Combinatorial operators and quadratic algebras

part II: the RSK correspondence from a combinatorial representation of the Heisenberg algebra $UD=DU+Id$

IMSc, Chennai

1 March 2012

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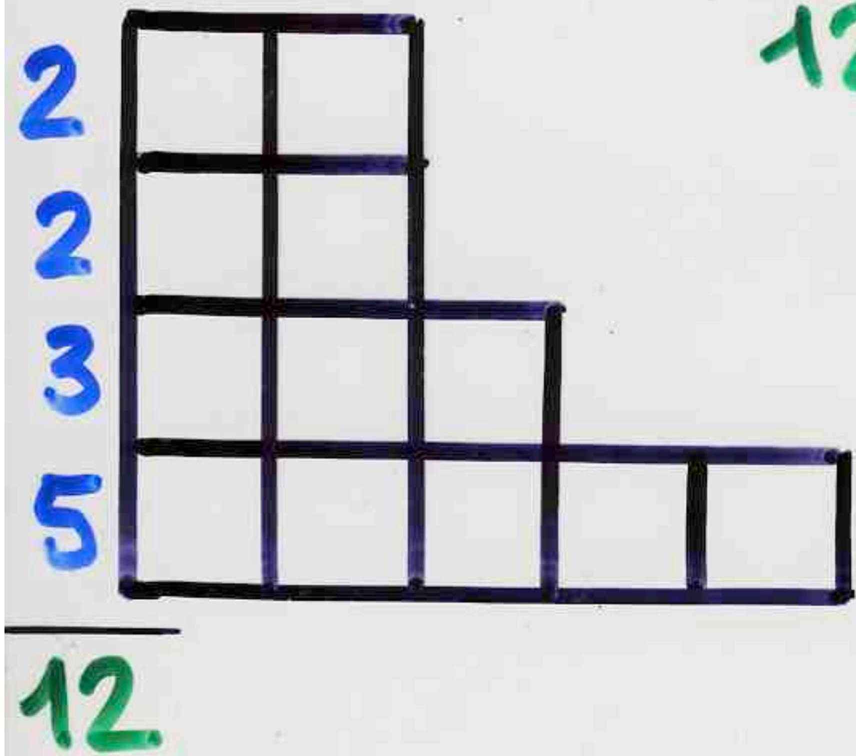
CNRS, Bordeaux, France

An introduction to RSK

G. de B. Robinson, 1938

C. Schensted, 1961

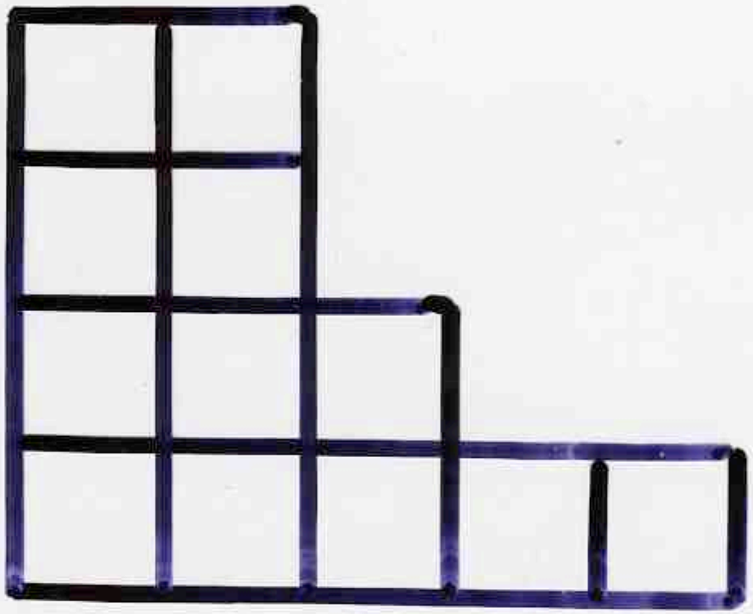
Young tableaux



$12 = n = 5 + 3 + 2 + 2$

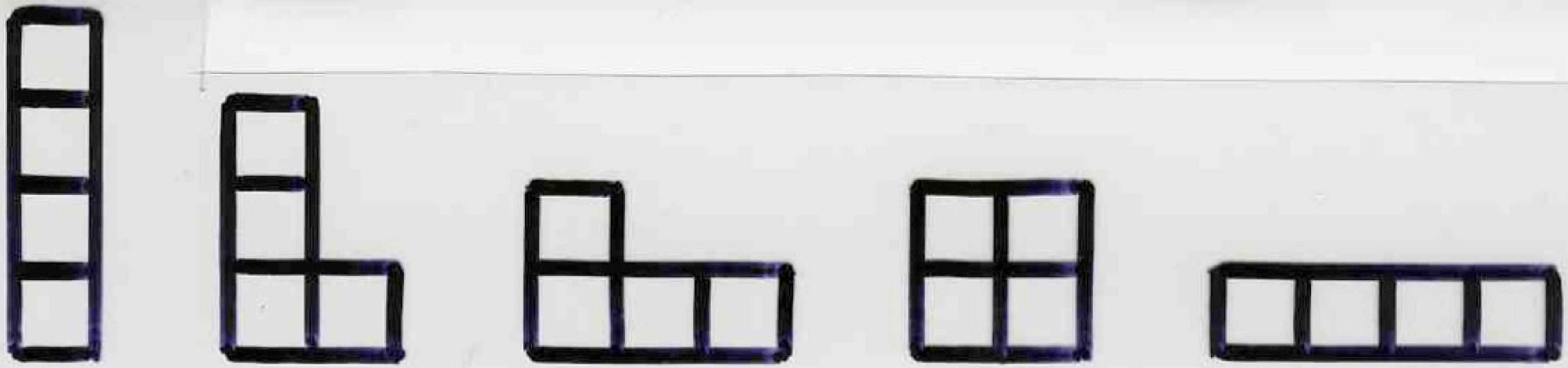
Ferrers
diagram

Partition of n

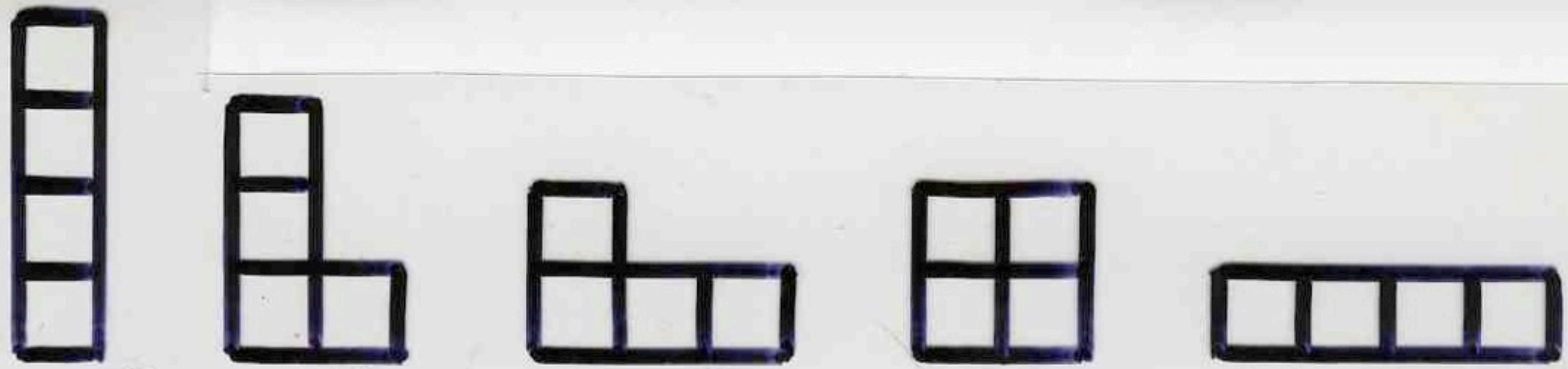


Young tableau

7	12			
6	10			
3	5	9		
1	2	4	8	11



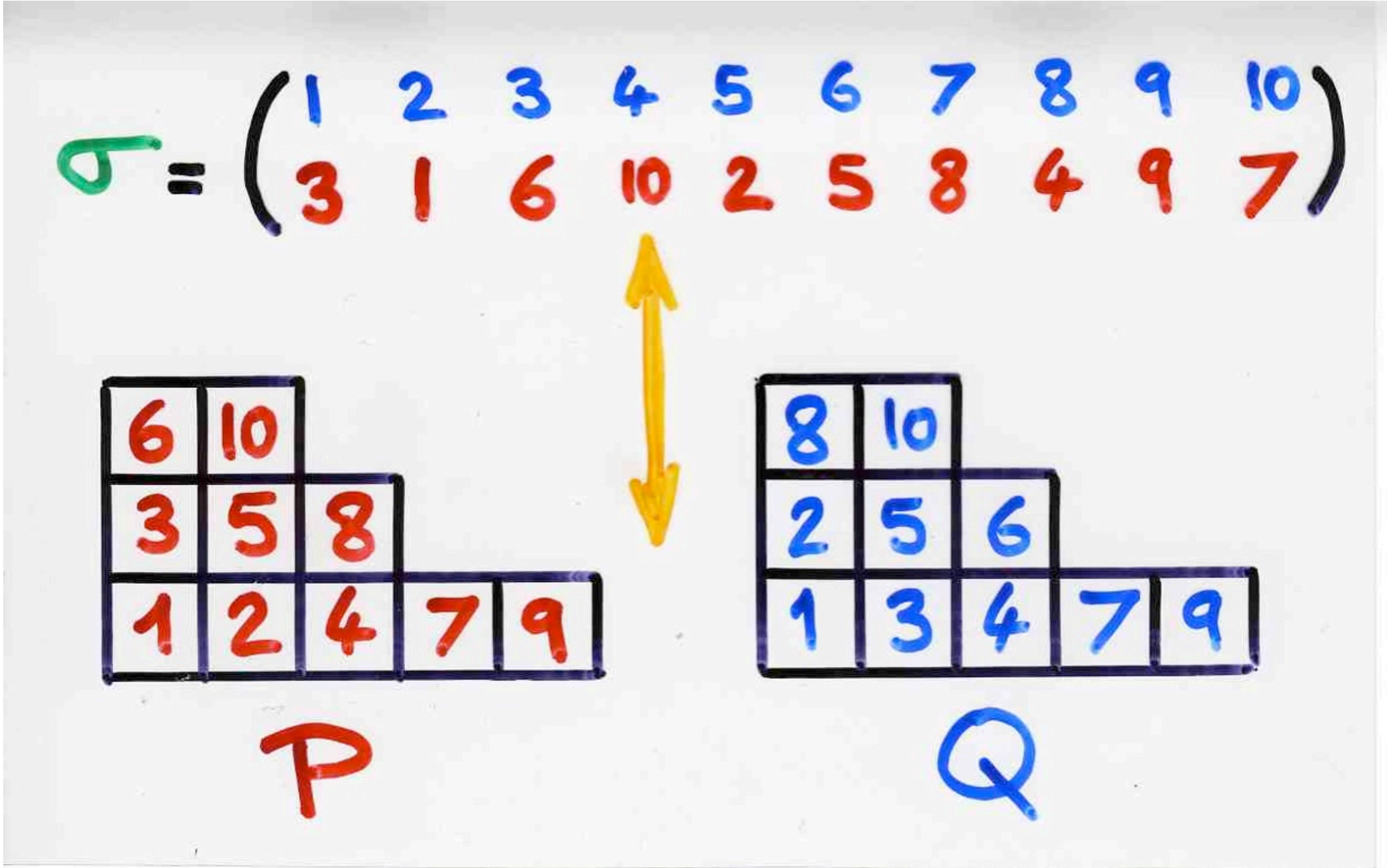
1 3 3 2 1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape nombre de permutations

$$n! = \sum_{\lambda} \left(f_{\lambda} \right)^2$$

λ : forme, n cases

f_{λ} : nombre de tableaux de Young de forme λ

G fini

$$|G| = \sum_{\varphi} \deg^2(\varphi)$$

φ représentation irréductible

$$n! = \sum_{\lambda} f_\lambda^2$$

$n!$: ordre groupe fini $\mathfrak{S}_n$

λ : représentations irréductibles

f_λ : degré

RSK with Schensted's insertions

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1					

3					

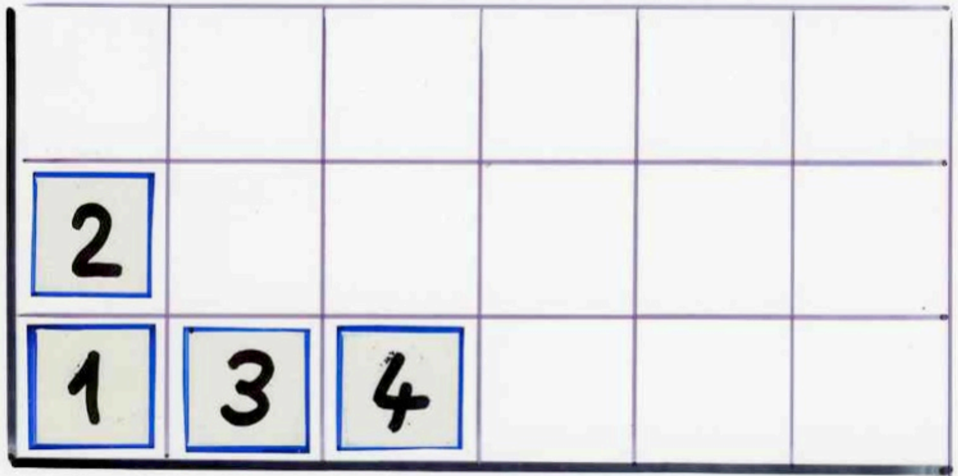
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

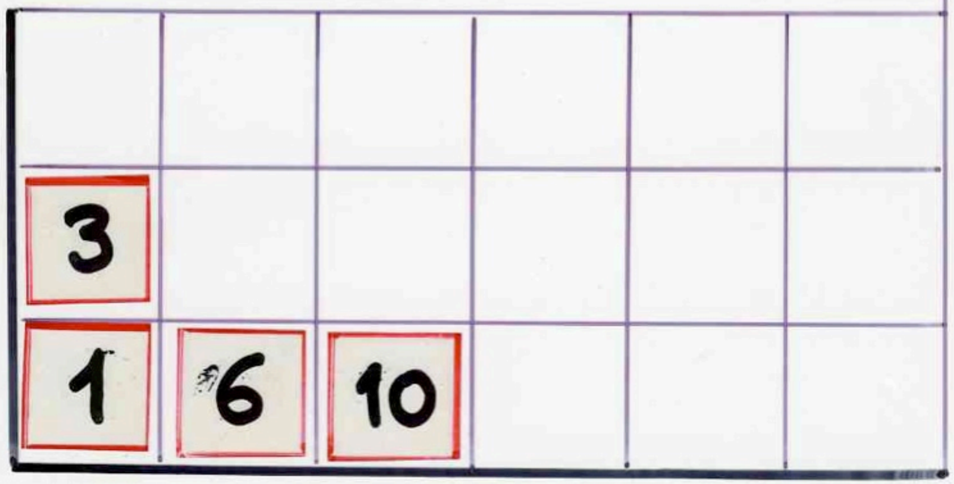
2					
1					

3					
1					

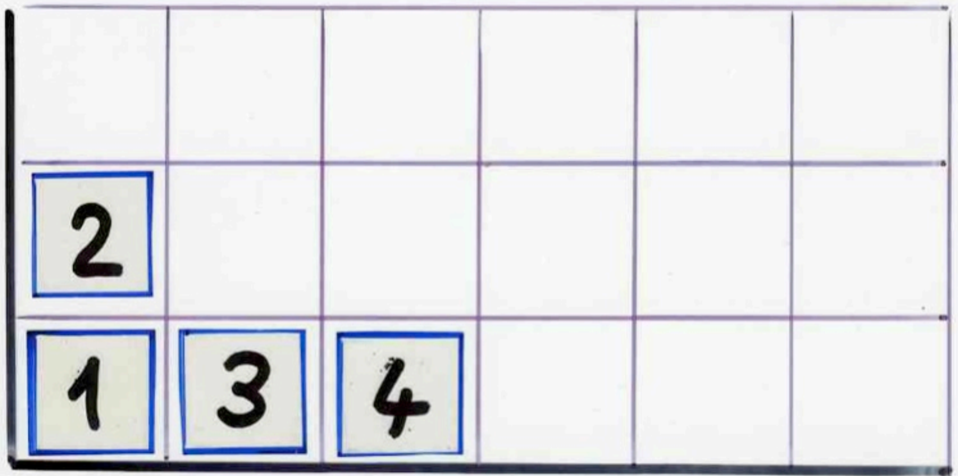
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

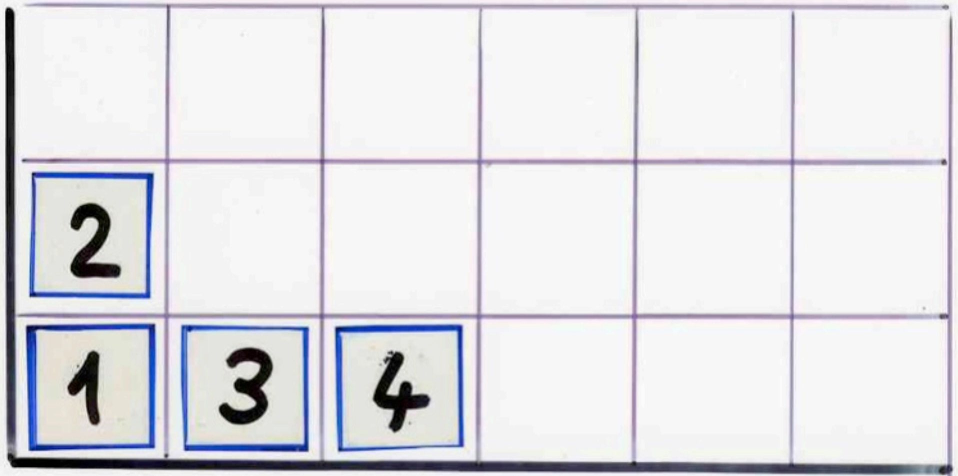


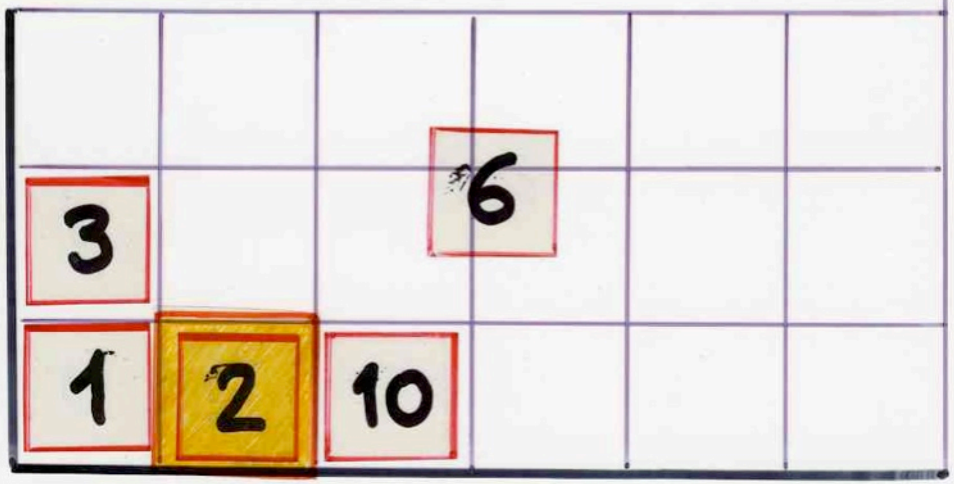


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

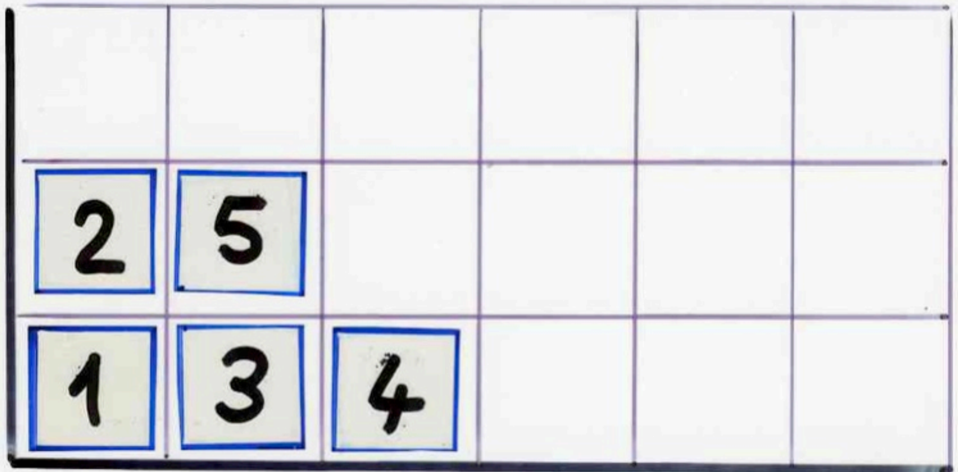


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

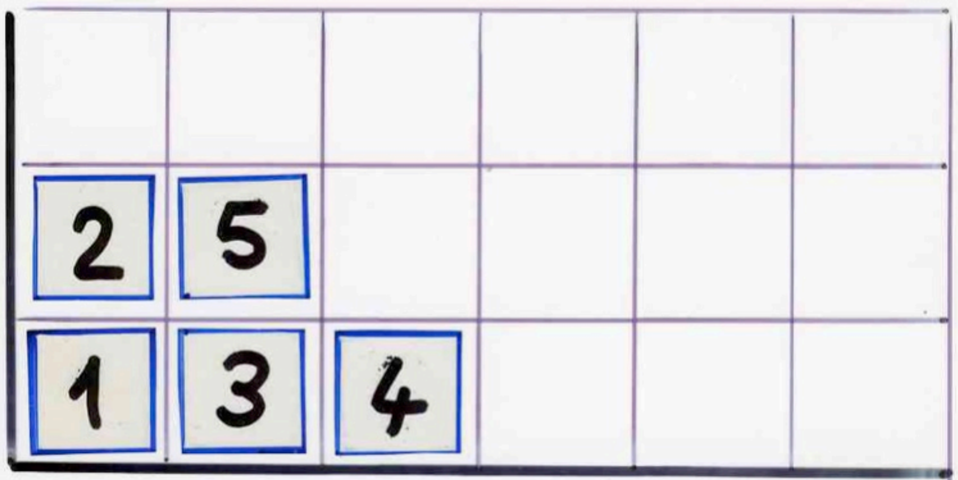


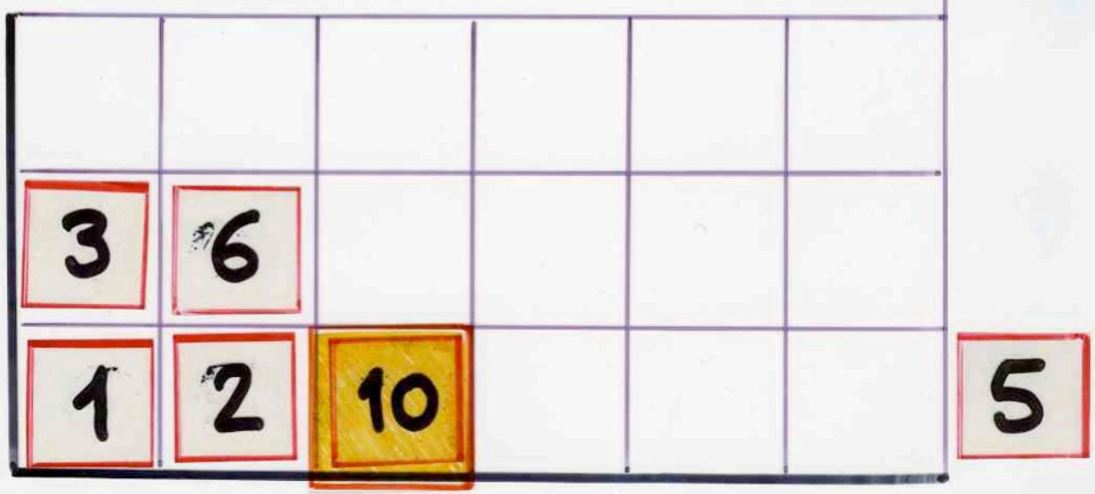


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

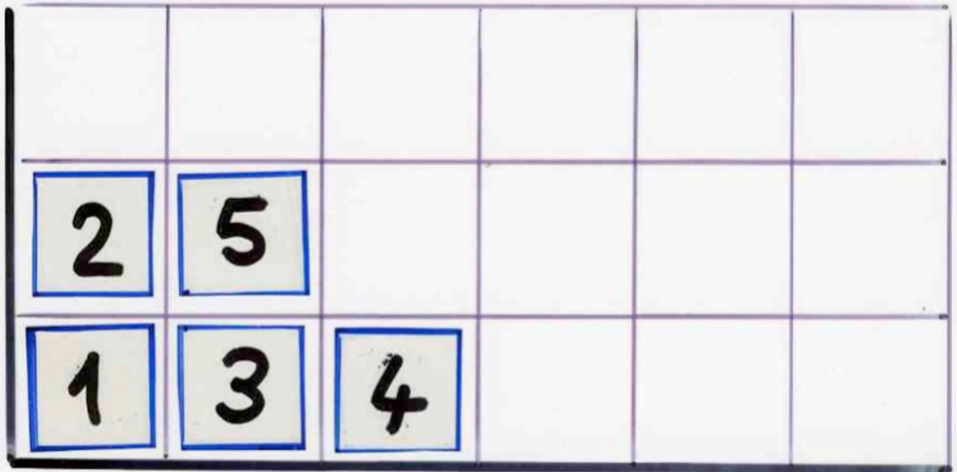


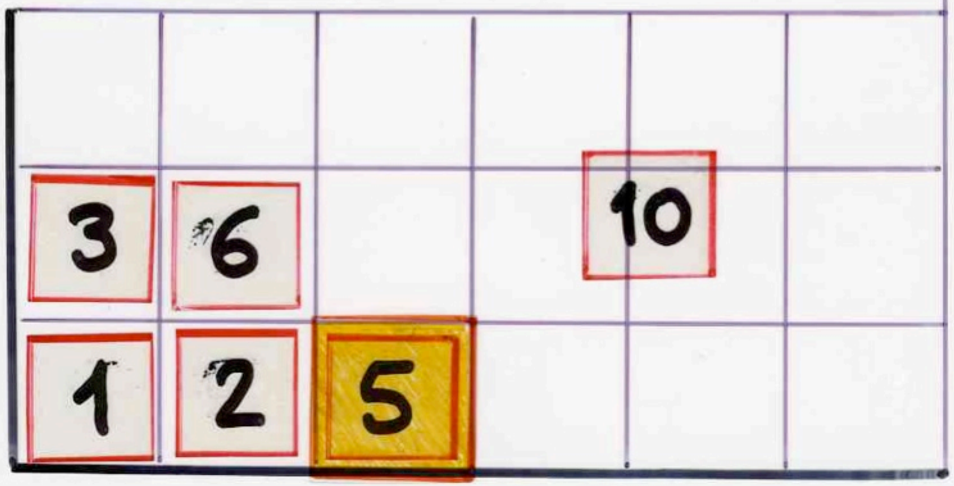
1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7





1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

2	5	6			
1	3	4			

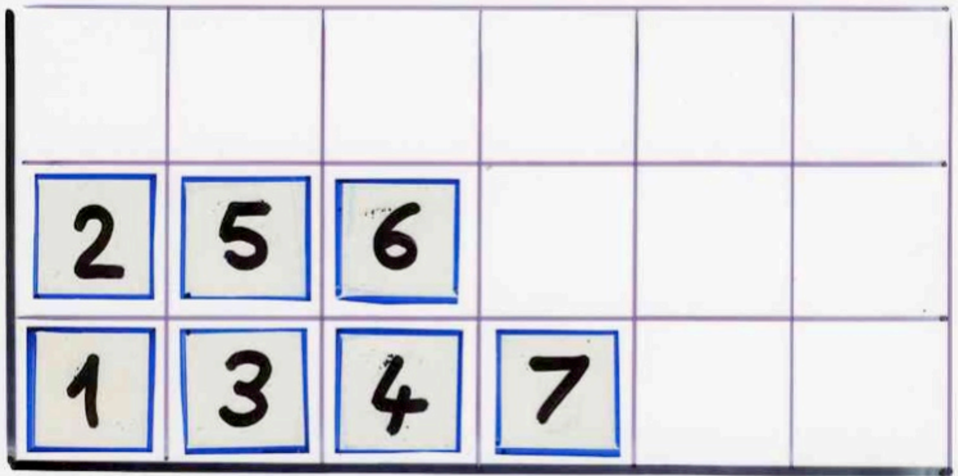
3	6	10			
1	2	5			

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

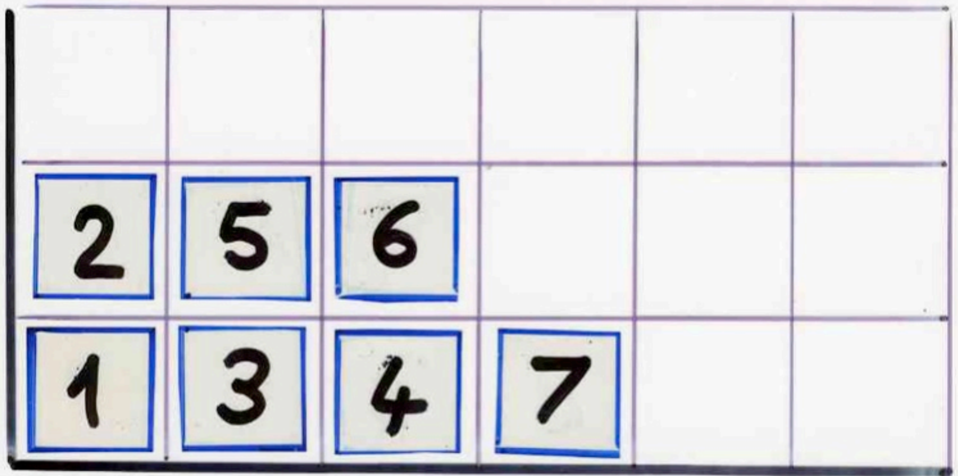
2	5	6			
1	3	4	7		

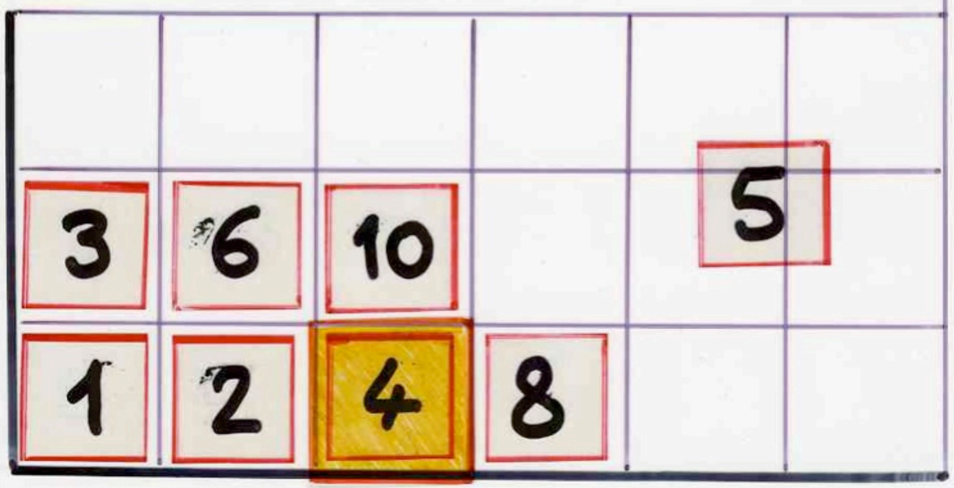
3	6	10			
1	2	5	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

