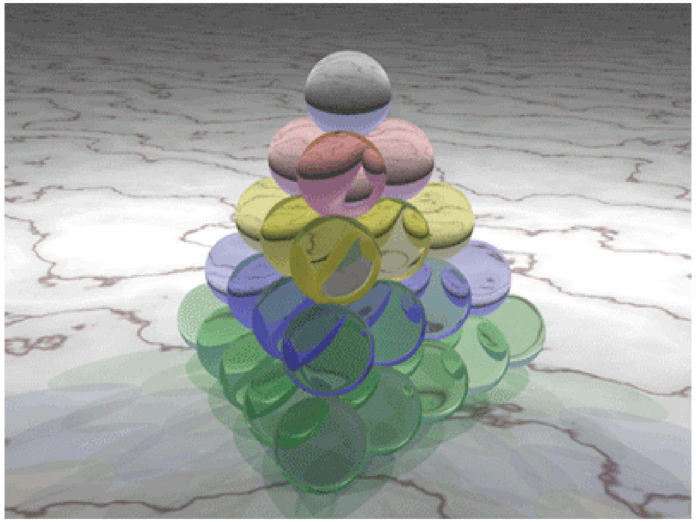
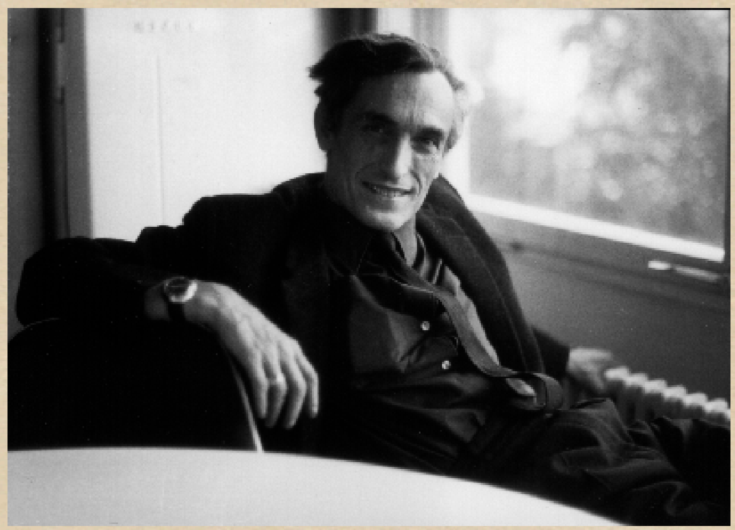
Métamorphoses algorithmiques et merveilleuses coïncidences mathématiques



xavier viennot
LaBRI, CNRS, Université Bordeaux 1

Jeu de taquin



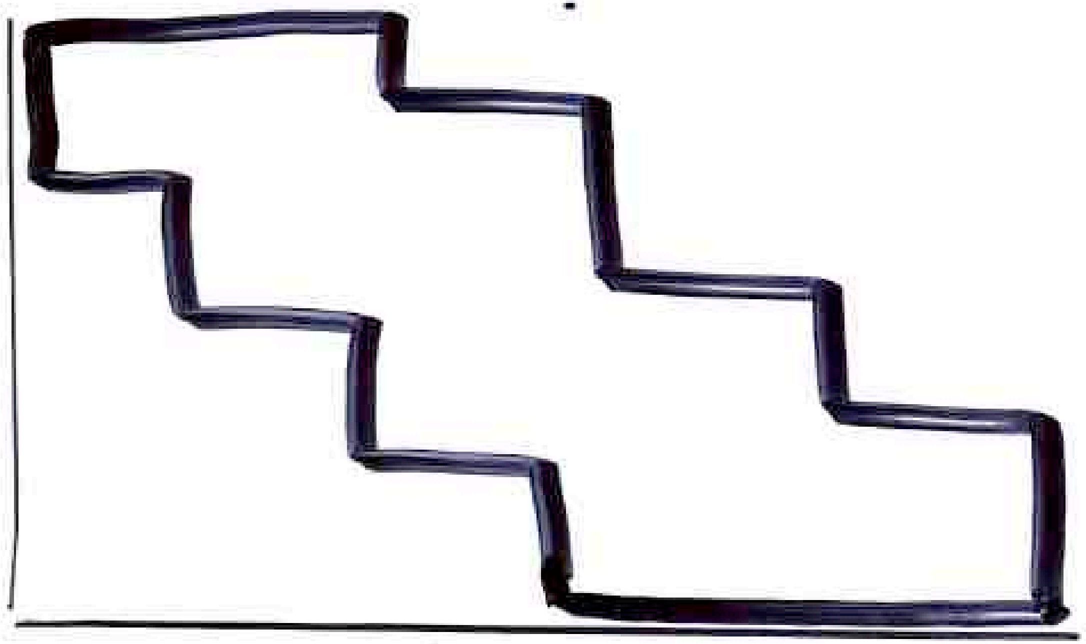
M.P. Schützenberger

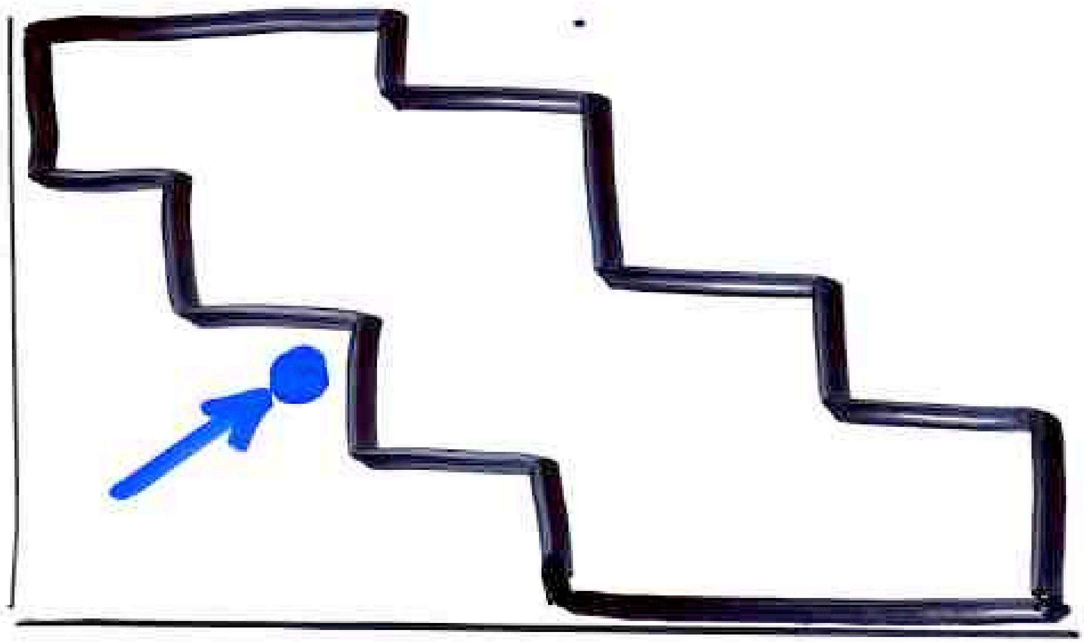
$$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$$

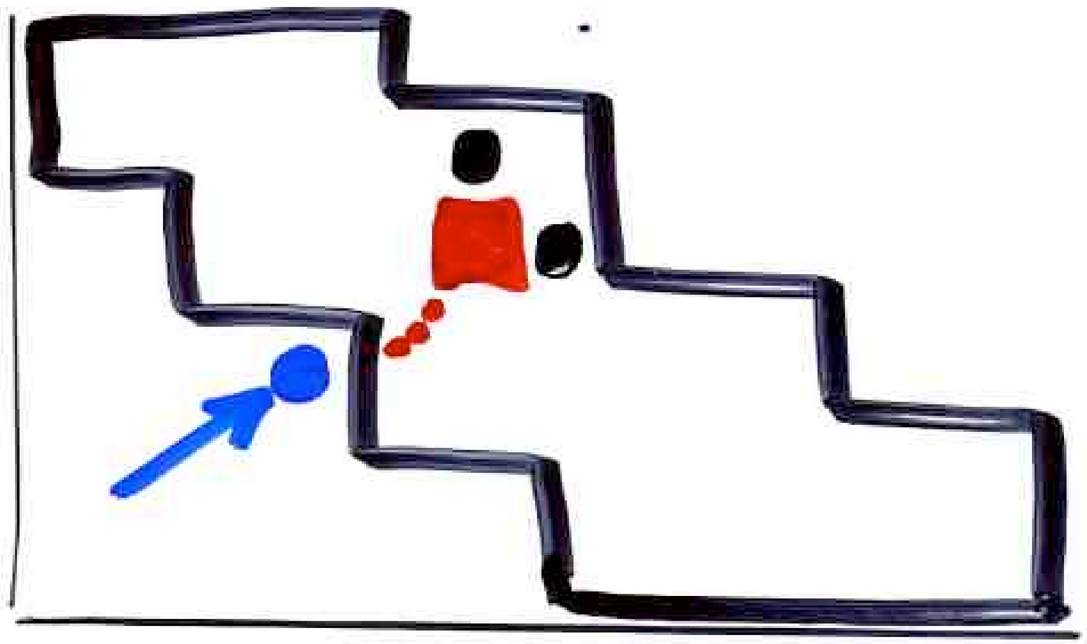
3					
1	6	10			
		2	5	8	
				4	9
					7

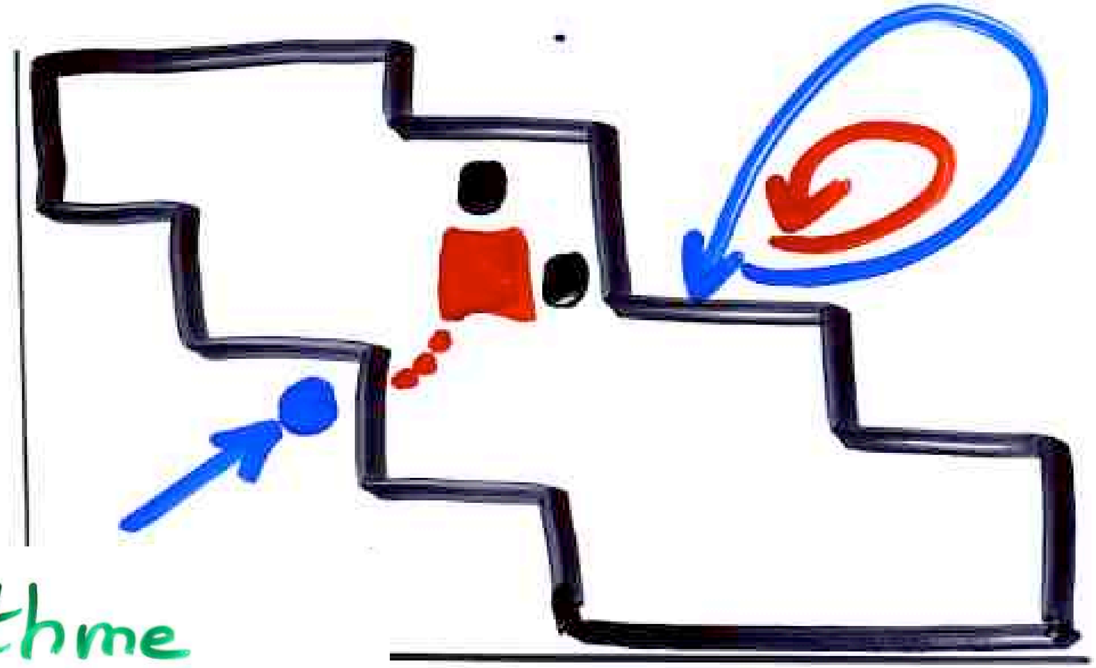
$\sigma = (3, 1, 6, 10, 2, 5, 8, 4, 9, 7)$

