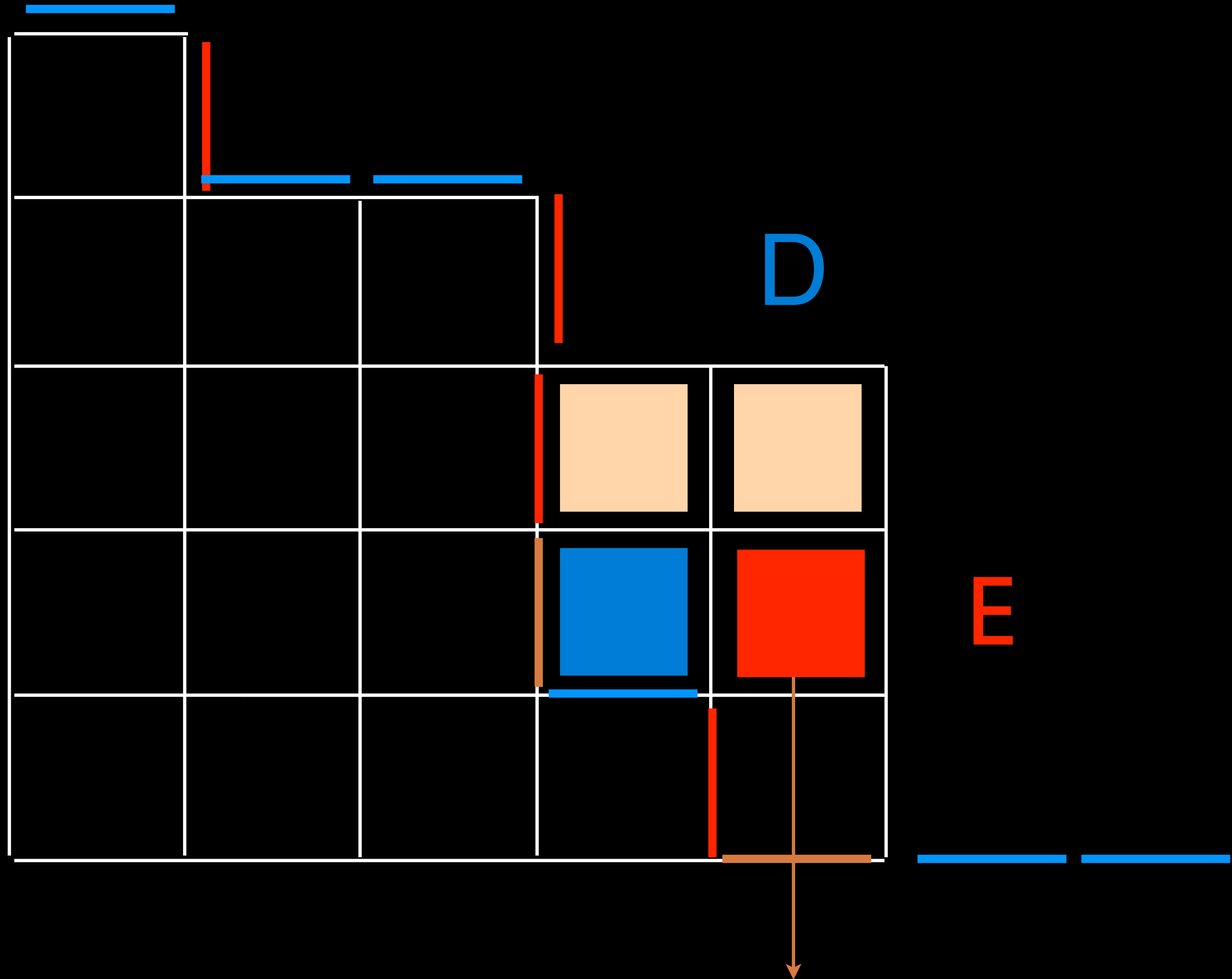
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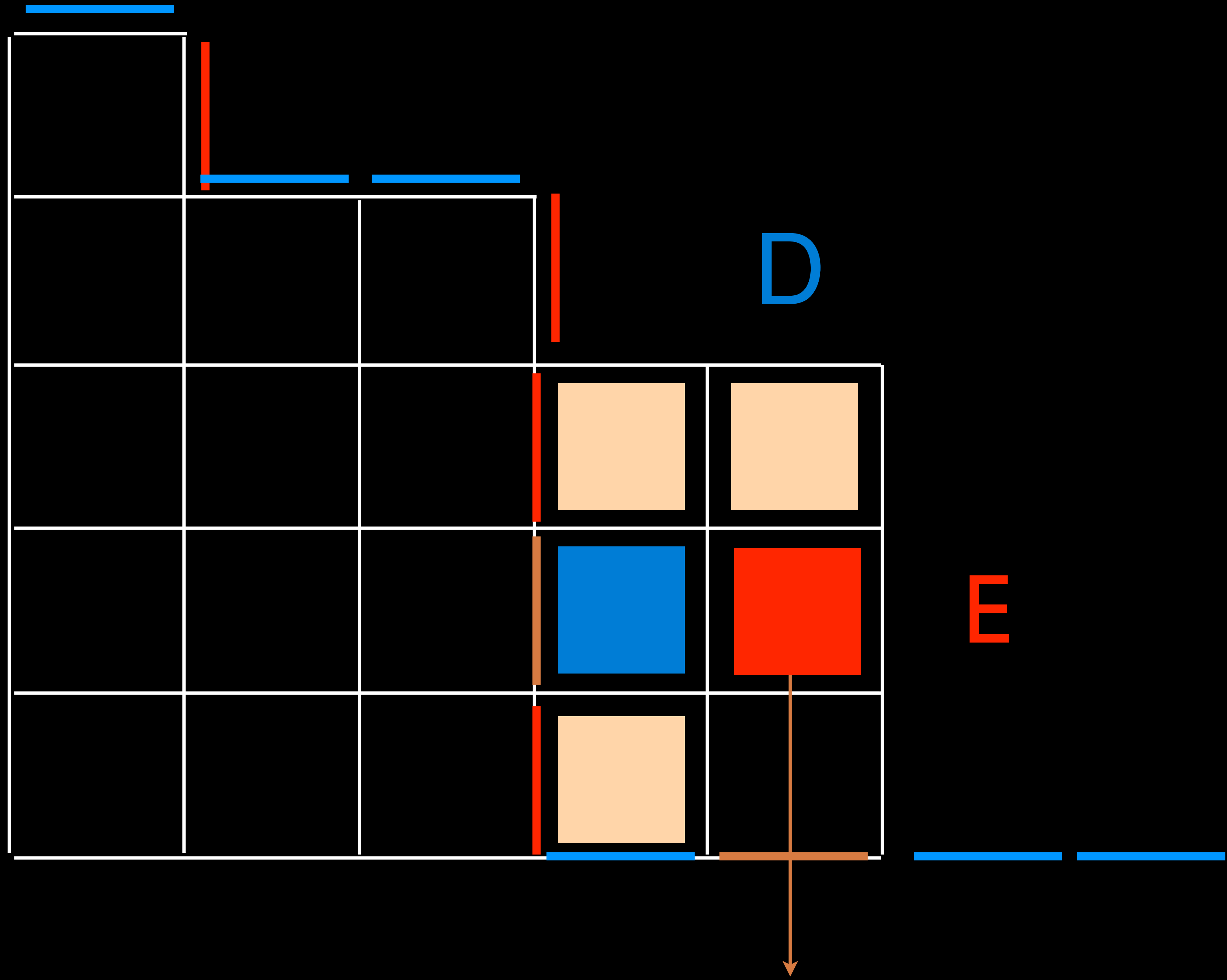
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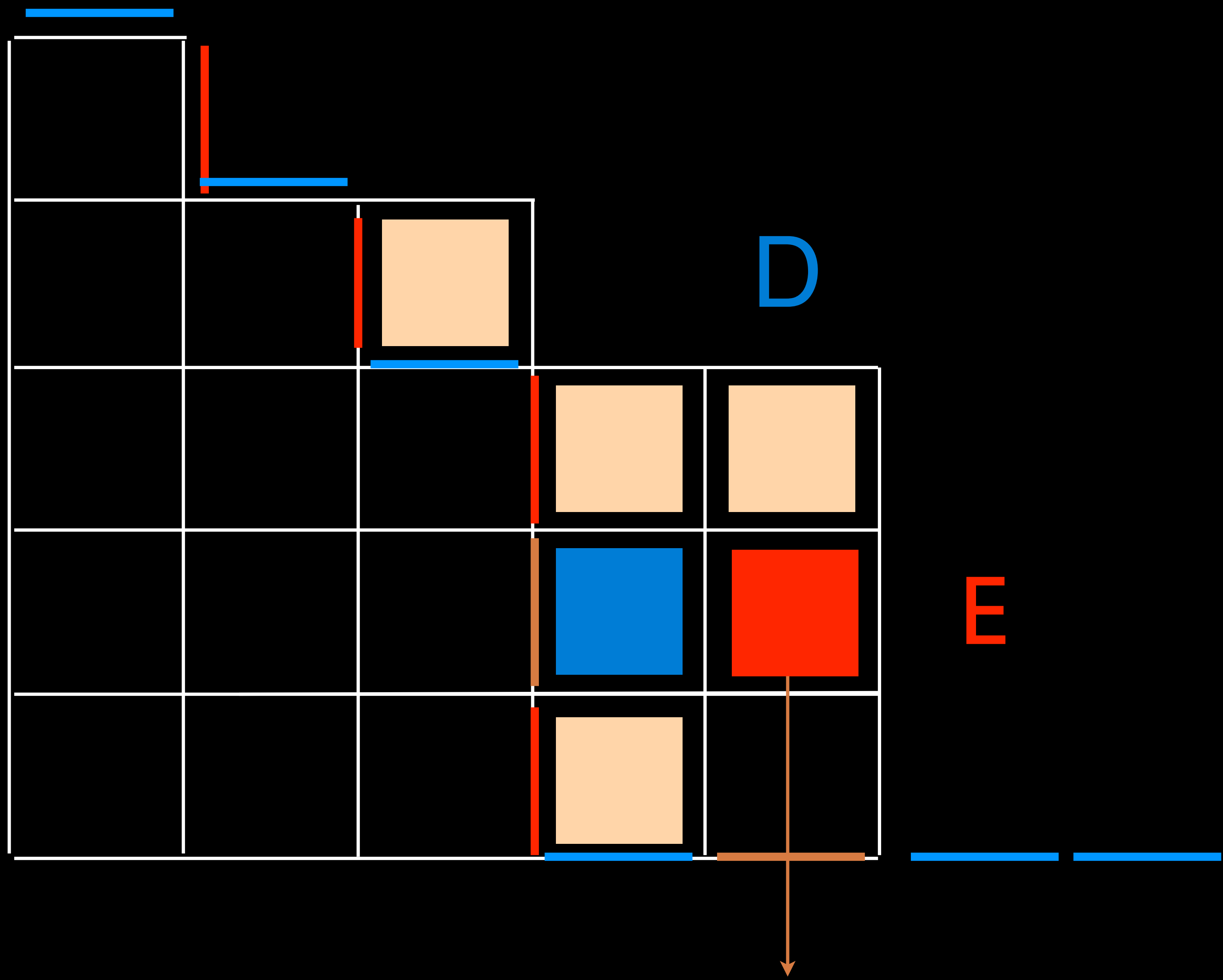