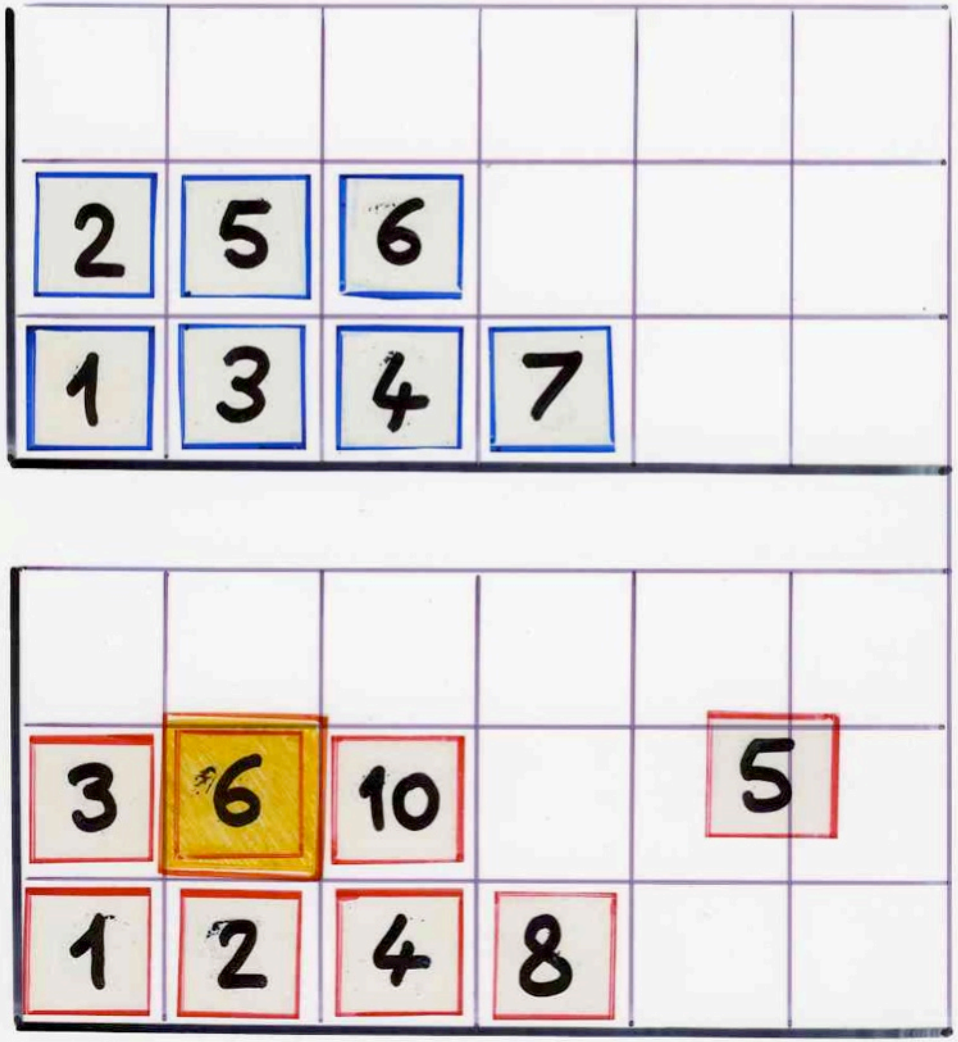


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

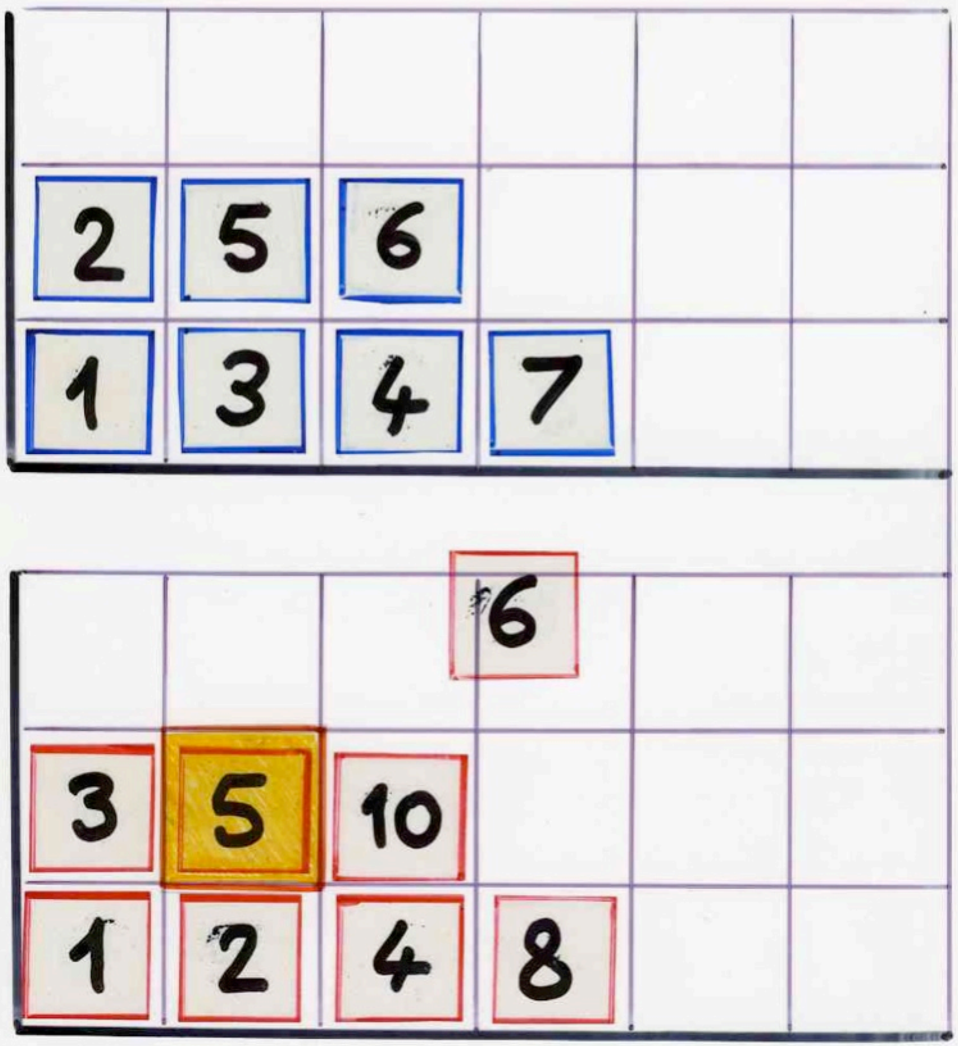




1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7		

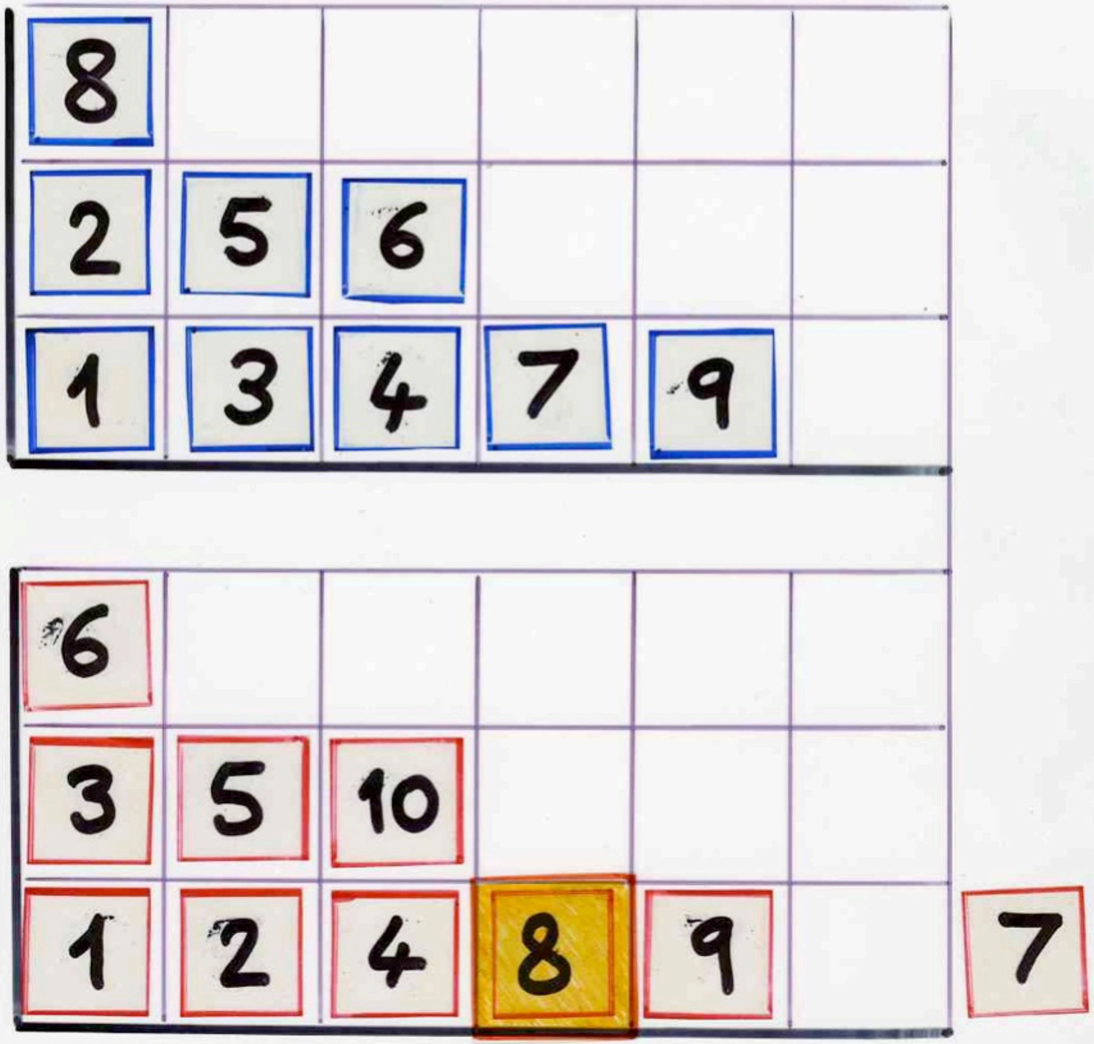
6					
3	5	10			
1	2	4	8		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8					
2	5	6			
1	3	4	7	9	

6					
3	5	10			
1	2	4	8	9	

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

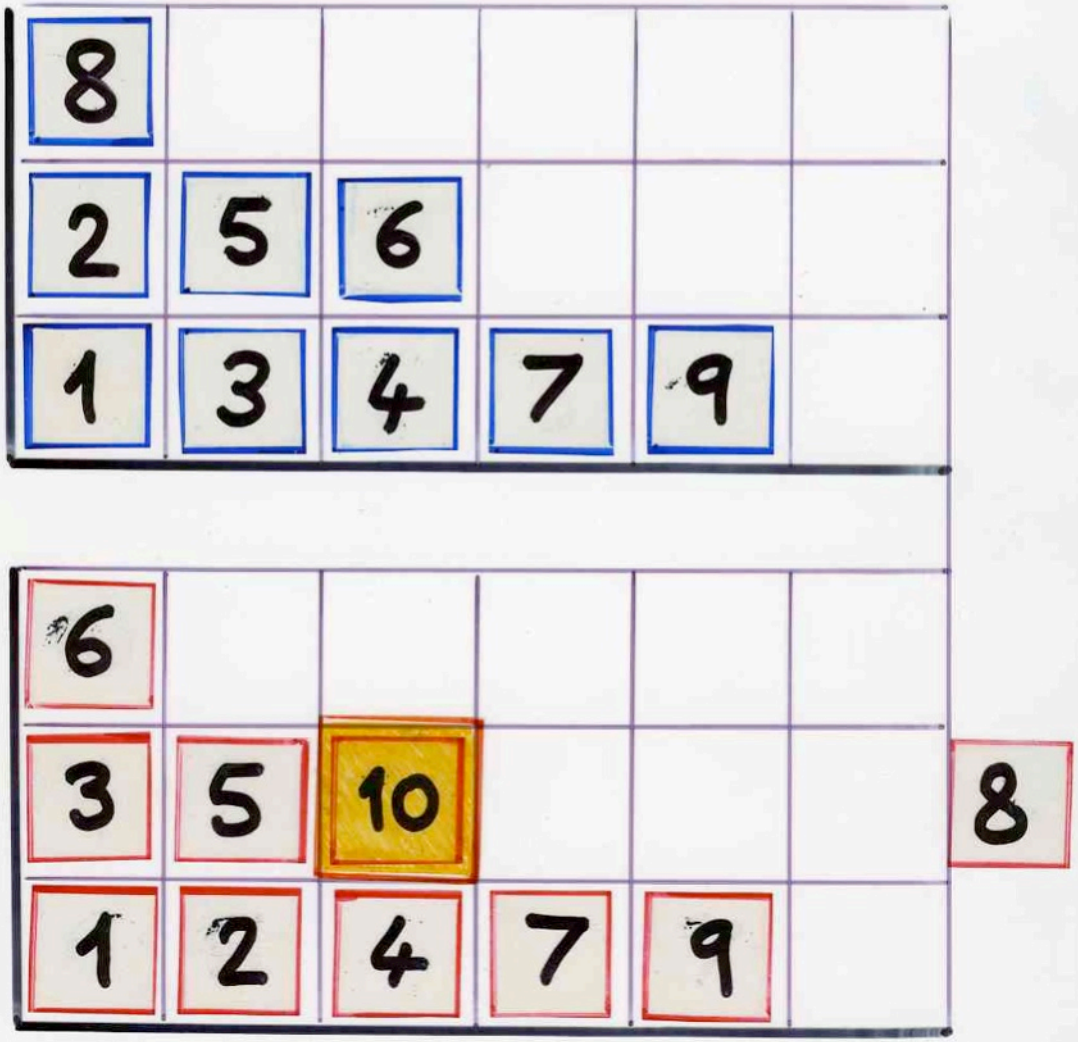


1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

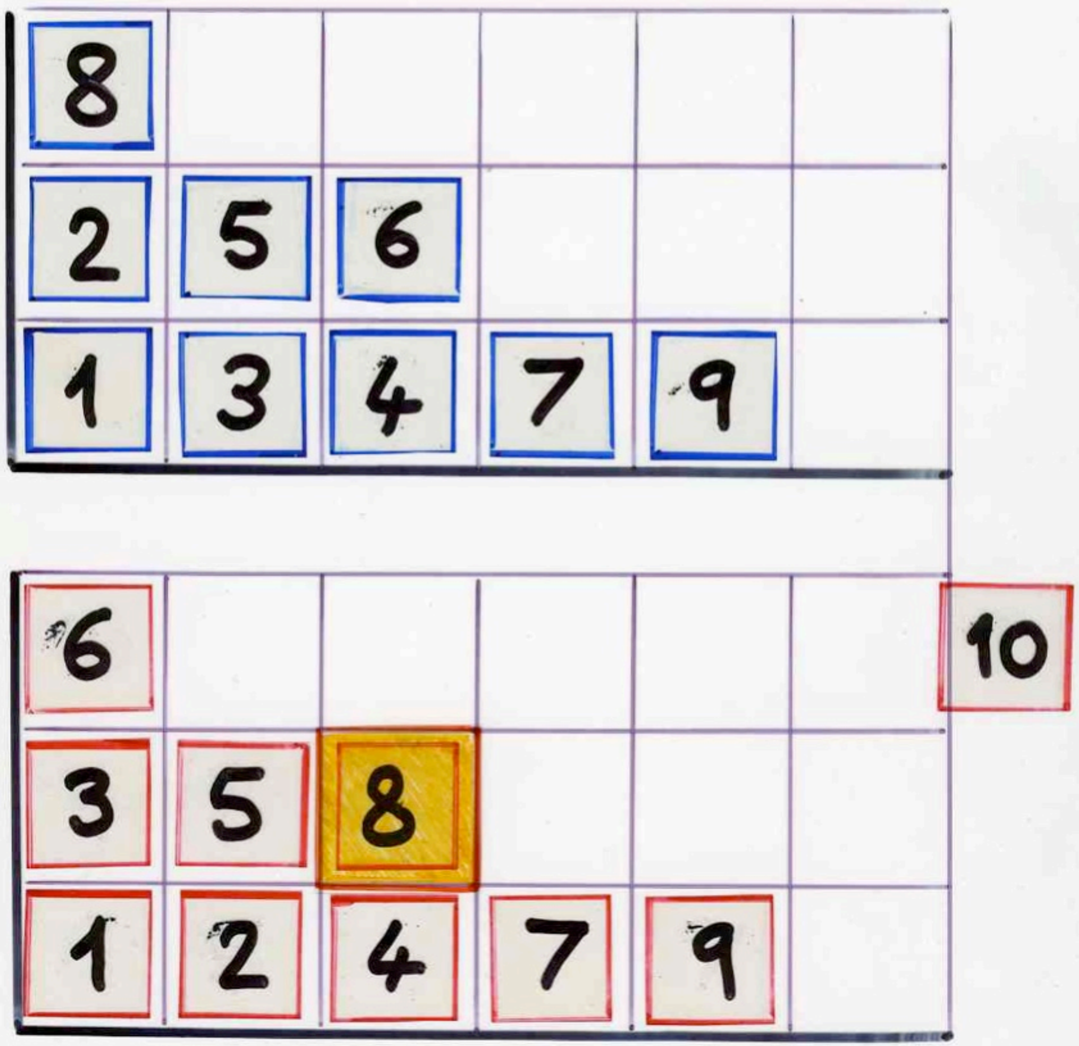
8					
2	5	6			
1	3	4	7	9	

6						
3	5	10				8
1	2	4	7	9		

1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7



1	2	3	4	5	6	7	8	9	10
3	1	6	10	2	5	8	4	9	7

8	10				
2	5	6			
1	3	4	7	9	

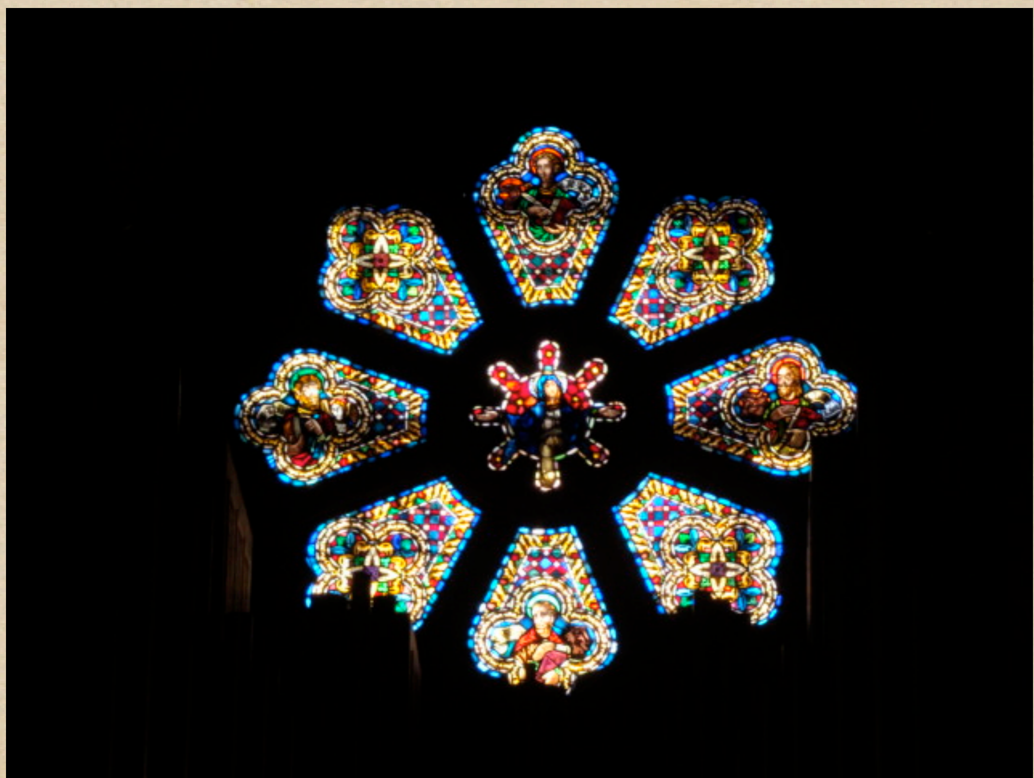
6	10				
3	5	8			
1	2	4	7	9	

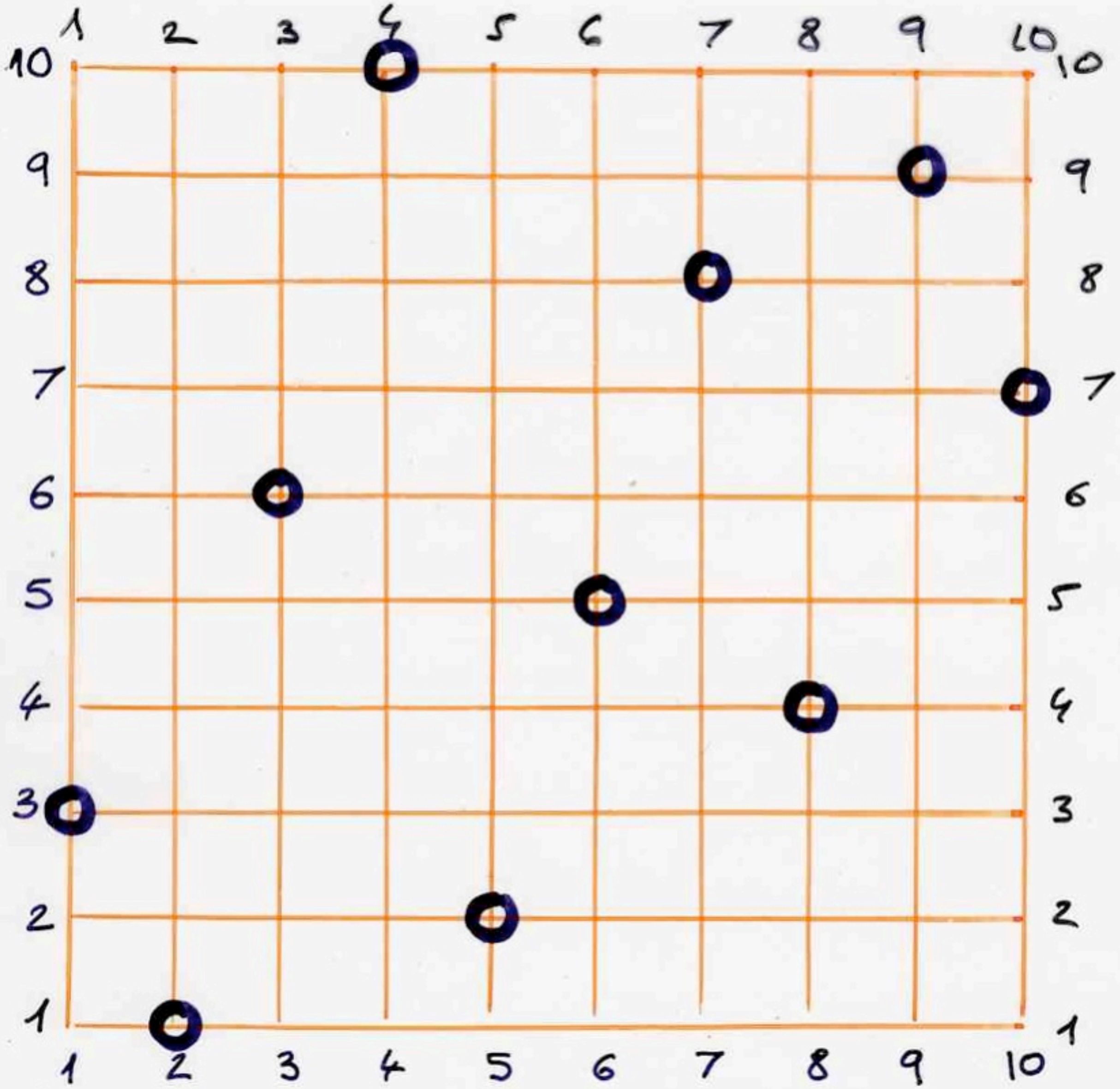
$$\sigma \longleftrightarrow (P, Q)$$

$$\sigma^{-1} \longleftrightarrow (Q, P)$$

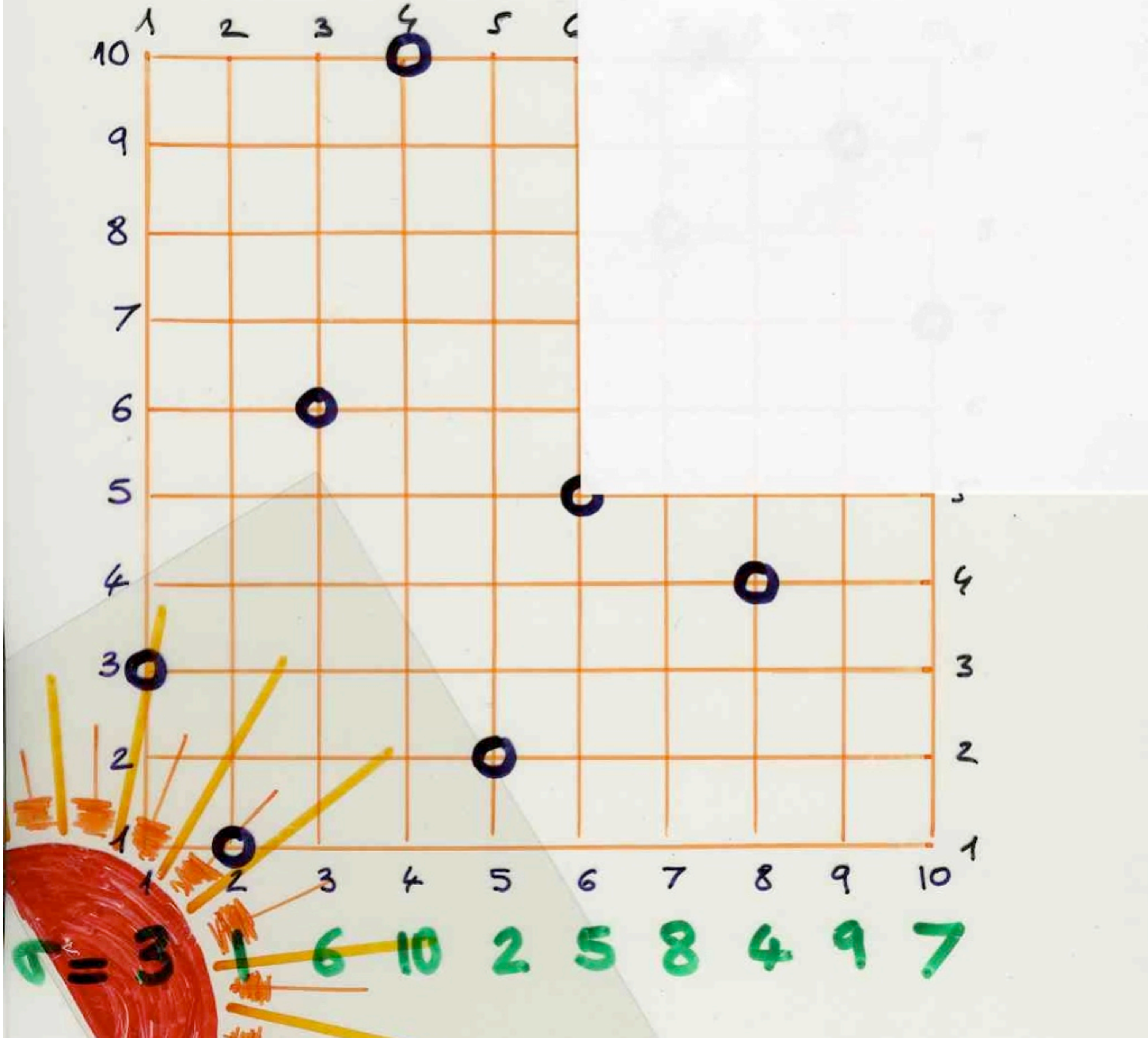
A geometric version of RSK with "light" and "shadow lines"

XGV, 1976

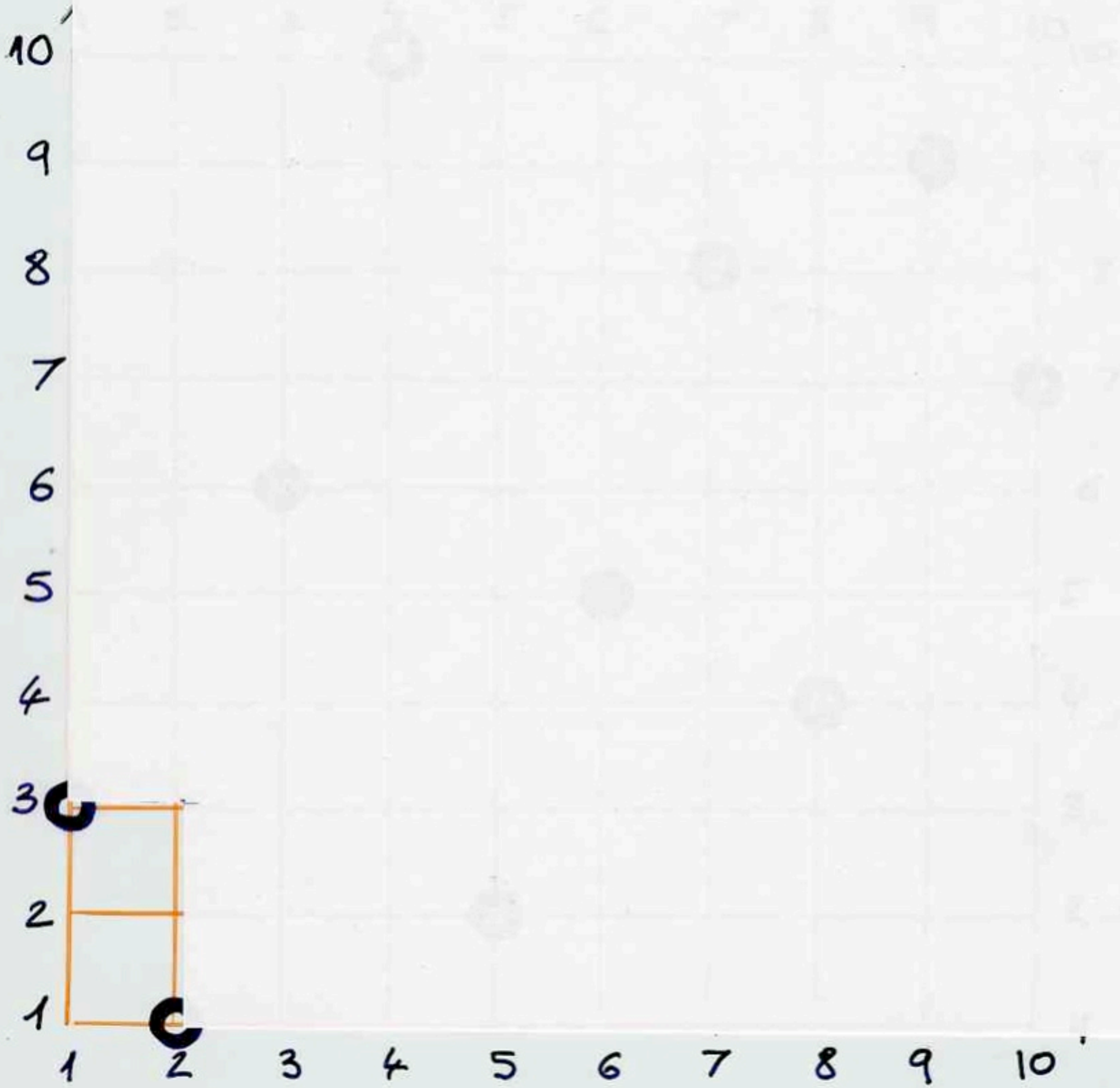




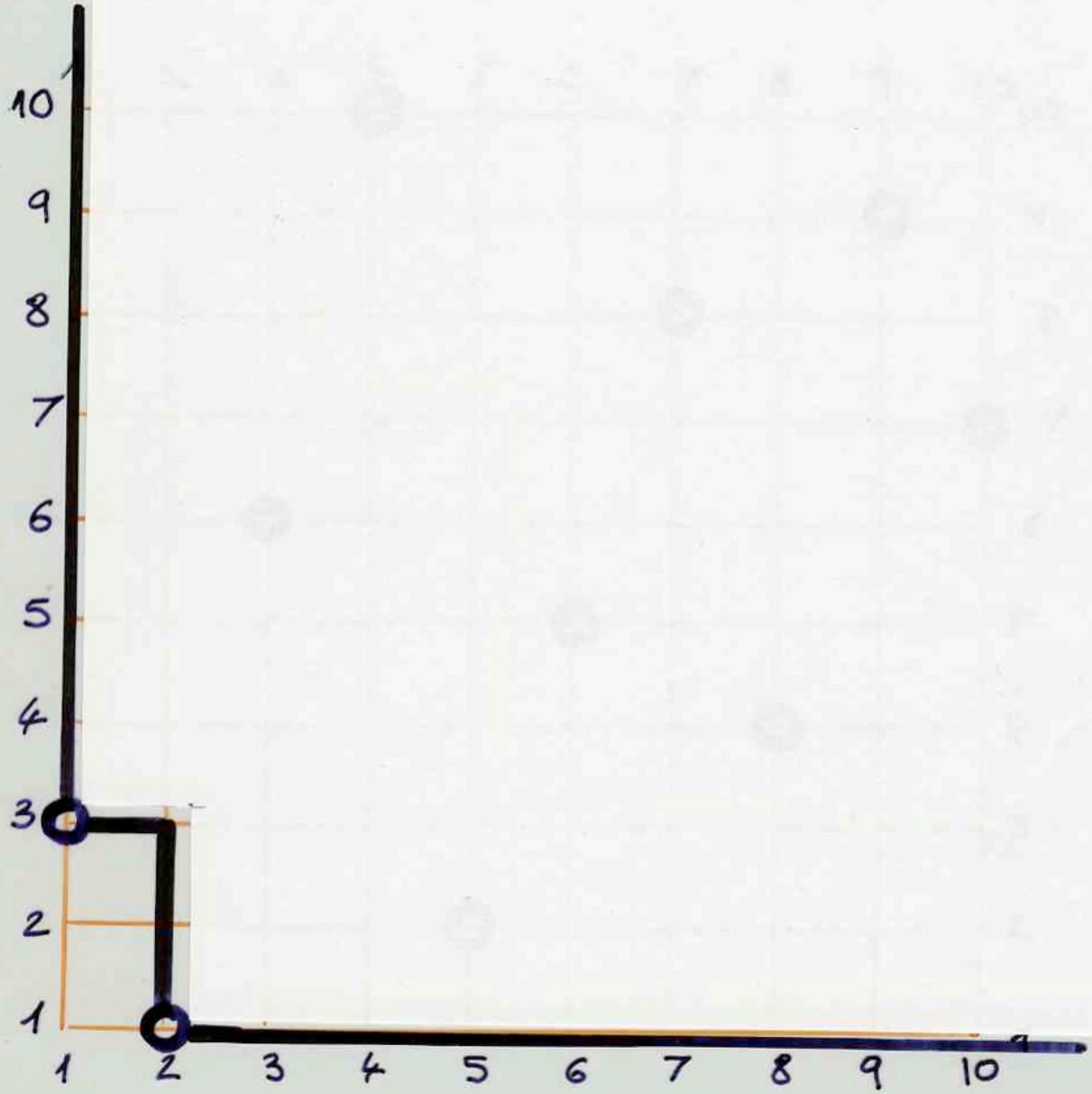
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