6	10				
3	5	8			
1	2	4	7	9	





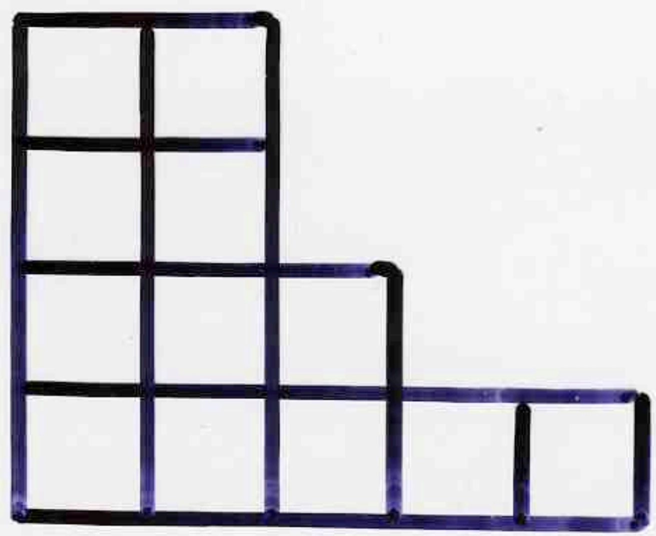


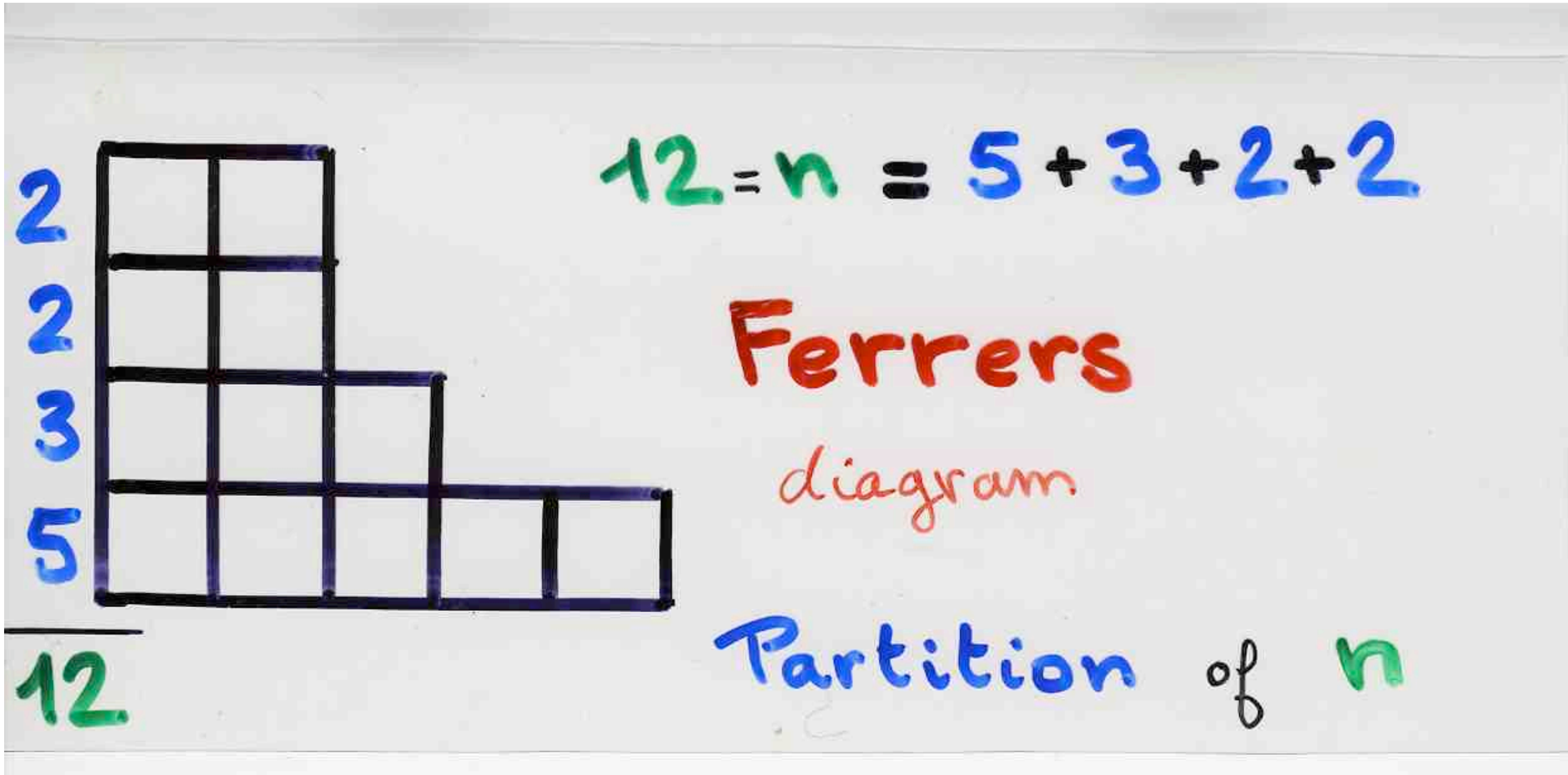


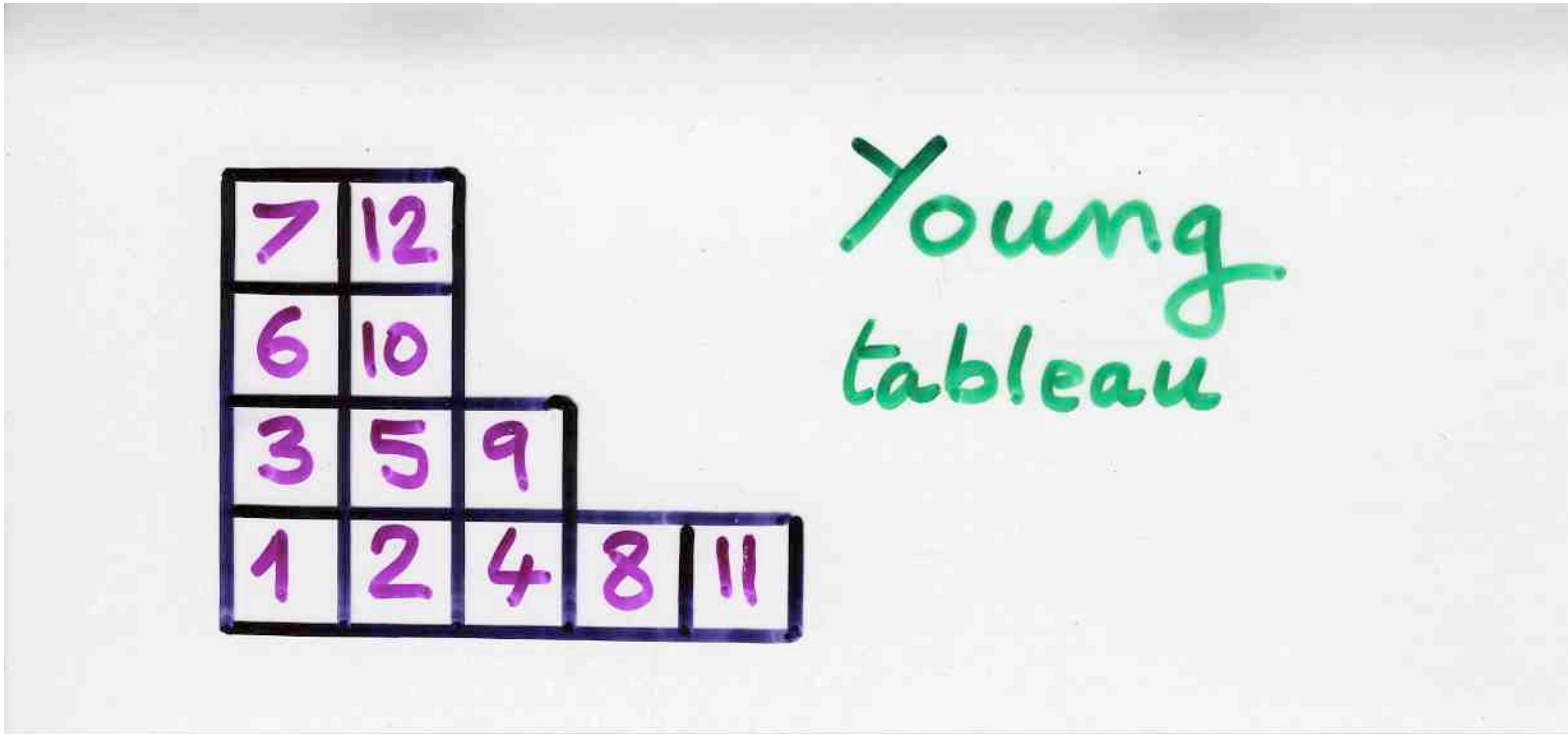
algorithme

non-déterminisme

Tableaux de Young

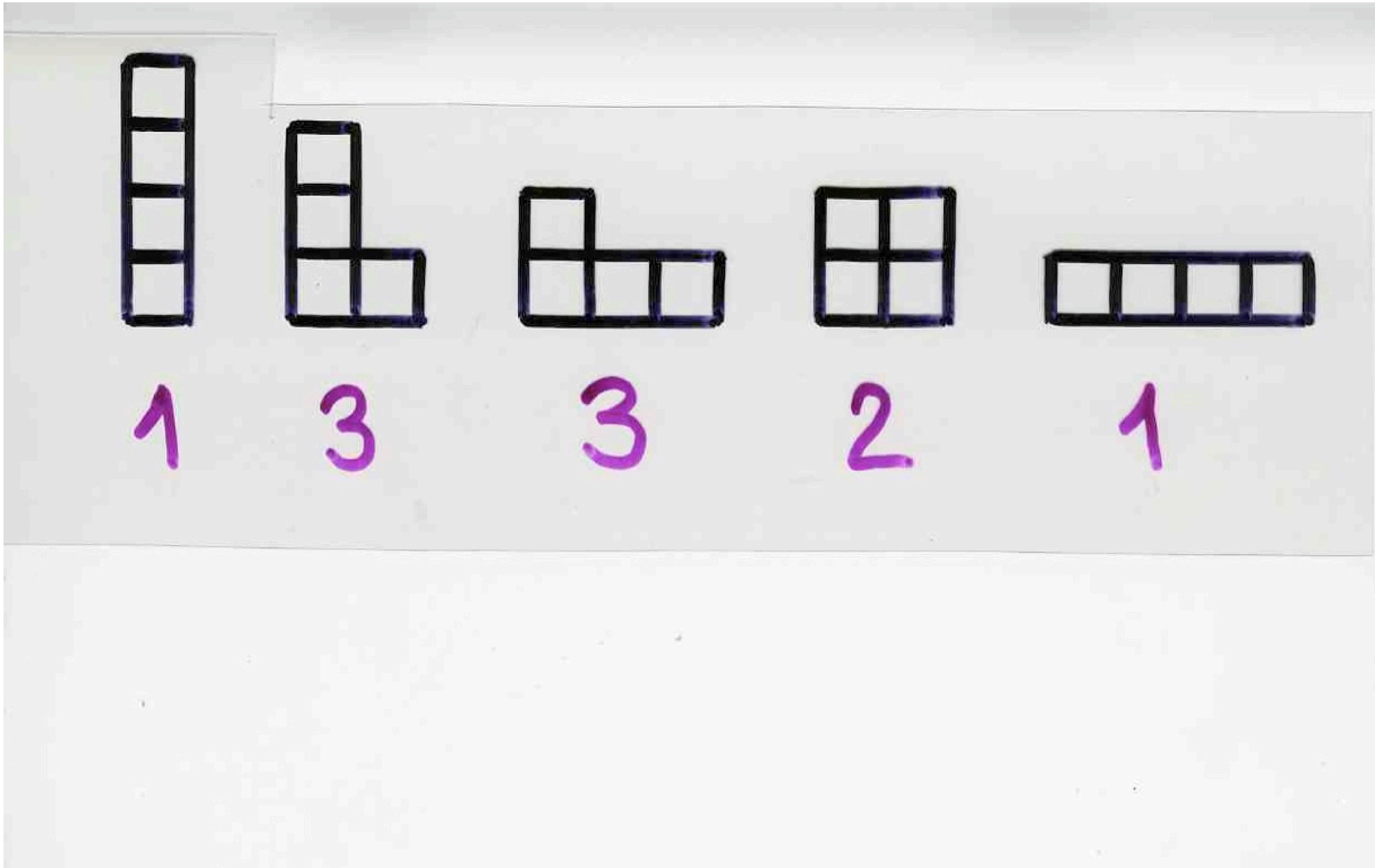


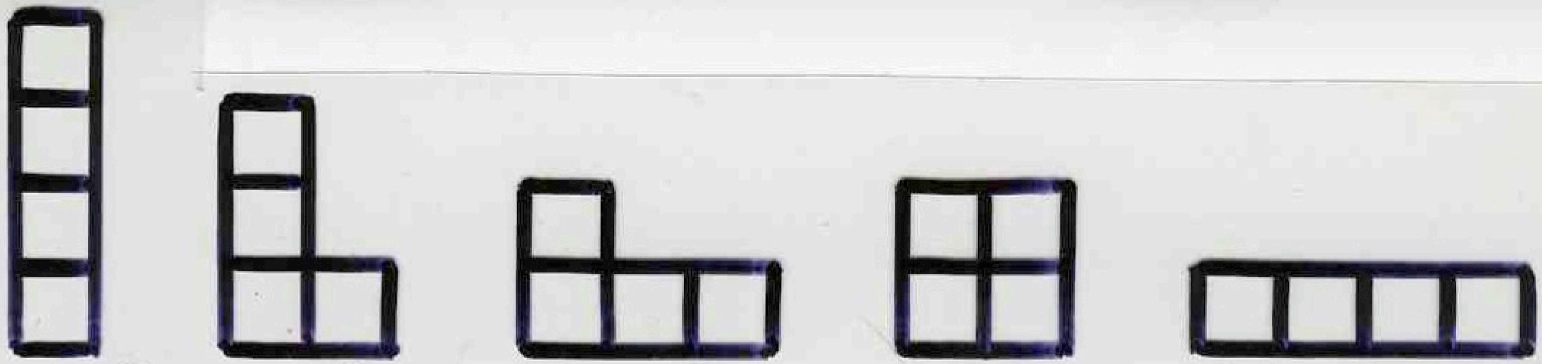

12 = n = 5 + 3 + 2 + 2
Ferrers
diagram
Partition of n
2
2
3
5
12


7 12
6 10
3 5 9
1 2 4 8 11
Young
tableau

Combien ?

Dénombrer ....