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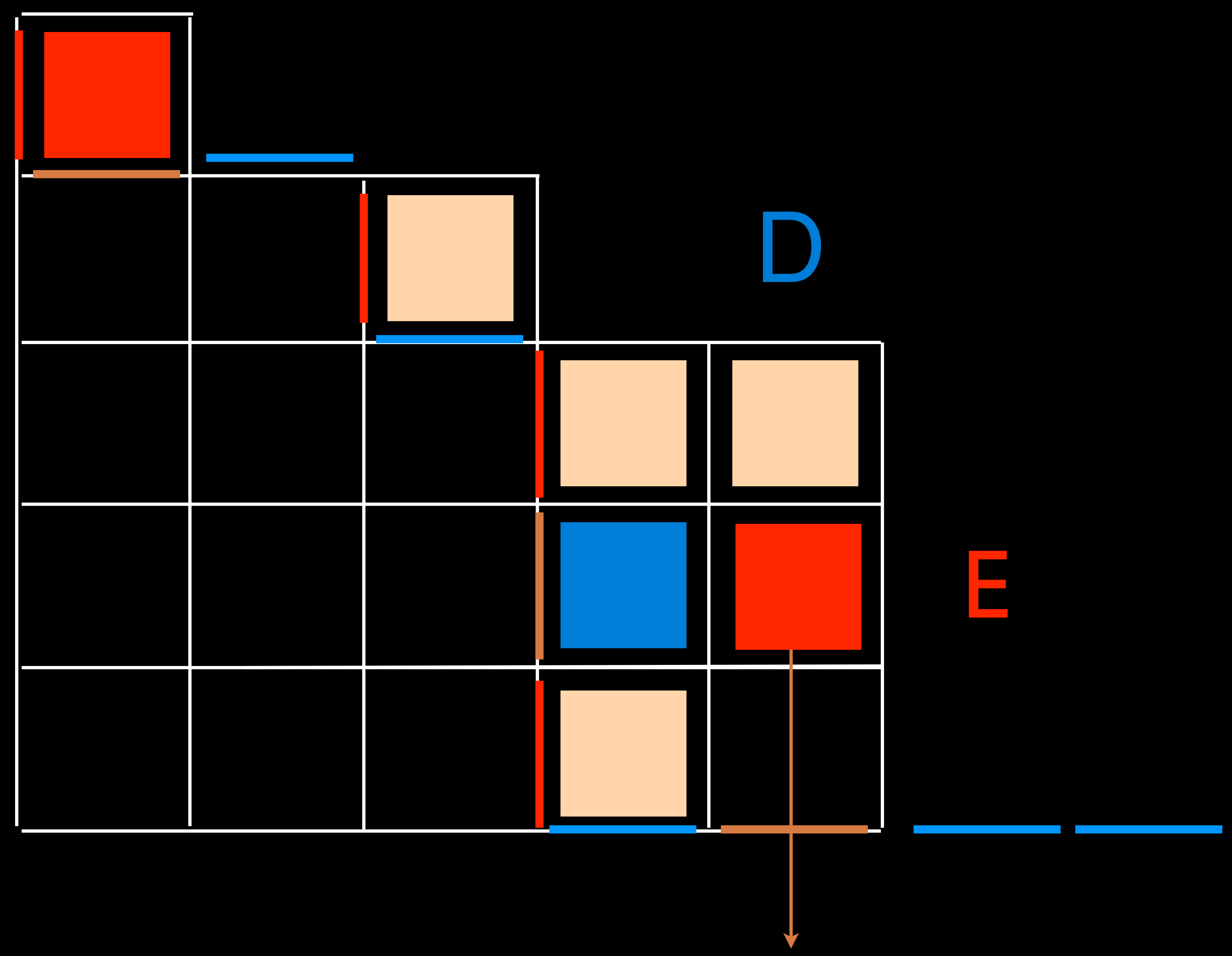
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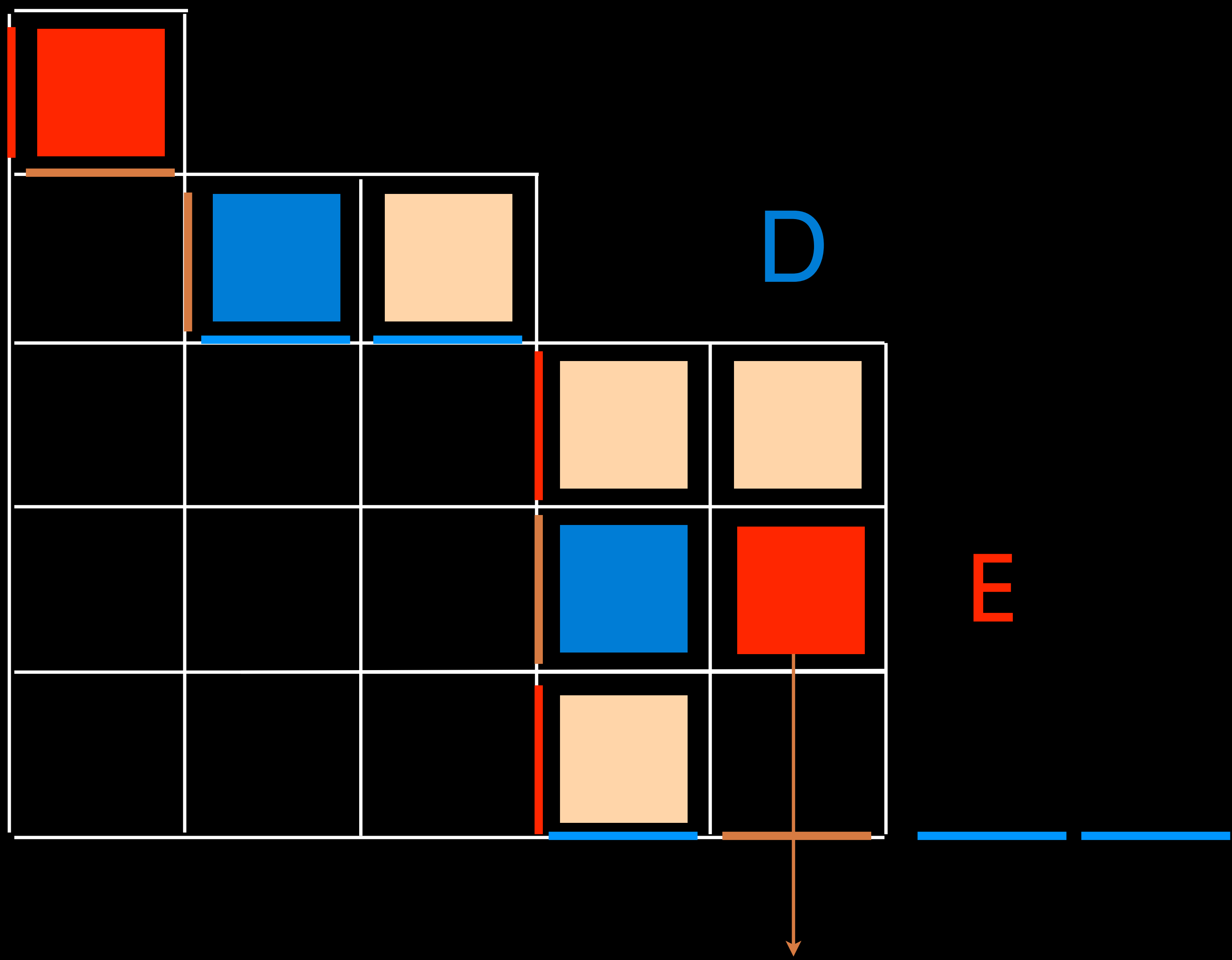