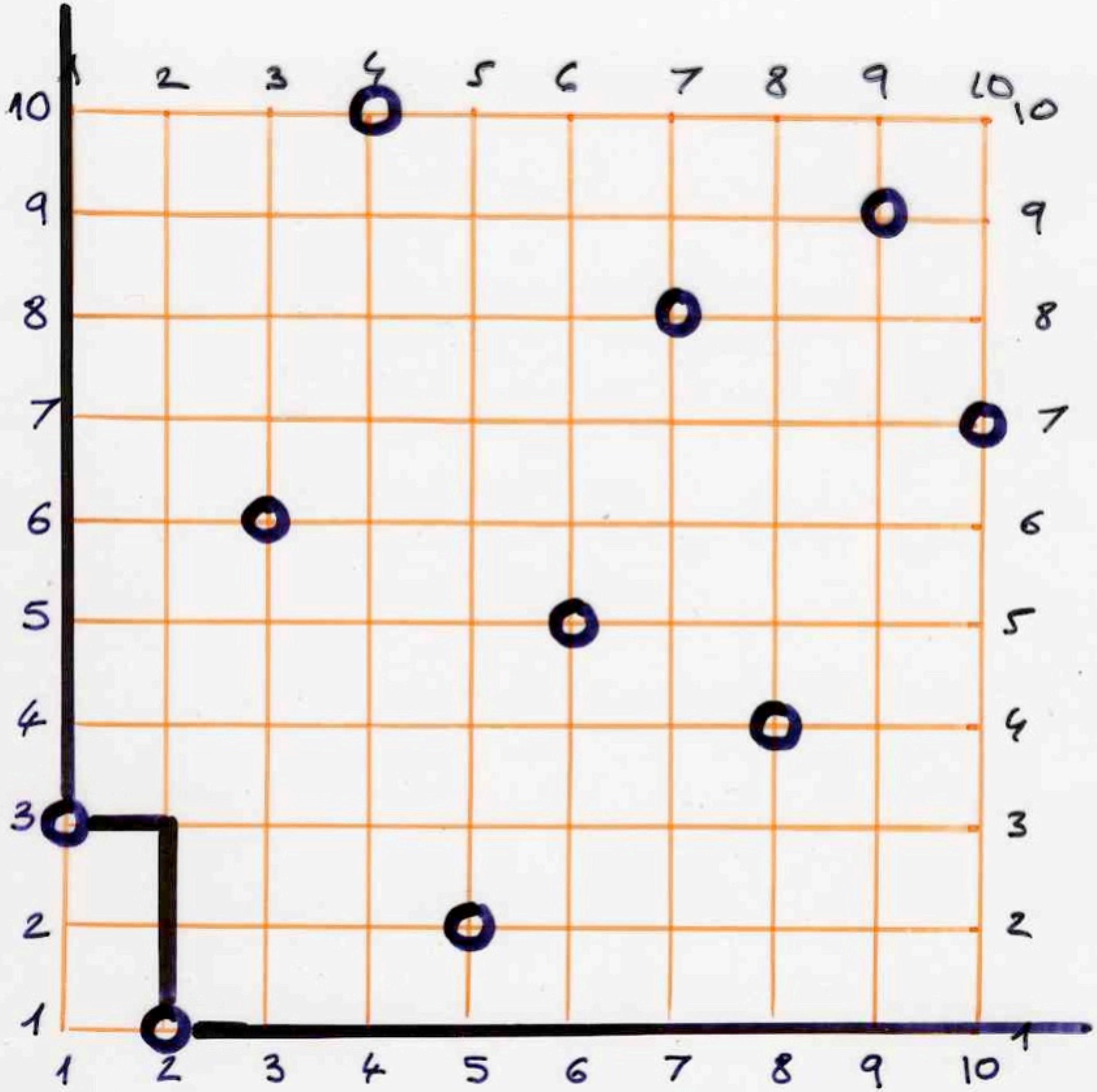
1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7



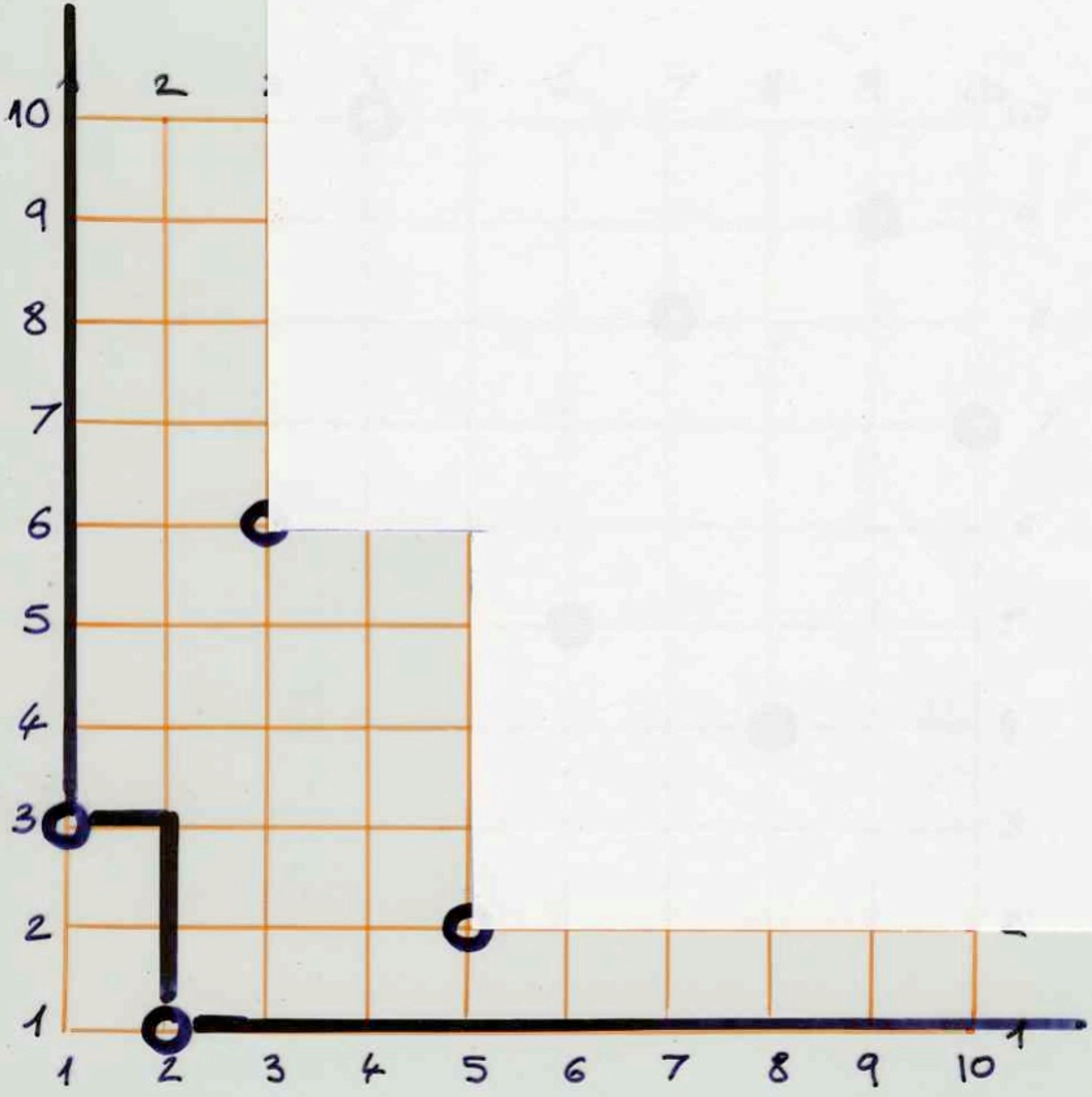
σ = 3 1 6 10 2 5 8 4 9 7



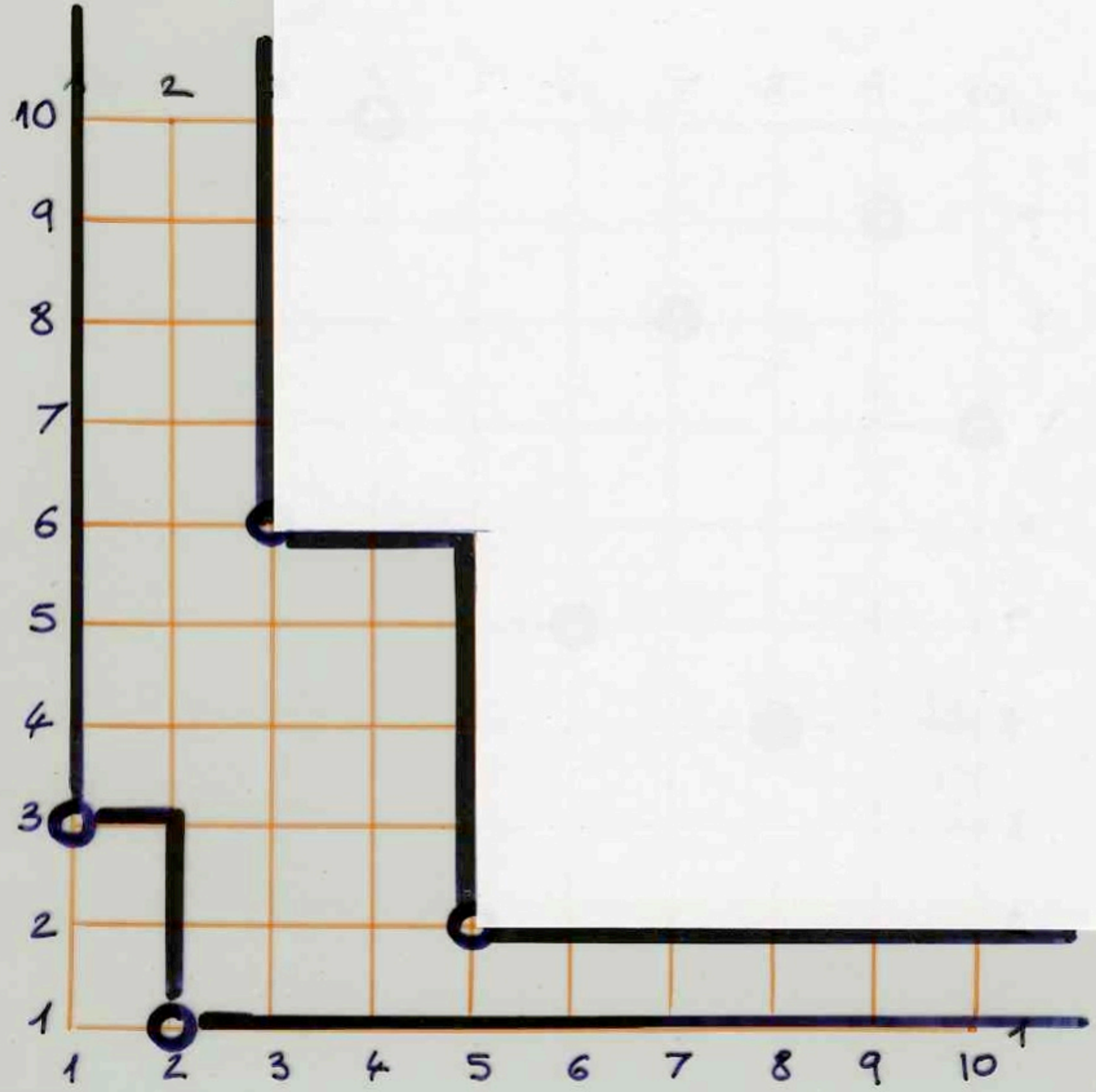
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



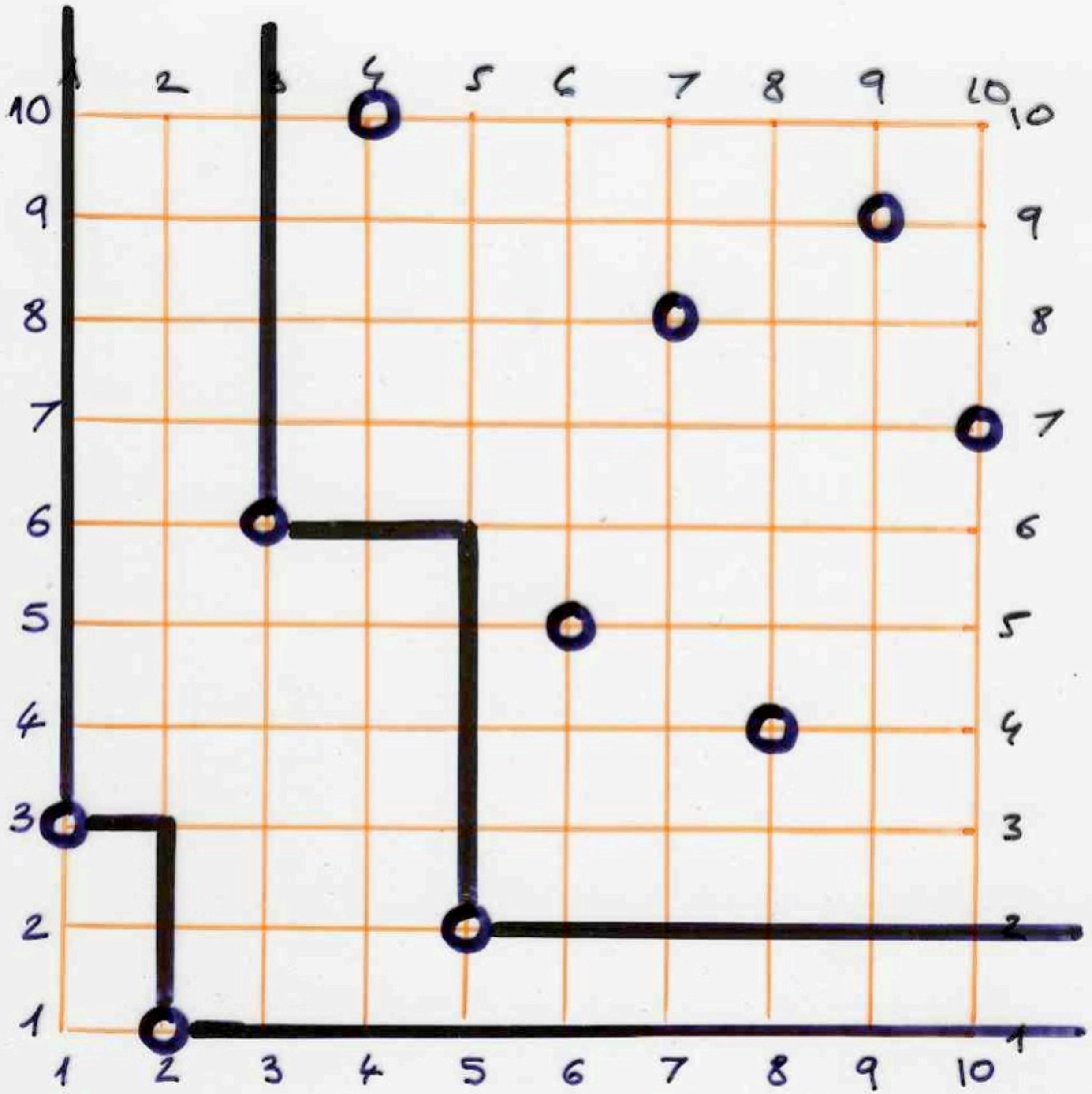
σ = 3 1 6 10 2 5 8 4 9 7



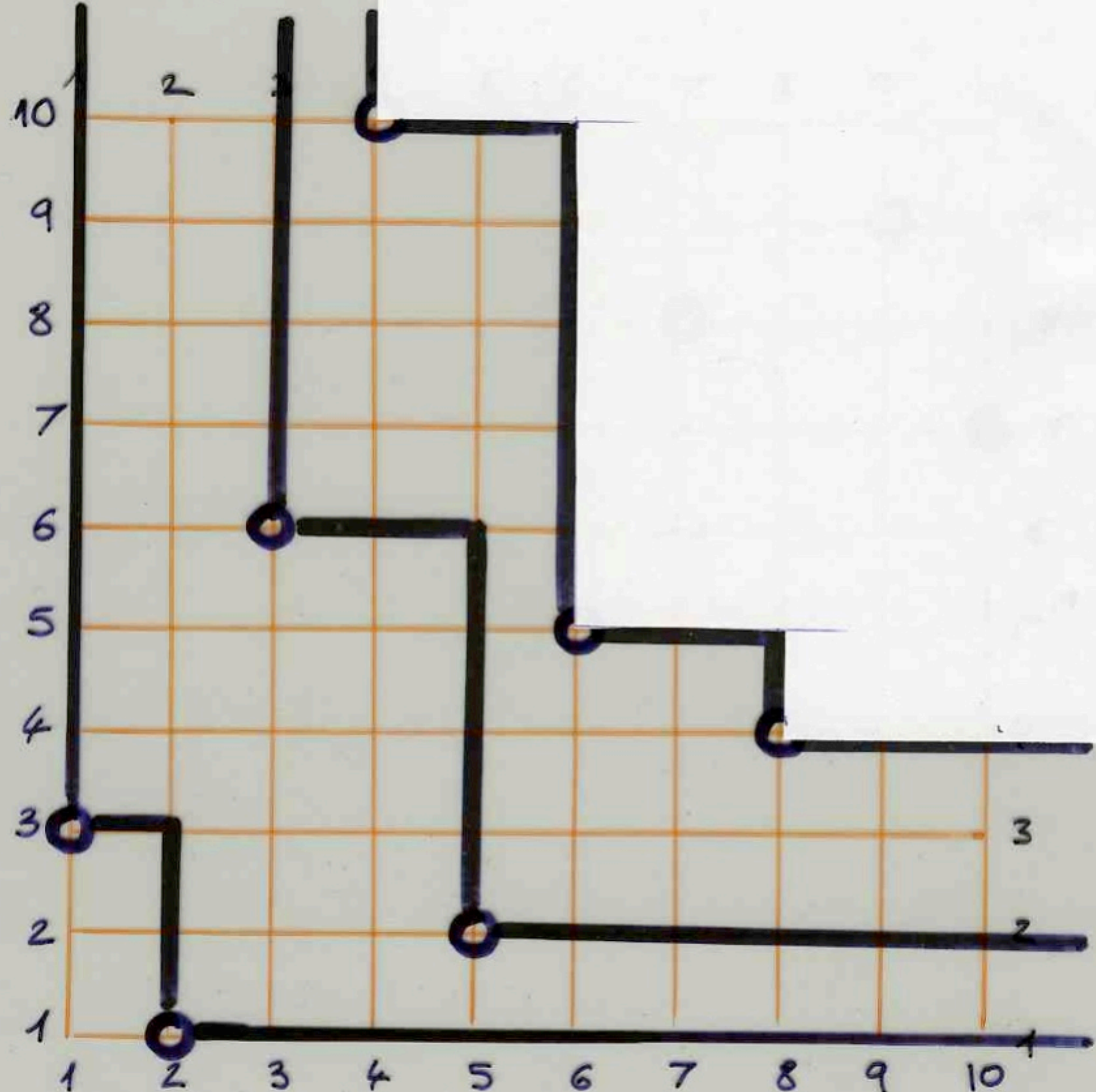
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



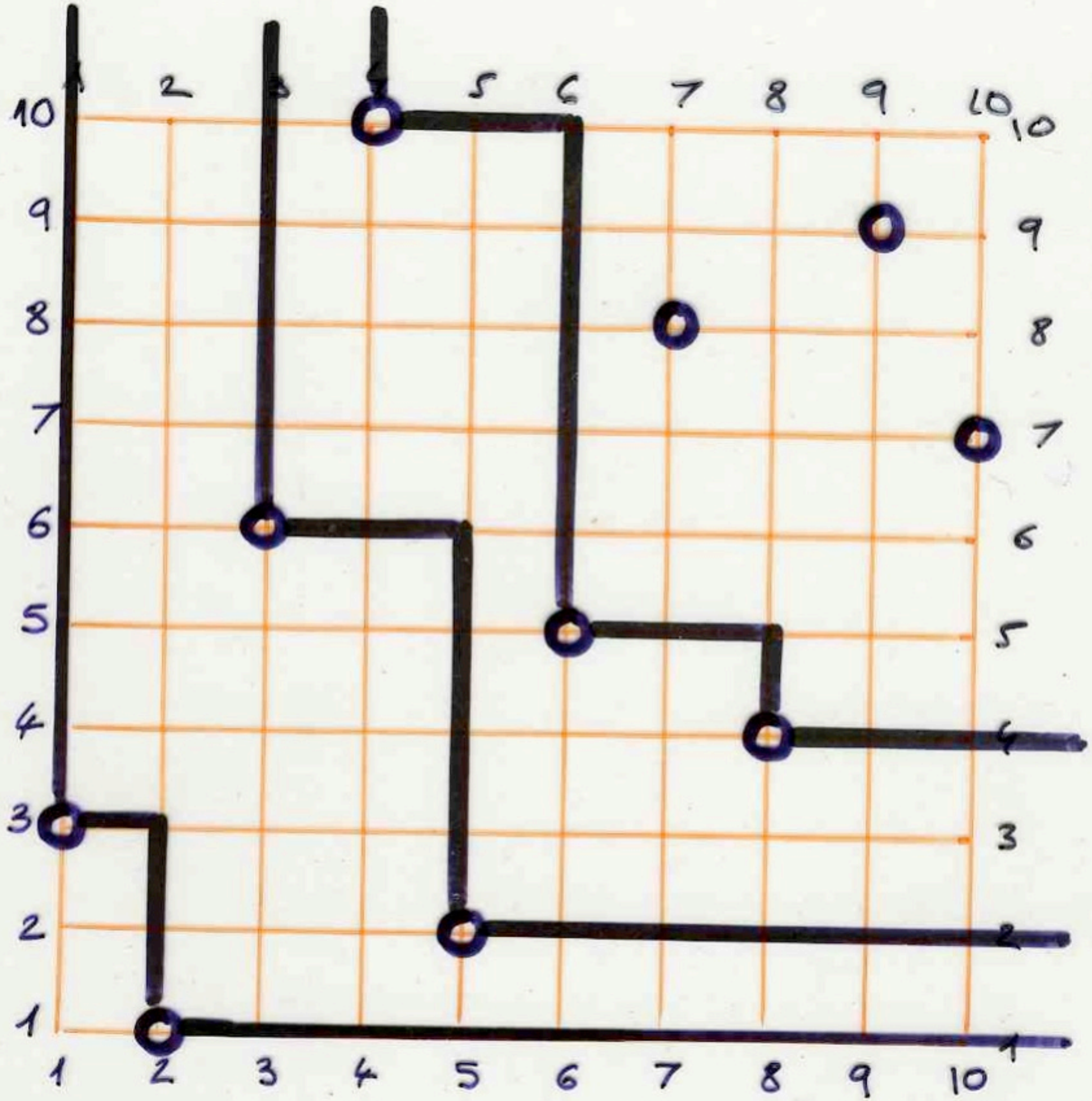
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



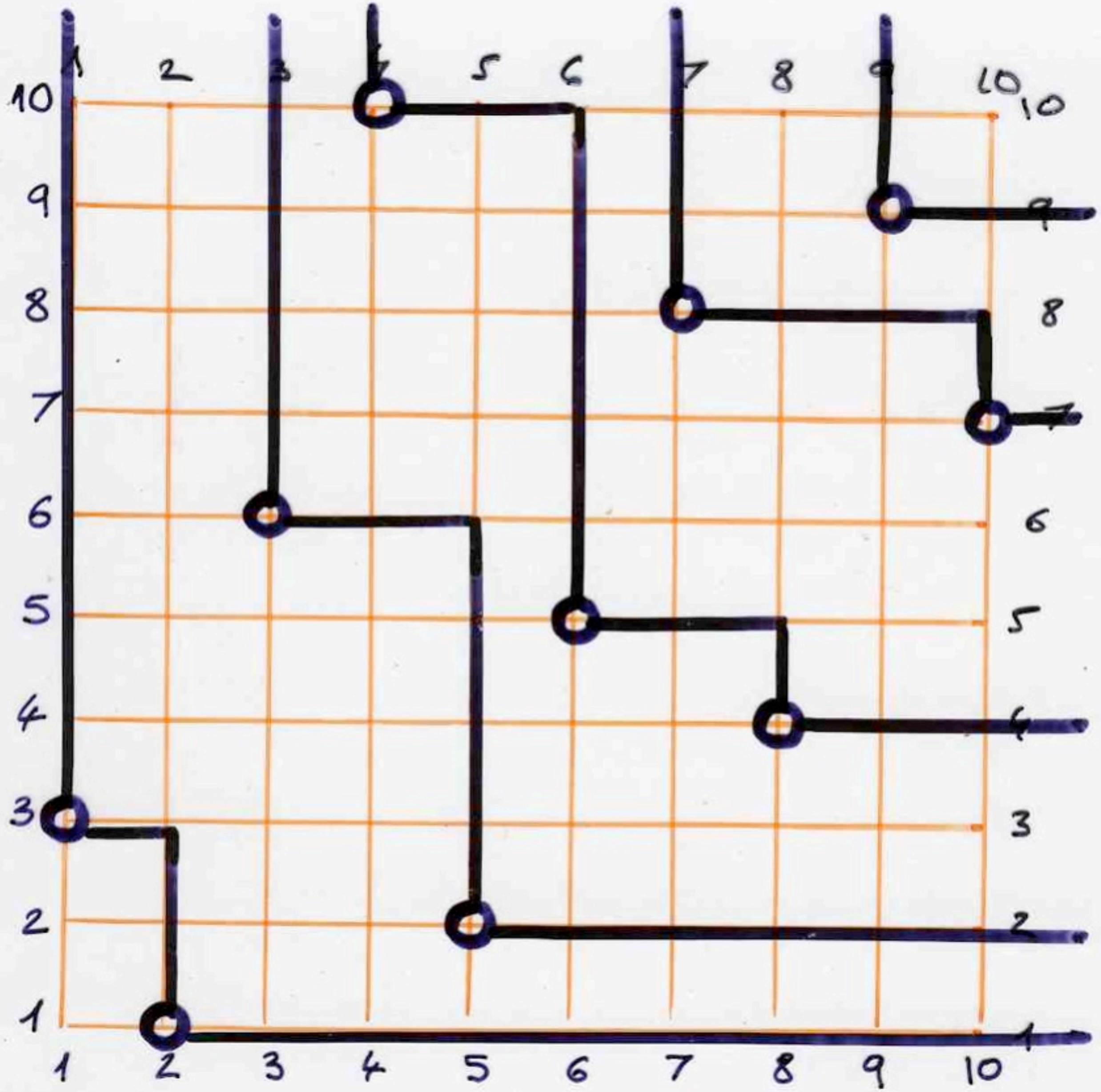
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



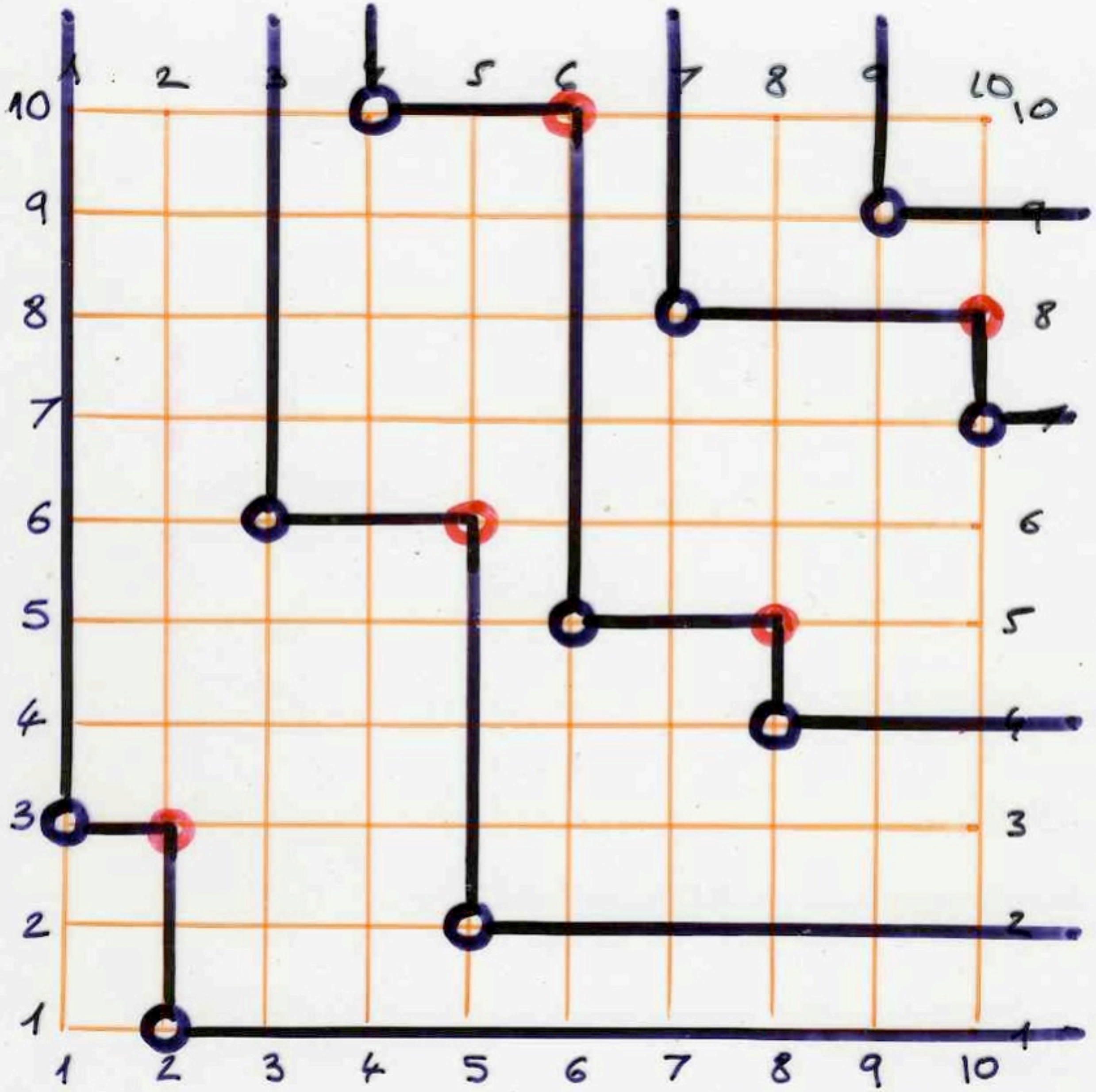
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