1
3
3
2
1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$

nombre de permutations

$$n! = \sum_{\lambda} \left(f_{\lambda} \right)^2$$

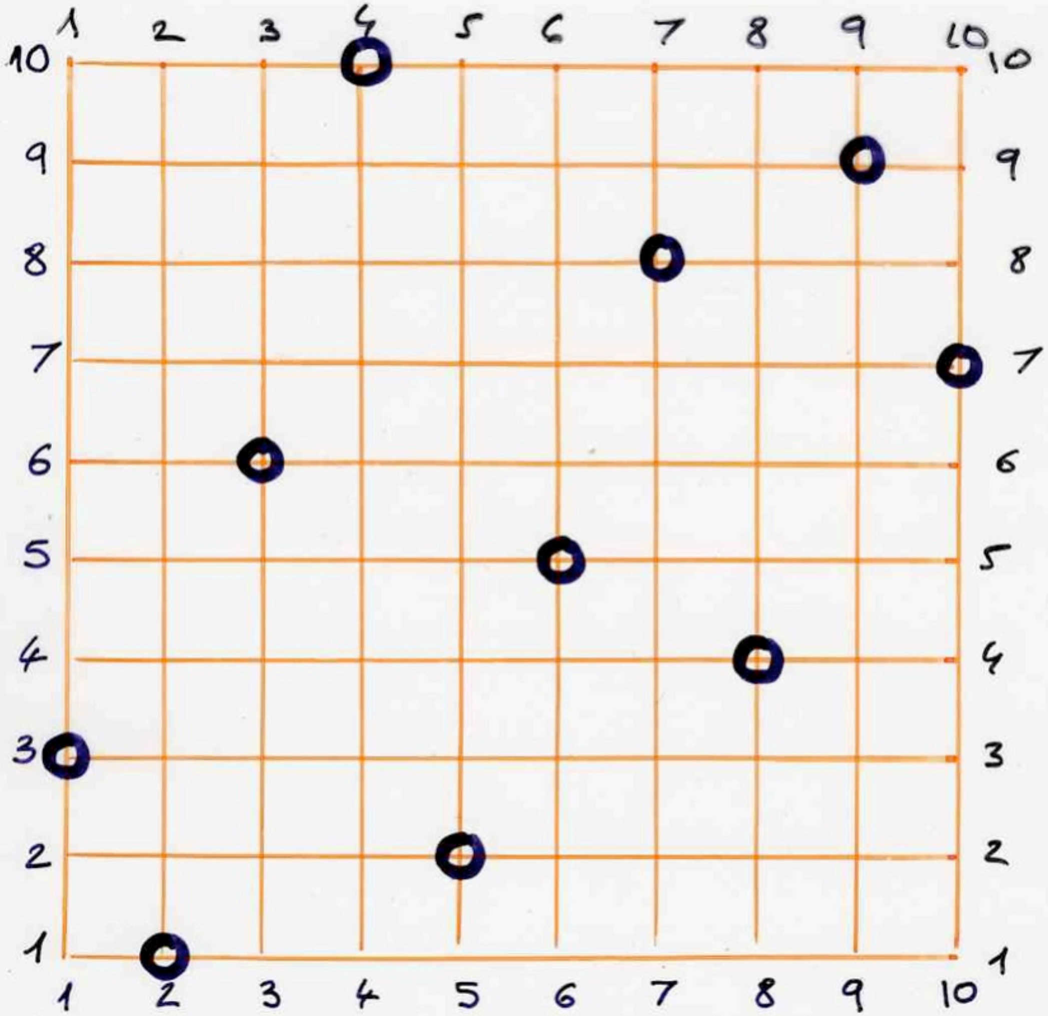
forme λ

n cases

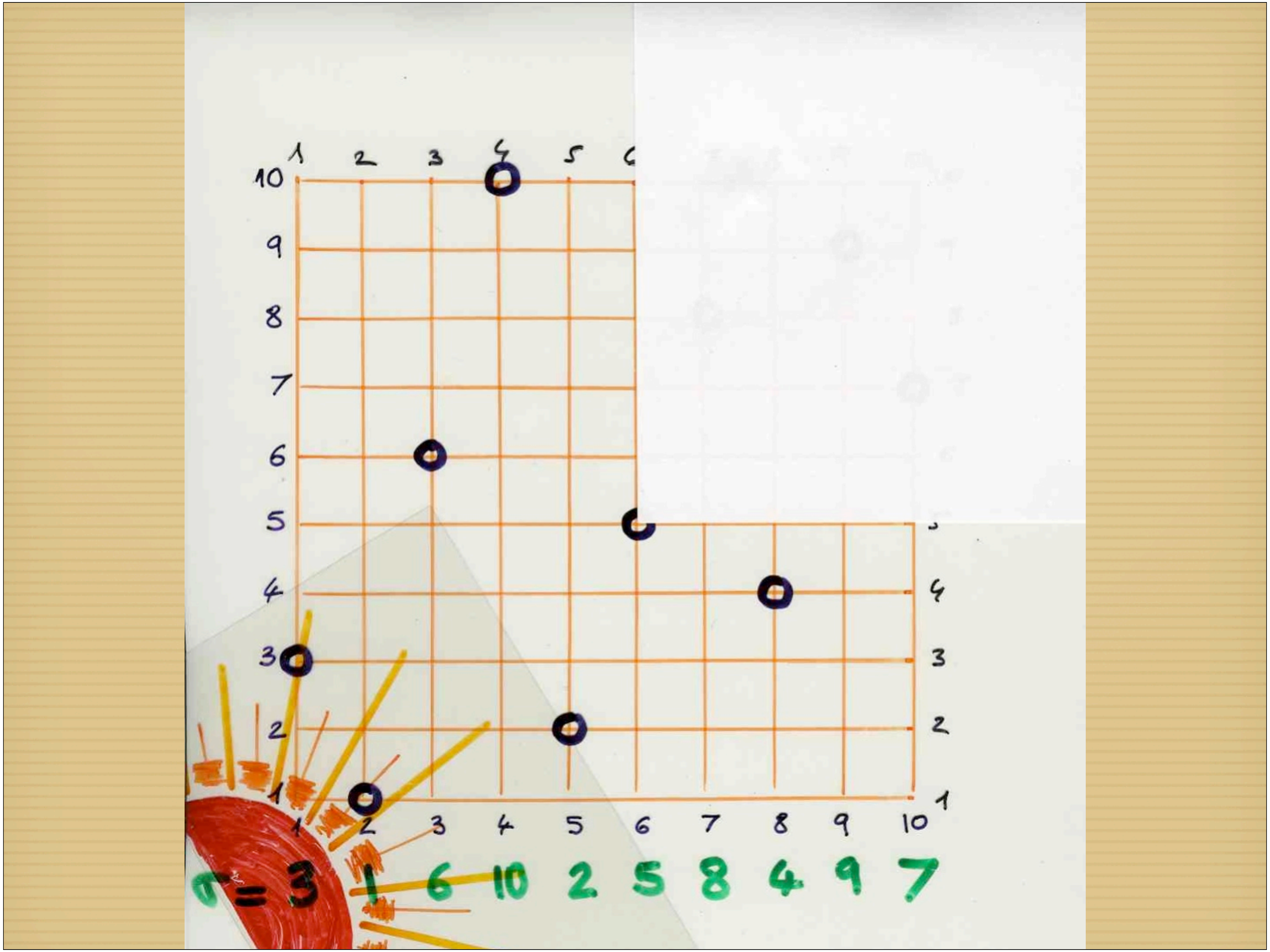
nombre de tableaux de Young de forme λ