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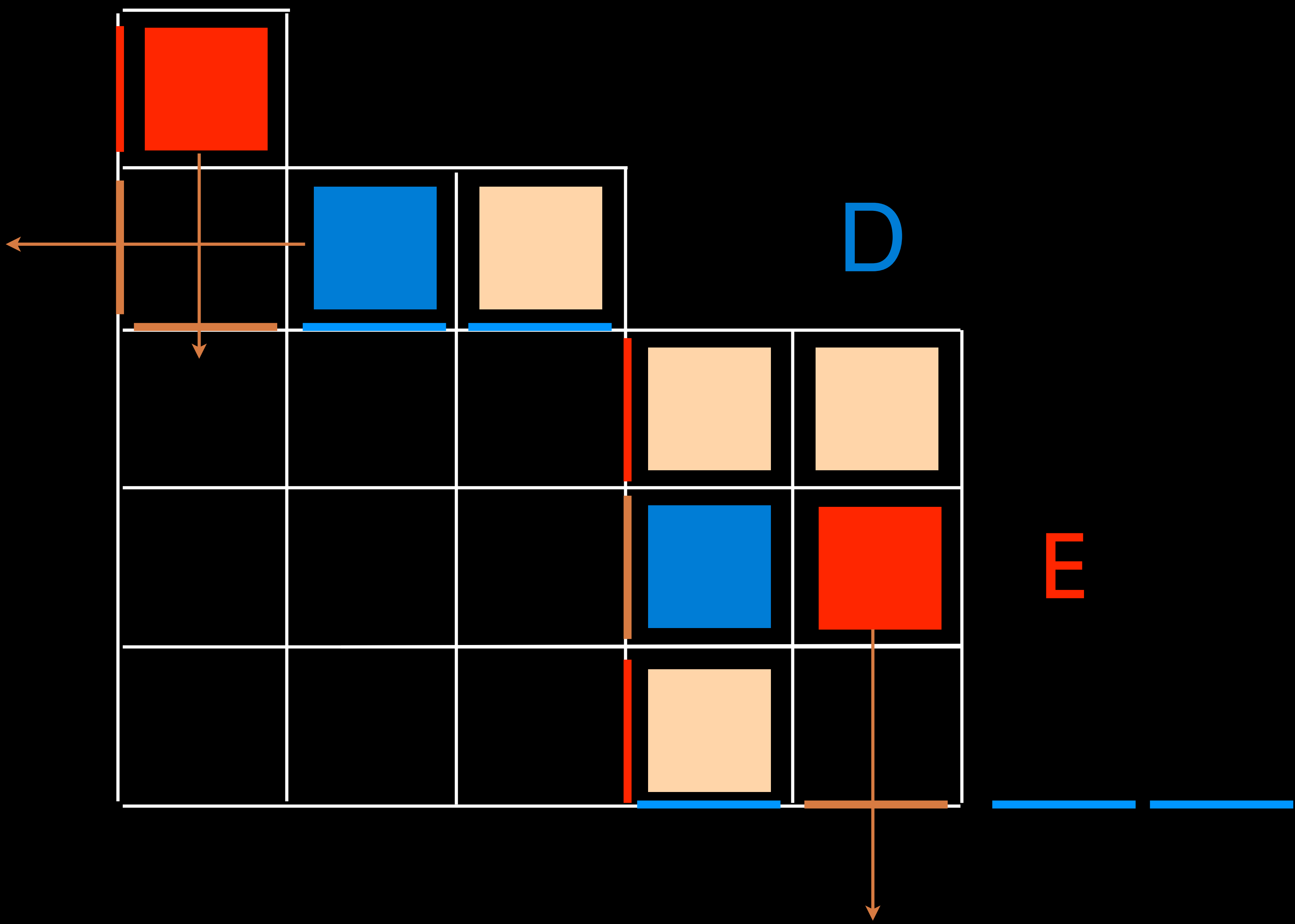
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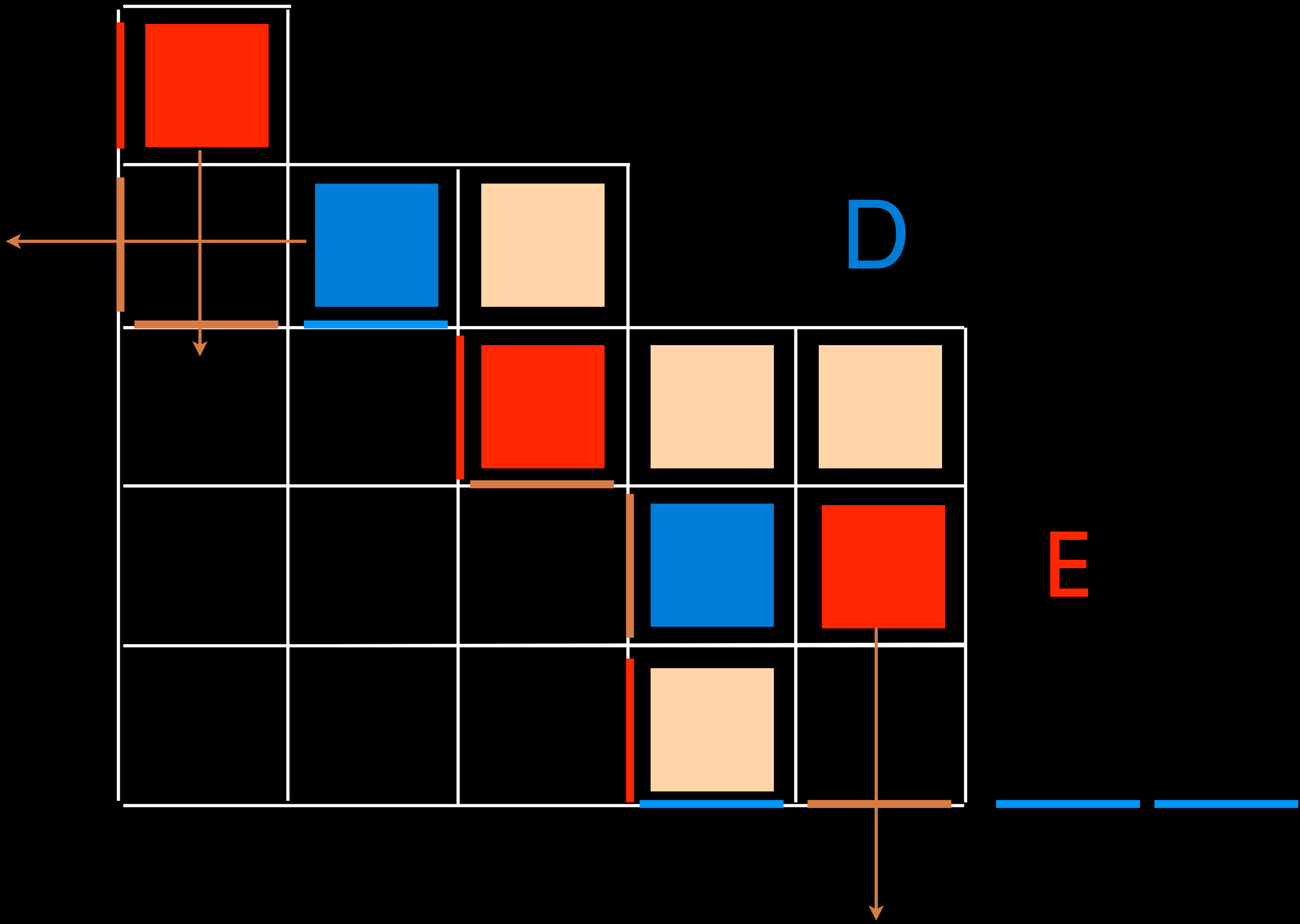
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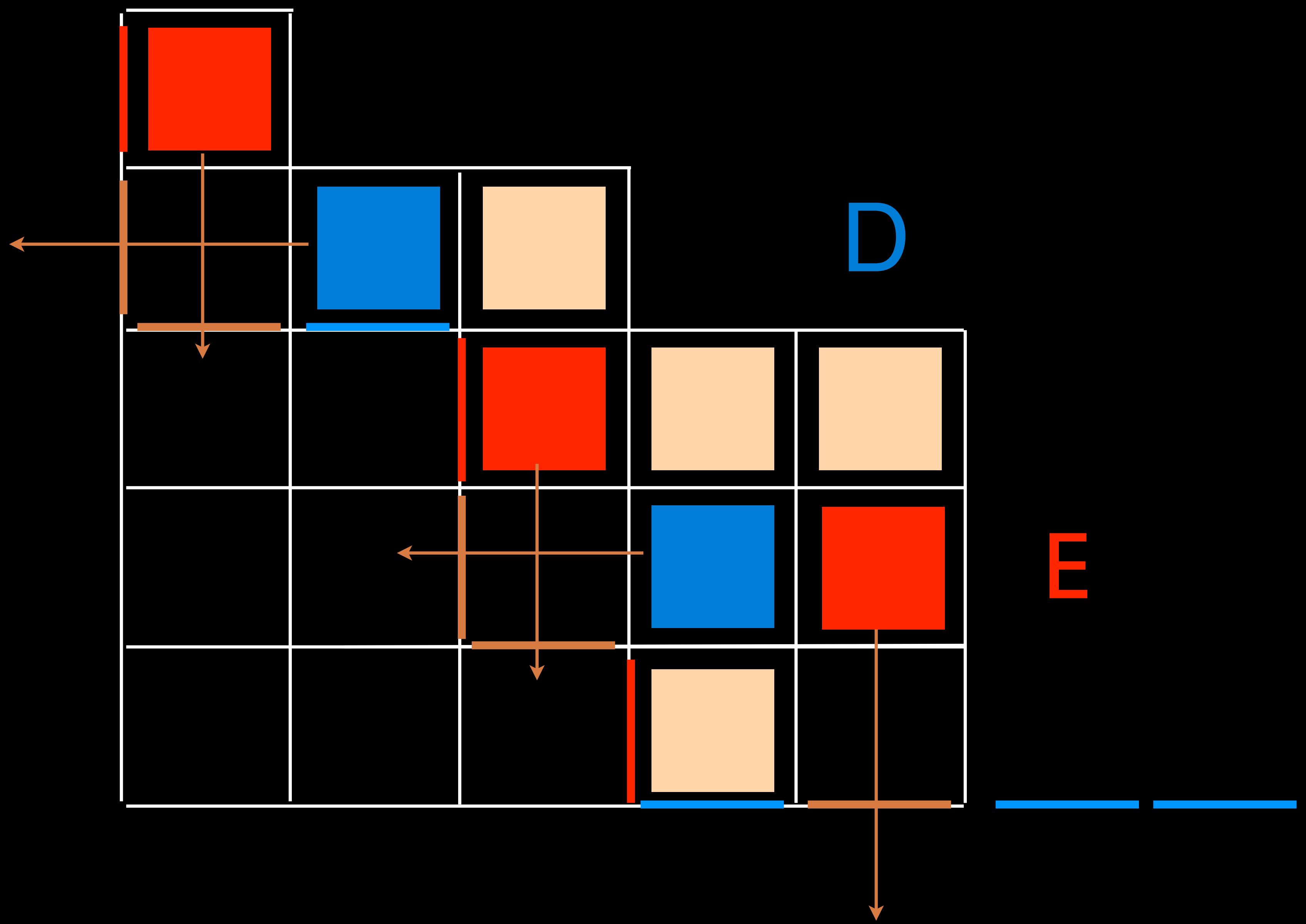