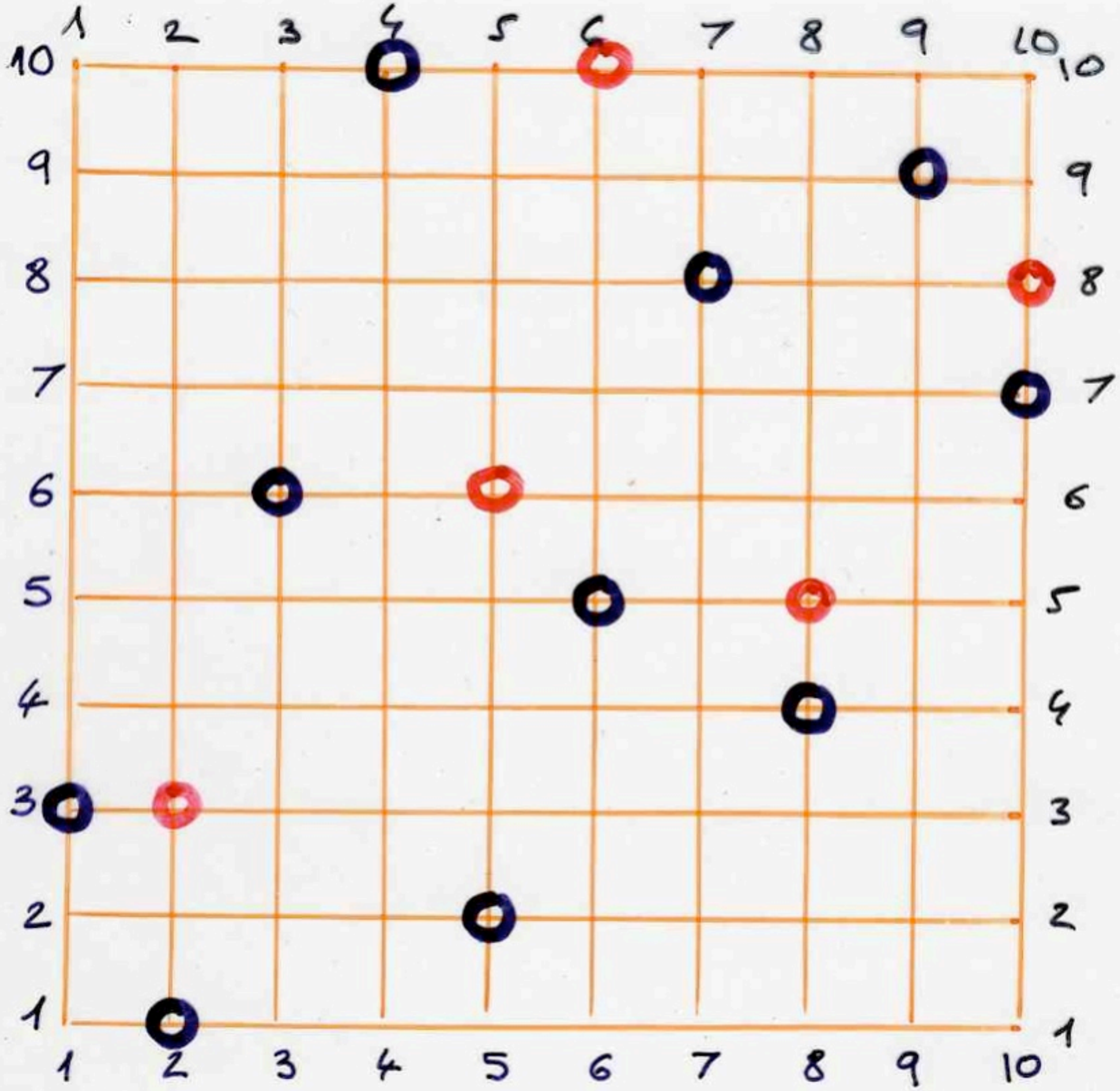
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



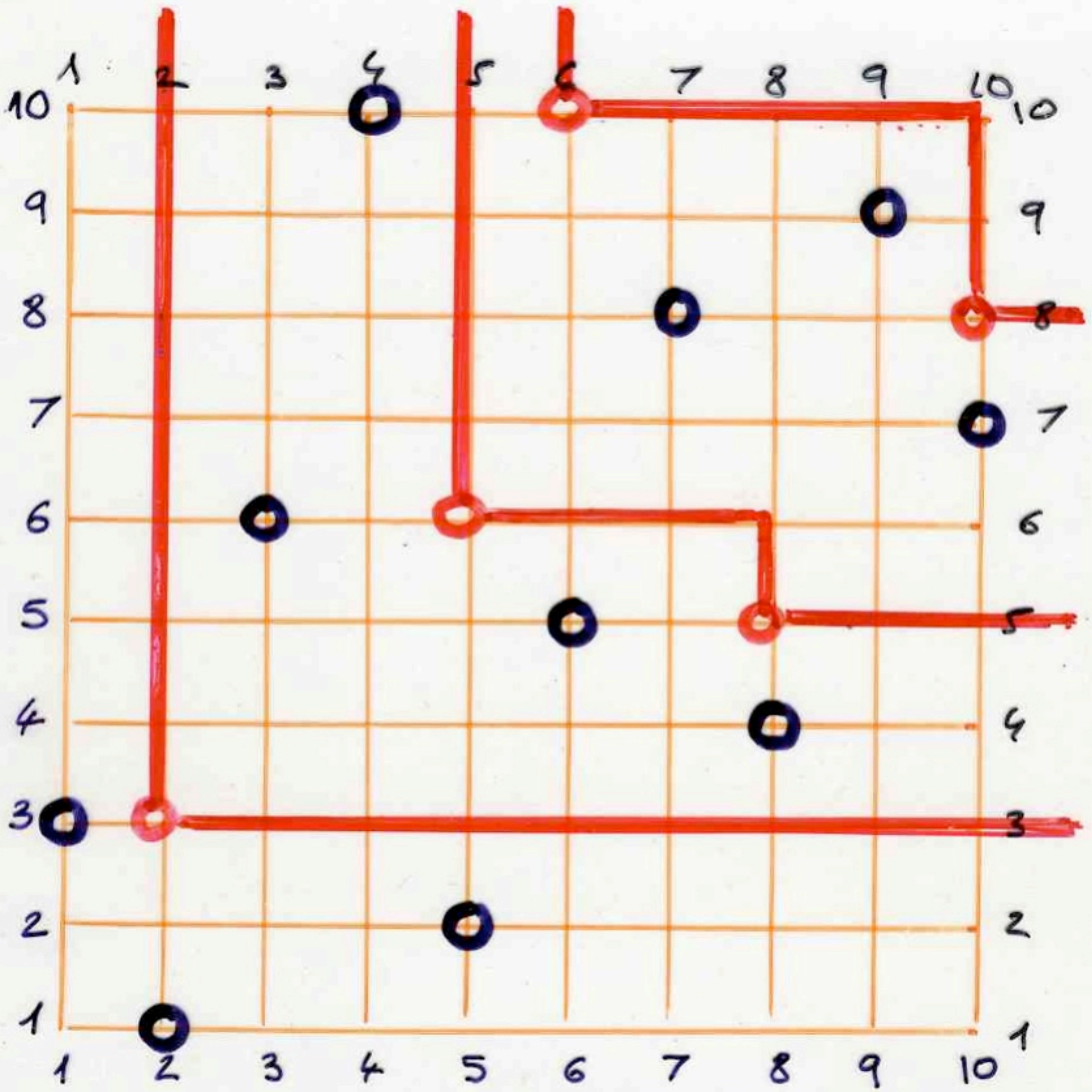
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



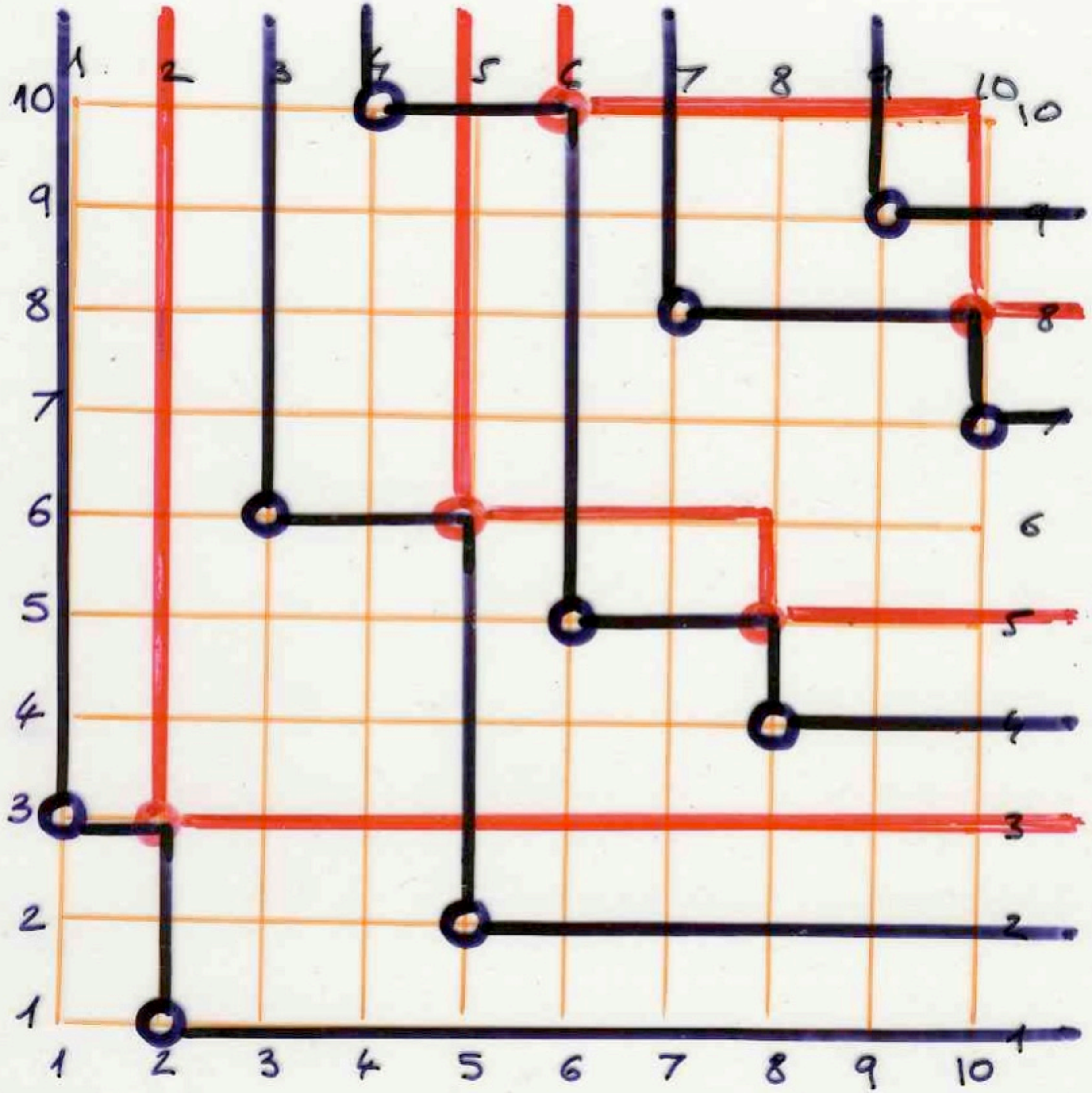
σ = 3 1 6 10 2 5 8 4 9 7



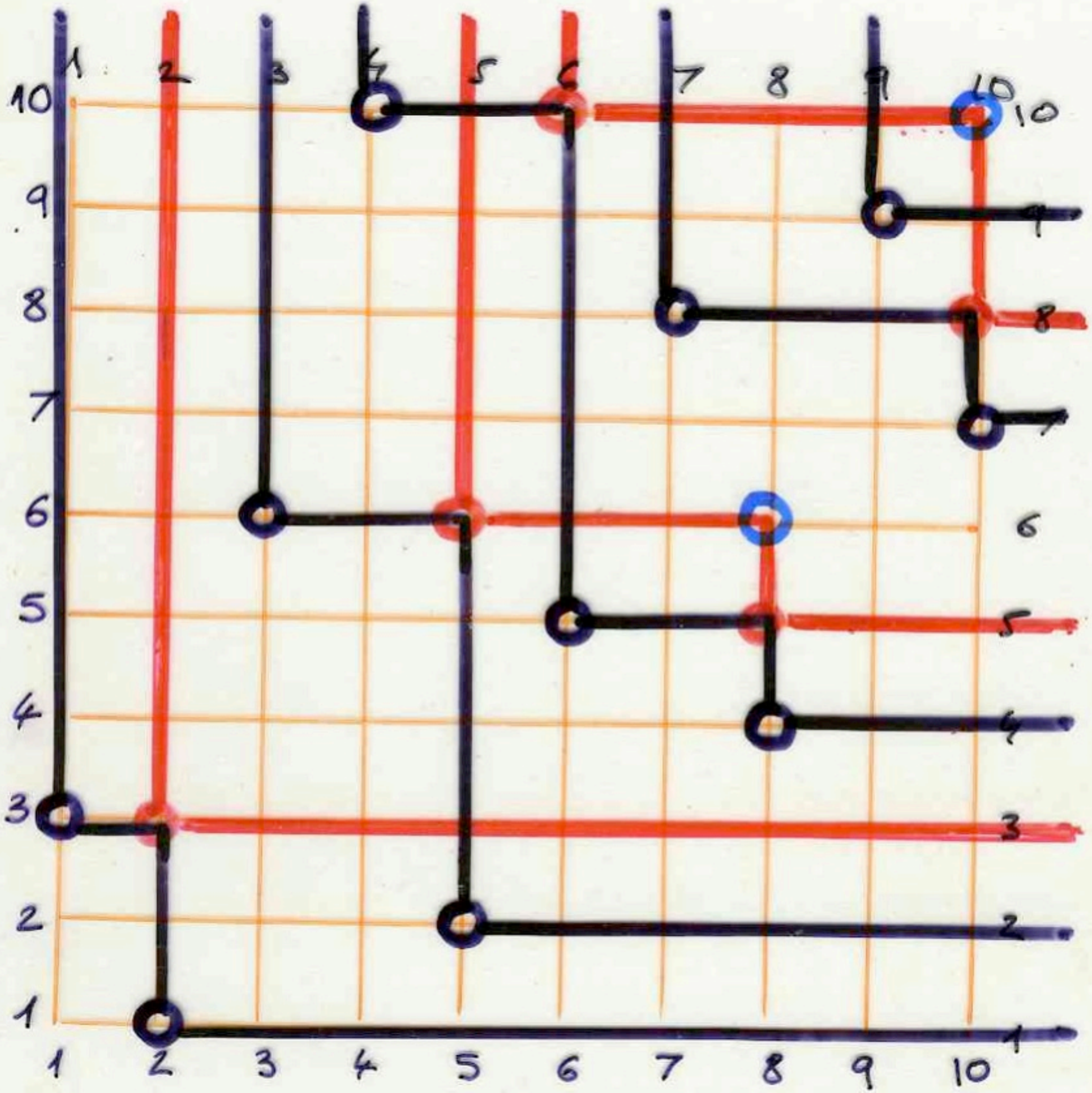
$\sigma = 3 \quad 1 \quad 6 \quad 10 \quad 2 \quad 5 \quad 8 \quad 4 \quad 9 \quad 7$



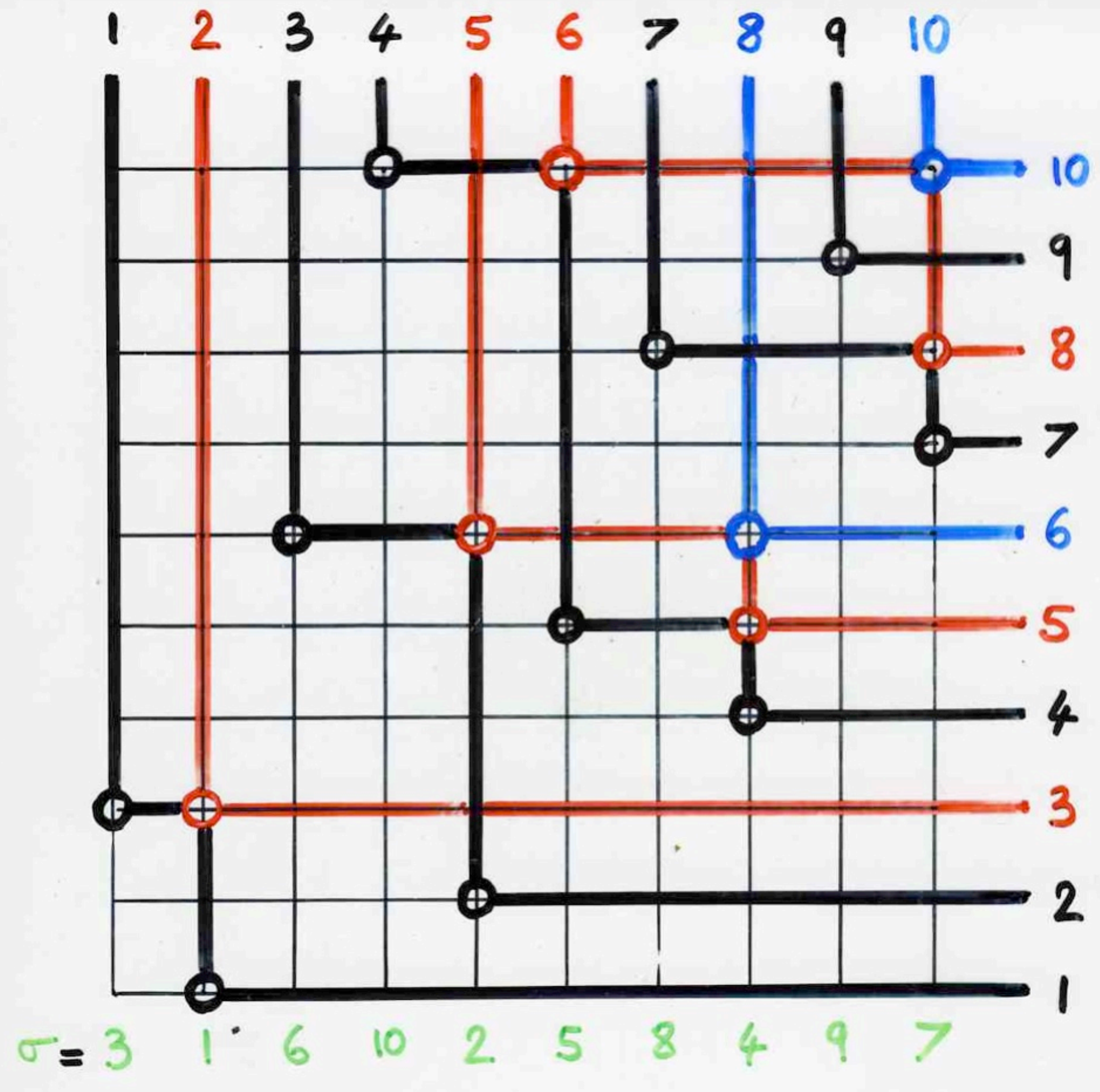
$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$



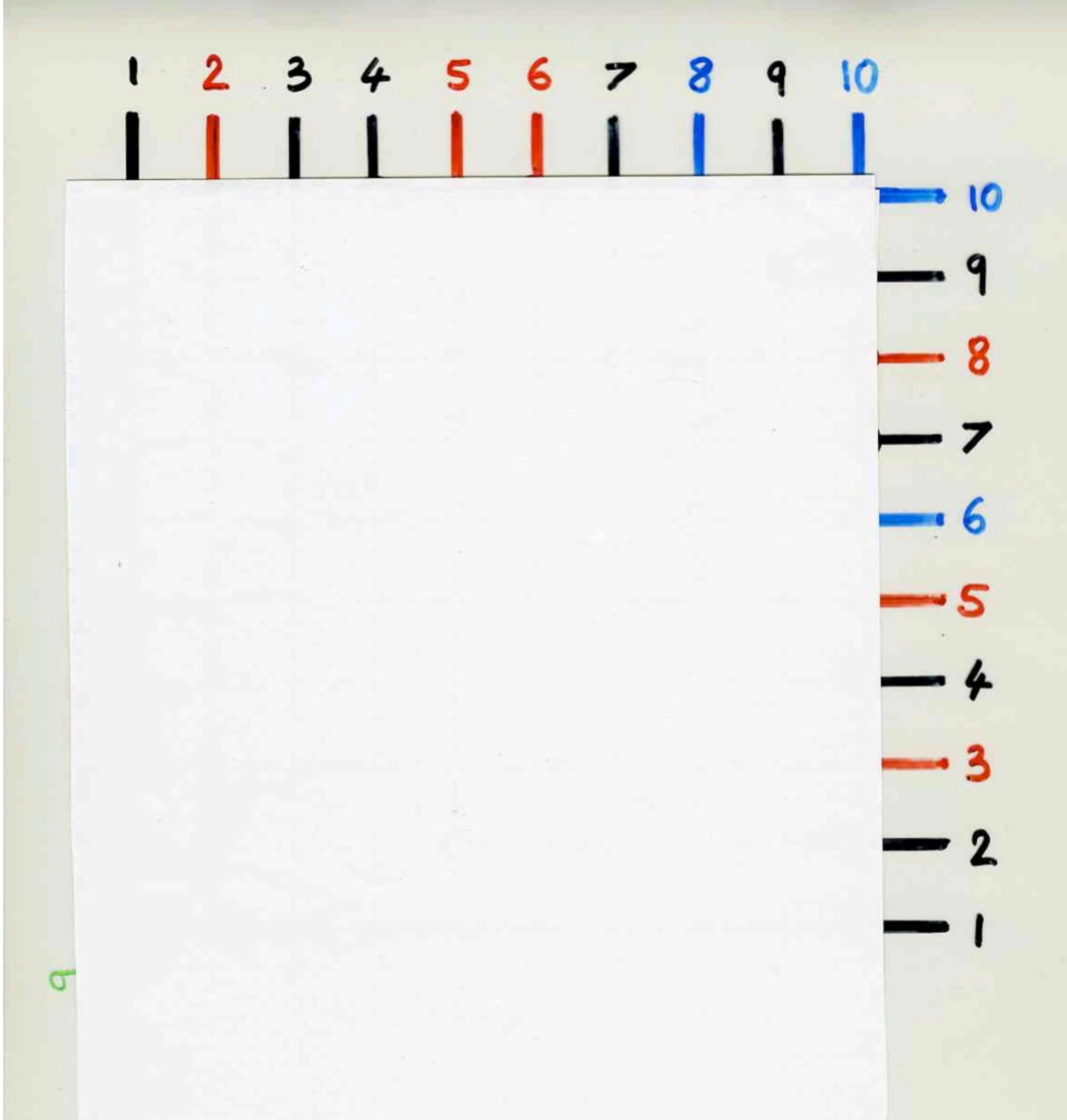
σ = 3 1 6 10 2 5 8 4 9 7



σ = 3 1 6 10 2 5 8 4 9 7



1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7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10
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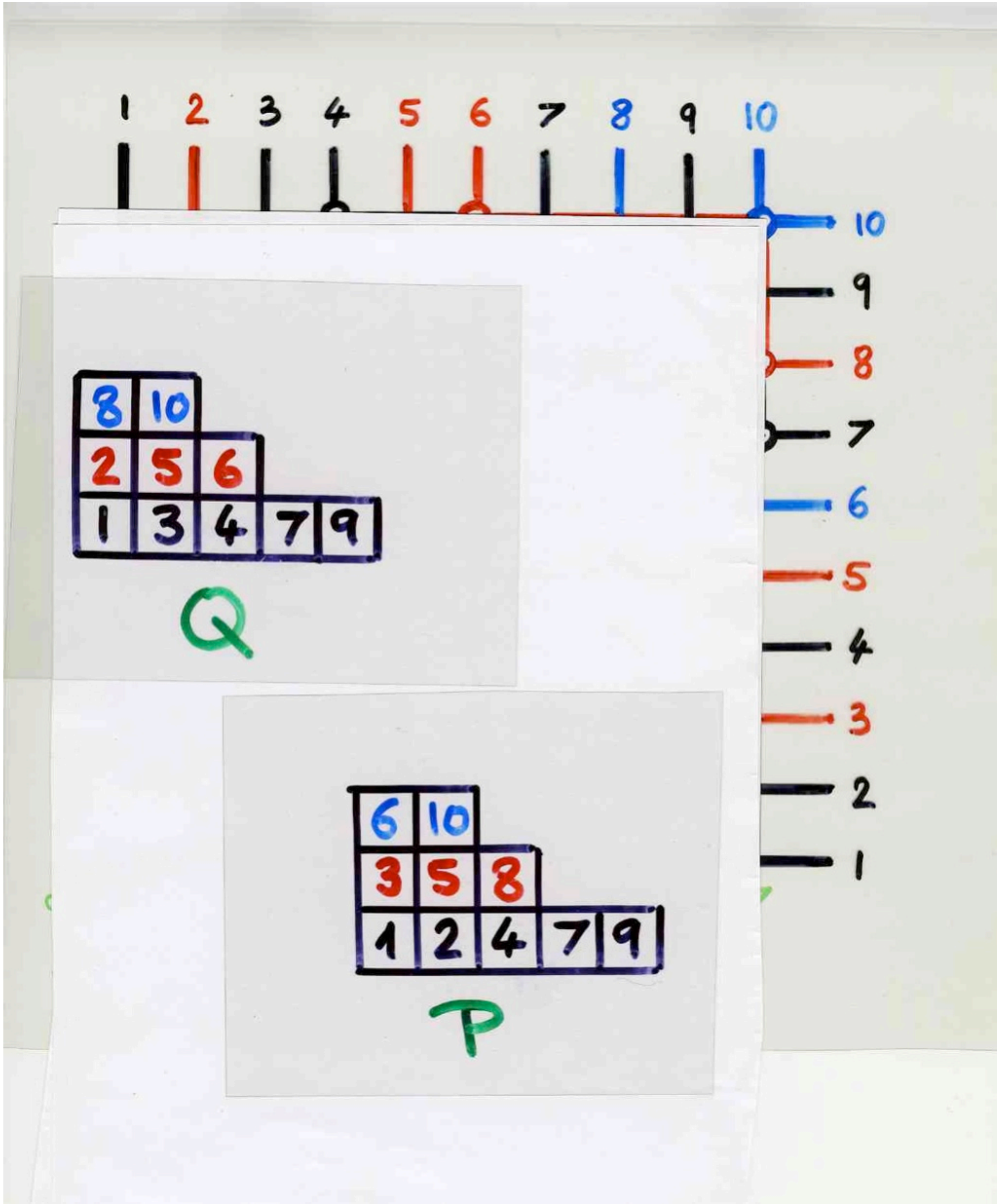
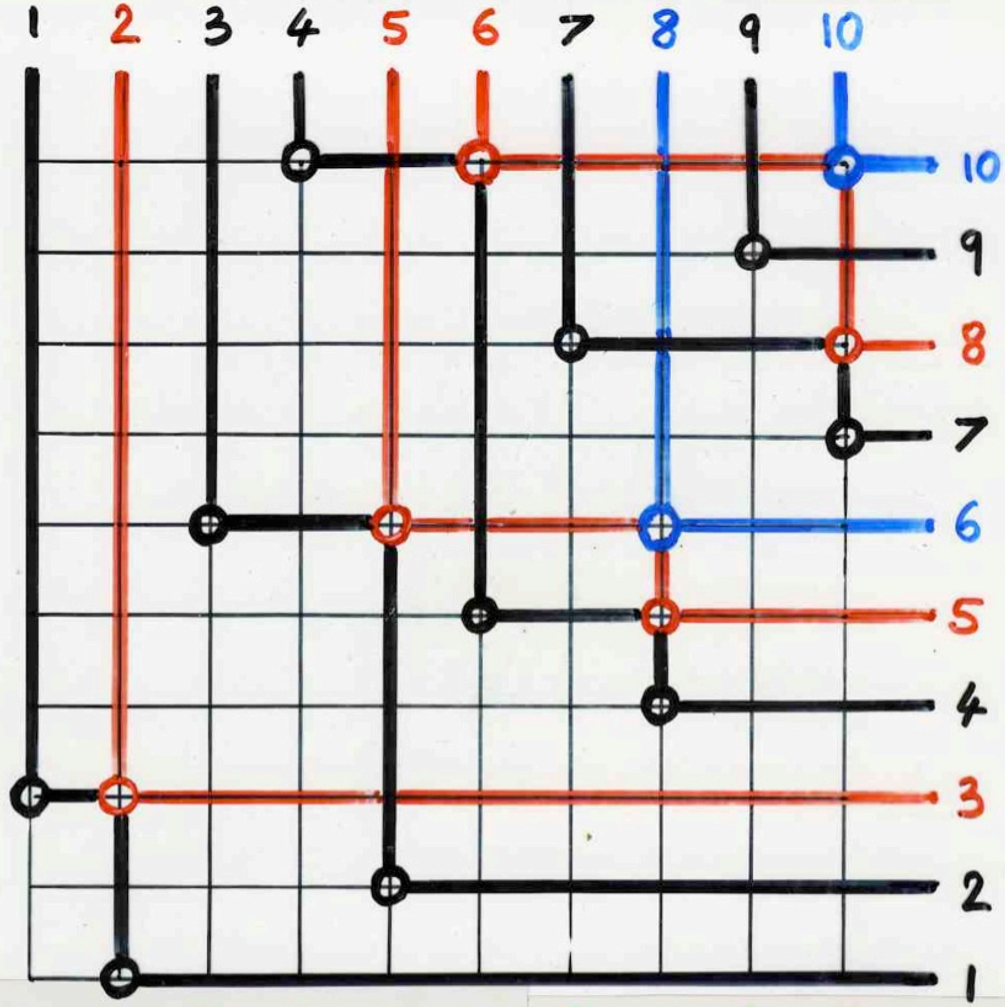
4
3
2
1


1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
8 10
2 5 6
1 3 4 7 9
Q
6 10
3 5 8
1 2 4 7 9
P



σ = 3 1 6 10 2 5 8 4 9 7

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

$$\sigma \longleftrightarrow (P, Q)$$

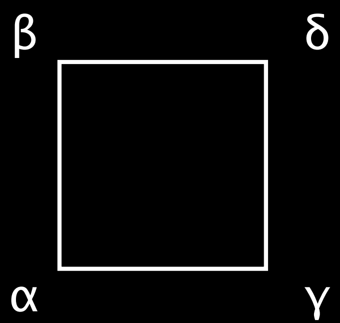
$$\sigma^{-1} \longleftrightarrow (Q, P)$$

"local" algorithm on a grid or "growth diagrams"

S. Fomin, 1986, 1994

M. van Leeuwen, 1996

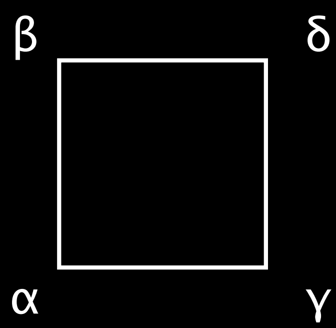
$\beta \neq \gamma$



$\delta = \beta \cup \gamma$

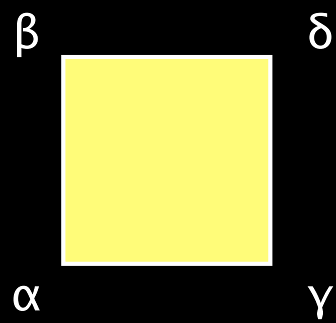
$\beta = \gamma$

$\beta = \gamma$
$\alpha \neq \beta$



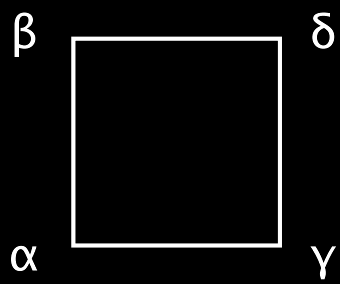
$\beta = \gamma = \alpha + (i)$
$\delta = \beta + (i+1)$