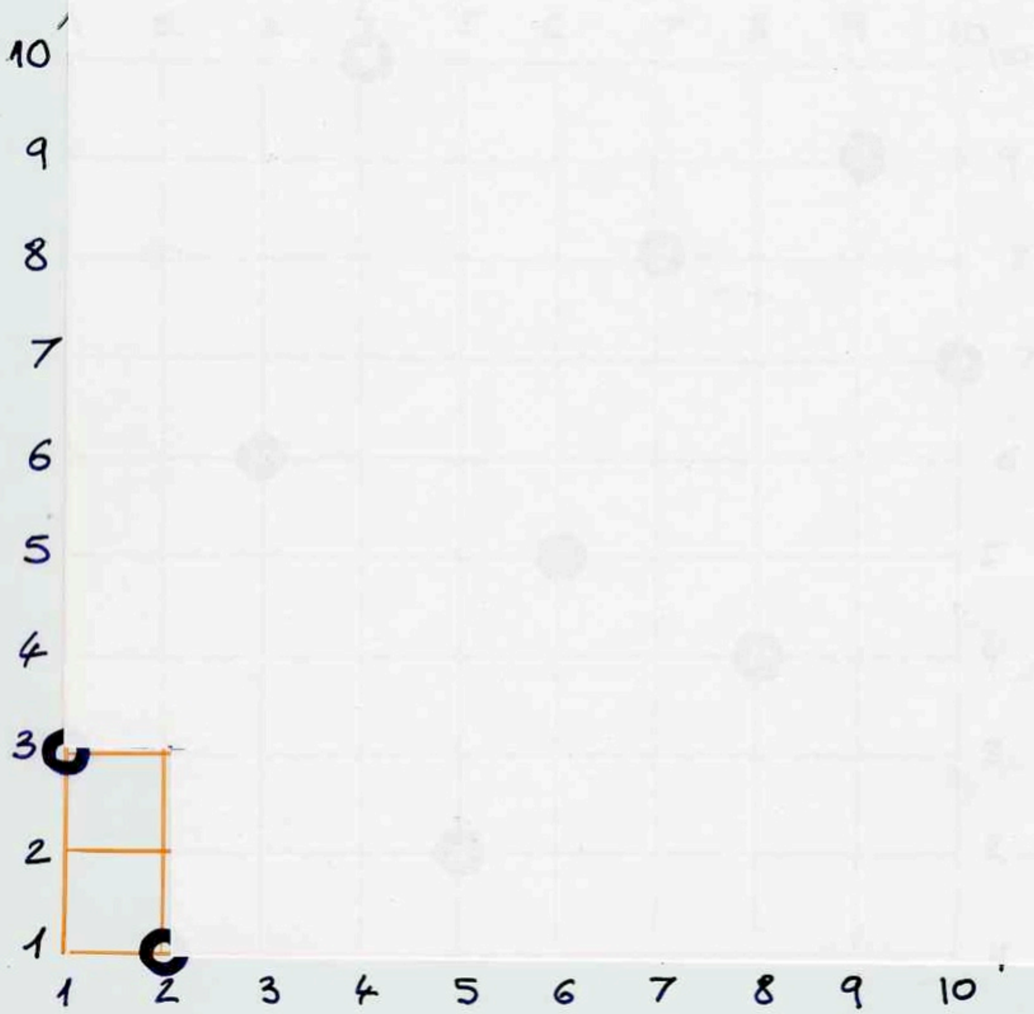
L'algorithme RSK

Robinson - Schensted - Knuth

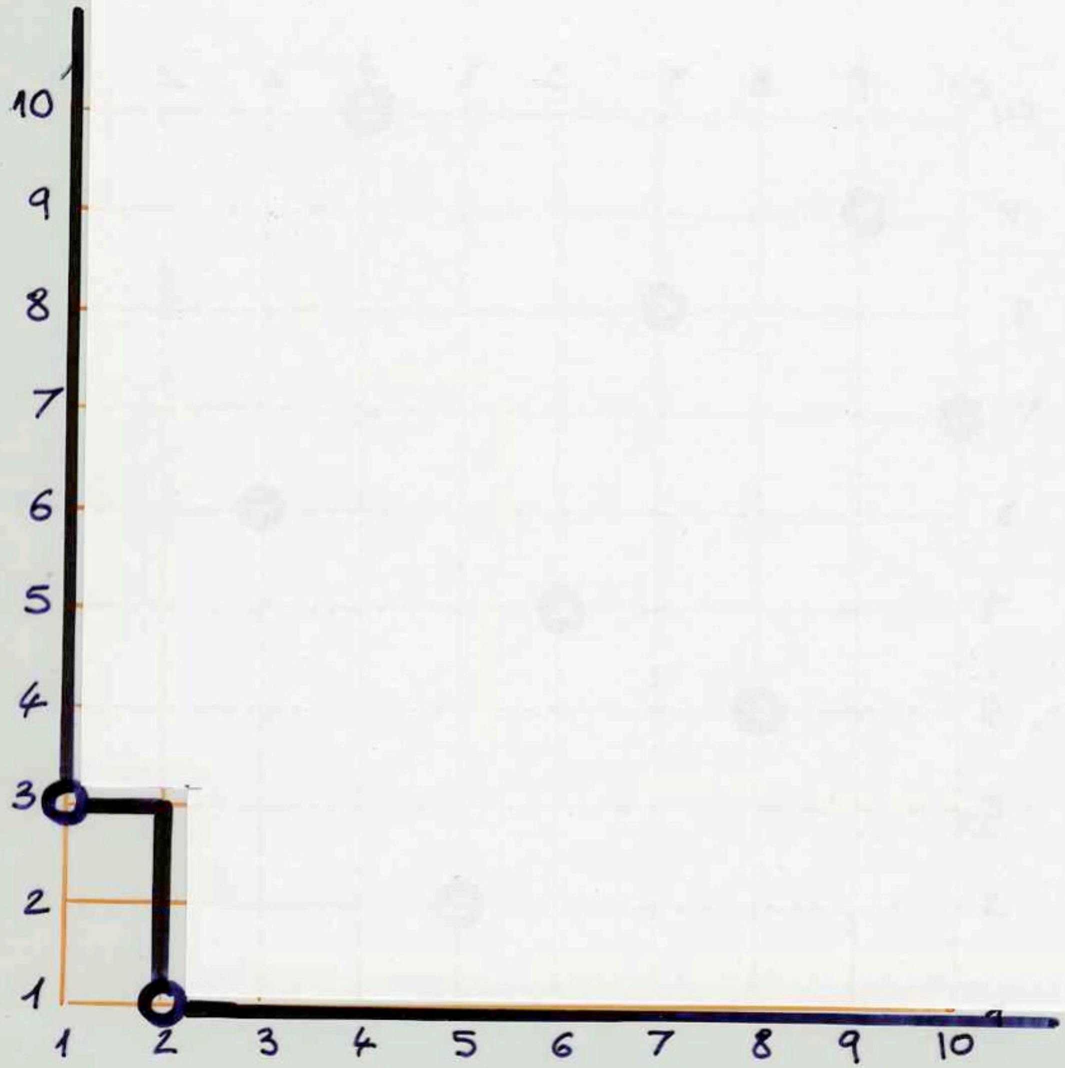


$\sigma = 3 \; 1 \; 6 \; 10 \; 2 \; 5 \; 8 \; 4 \; 9 \; 7$

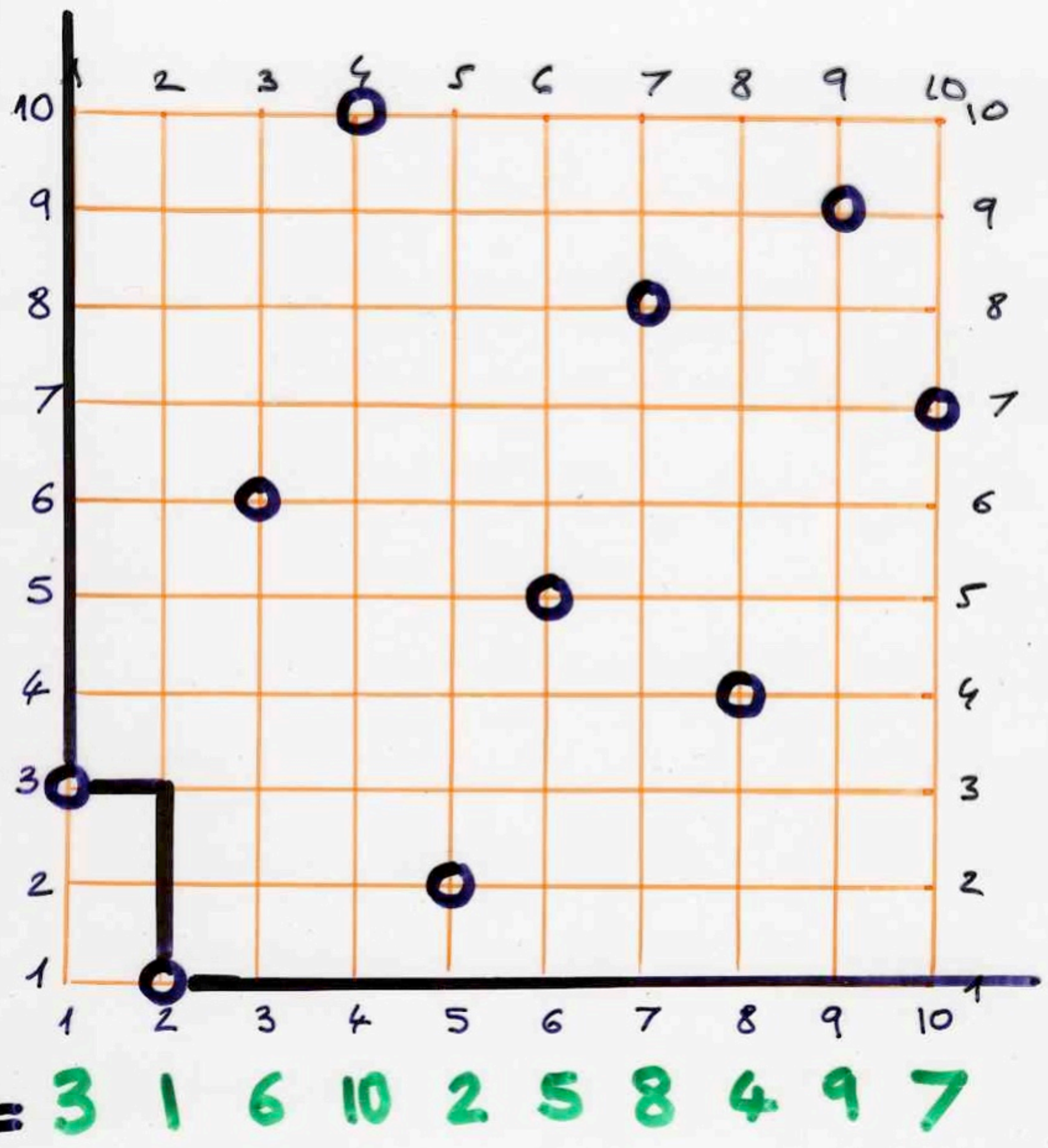

1 2 3 4 5 6 7 8 9 10
σ = 3 1 6 10 2 5 8 4 9 7

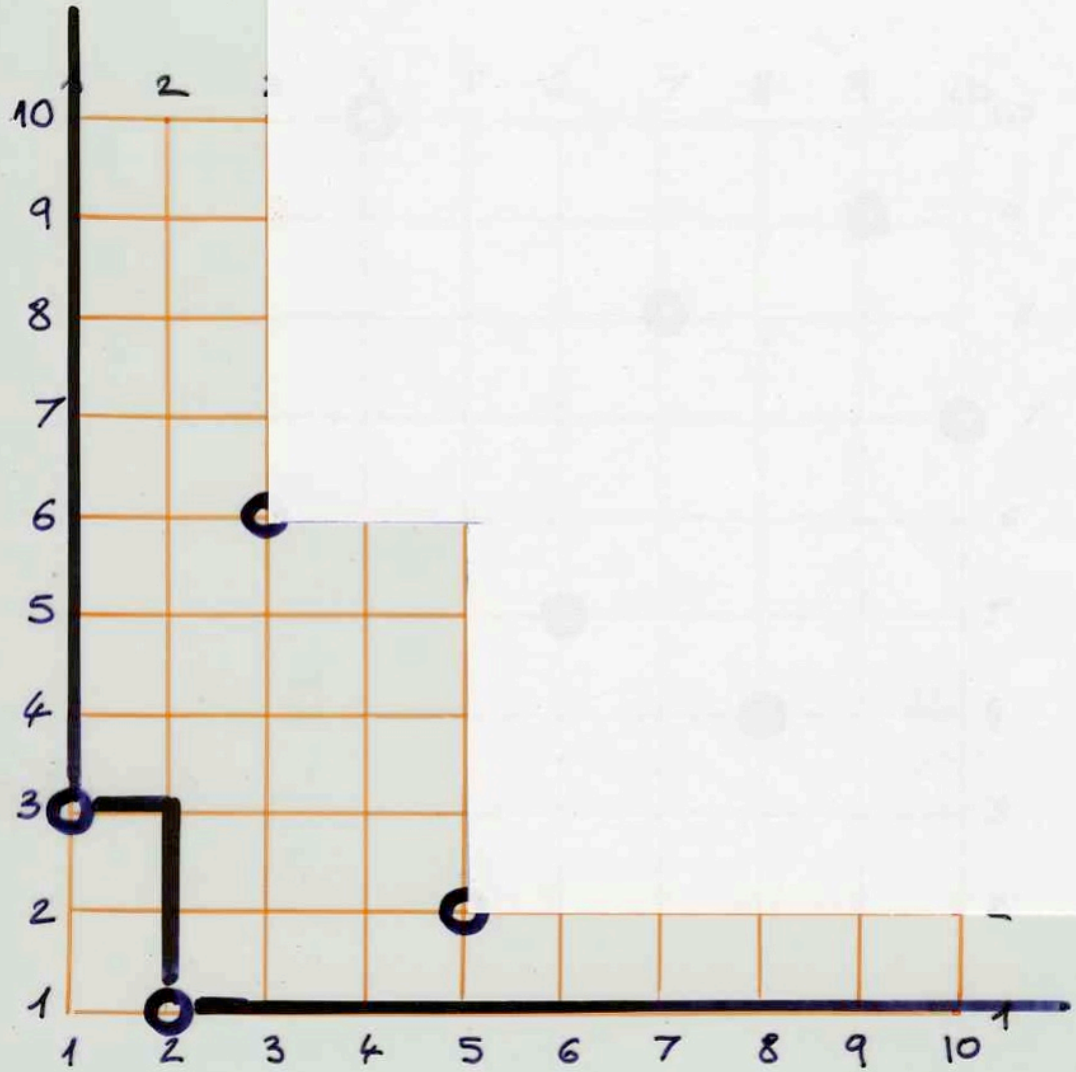