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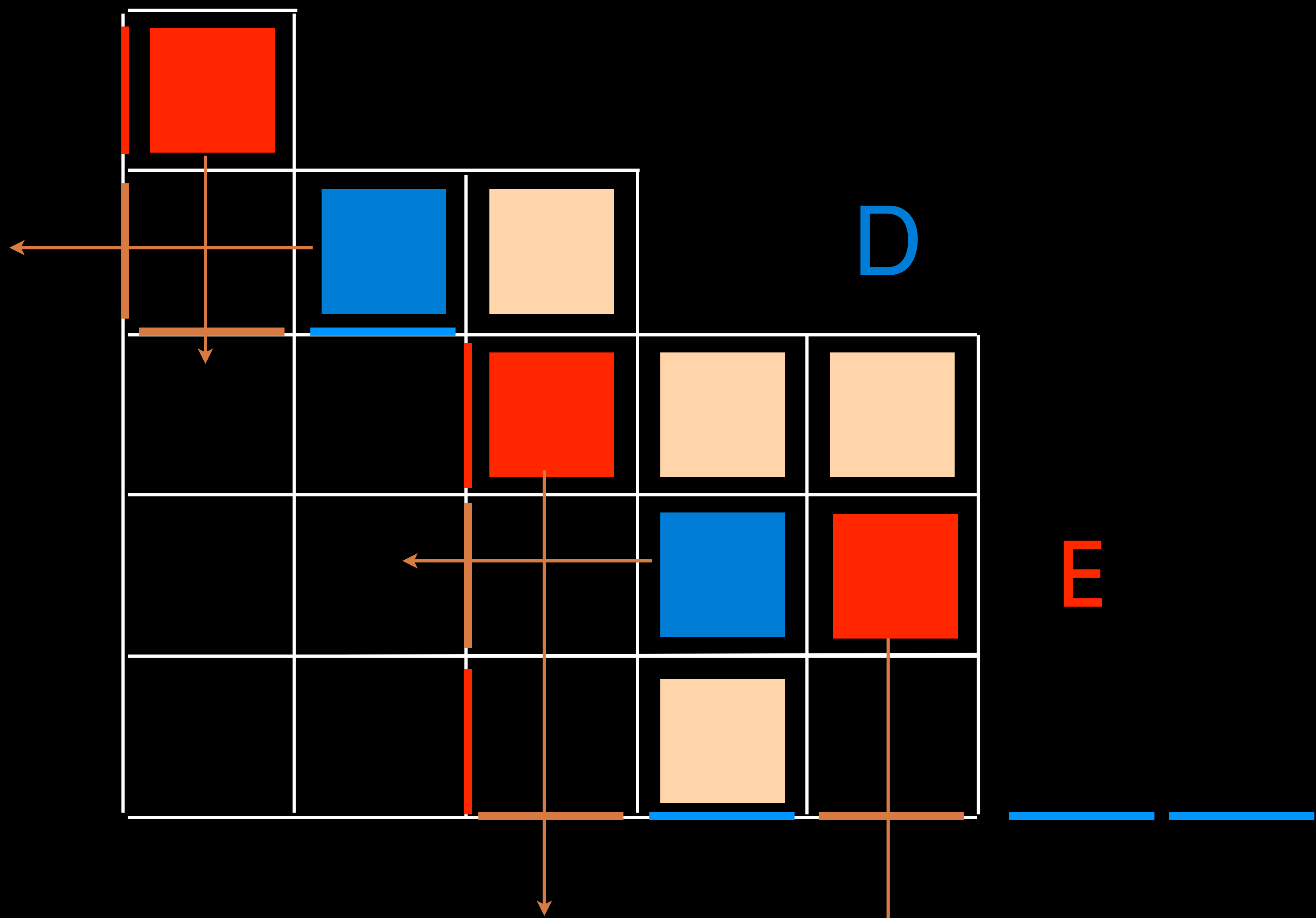
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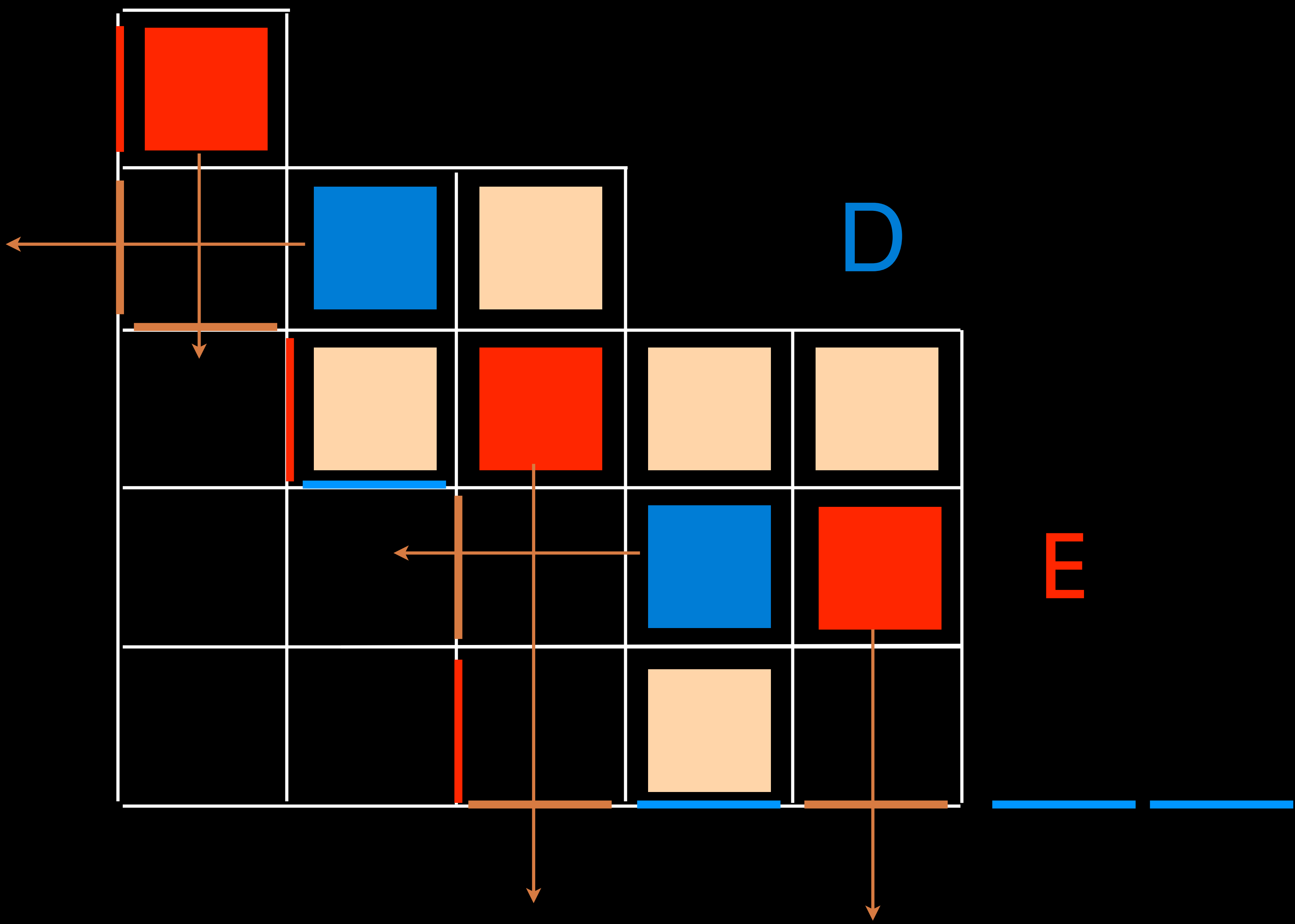