$\alpha = \beta = \gamma$

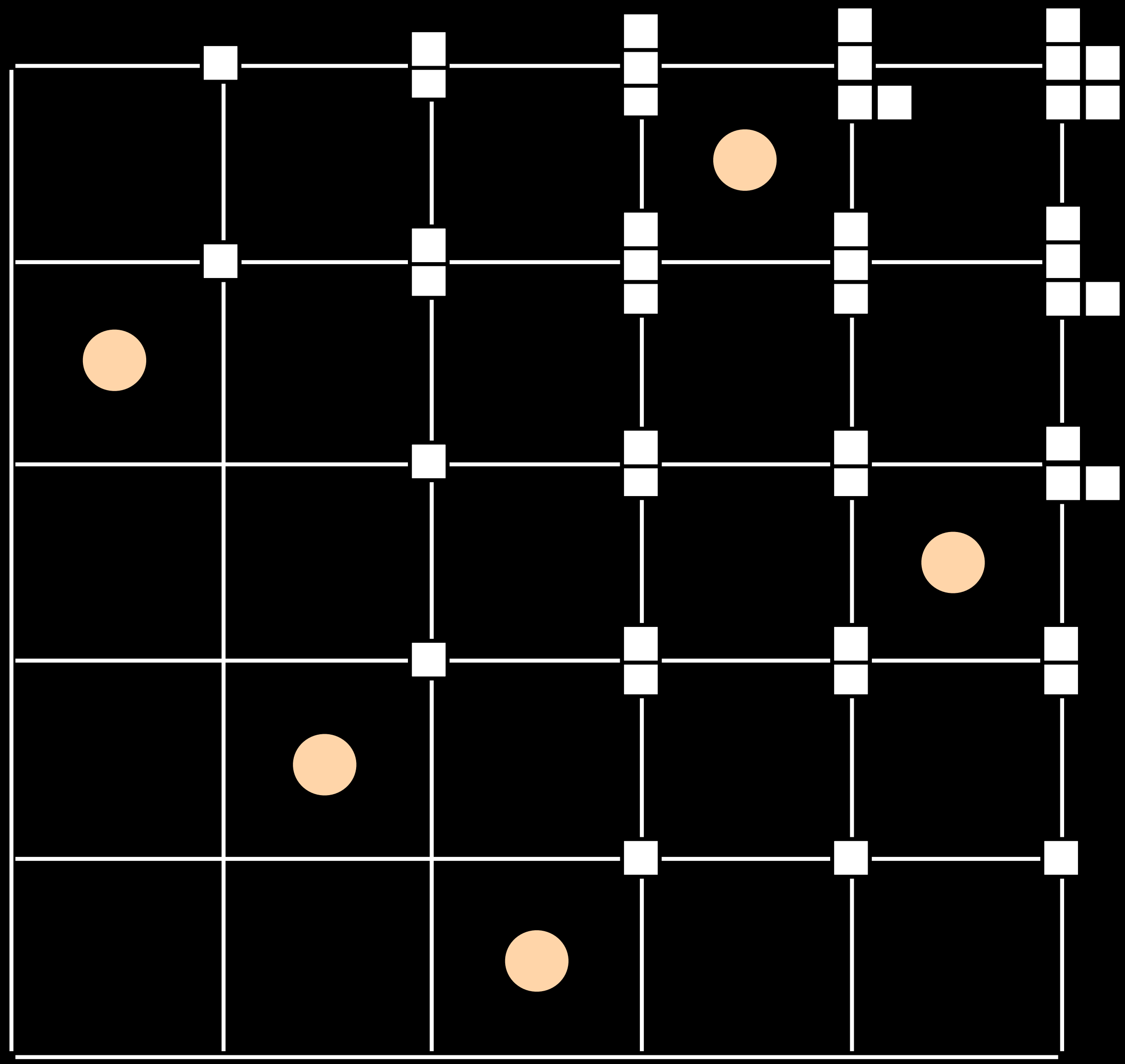


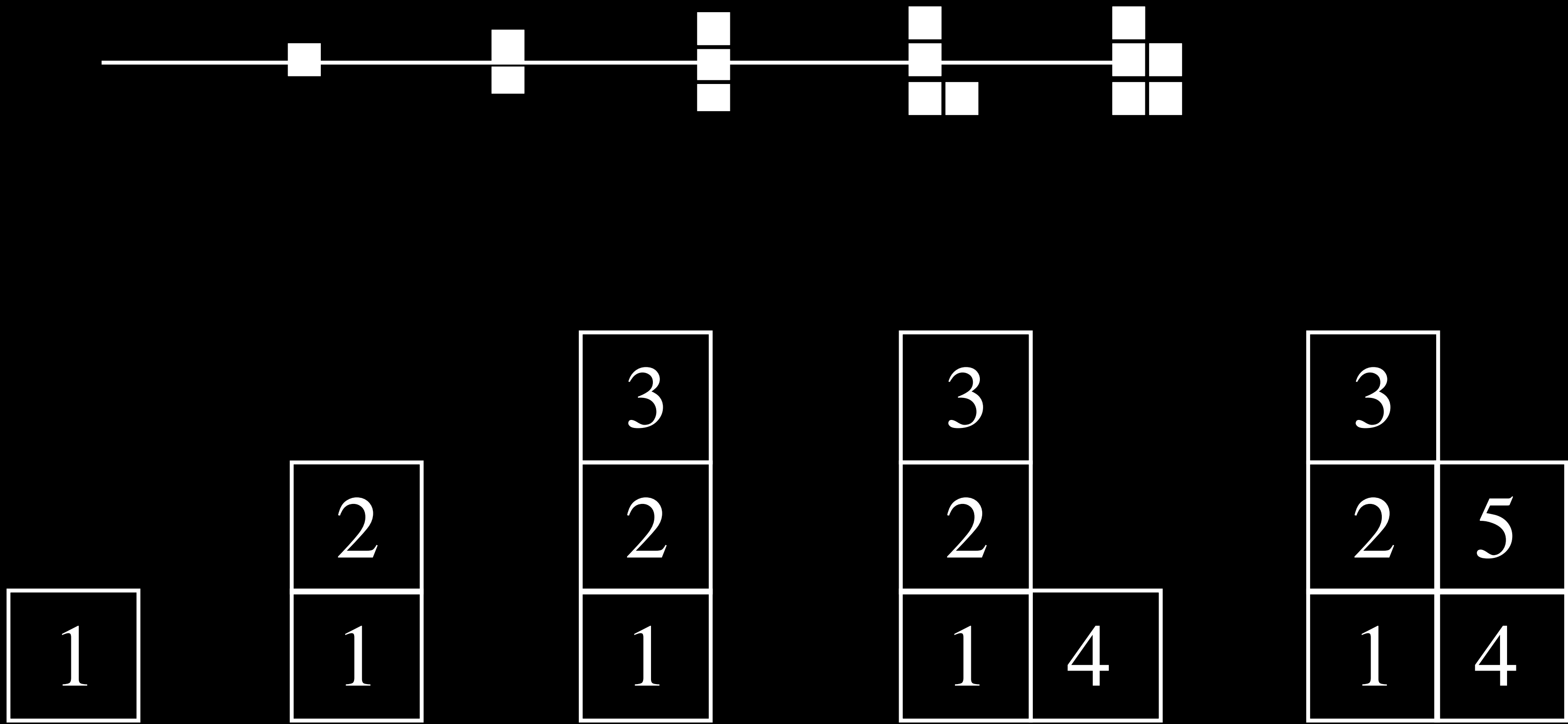
$\delta = \alpha + (1)$

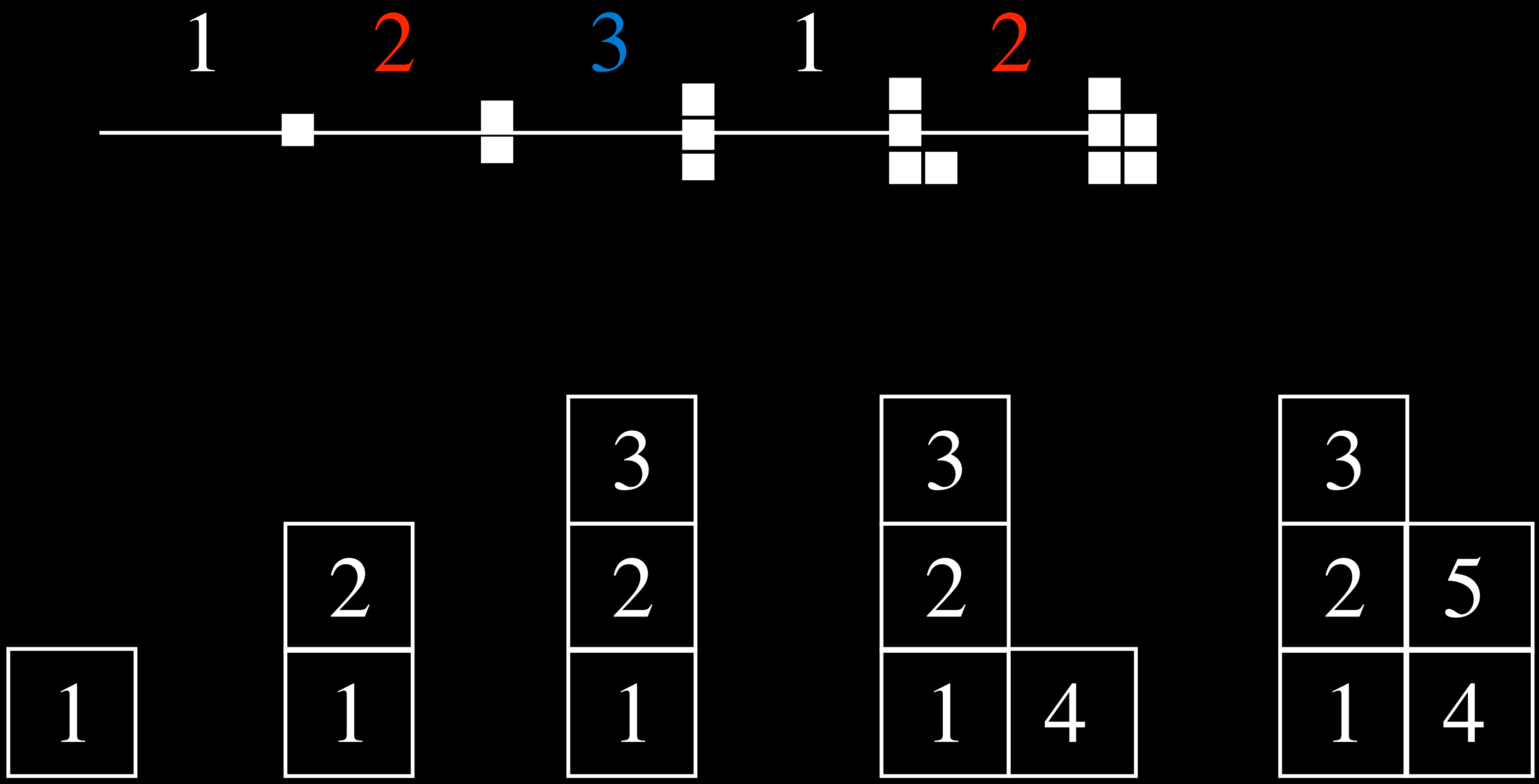
$\alpha = \beta = \gamma$



$\delta = \alpha = \beta = \gamma$

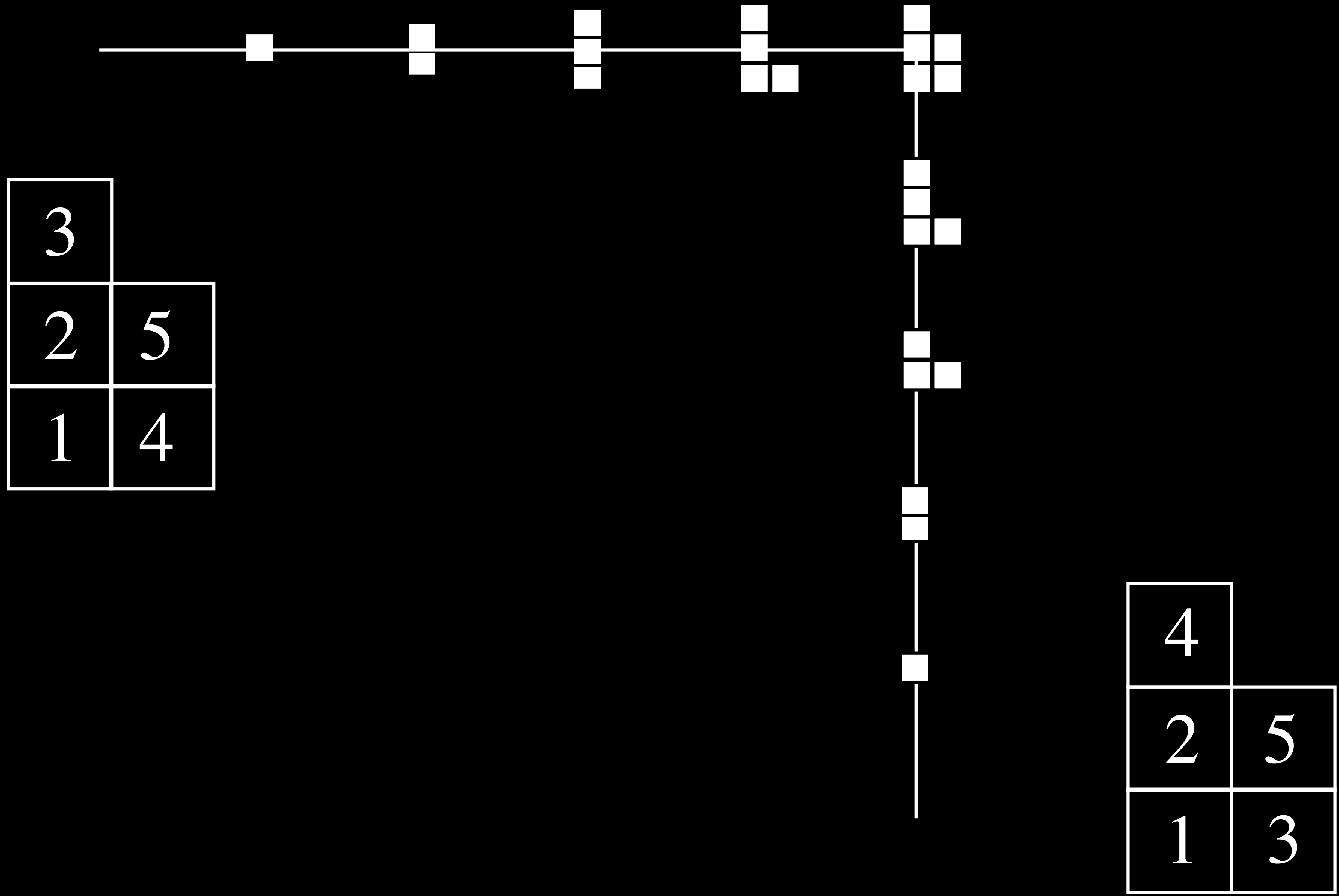


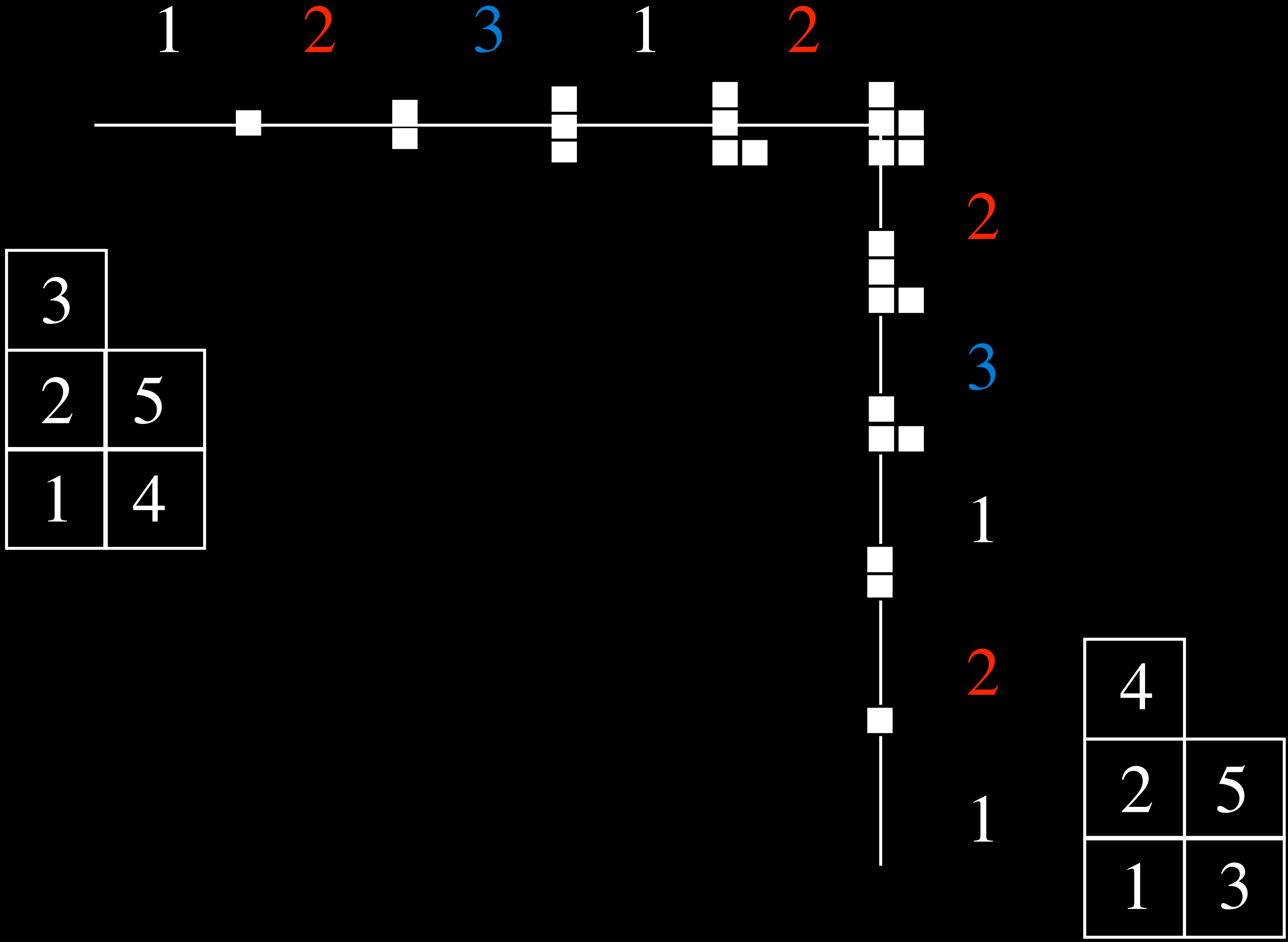

1
2
1
3
2
1
3
2
1 4
3
2 5
1 4



w = 1 2 3 1 2

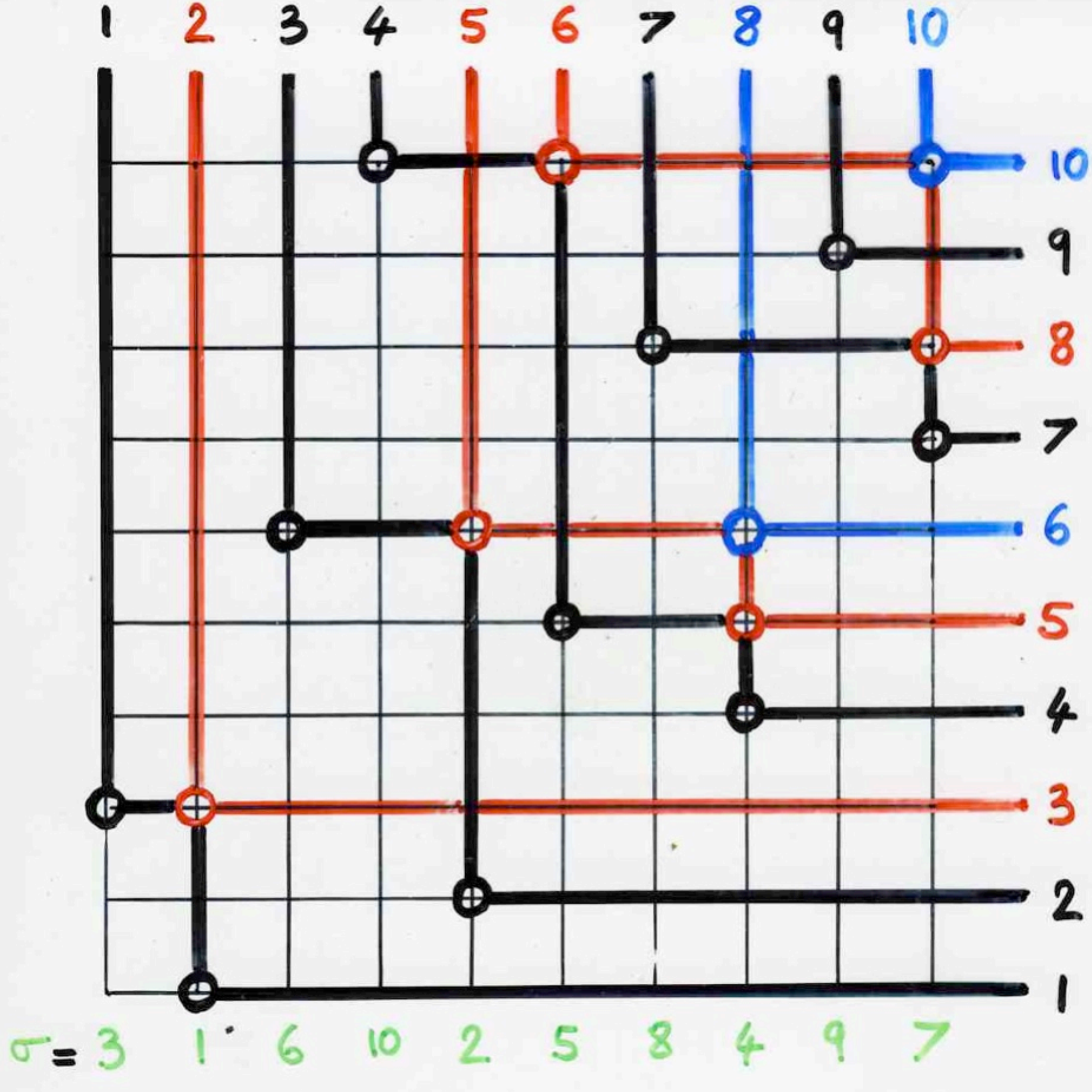
Yamanuchi word

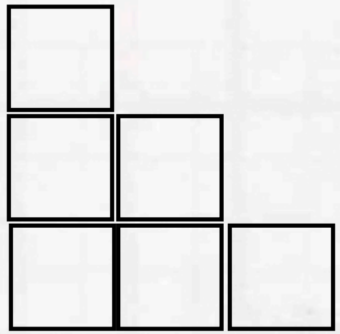

3
2 5
1 4
4
2 5
1 3

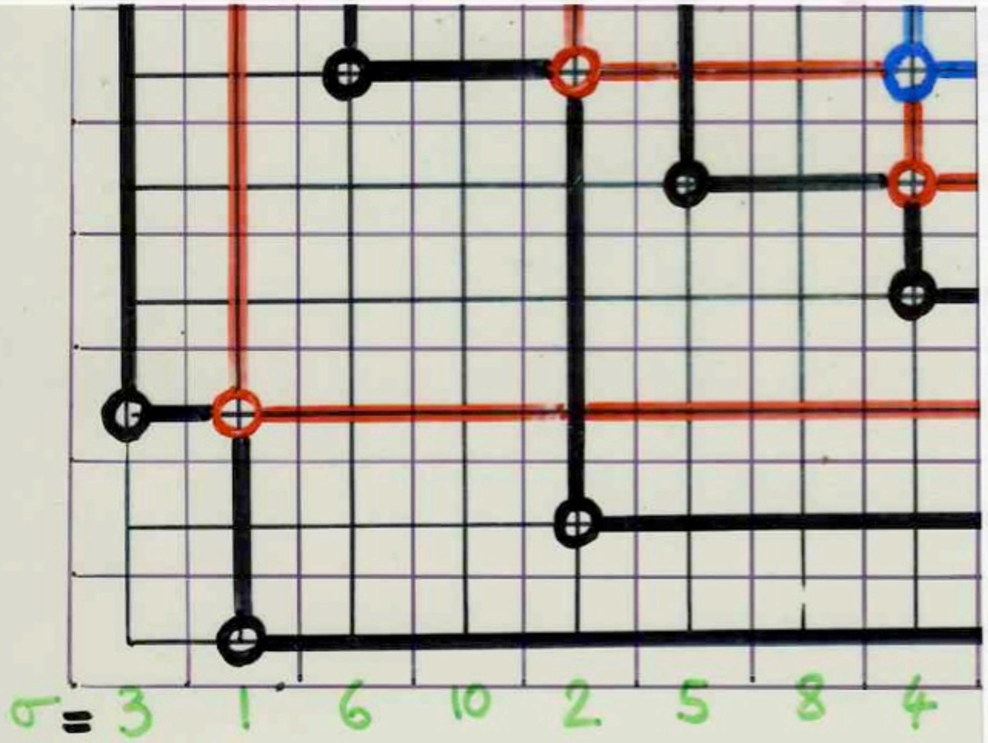

1 2 3 1 2
2
3
1
2
1
3
2 5
1 4
4
2 5
1 3

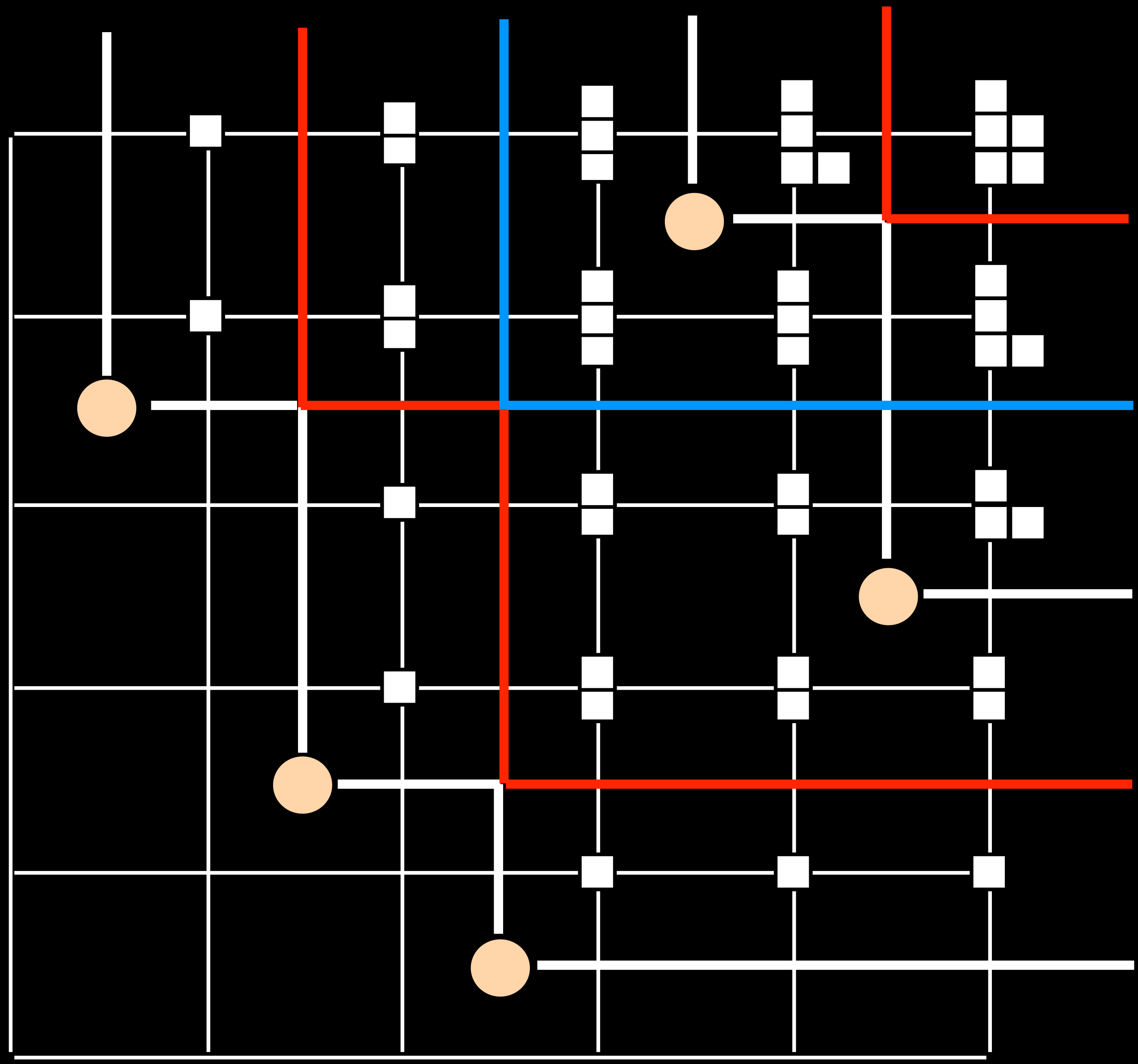
equivalence
local RSK and geometric RSK

(the geometric construction with "light" and "shadow" for RSK leads to a simple proof of the fact that RSK and the "local rules" give the same bijection)