$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

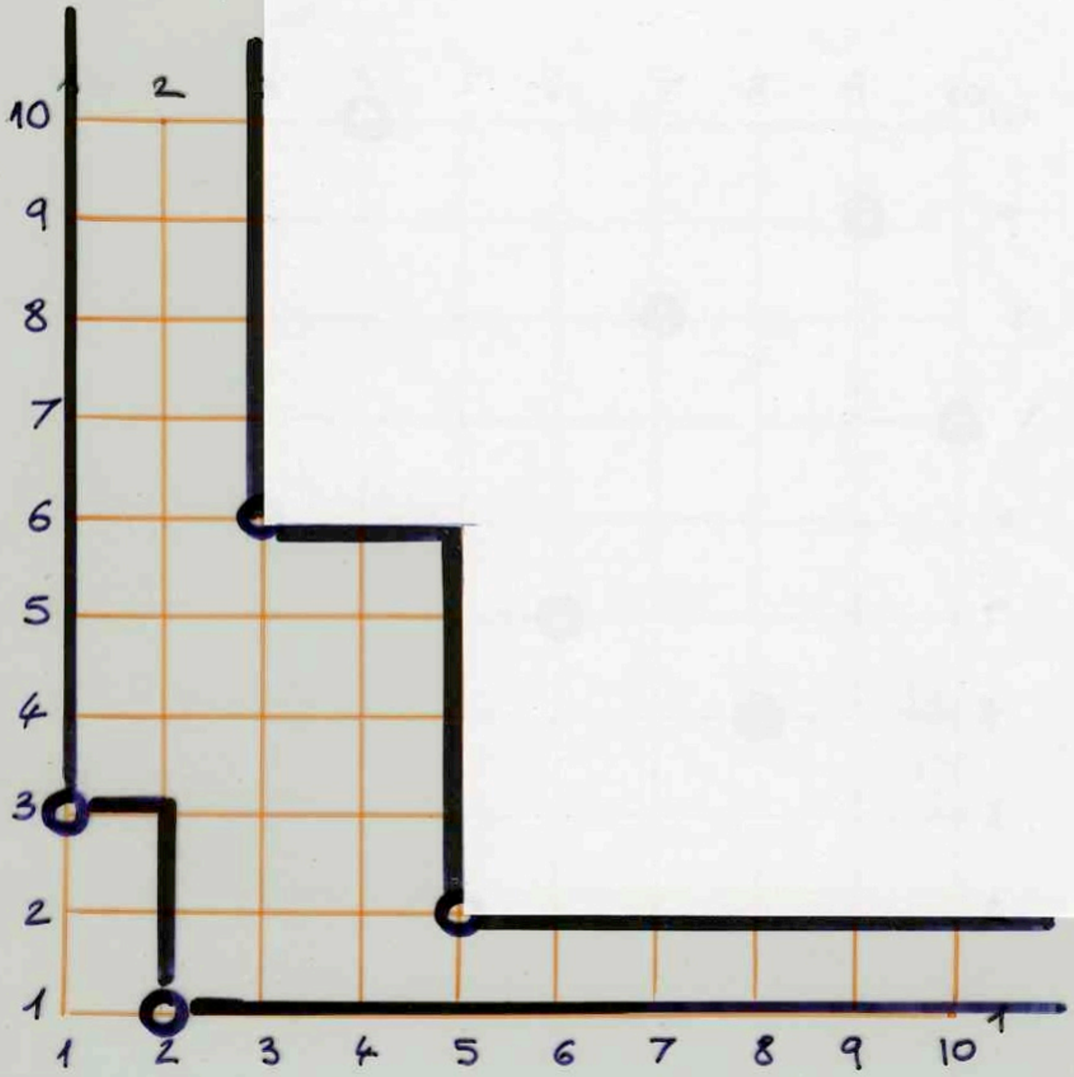


σ = 3 1 6 10 2 5 8 4 9 7

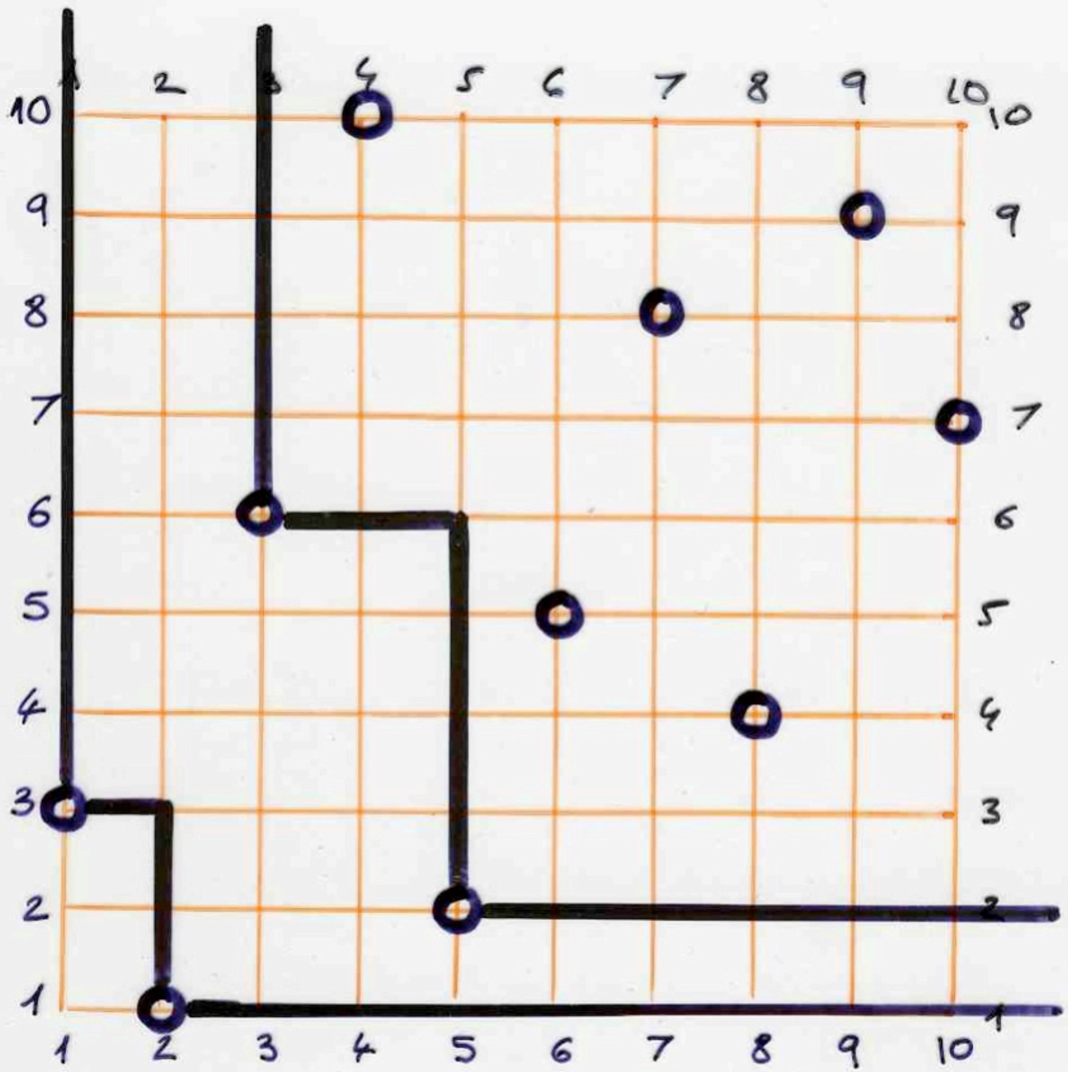

1 2 3 4 5 6 7 8 9 10
10 9 8 7 6 5 4 3 2 1
σ = 3 1 6 10 2 5 8 4 9 7



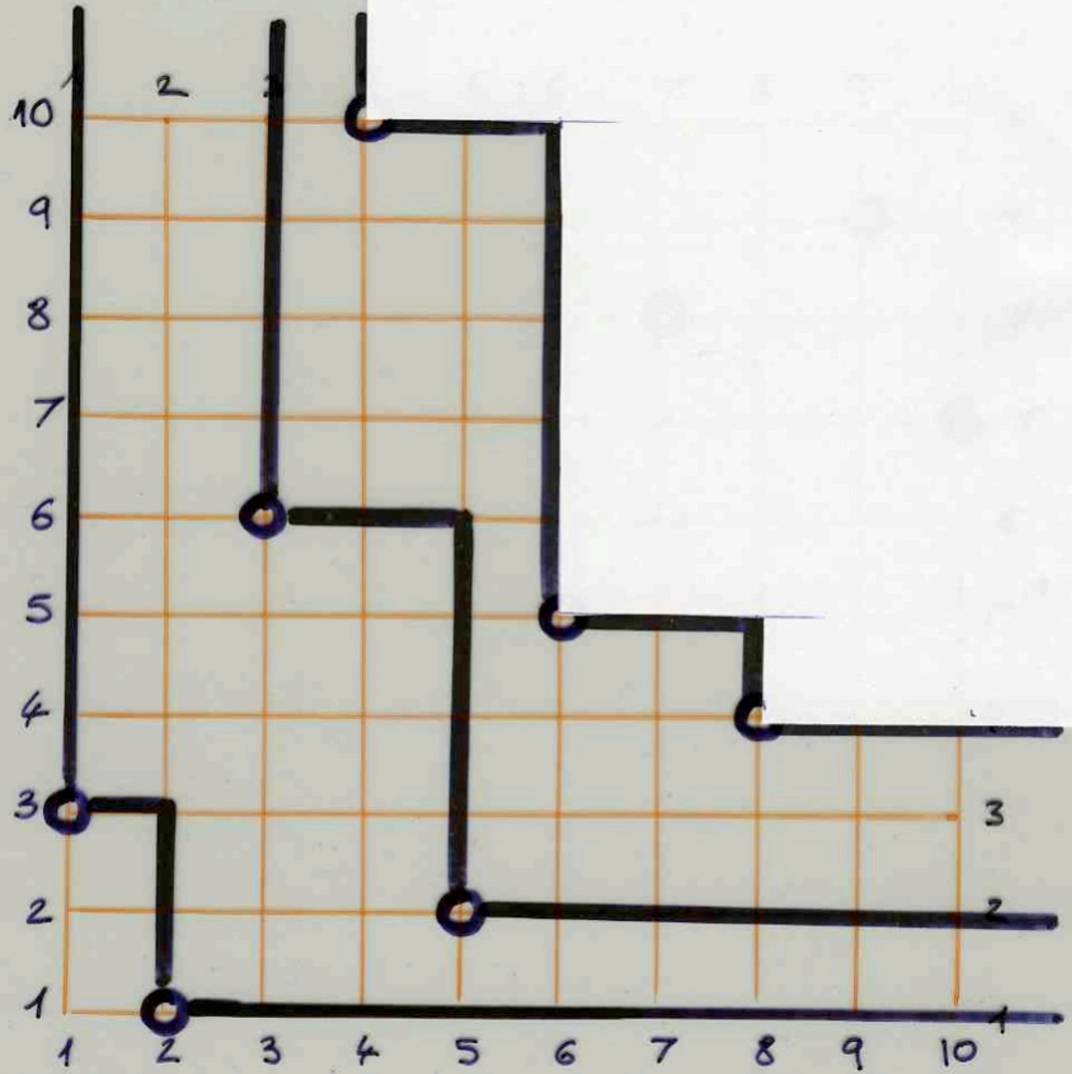