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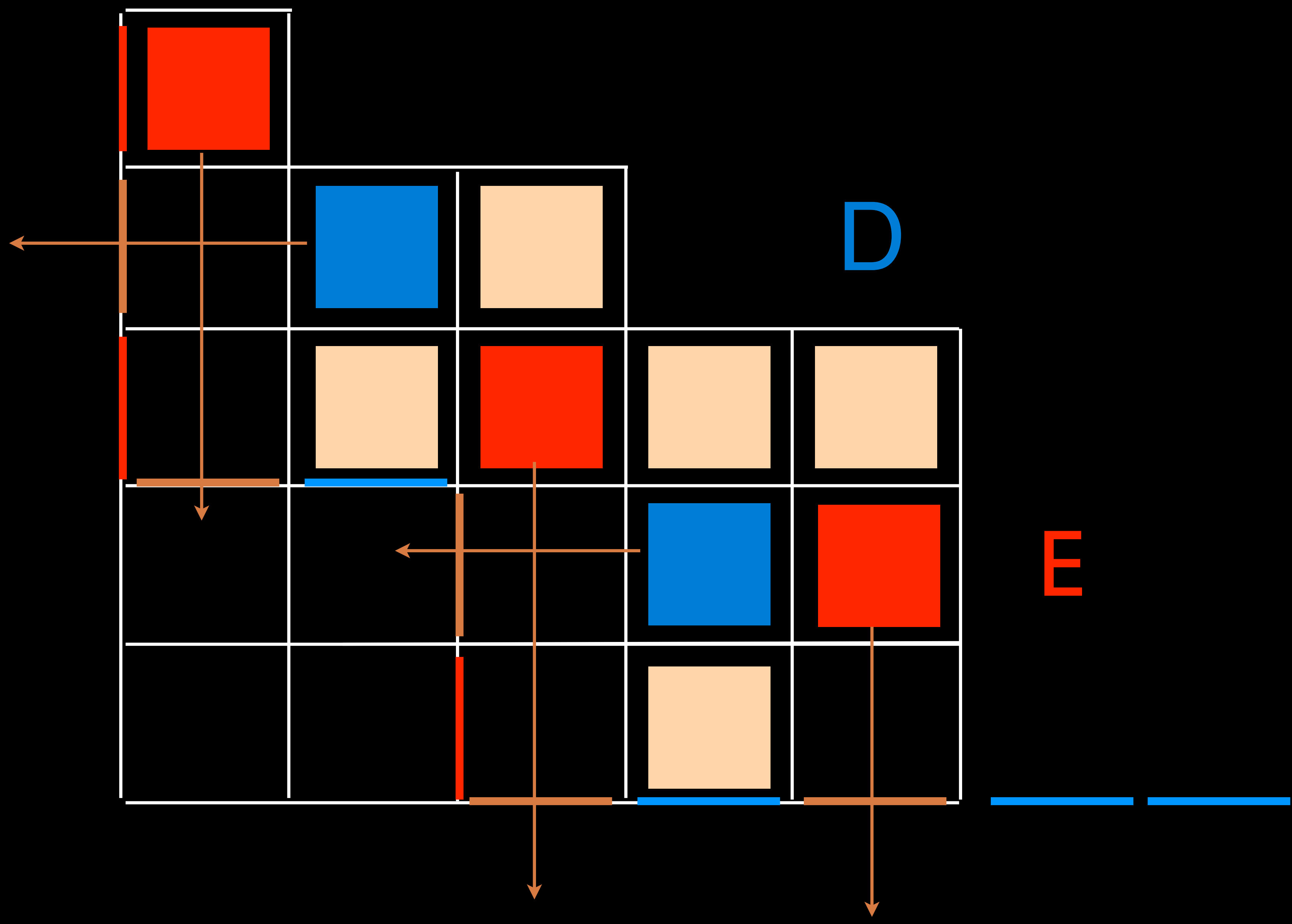
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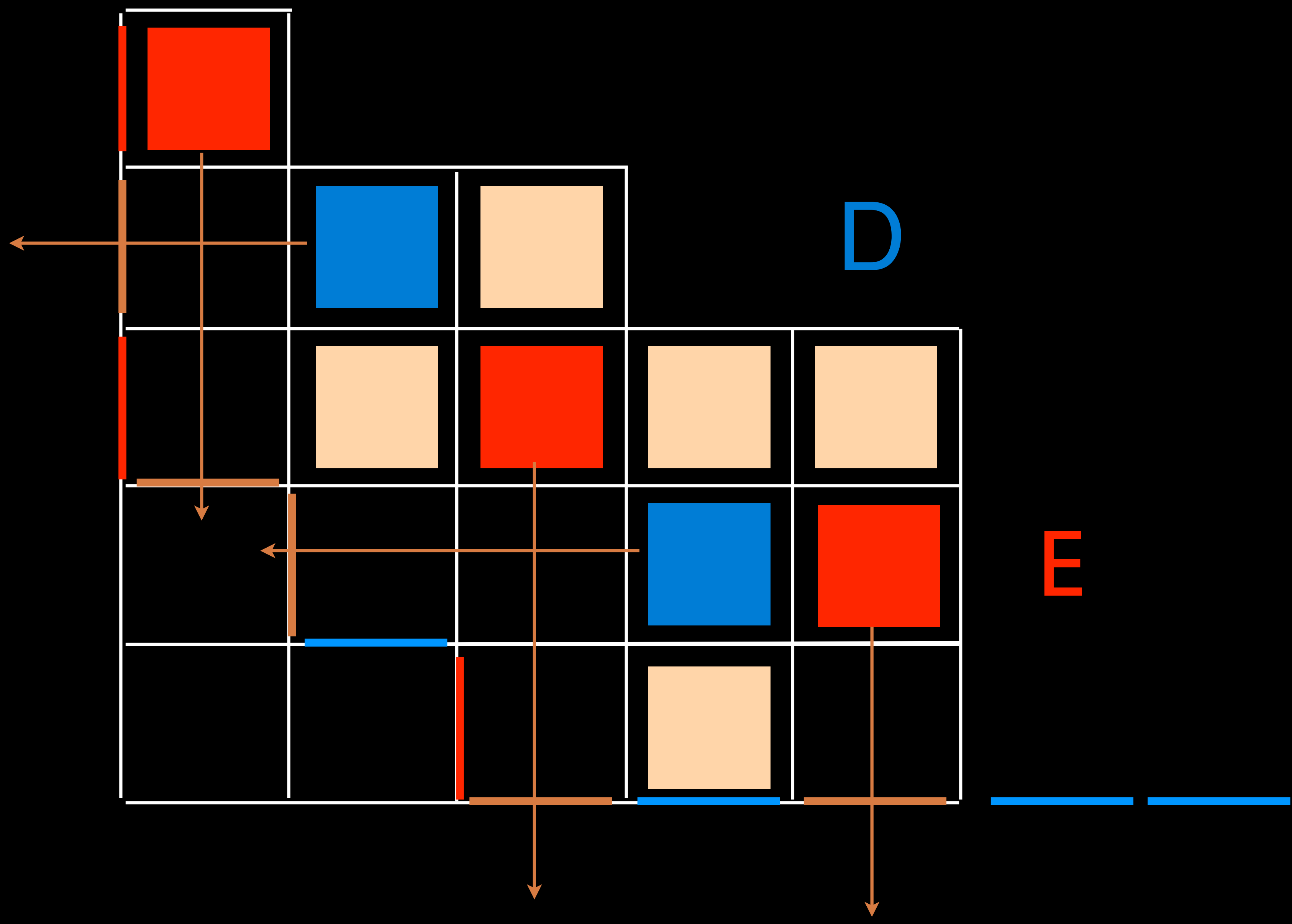
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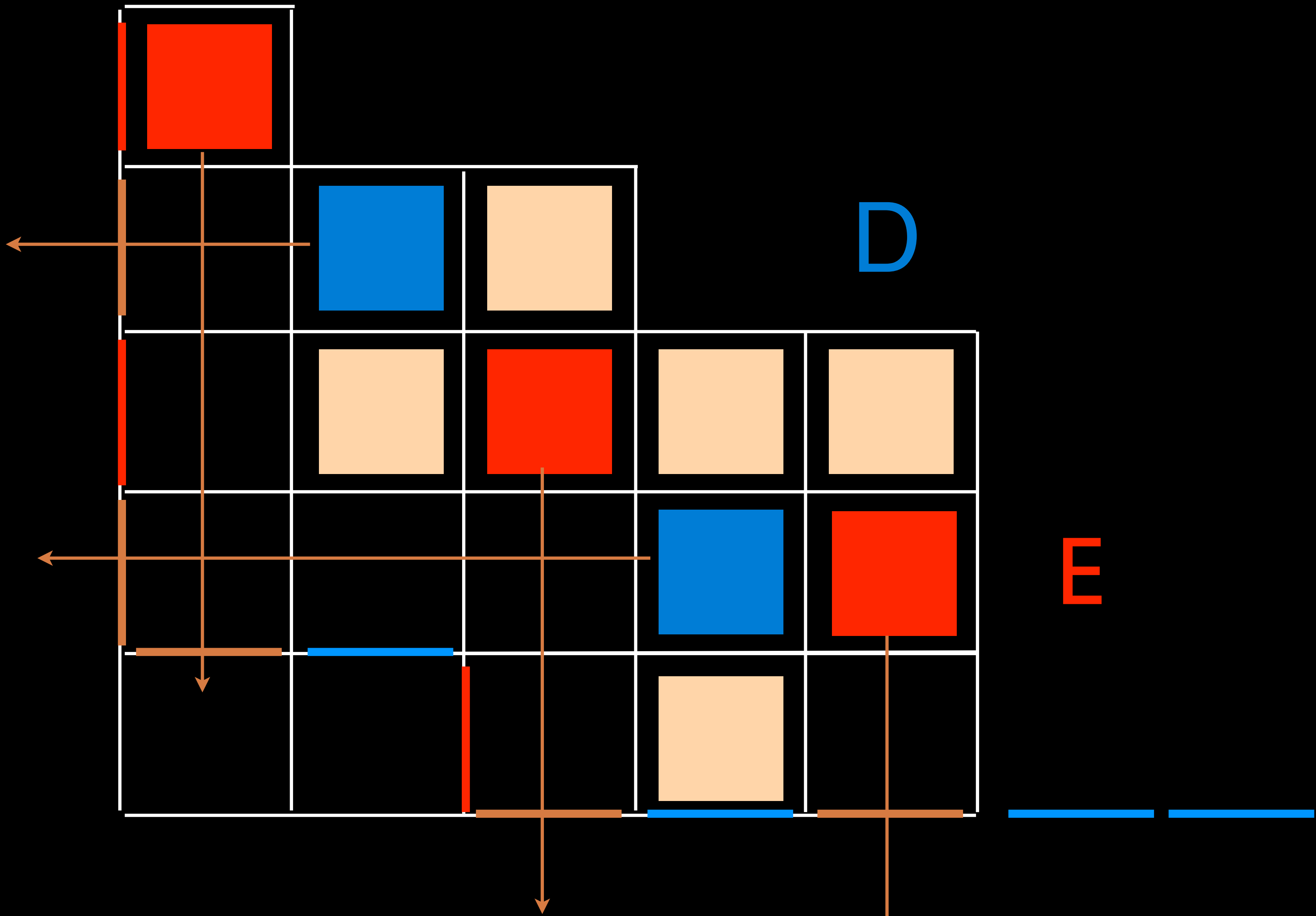