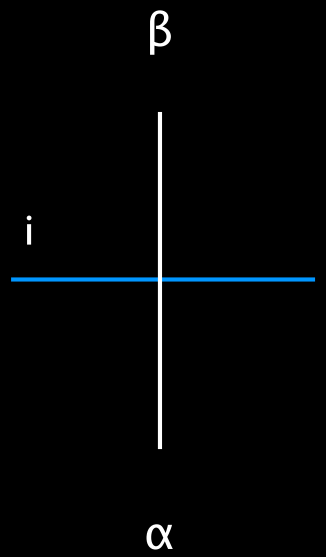

1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7

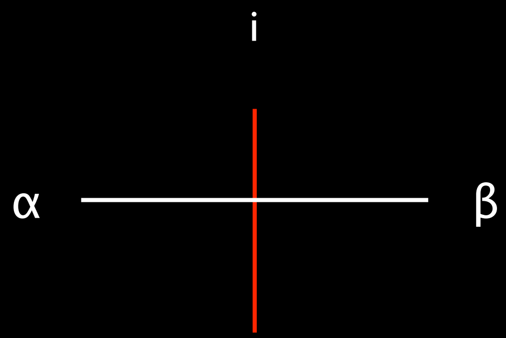


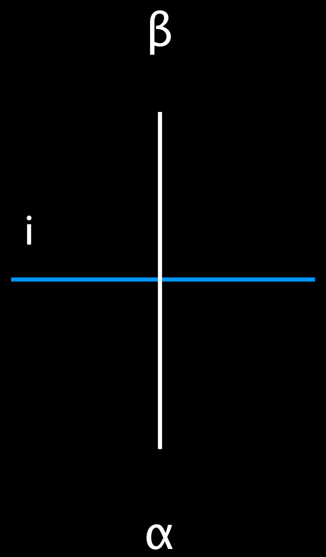

σ = 3 1 6 10 2 5 8 4



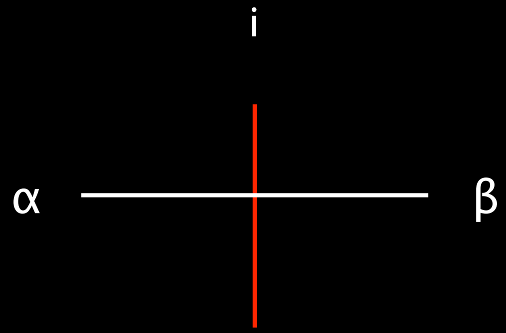
4 2 1 5 3

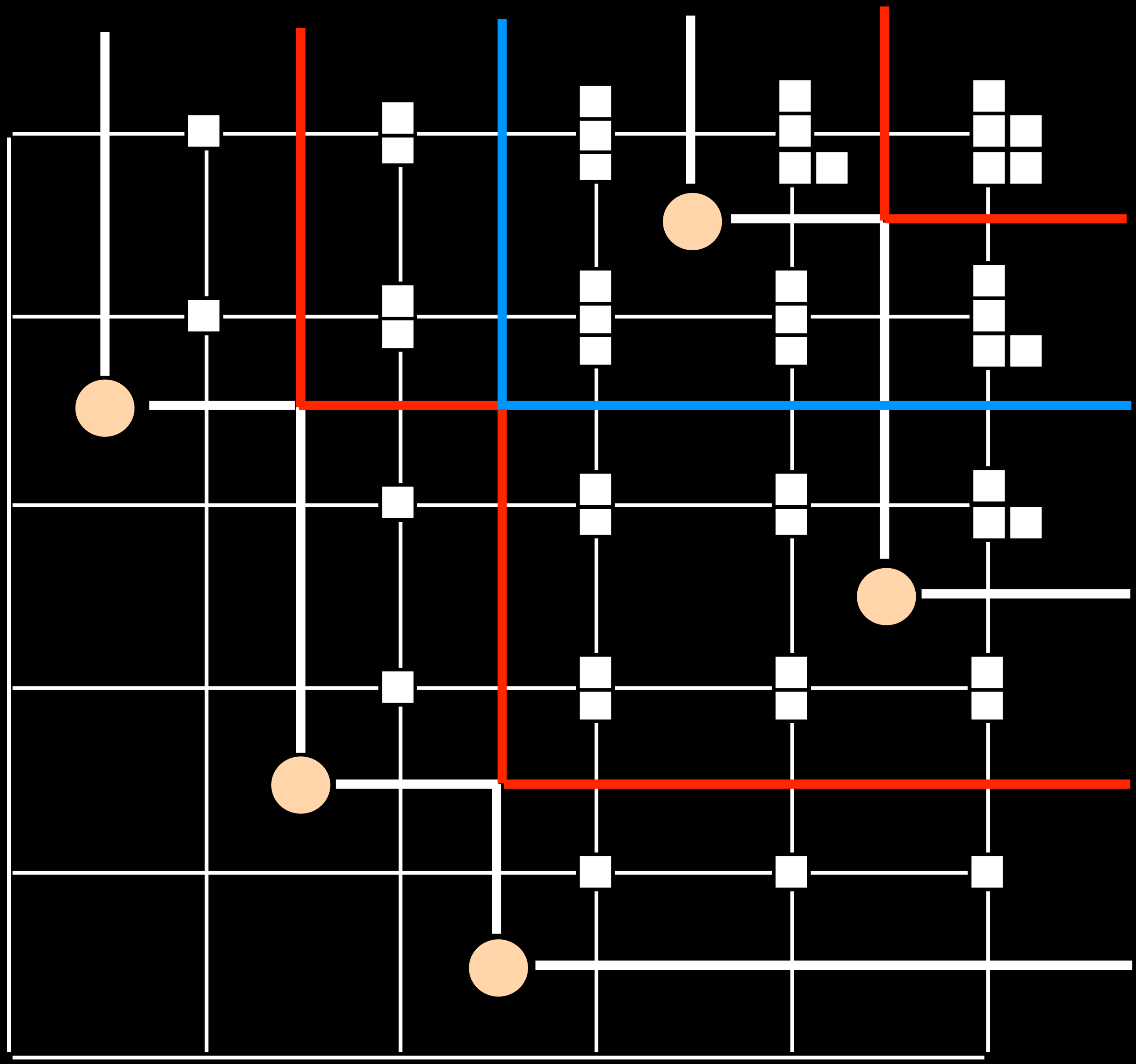

β
i
α

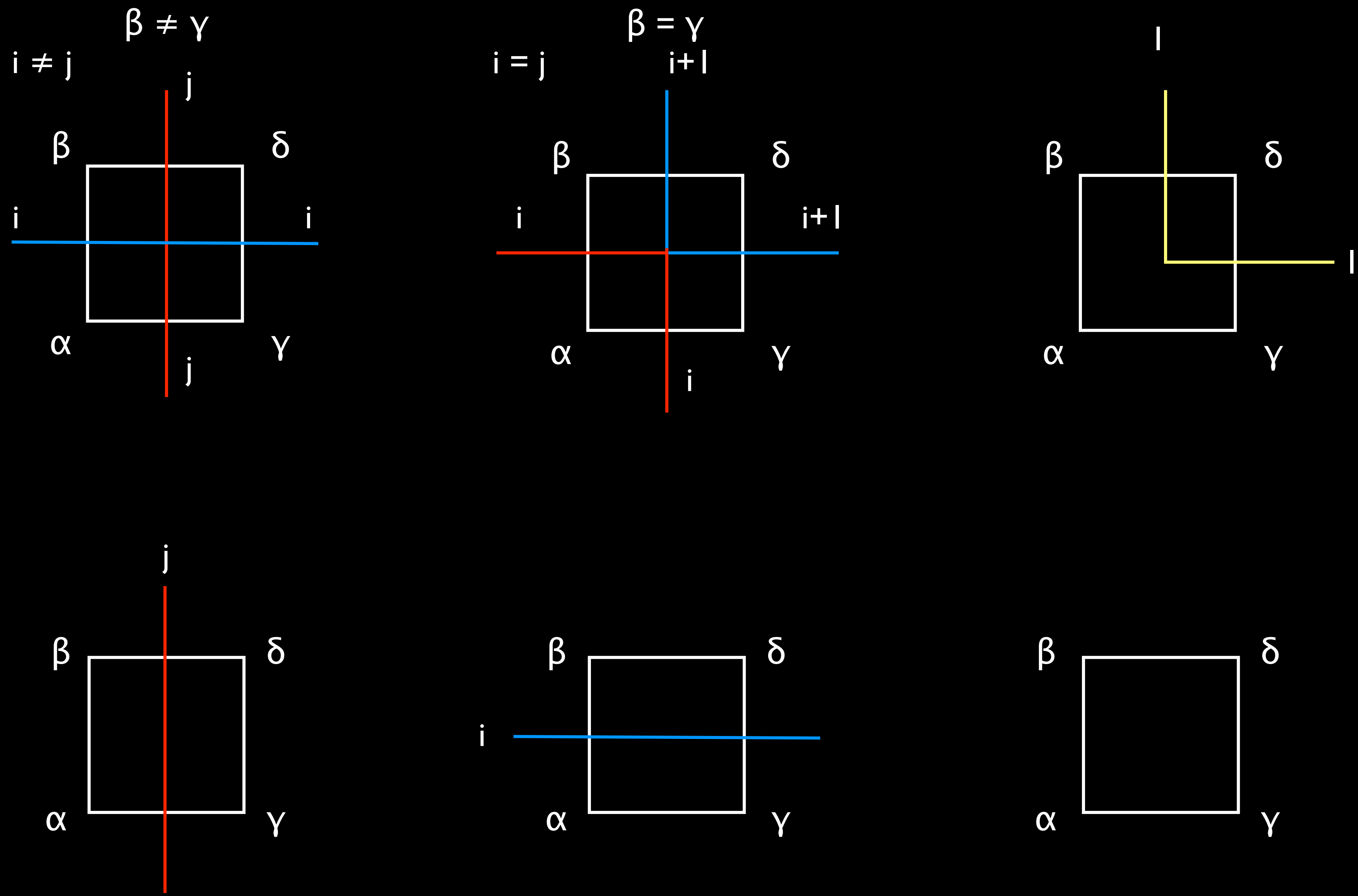

i
α
β

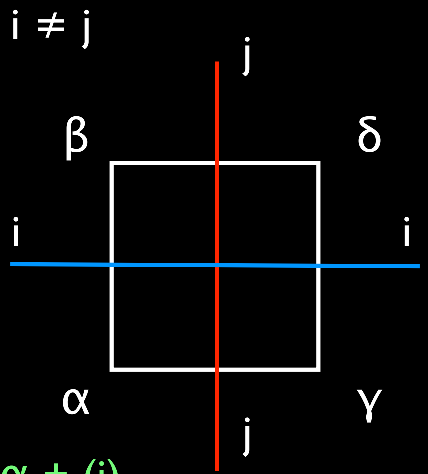


$\beta = \alpha + (i)$

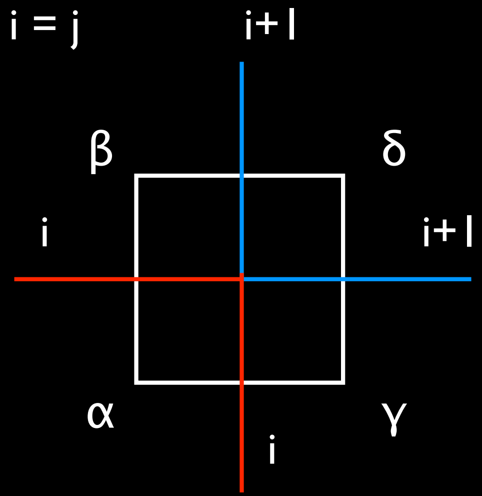




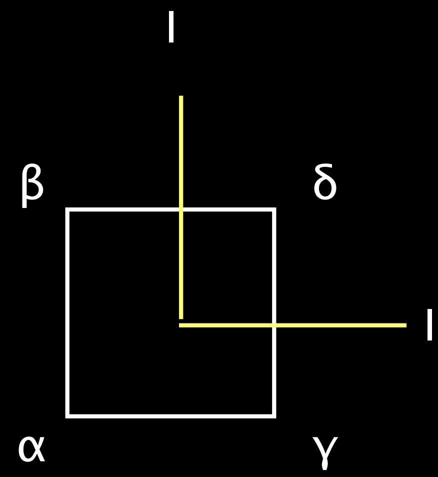

β ≠ γ
i ≠ j
j
β
δ
i
i
α
γ
j
β = γ
i = j
i+1
β
δ
i
i+1
α
γ
i
1
β
δ
1
α
γ
j
β
δ
α
γ
β
δ
i
α
γ
β
δ
α
γ



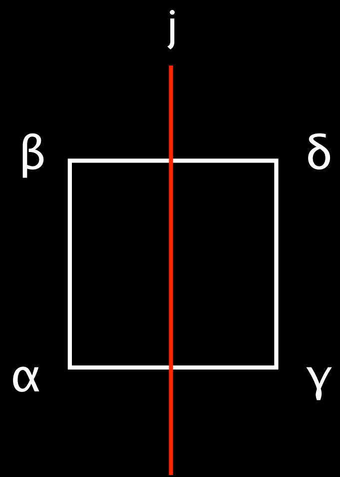
$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$



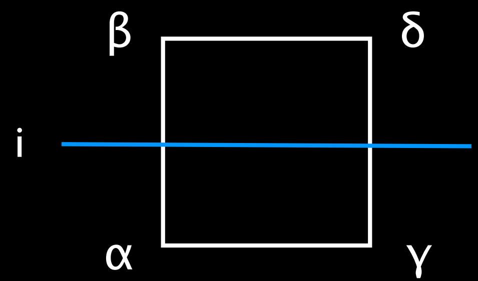
$\beta = \gamma = \alpha + (i)$
$\delta = \alpha + (i) + (i+1)$



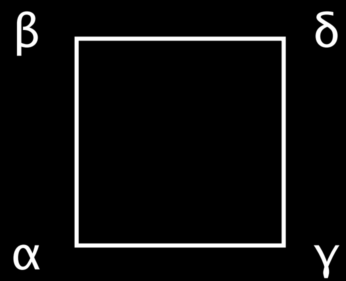
$\beta = \gamma = \alpha$
$\delta = \alpha + (1)$



$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$

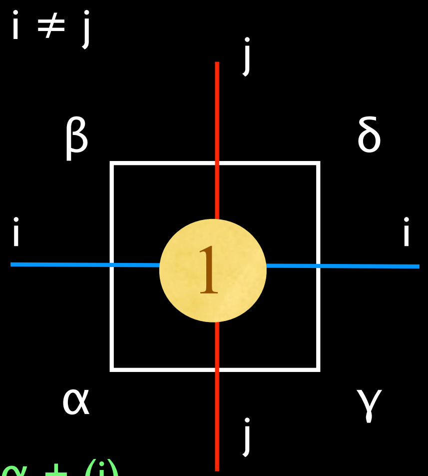


$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$



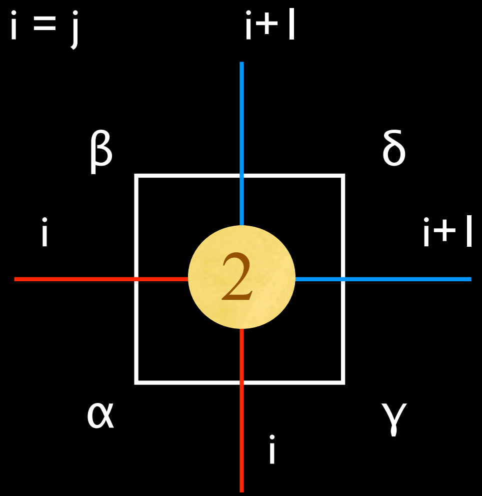
$\delta = \beta = \gamma = \alpha$

$\beta \neq \gamma$

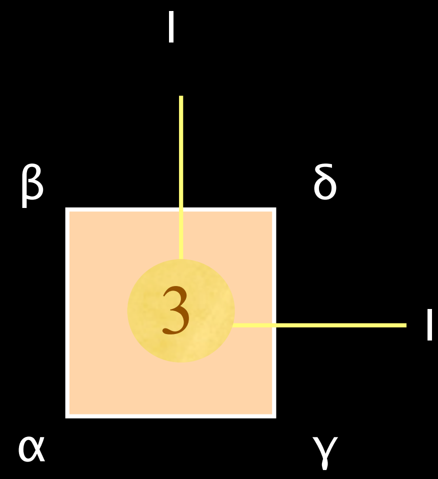


$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$

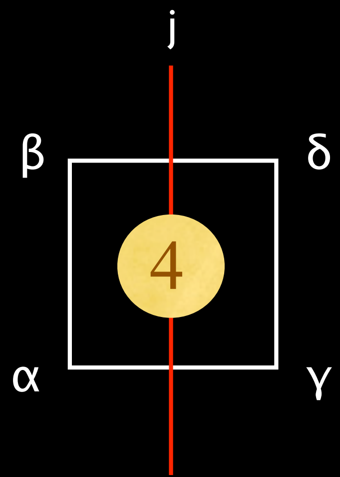
$\beta = \gamma$



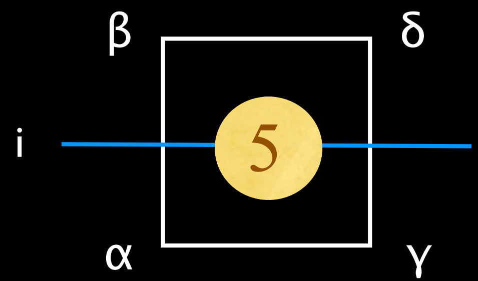
$\beta = \gamma = \alpha + (i)$
$\delta = \alpha + (i) + (i+1)$



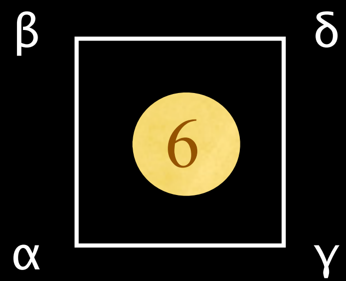
$\beta = \gamma = \alpha$
$\delta = \alpha + (1)$



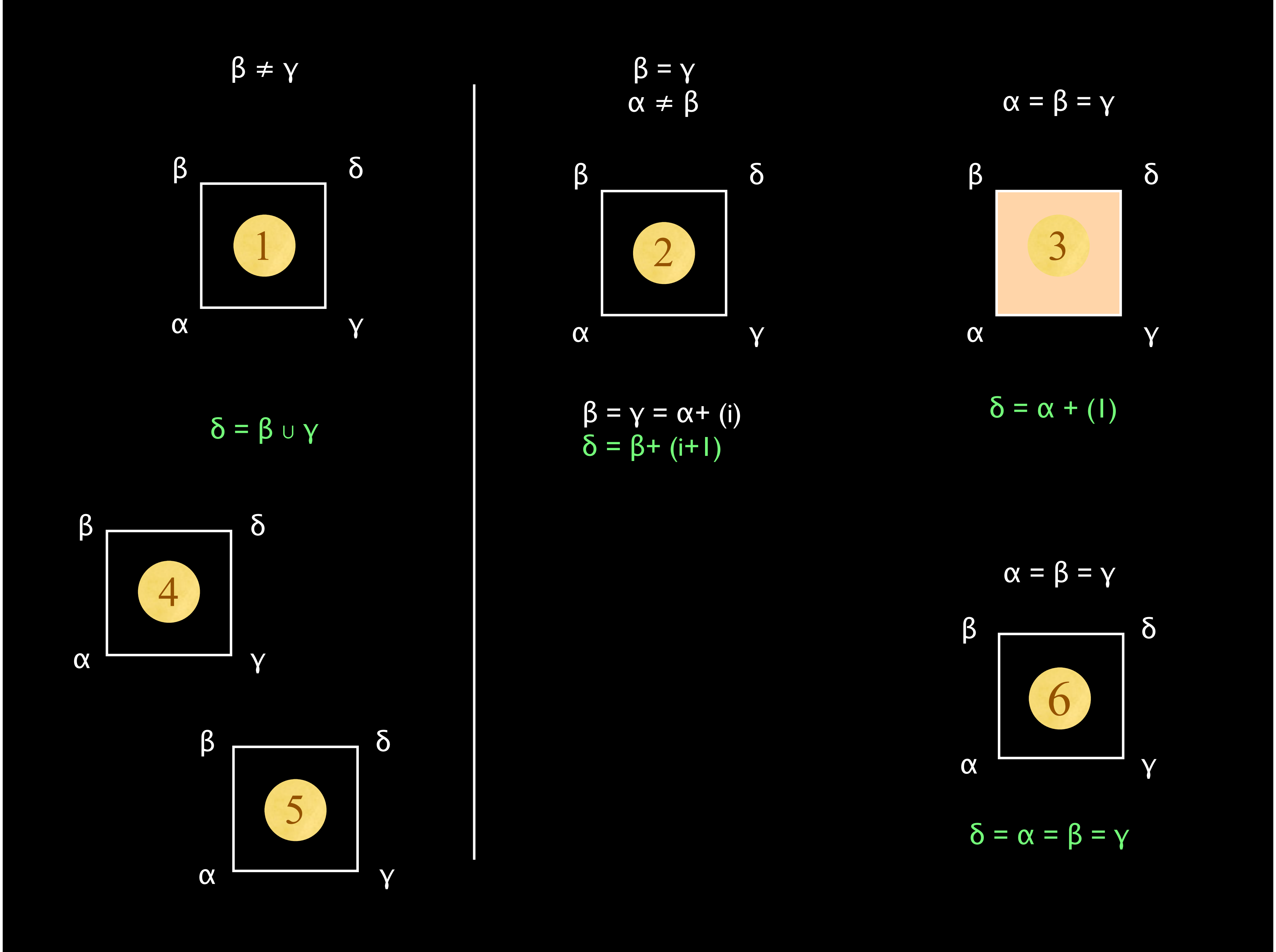
$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$



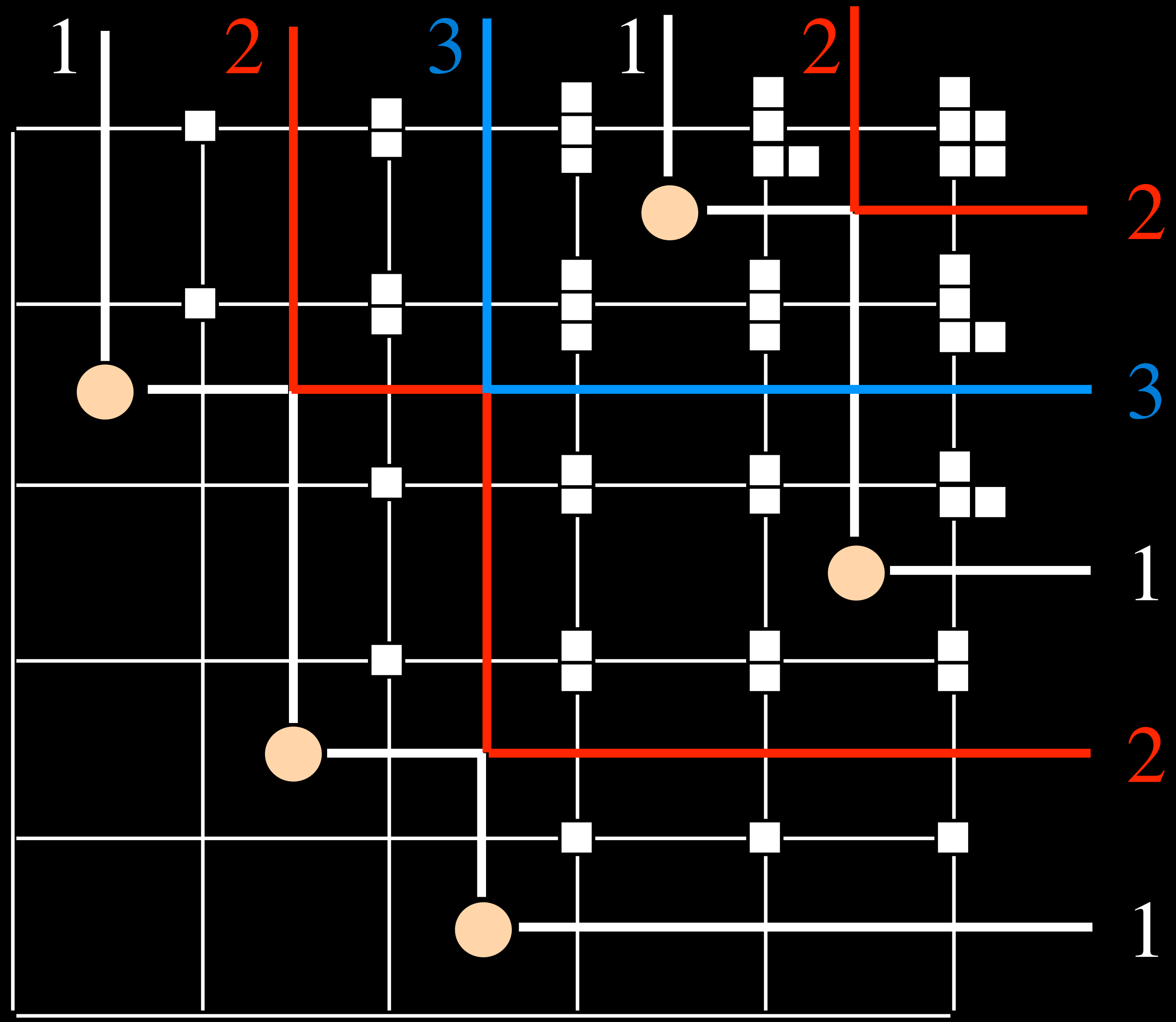
$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$



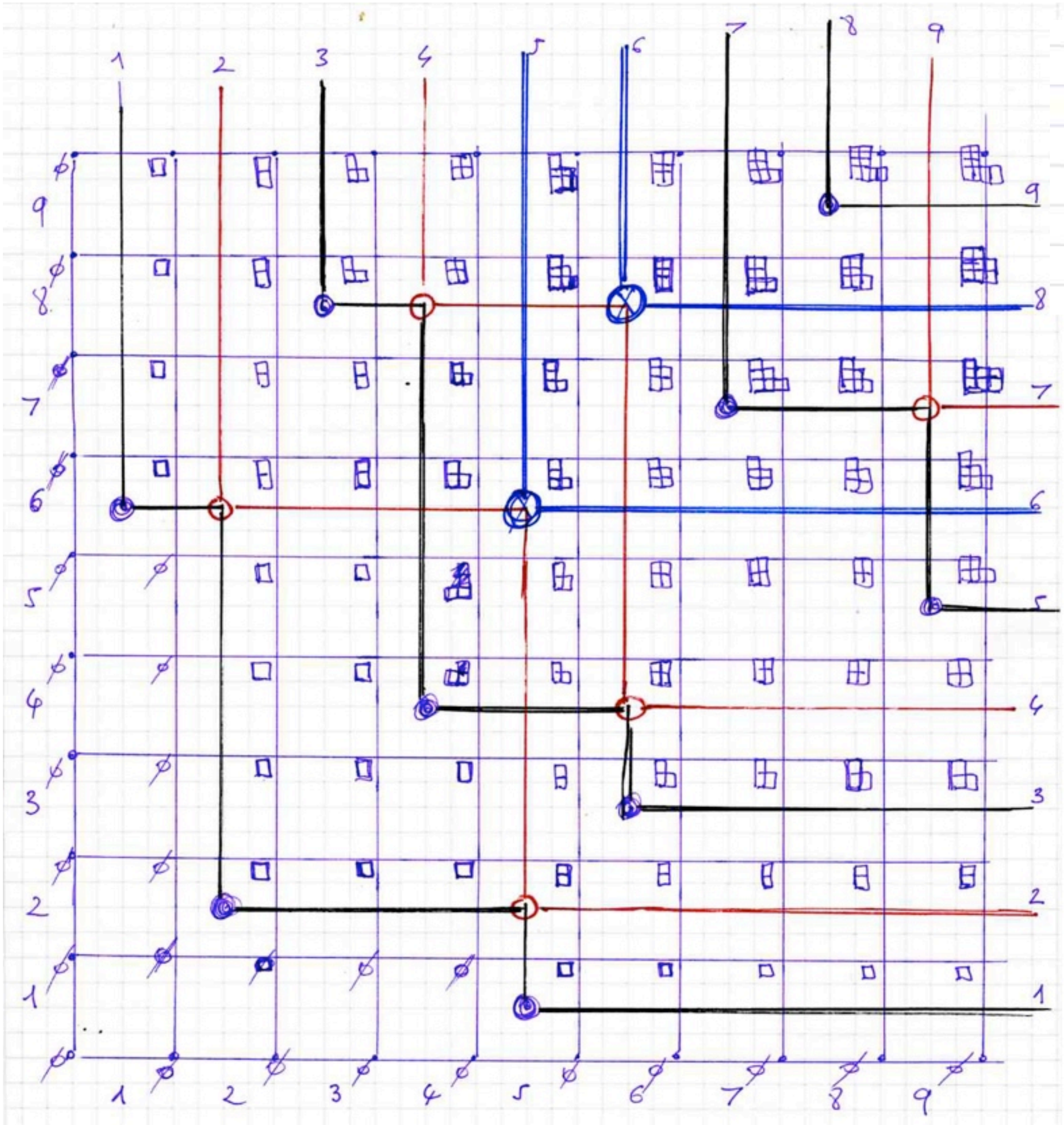
$\delta = \beta = \gamma = \alpha$



β ≠ γ
β δ
1
α γ
δ = β ∪ γ
β δ
4
α γ
β δ
5
α γ
β = γ
α ≠ β
β δ
2
α γ
β = γ = α+ (i)
δ = β+ (i+1)
α = β = γ
β δ
3
α γ
δ = α + (1)
α = β = γ
β δ
6
α γ
δ = α = β = γ



1
2
3
1
2
2
3
1
2
1



5	6		
2	4	9	
1	3	7	8

Q

another example with
6 2 8 4 1 3 7 9 5

6	8		
2	4	7	
1	3	5	9

P

The cellular Ansatz

guided construction
of a bijection
from a representation of U and D

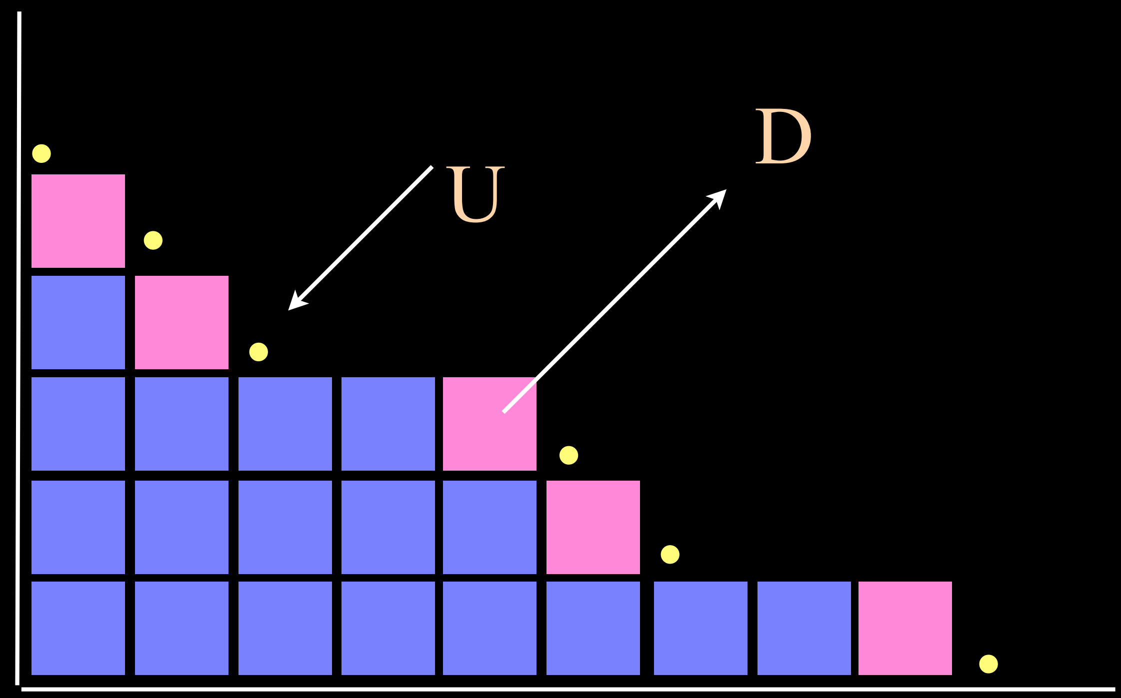
$UD = DU + 1$

representation
of the operators:
U, D acting
on Ferrers diagrams

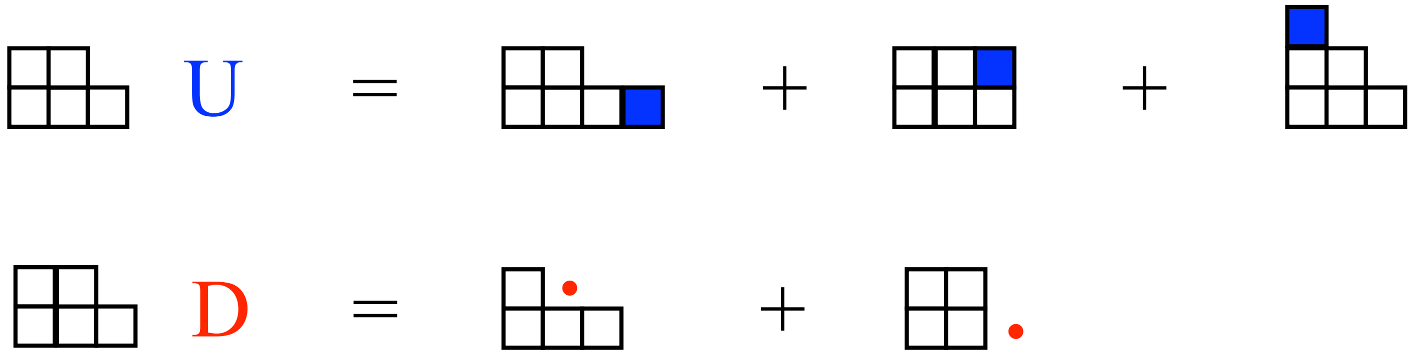
Sergey Fomin

Operators U and D

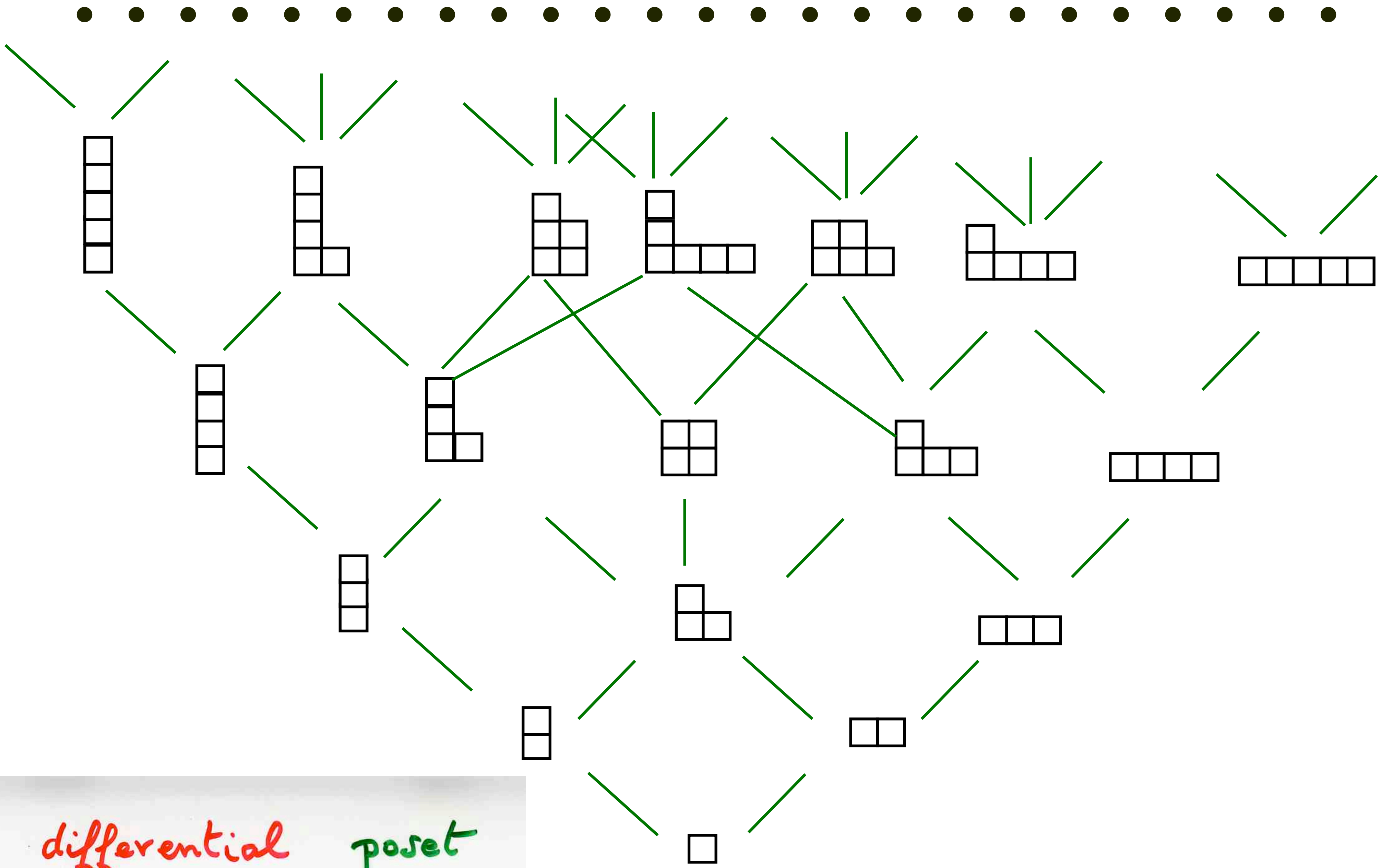
adding or deleting a cell in a Ferrers diagram