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



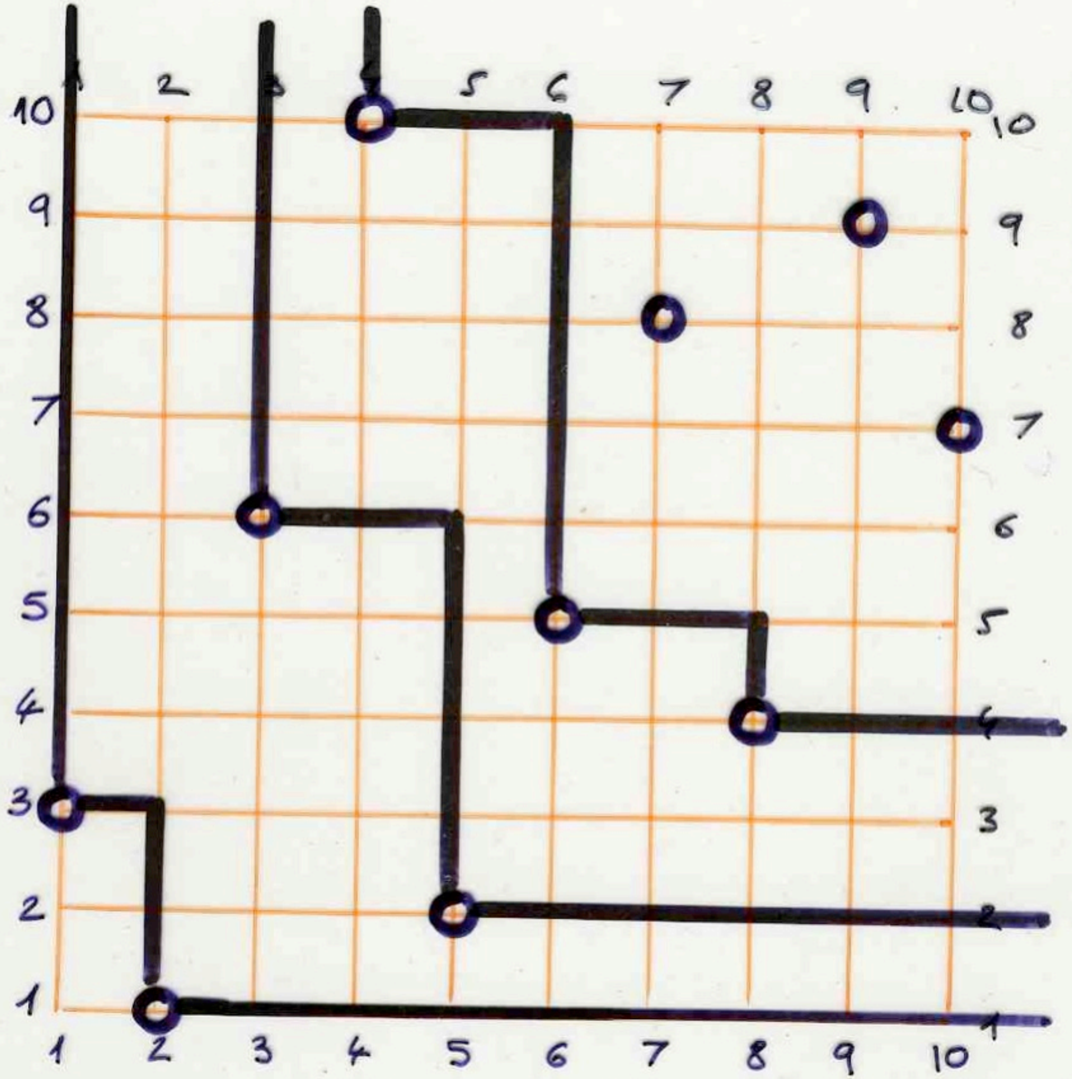
σ = 3 1 6 10 2 5 8 4 9 7



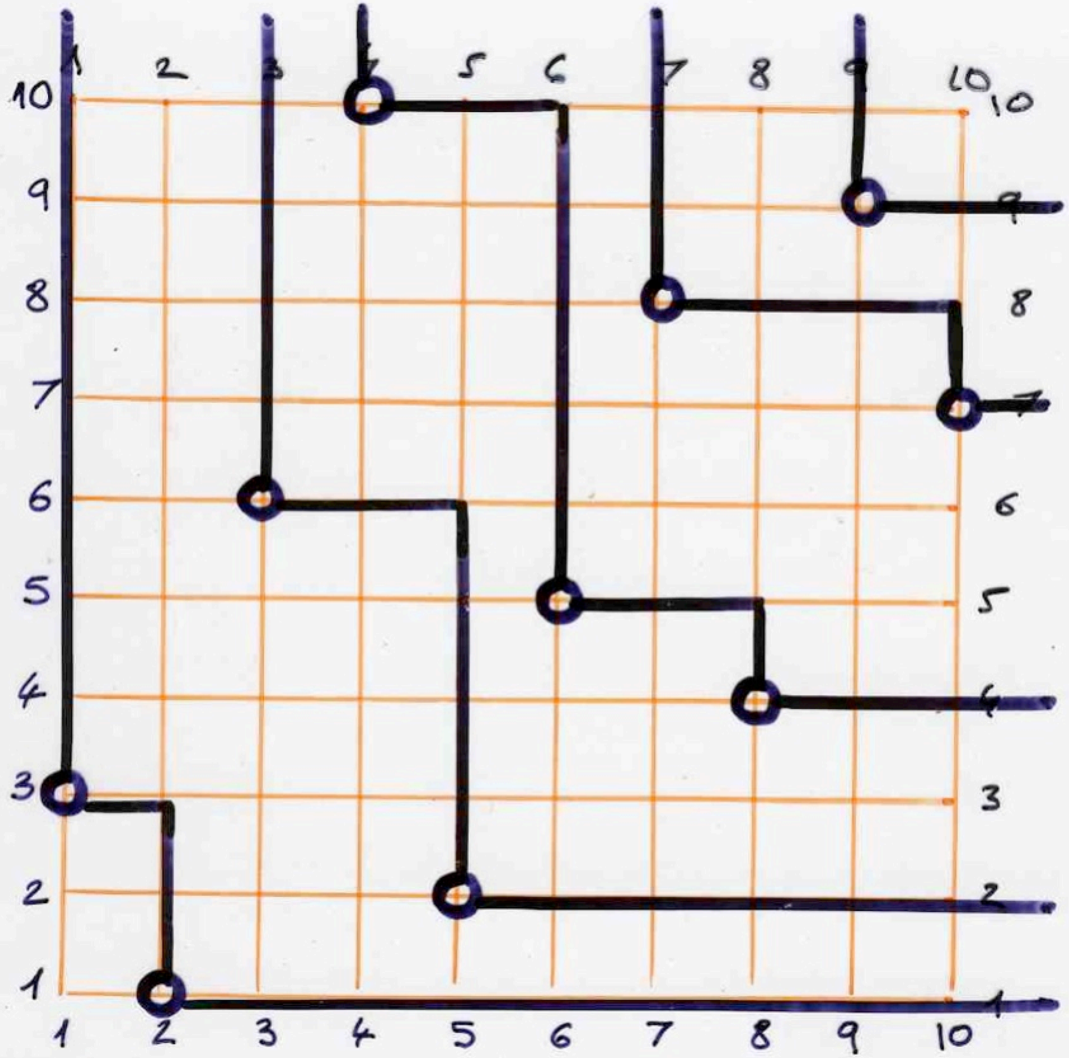
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



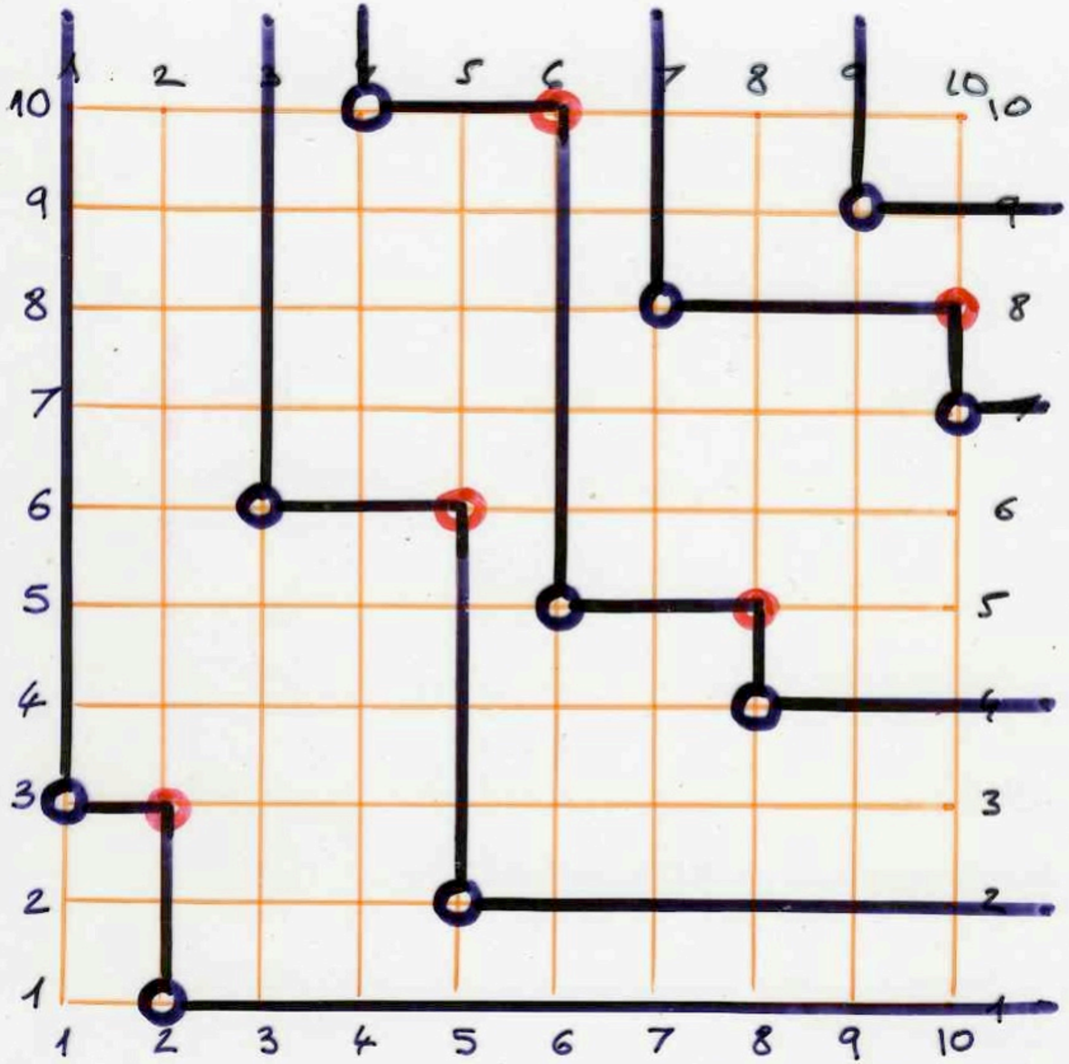
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