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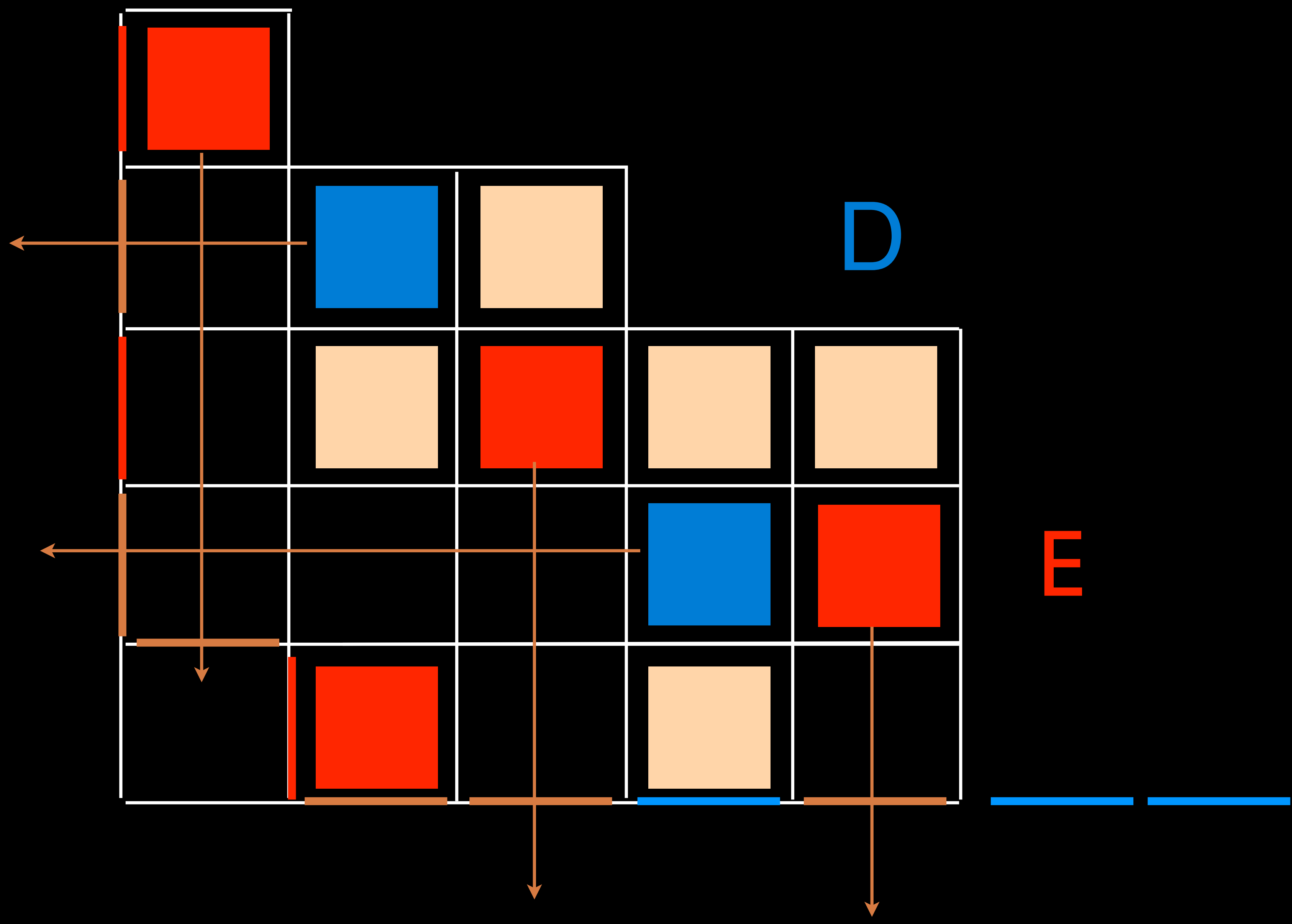
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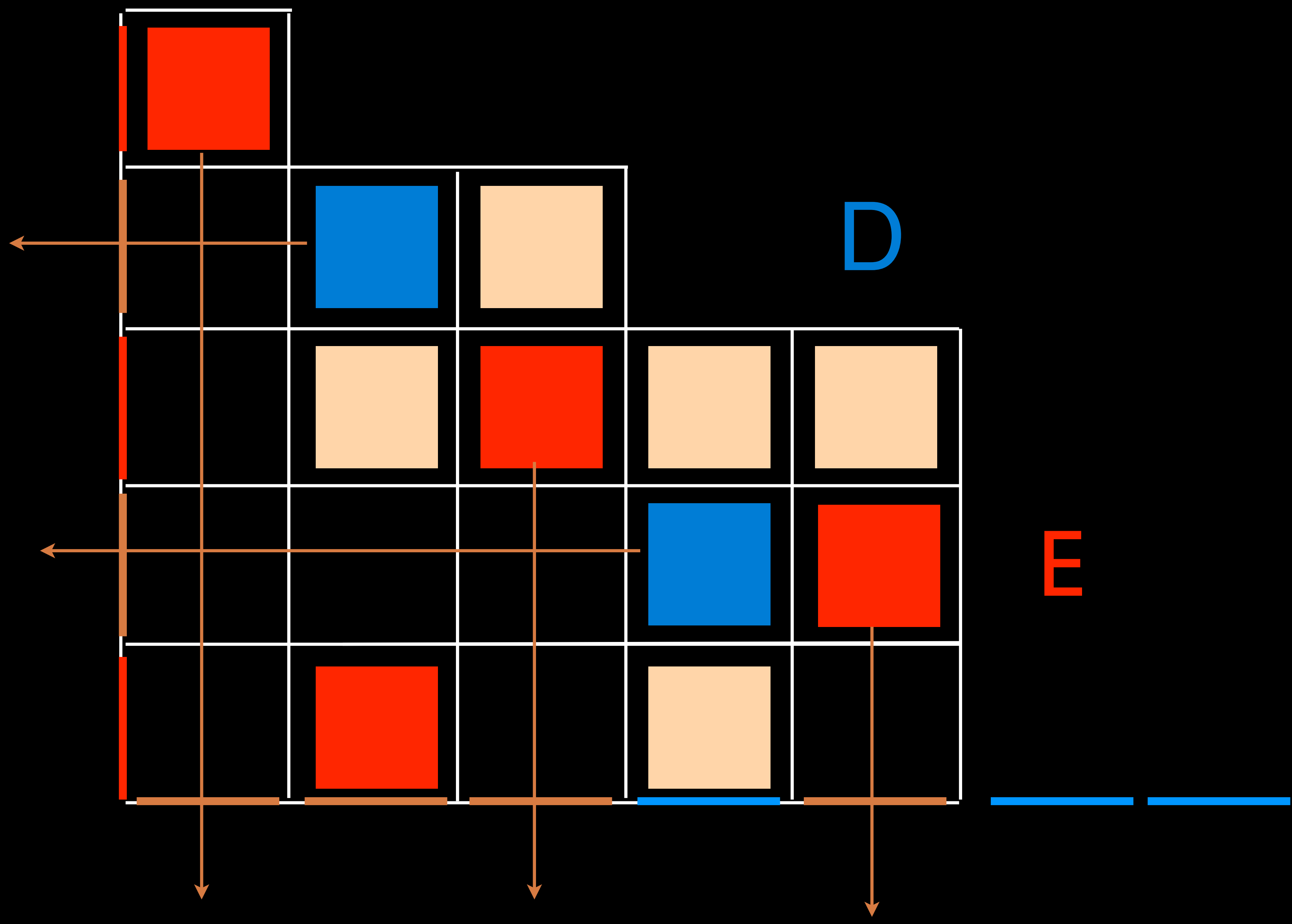
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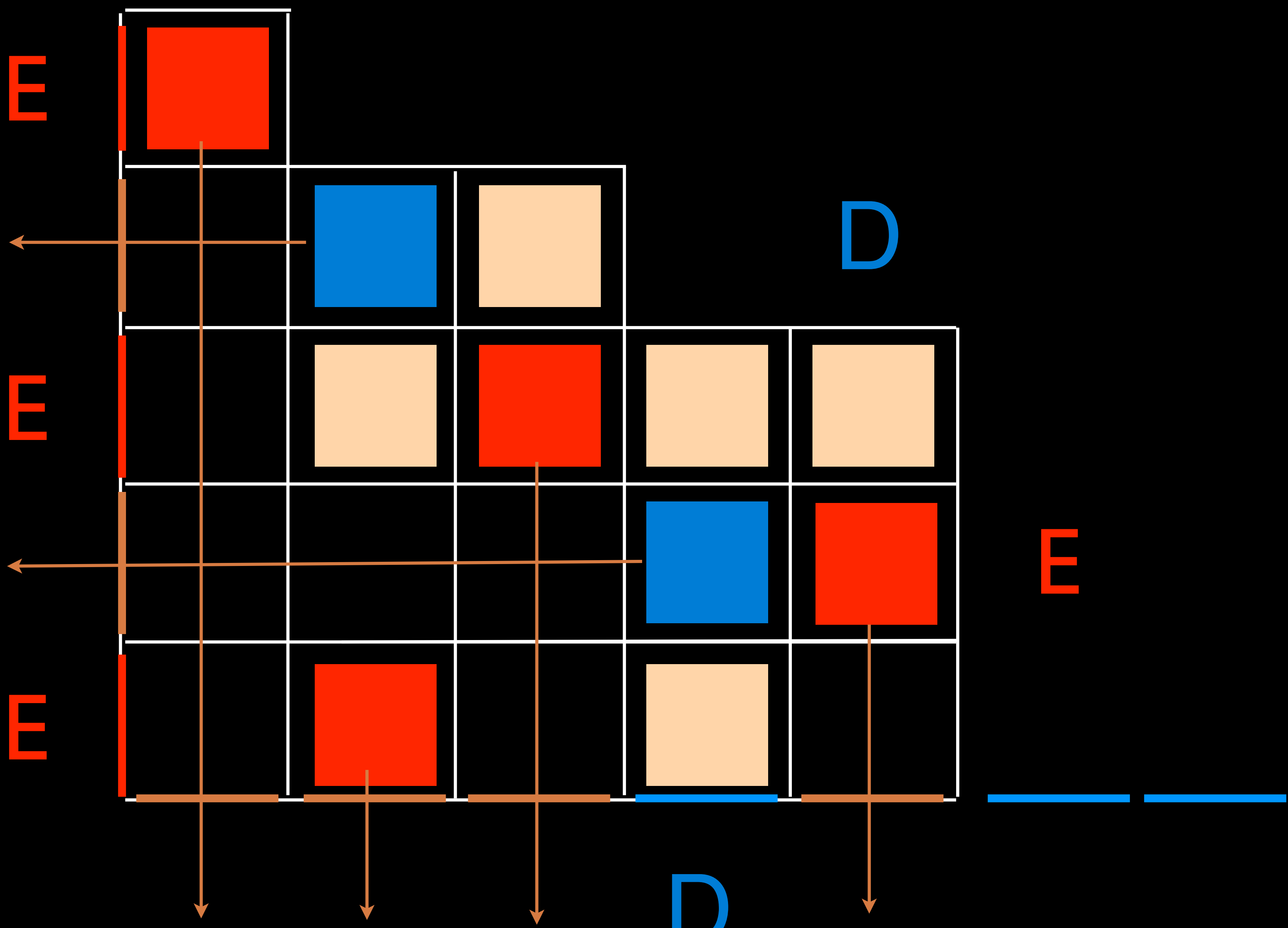
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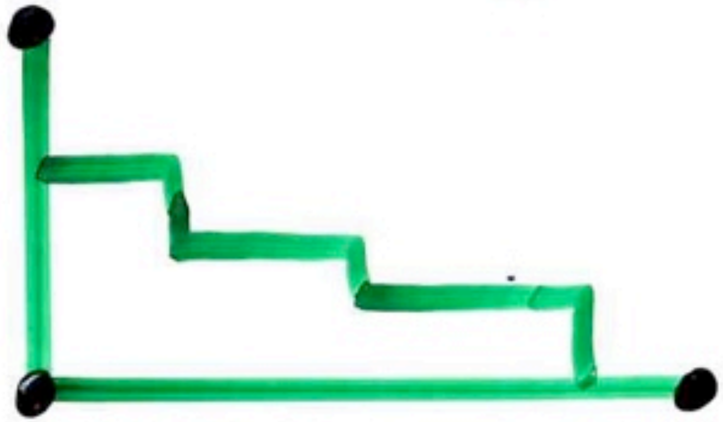
alternative tableaux