Young lattice

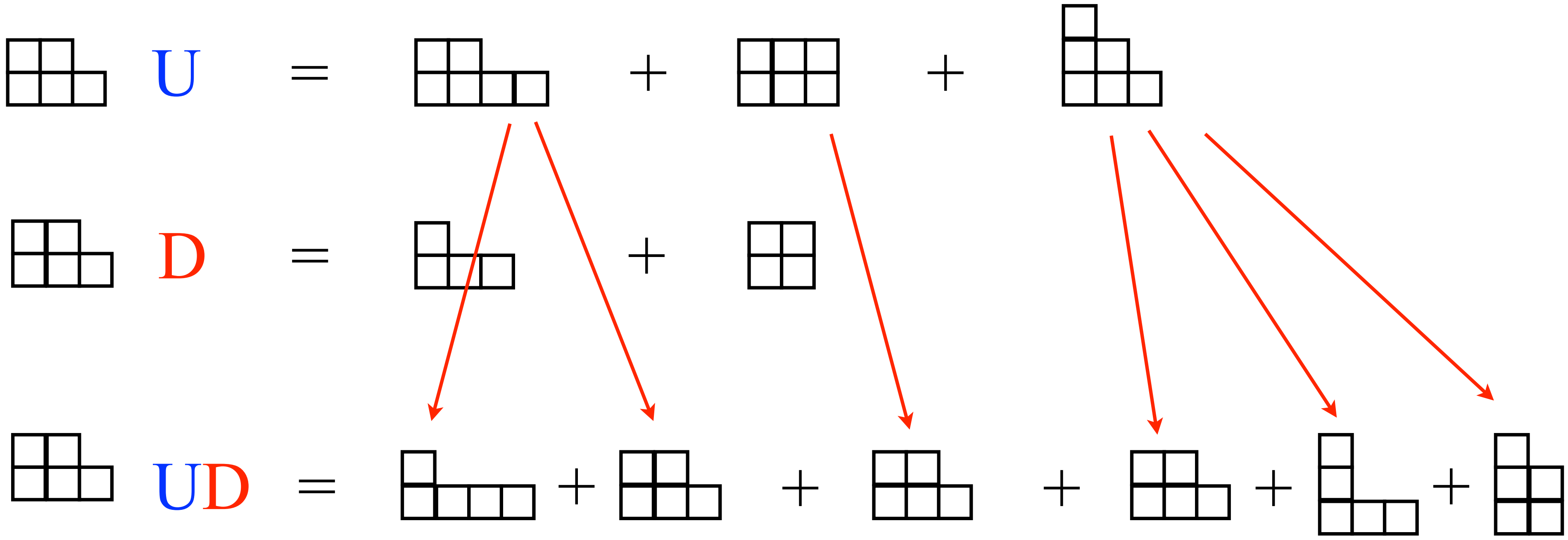

U
D

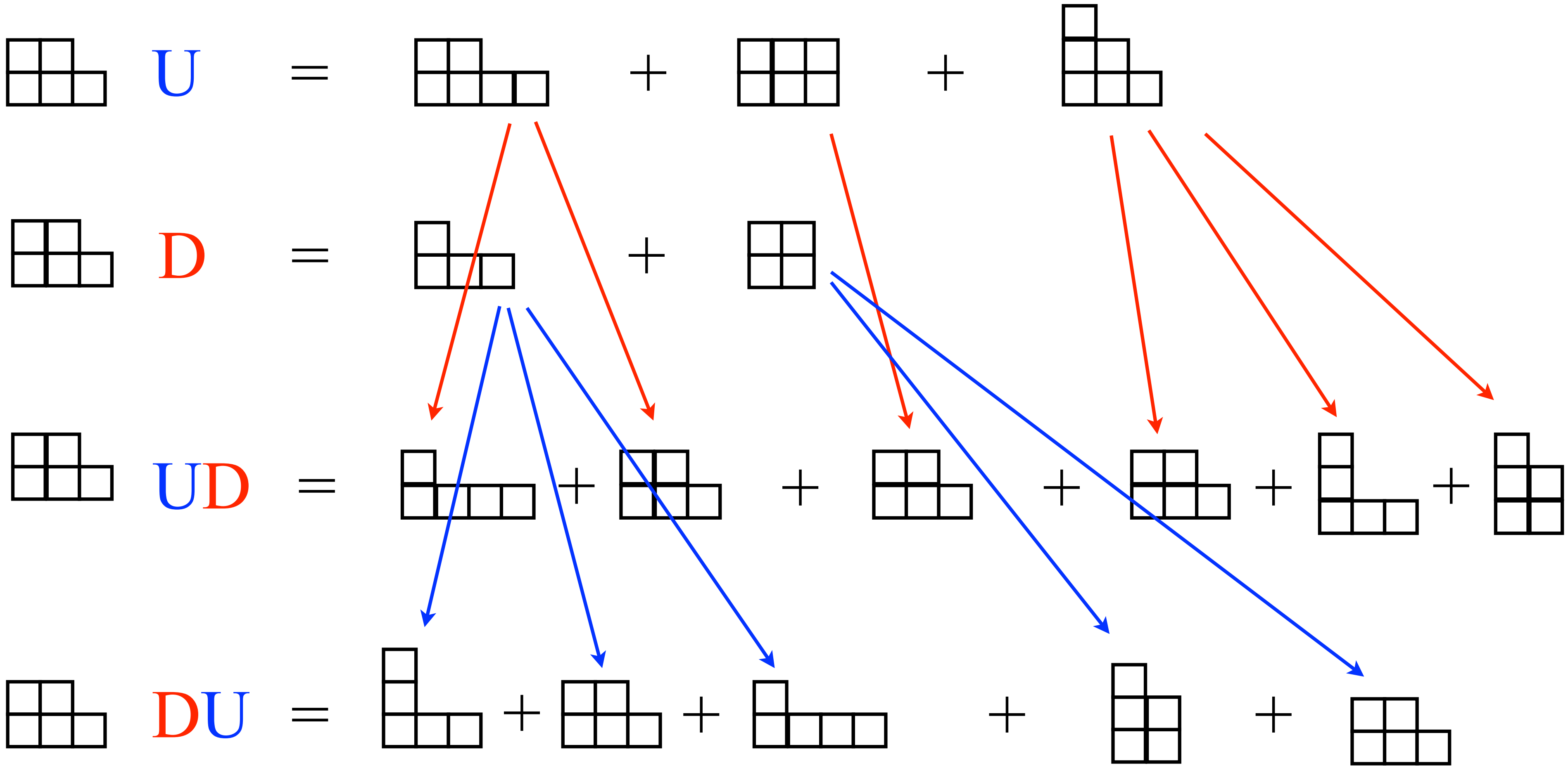
Young lattice

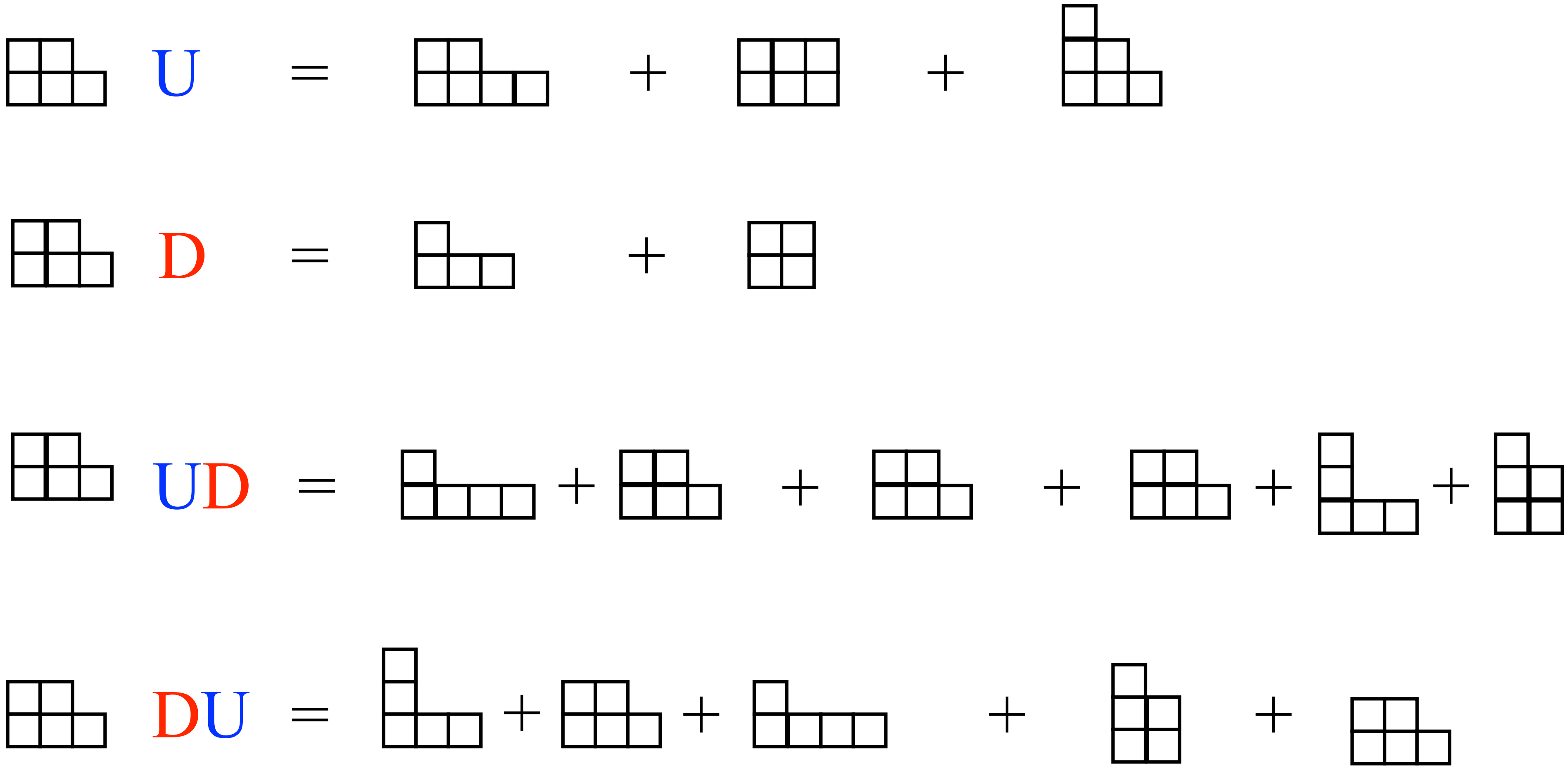


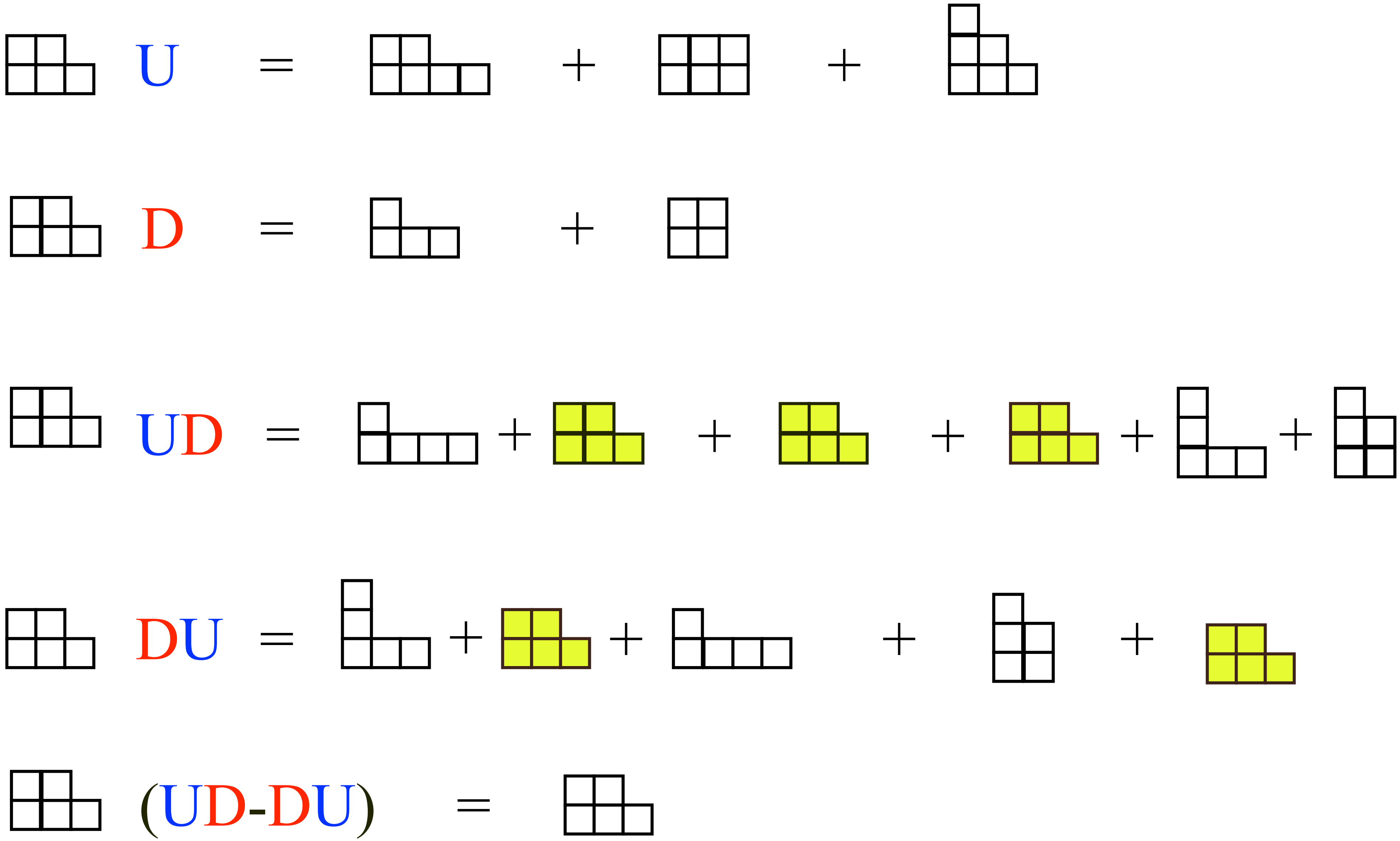
differential poset

Fomin, Stanley

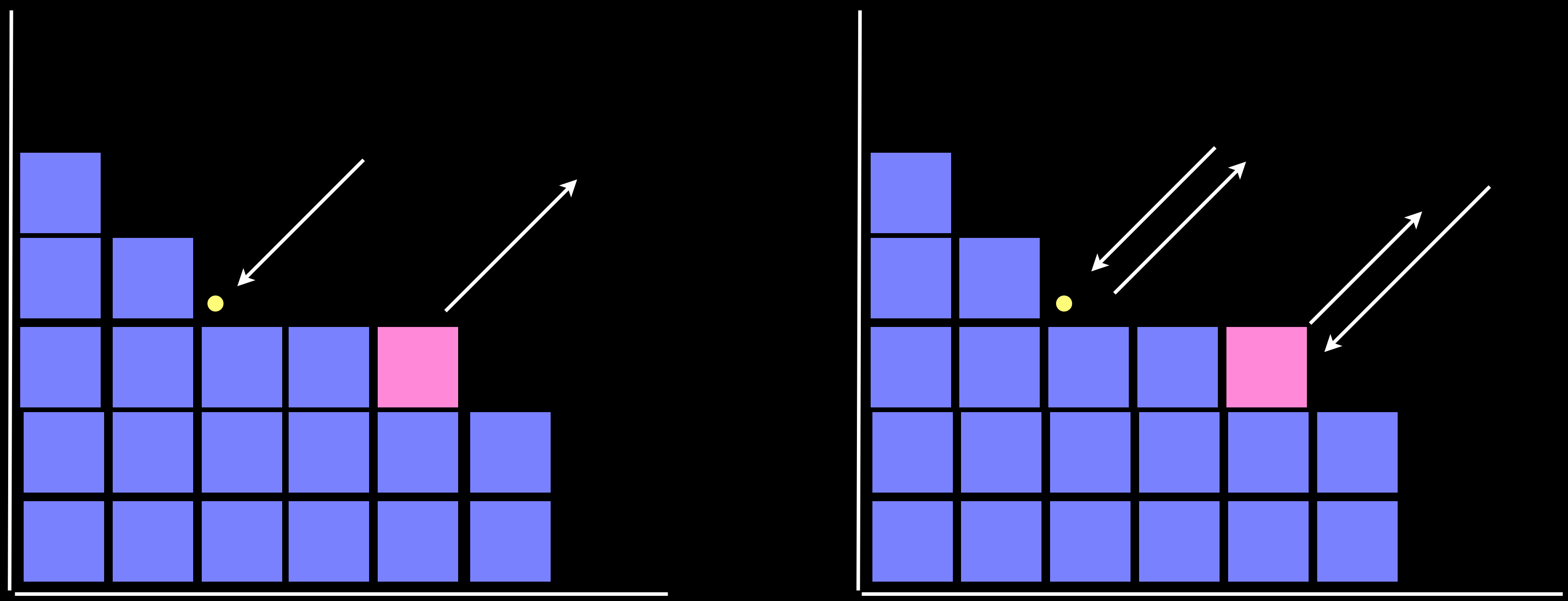

U
D
UD


U =
D =
UD =
DU =

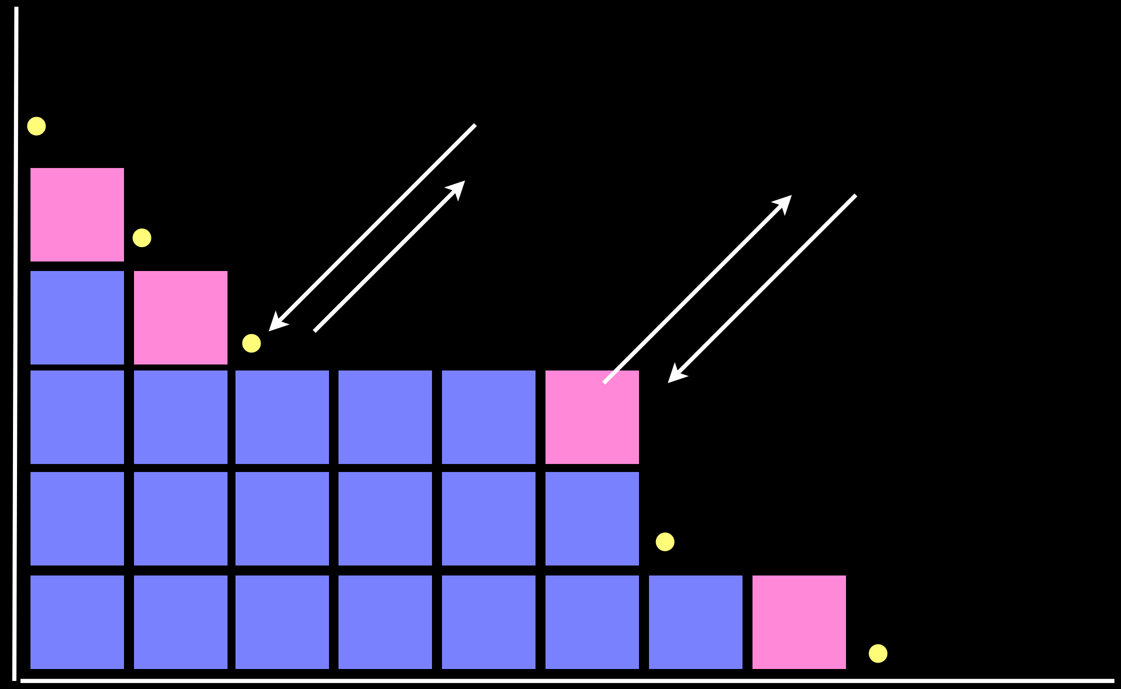

U
D
UD
DU


U =
D =
UD =
DU =
(UD-DU) =

$$UD = DU + I$$

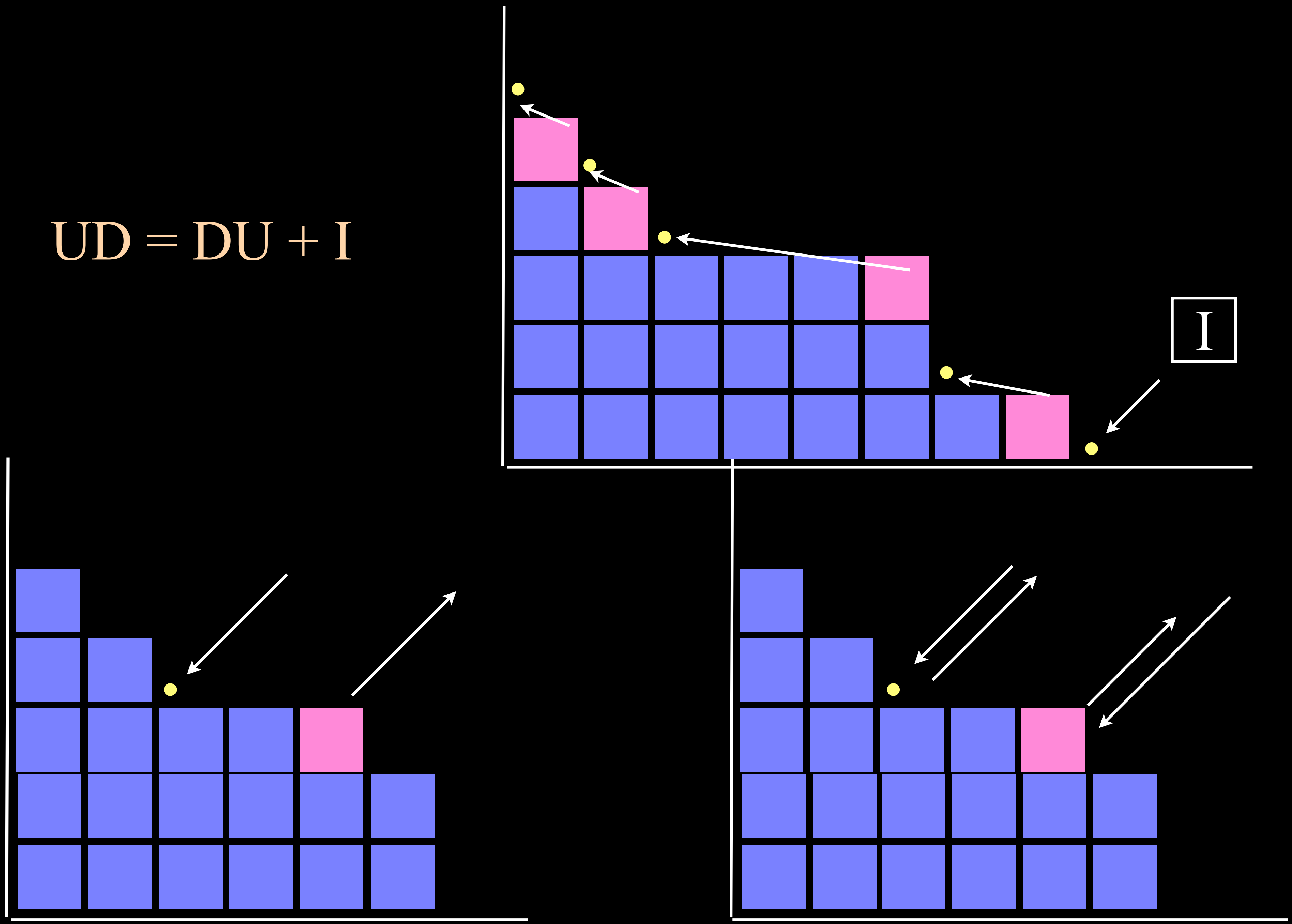


$$UD = DU + I$$

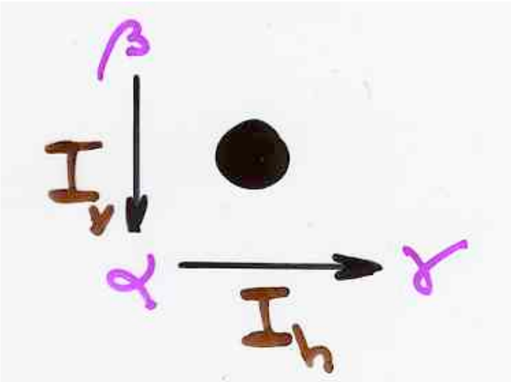


combinatorial "representation" of the commutation relation UD = DU + I

UD = DU + I

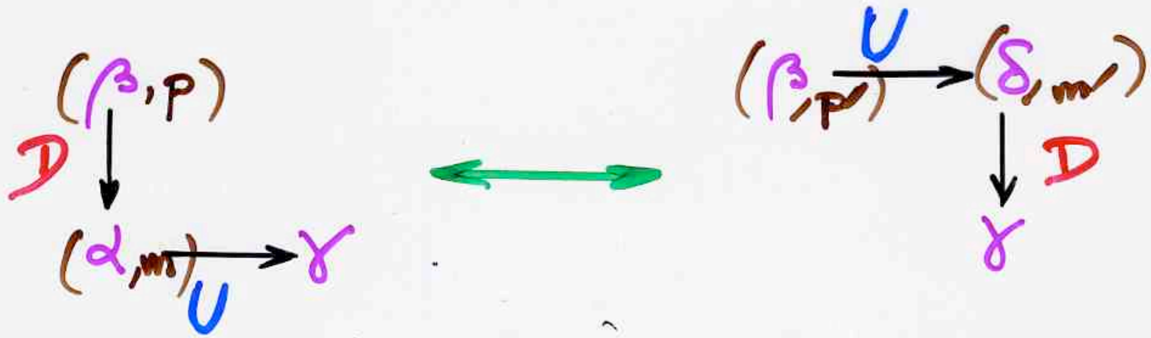


$$UD = DU + I_v I_h$$



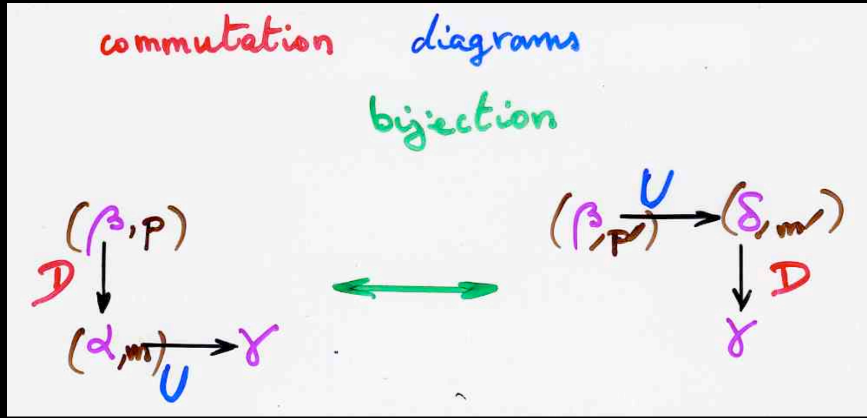
commutation diagrams

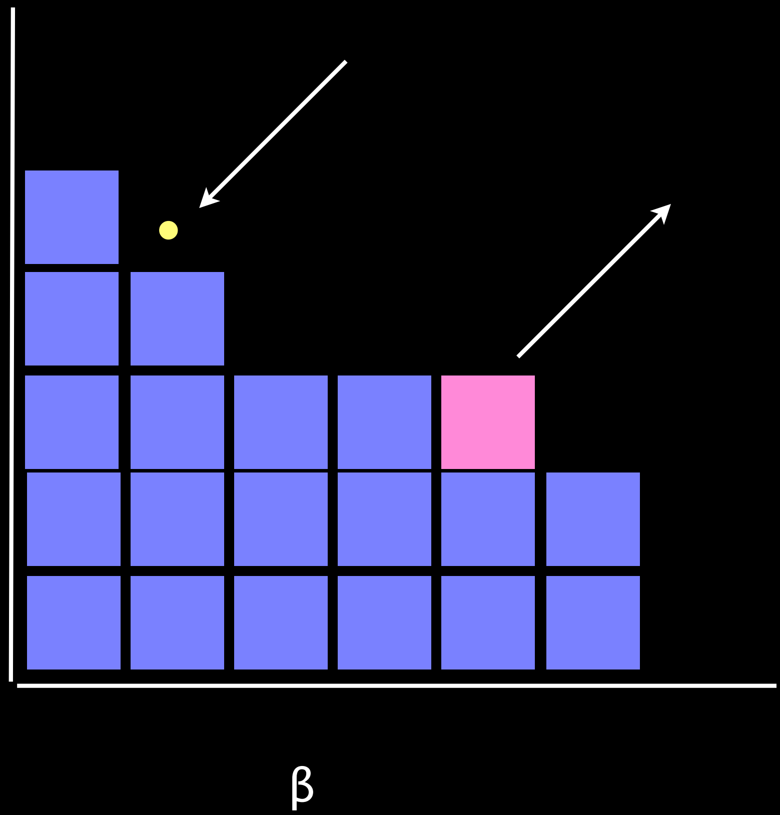
bijection

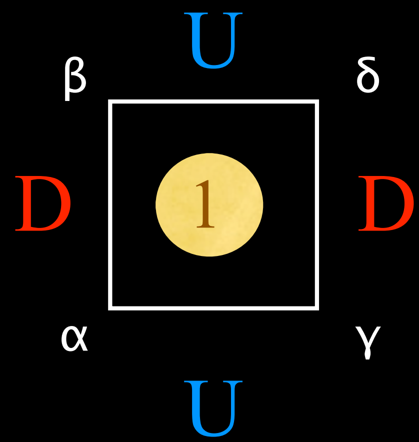


p, m, p', m' "positions"

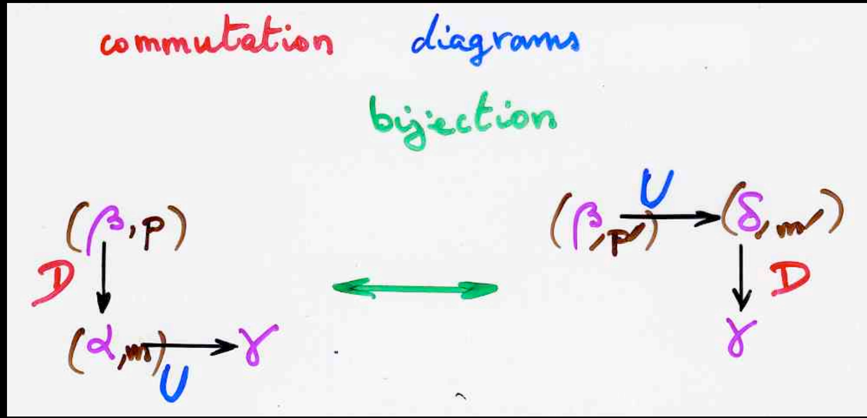
in $\beta, \alpha, \beta, \delta$ resp.