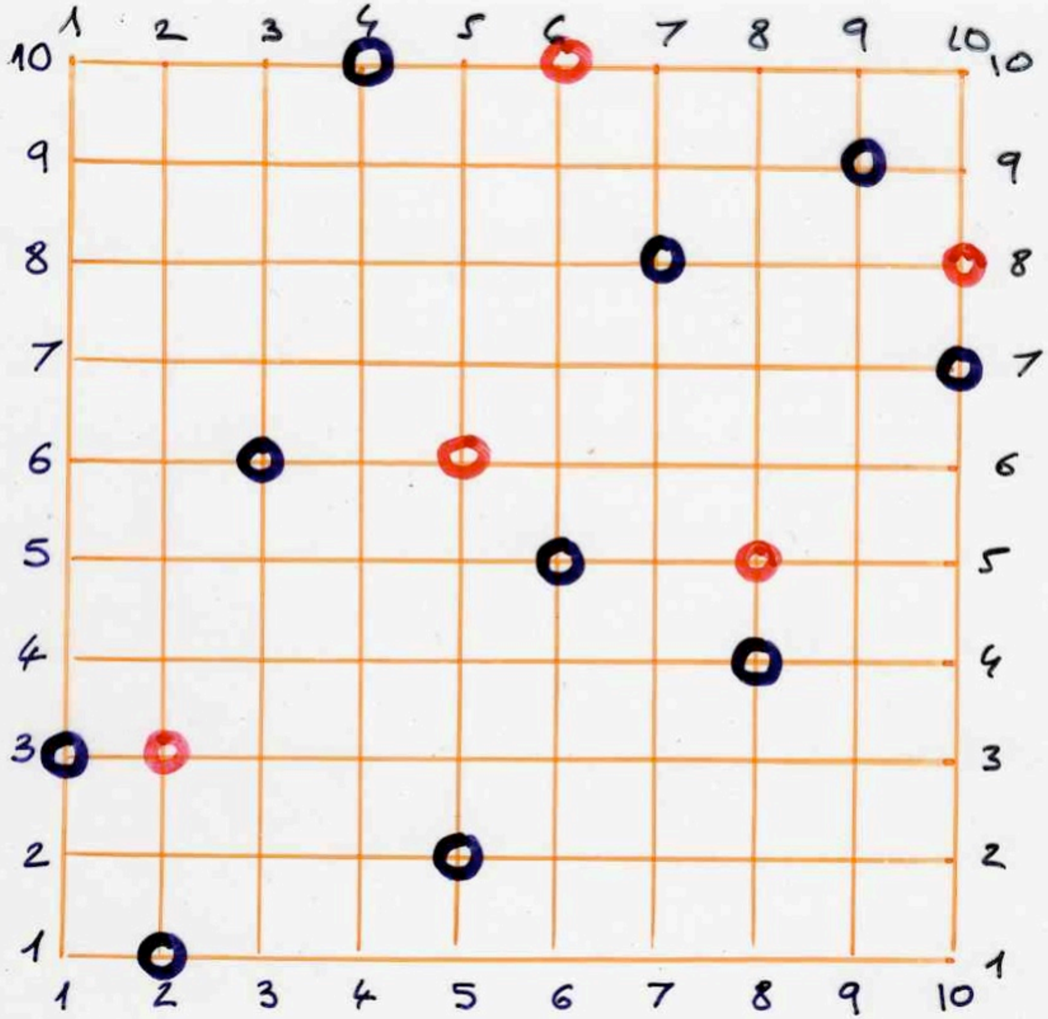
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



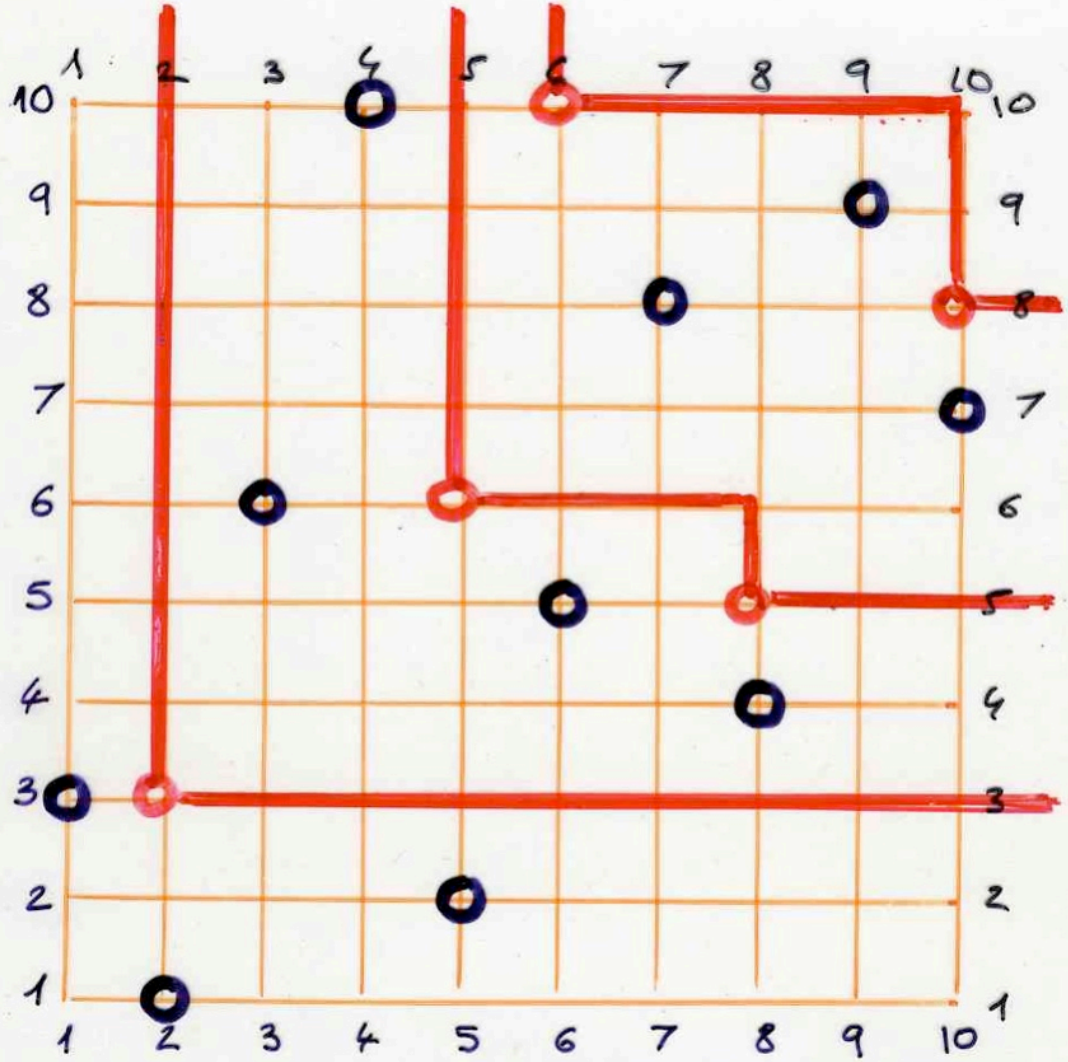
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



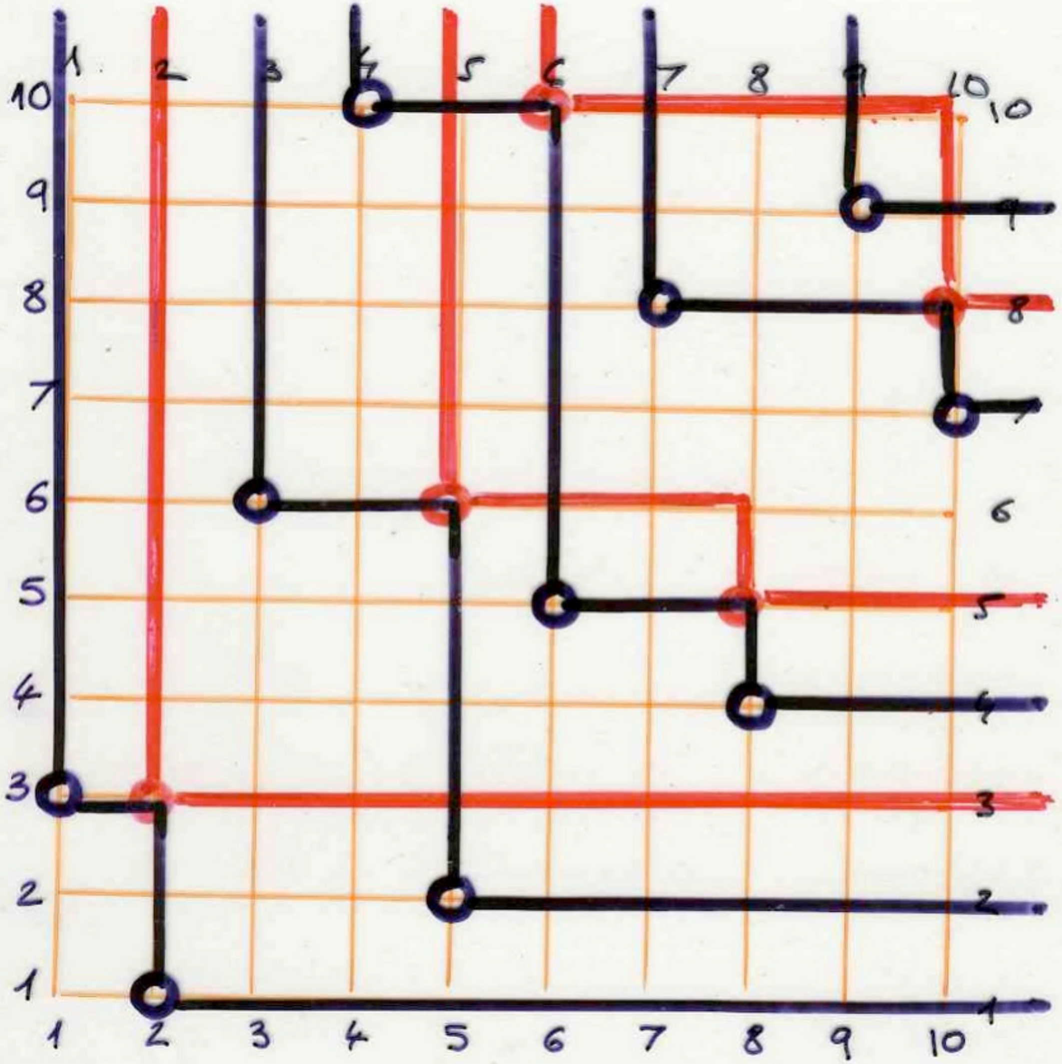
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