alternative tableau

- Ferrers diagram F

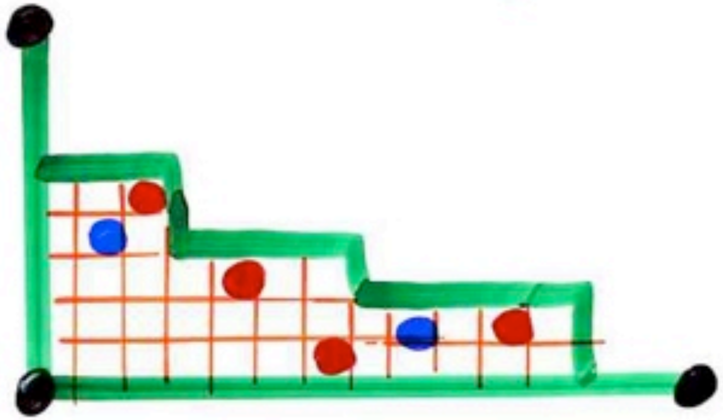


(possibly empty rows or column)

(nb of rows) + (nb of columns) = n

alternative tableau

- Ferrers diagram F



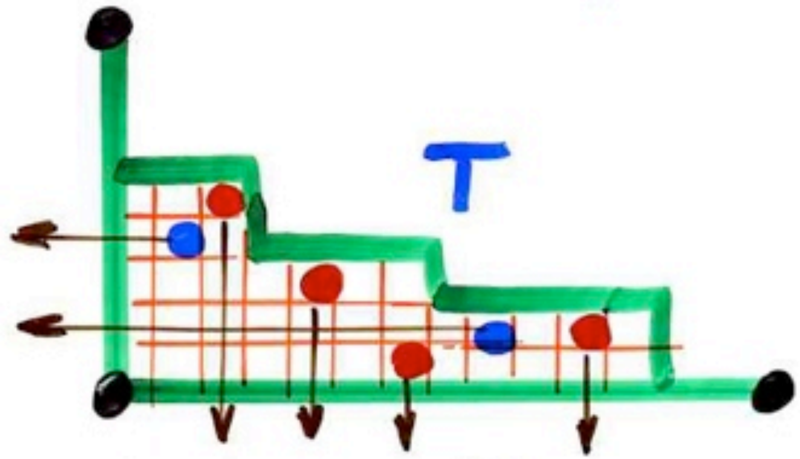
(possibly empty rows or column)

(nb of rows) + (nb of columns) = n

- some cells are coloured **red** or **blue**

alternative tableau T

– Ferrers diagram F (possibly empty rows or column)



(nb of rows) + (nb of columns) $= n$

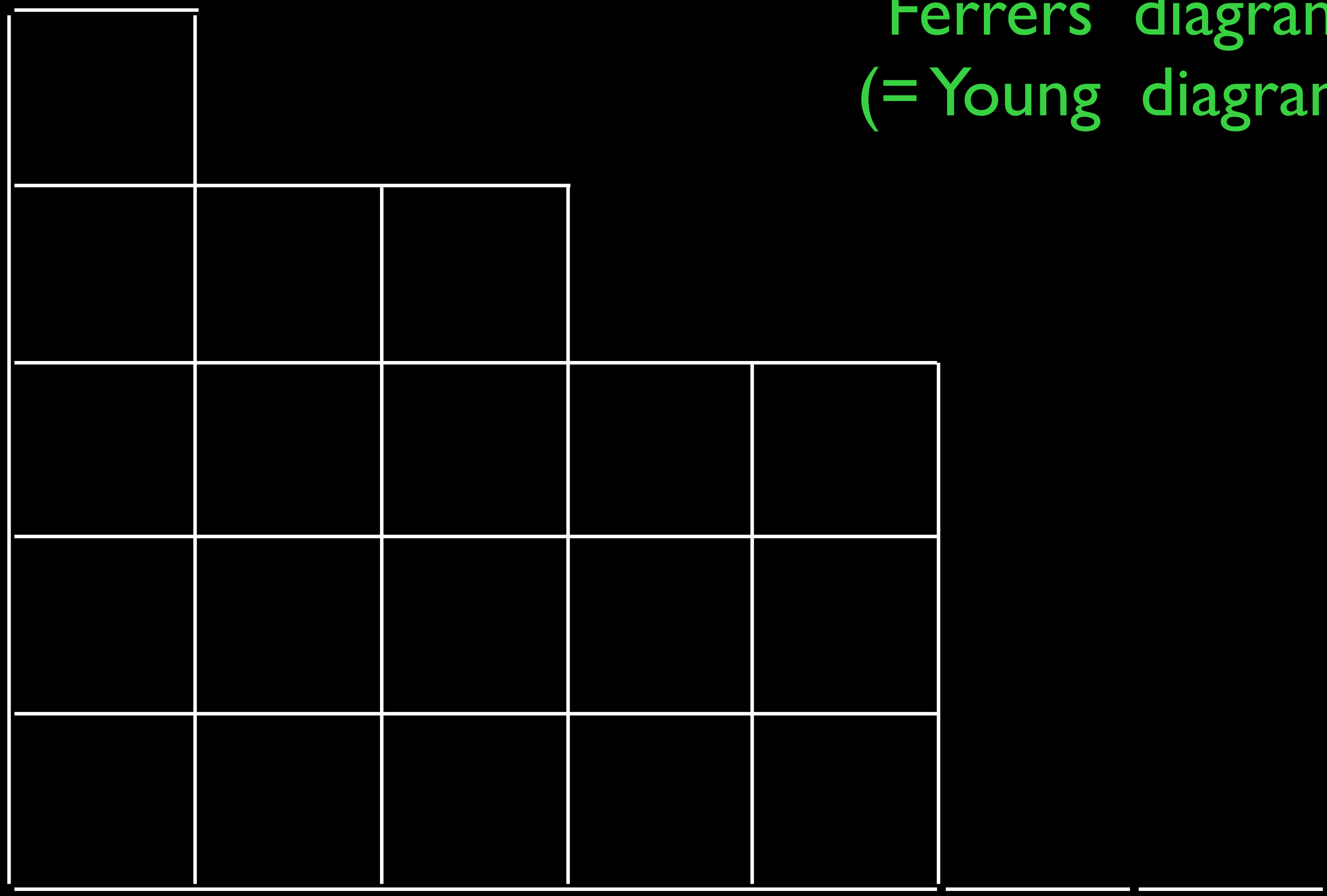
– some cells are coloured **red** or **blue**

– { no coloured cell at the left of (blue cell)
  { no coloured cell below (red cell)