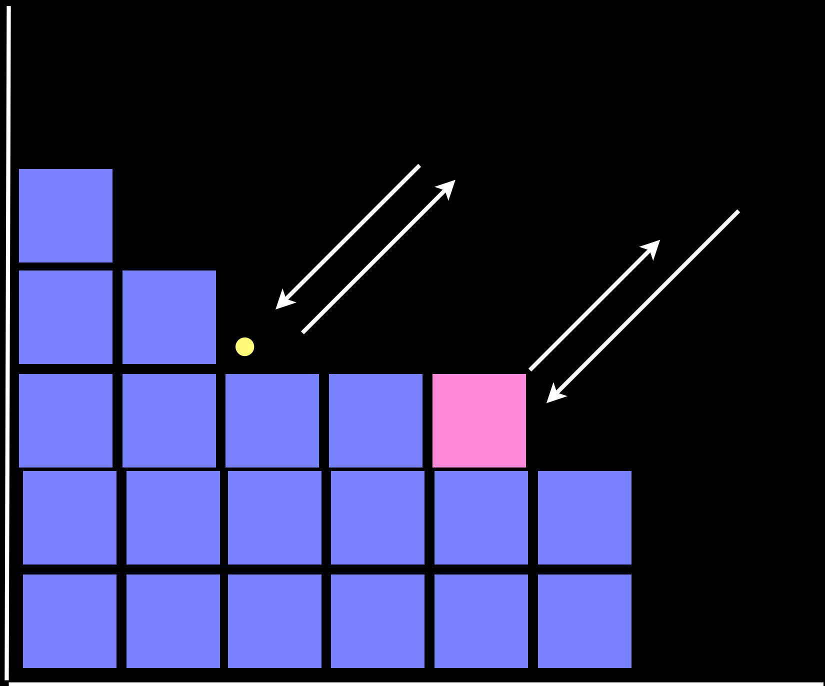


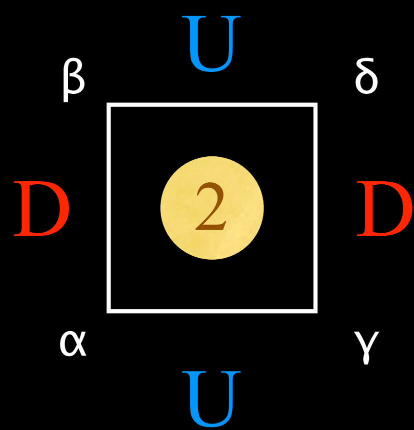




$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$

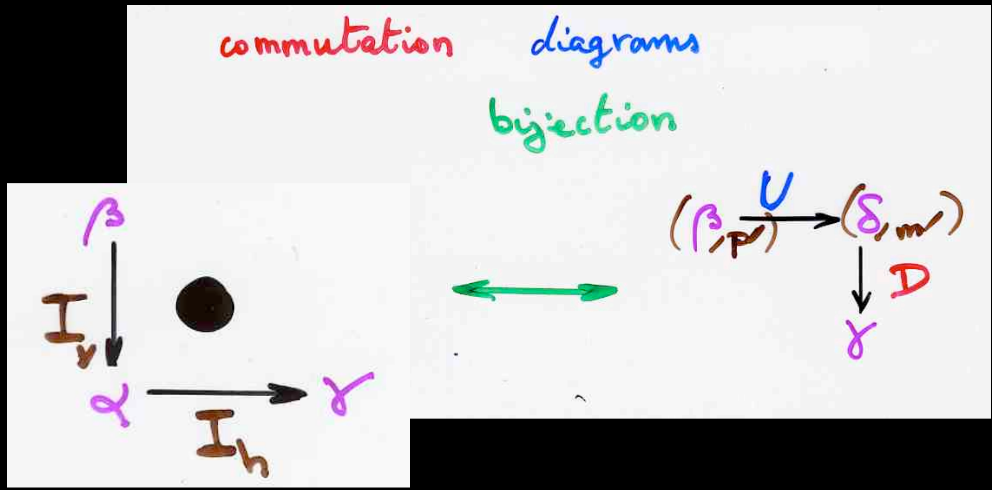


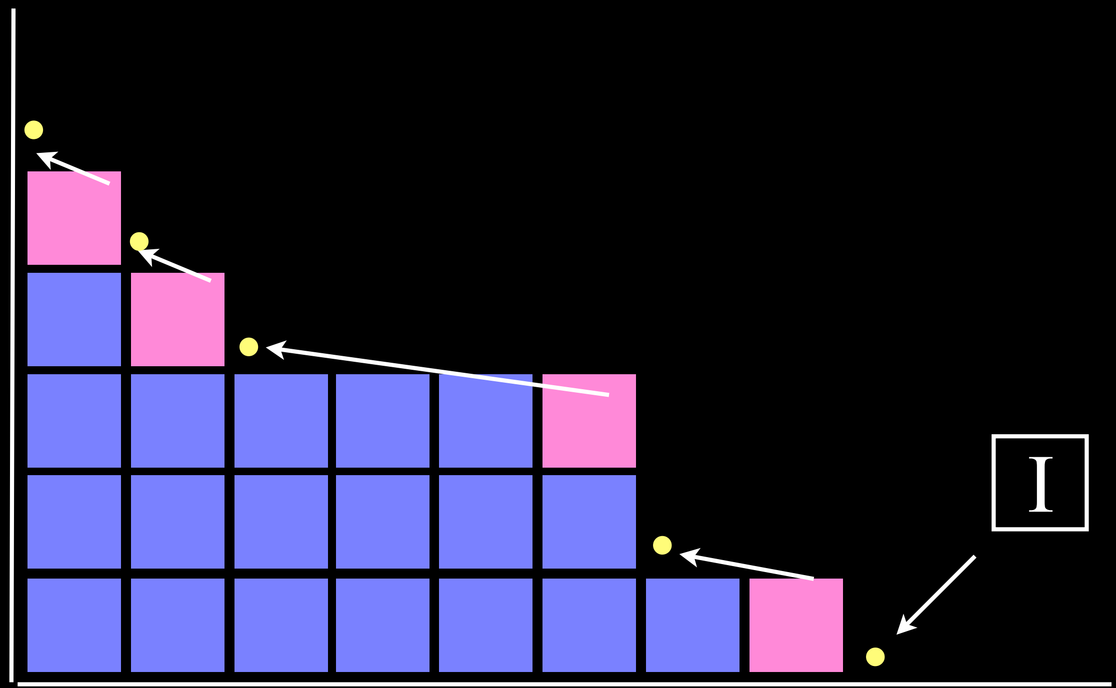


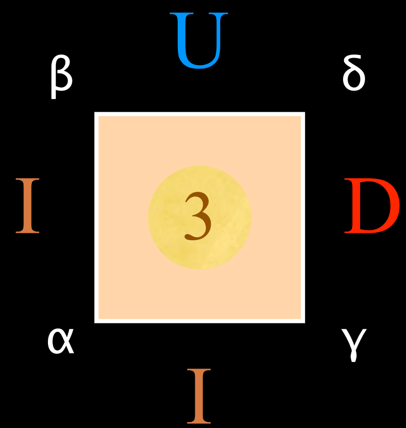


$\beta = \gamma = \alpha + (i)$

$\delta = \beta + (i+1)$

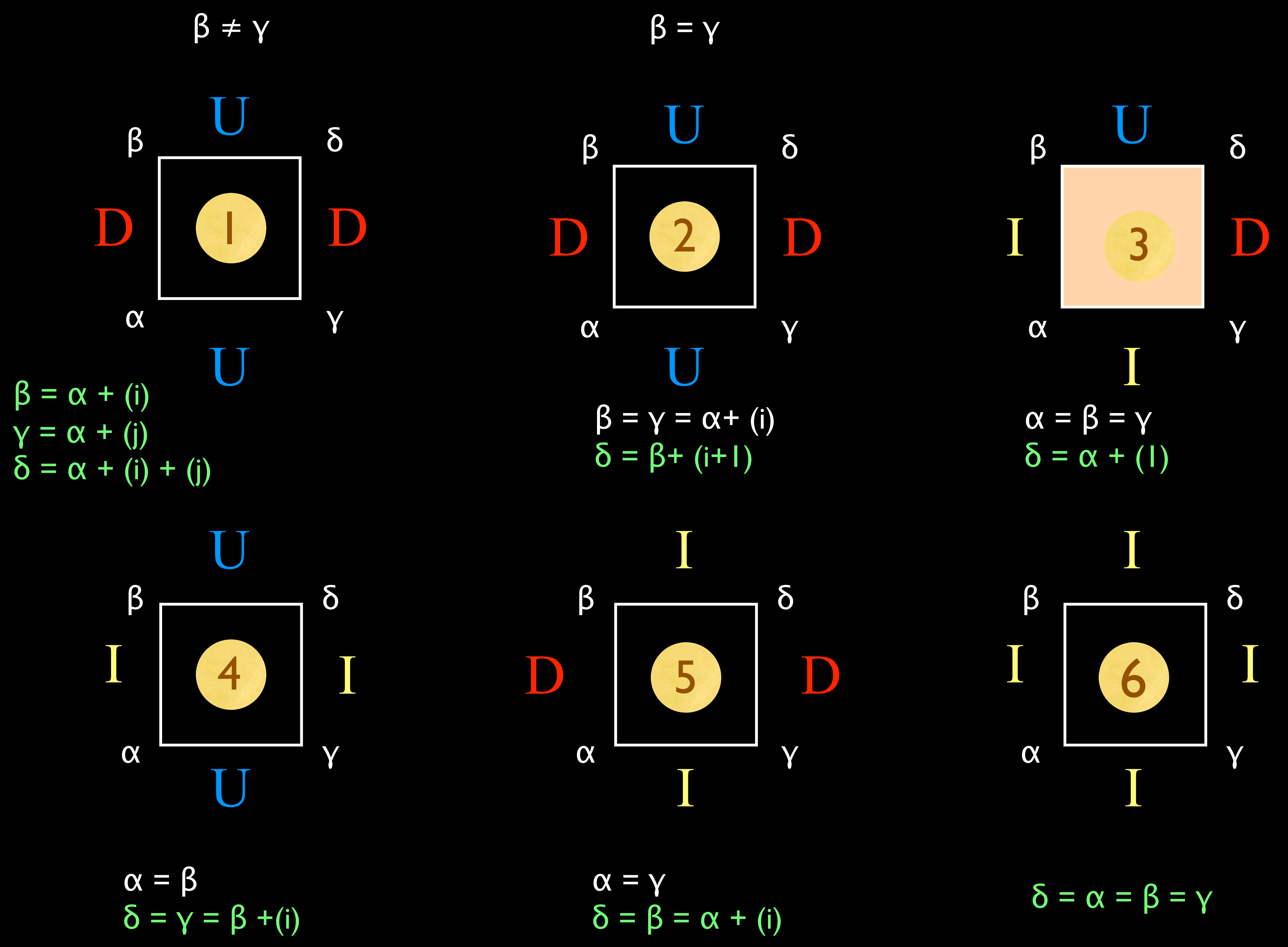


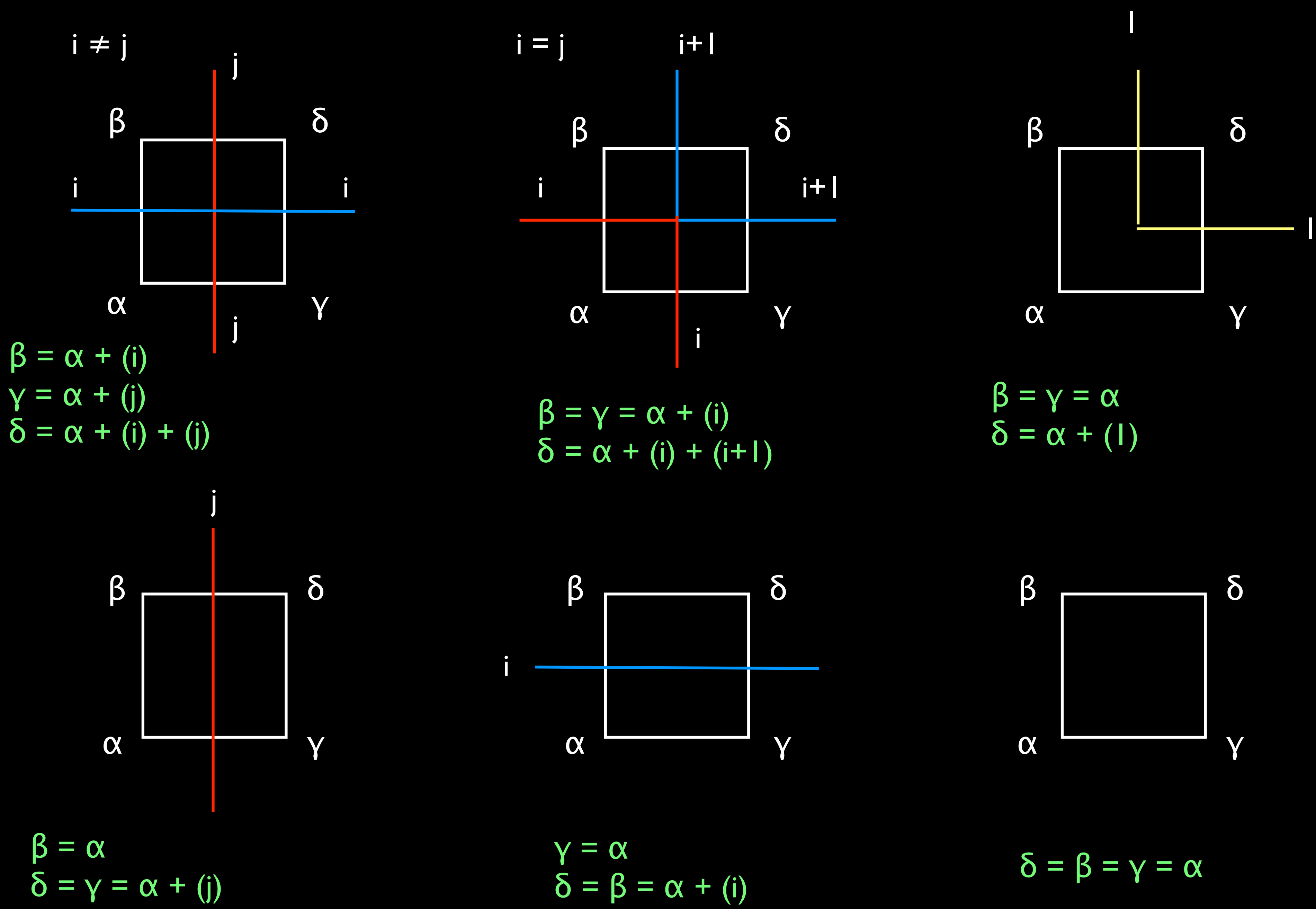




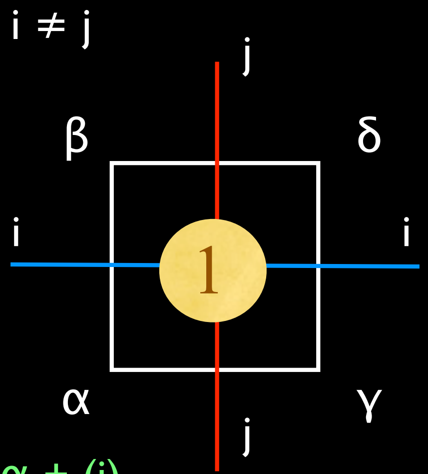
$\beta = \gamma = \alpha$
$\delta = \alpha + (1)$

$$
\begin{cases}
UD = DU + I_v I_h \\
U I_v = I_v U \\
I_h D = D I_h \\
I_h I_v = I_v I_h
\end{cases}
$$


β ≠ γ
β = γ
U
β
δ
D
1
D
α
γ
U
β = α + (i)
γ = α + (j)
δ = α + (i) + (j)
U
β
δ
D
2
D
α
γ
U
β = γ = α+ (i)
δ = β+ (i+1)
U
β
δ
I
3
D
α
γ
I
α = β = γ
δ = α + (1)
U
β
δ
I
4
I
α
γ
U
α = β
δ = γ = β +(i)
I
β
δ
D
5
D
α
γ
I
α = γ
δ = β = α + (i)
I
β
δ
I
6
I
α
γ
I
δ = α = β = γ

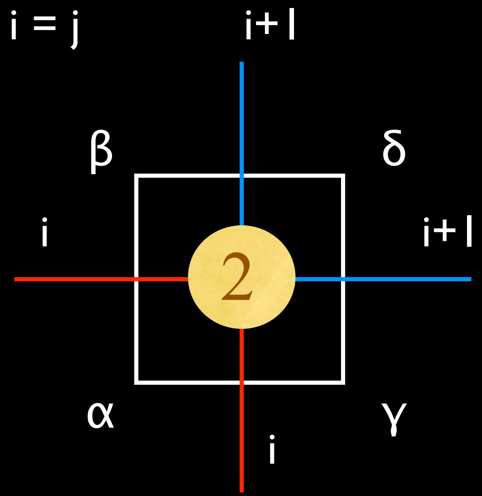

i ≠ j
j
β
δ
i
i
α
γ
j
β = α + (i)
γ = α + (j)
δ = α + (i) + (j)
i = j
i+1
β
δ
i
i+1
α
γ
i
β = γ = α + (i)
δ = α + (i) + (i+1)
1
β
δ
1
α
γ
β = γ = α
δ = α + (1)
j
β
δ
α
γ
β = α
δ = γ = α + (j)
β
δ
i
α
γ
γ = α
δ = β = α + (i)
β
δ
α
γ
δ = β = γ = α

$\beta \neq \gamma$

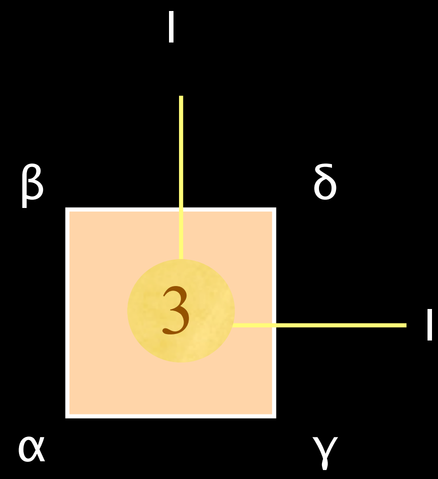


$\beta = \alpha + (i)$
$\gamma = \alpha + (j)$
$\delta = \alpha + (i) + (j)$

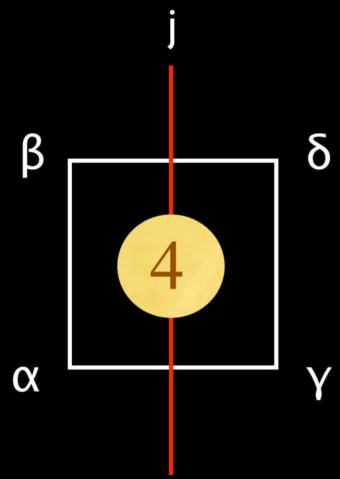
$\beta = \gamma$



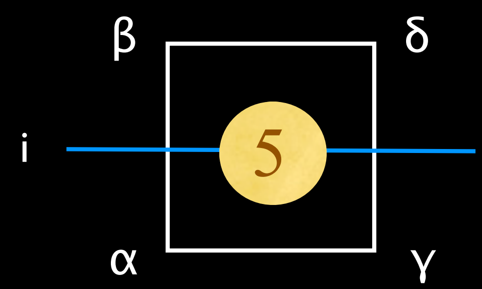
$\beta = \gamma = \alpha + (i)$
$\delta = \alpha + (i) + (i+1)$



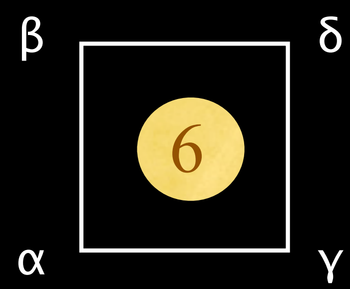
$\beta = \gamma = \alpha$
$\delta = \alpha + (1)$



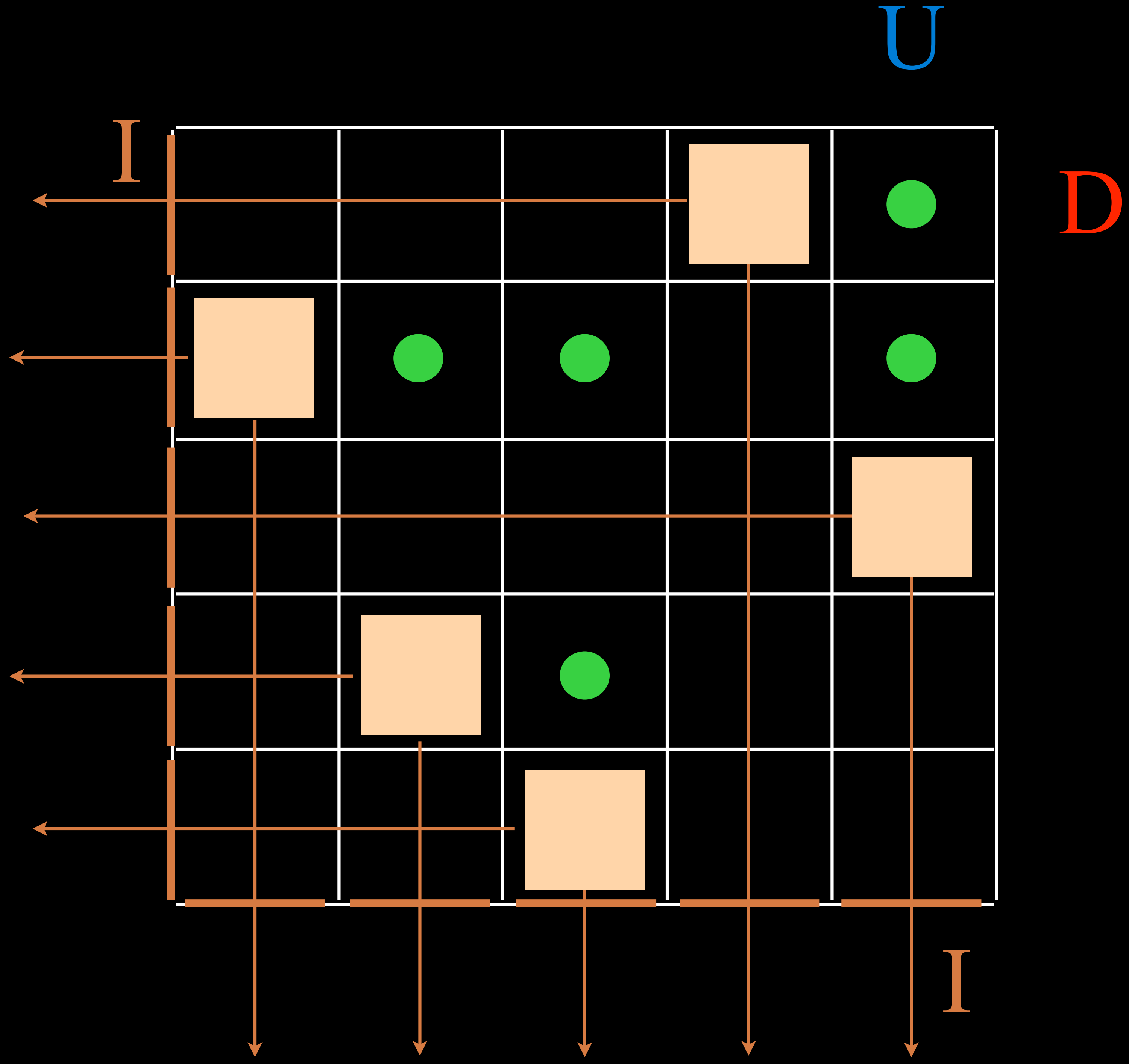
$\beta = \alpha$
$\delta = \gamma = \alpha + (j)$

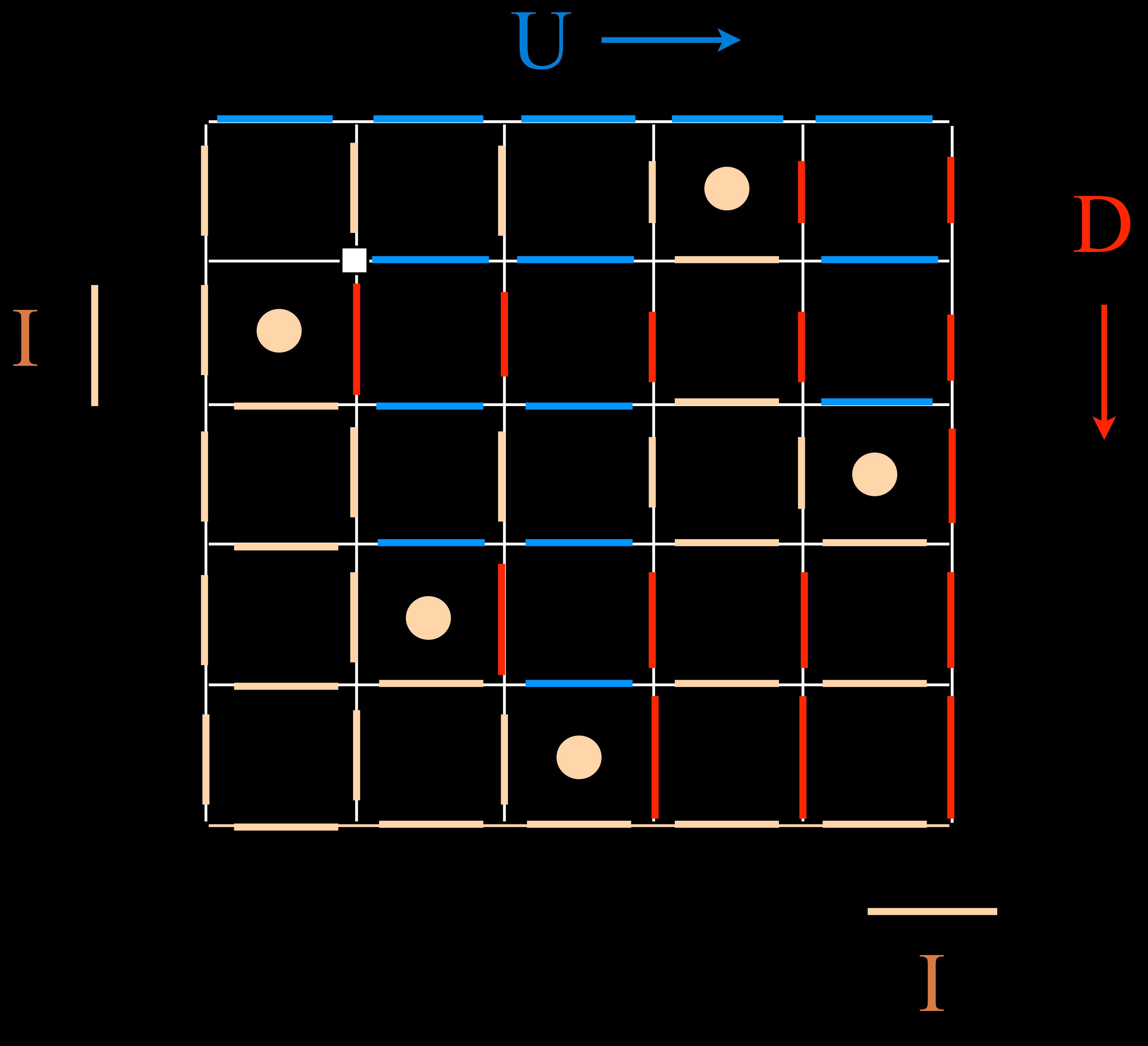


$\gamma = \alpha$
$\delta = \beta = \alpha + (i)$

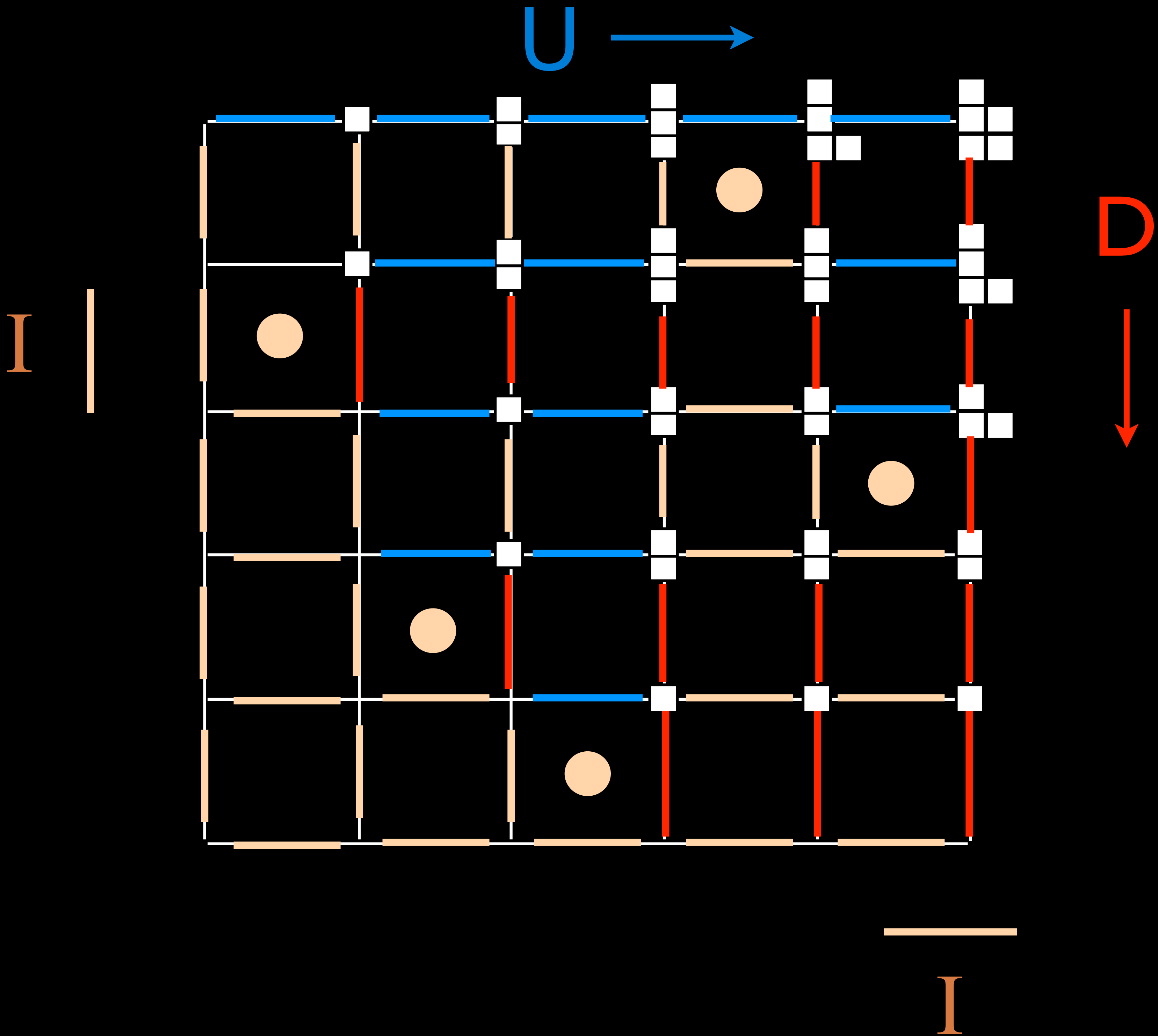


$\delta = \beta = \gamma = \alpha$

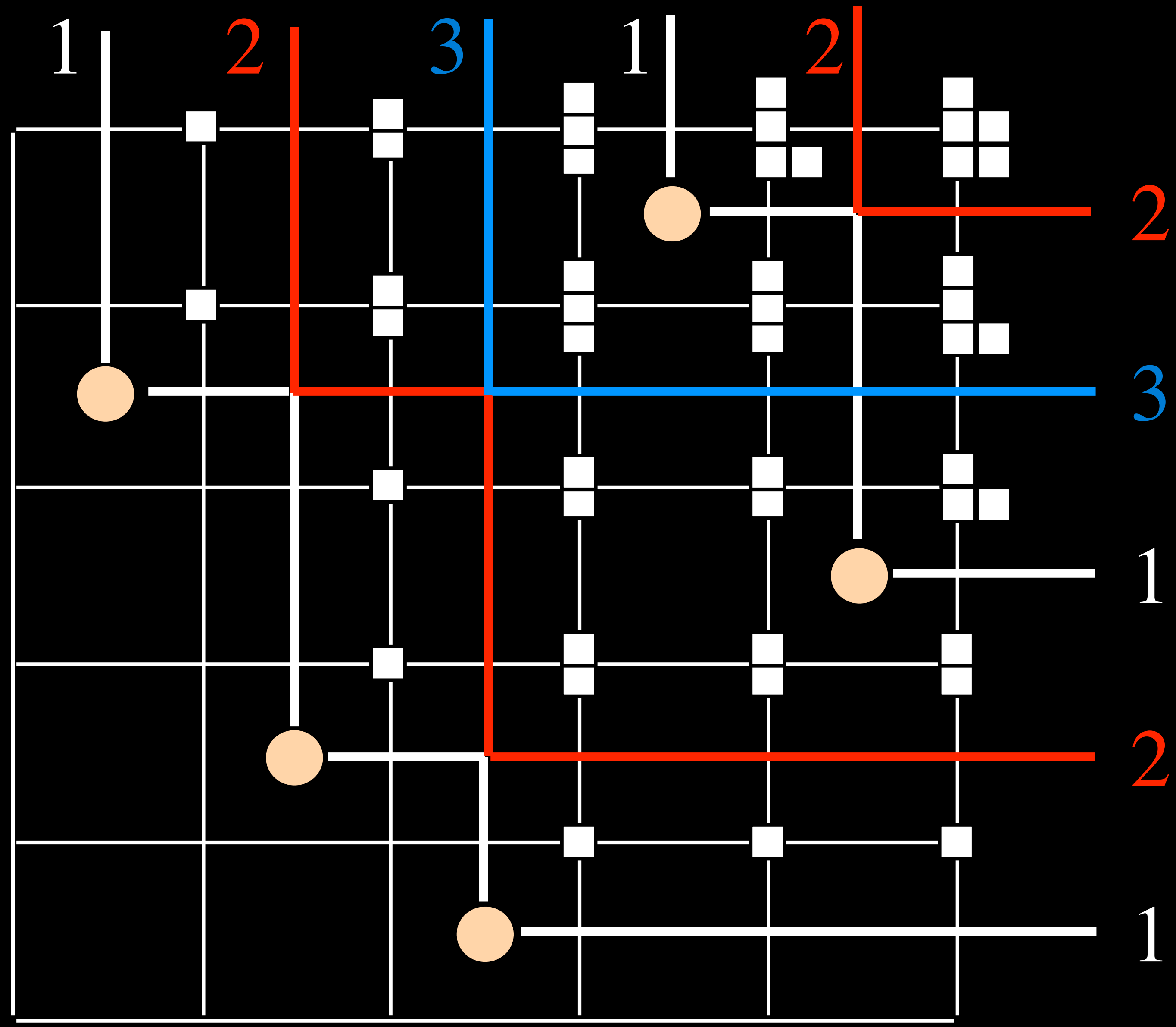

U
I
D
I



U
D
I
I



U
D
I
I


1
2
3
1
2
2
3
1
2
1