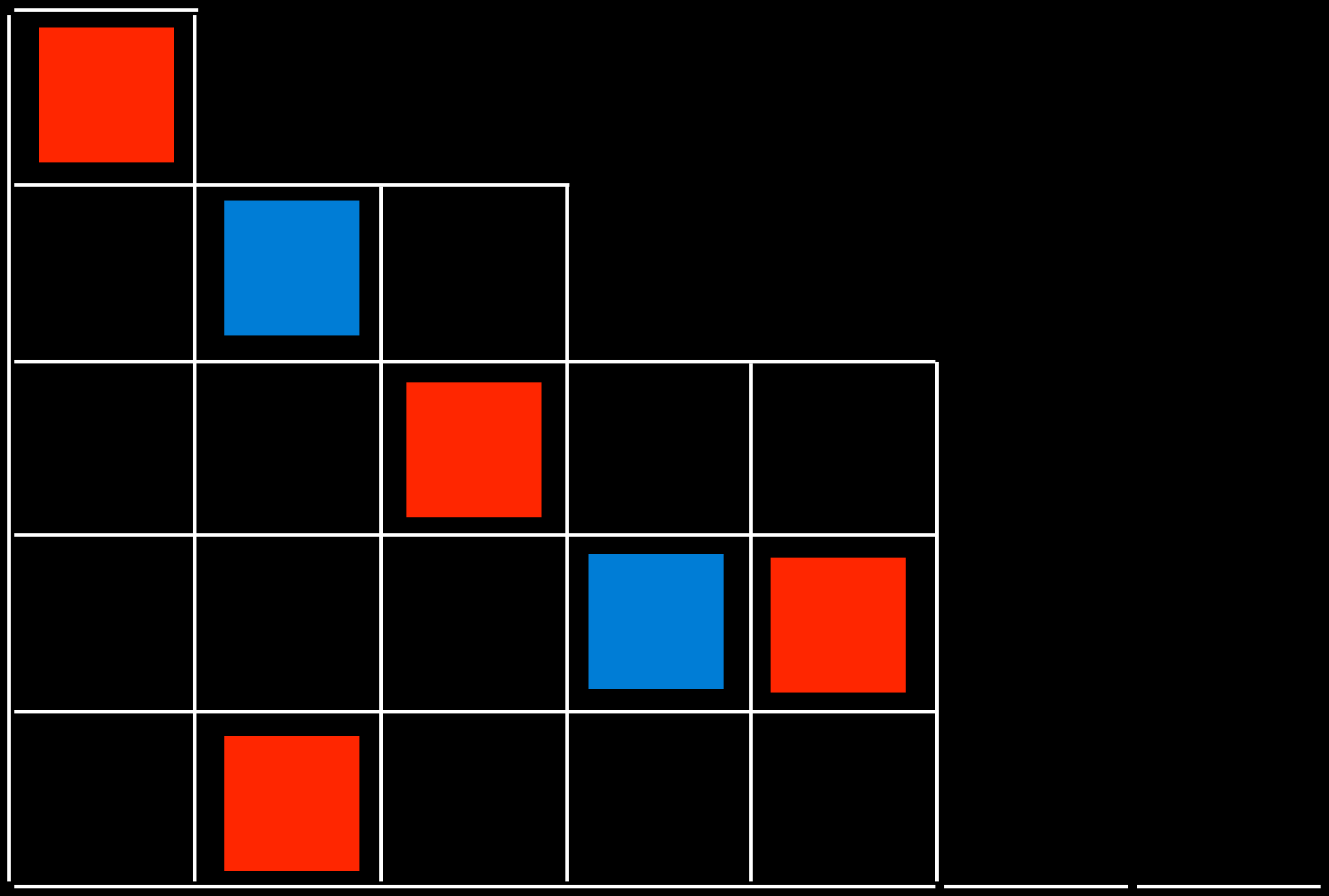
n size of T

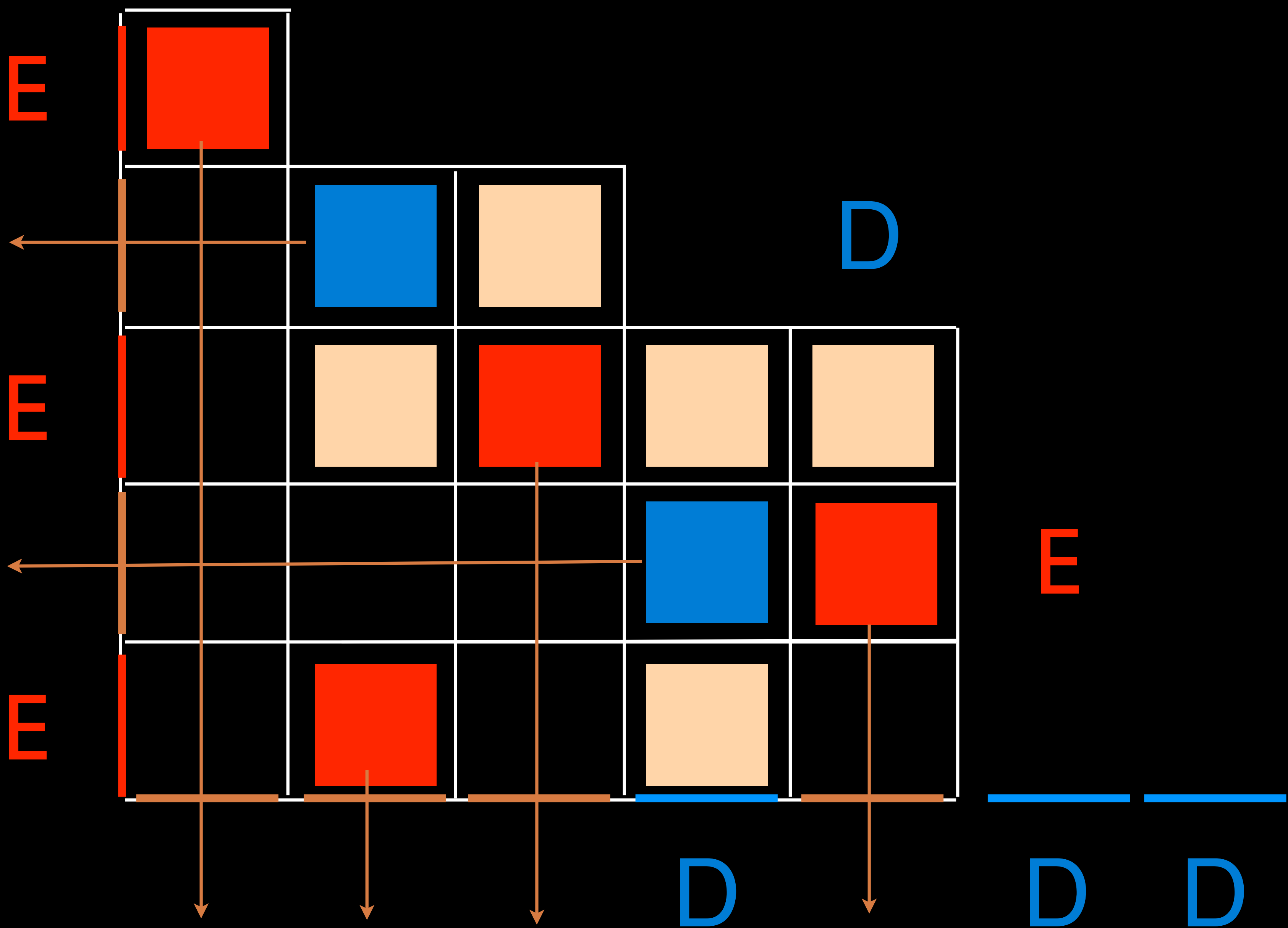
alternative tableau

Ferrers diagram
(= Young diagram)



alternative tableau




E
D
E
E
E
D
D
D

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

unicity

T alternative tableau with profile w

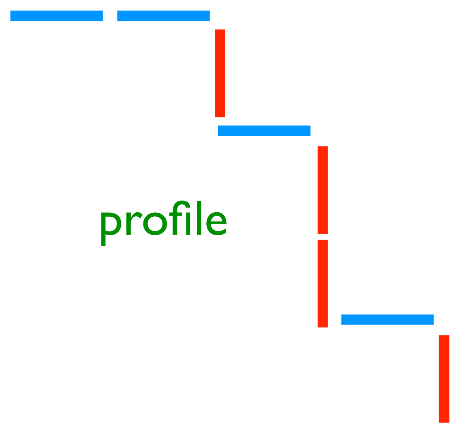
$k(T)$ = nb of □

$i(T)$ = nb of rows without blue cell

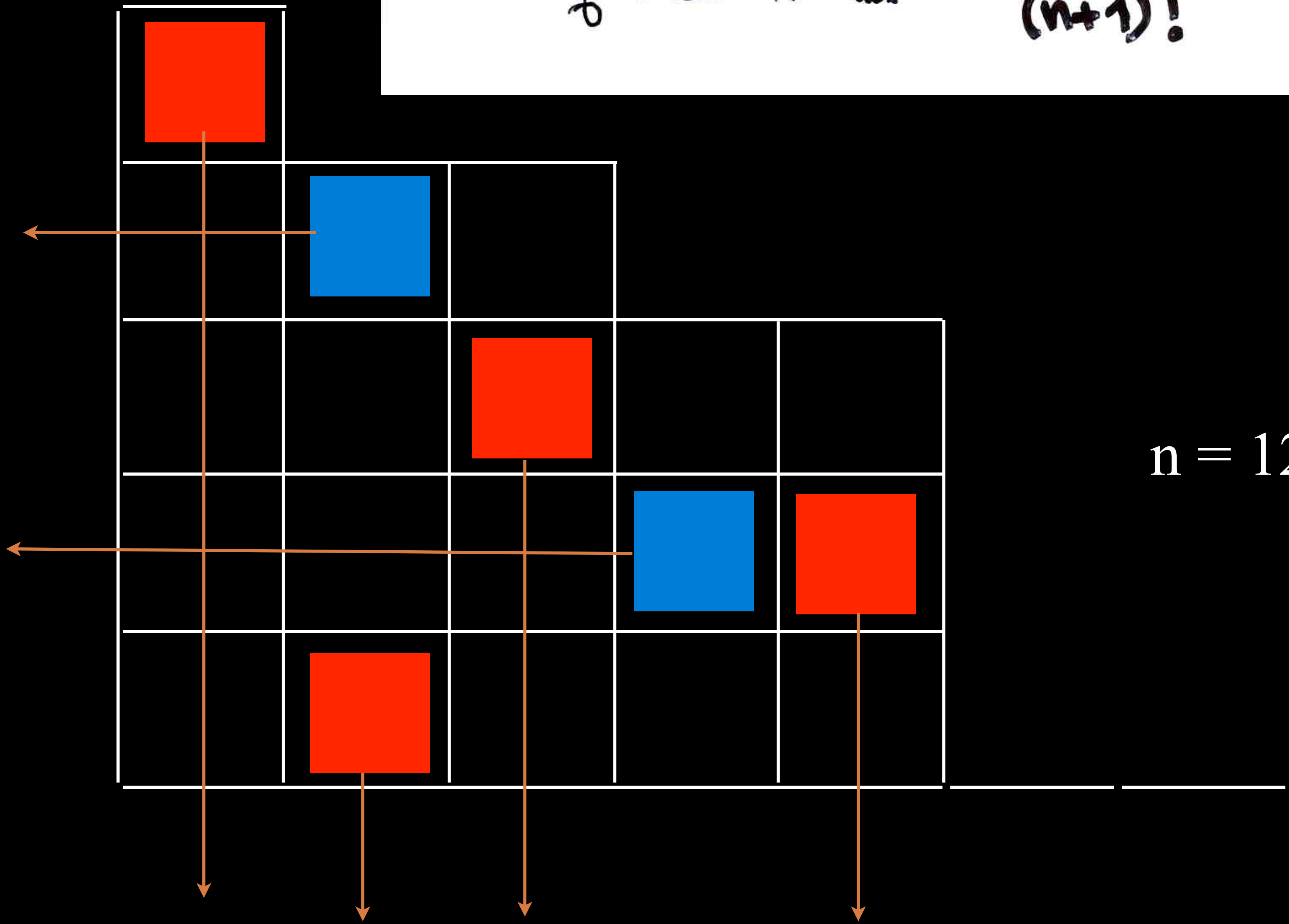
$j(T)$ = nb of columns without red cell

w = D D E D E E D E ⟶

profile



Prop. The number of alternative tableaux of size n is (n+1)!



n = 12

The cellular Ansatz

Construction of a bijection
alternative tableaux --- permutations
from a combinatorial representation
of the PASEP algebra DE=qED+E+D

analog to:
a combinatorial representation of the
Heisenberg algebra UD=DU+Id
gives the RSK correspondence