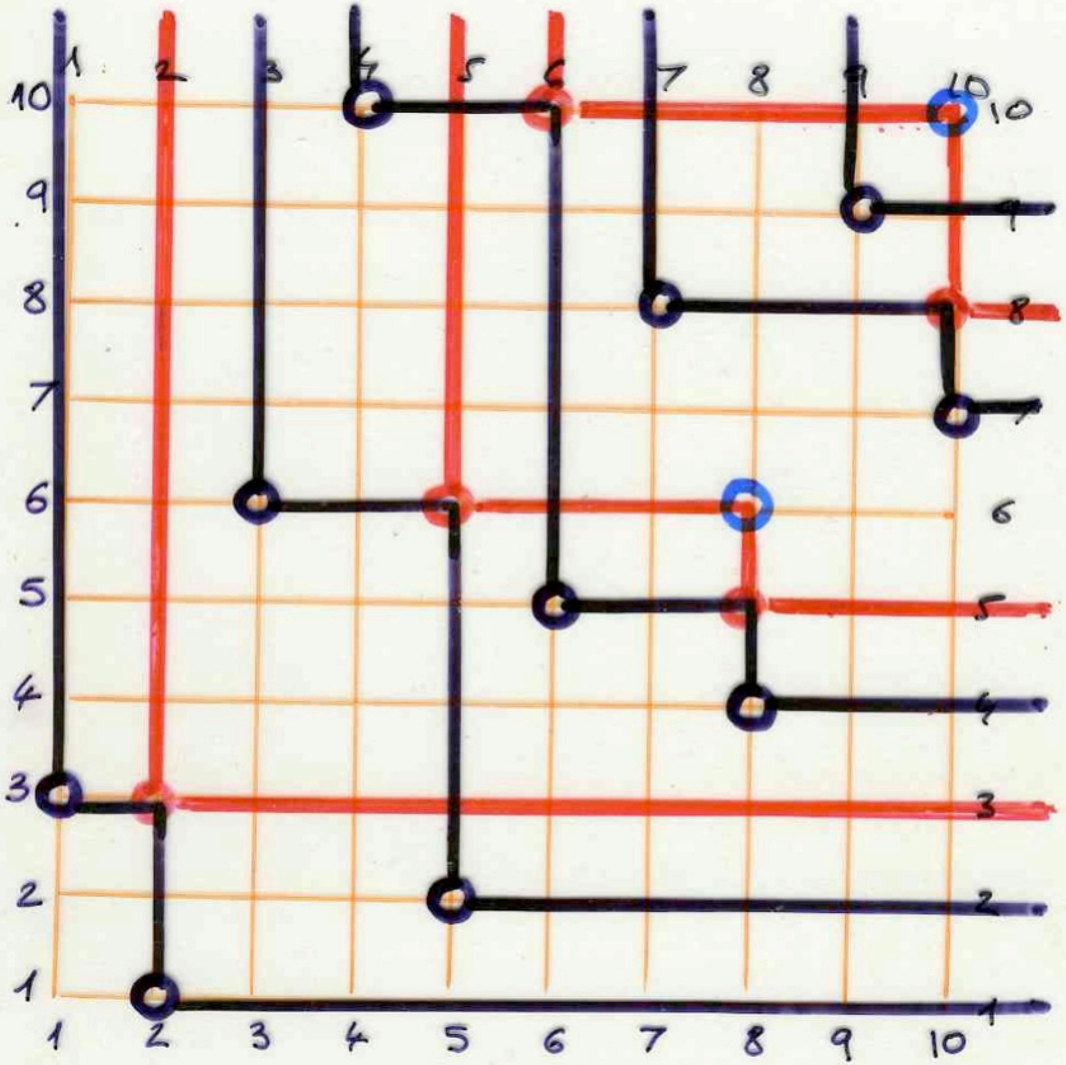
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



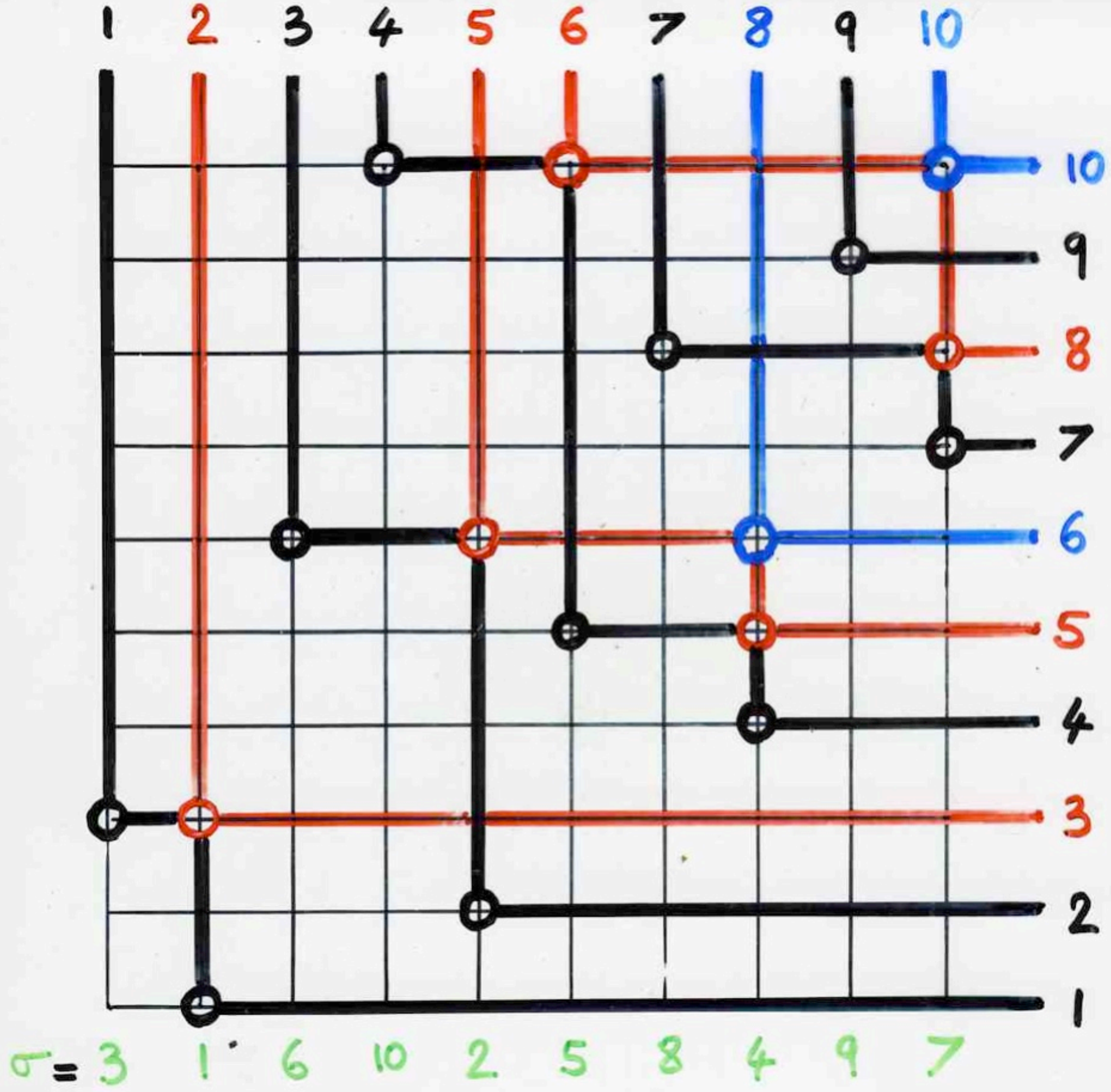
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$



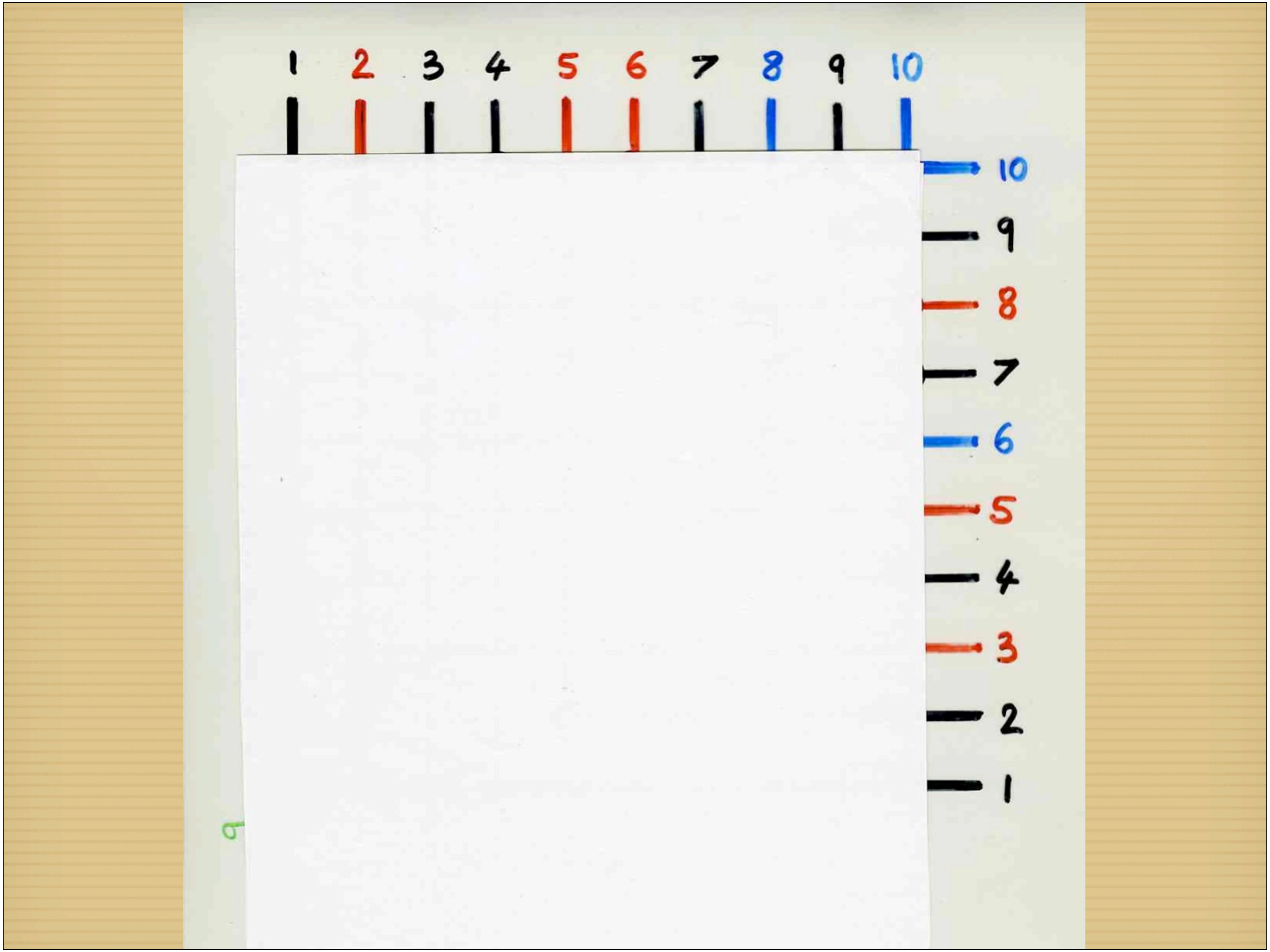
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

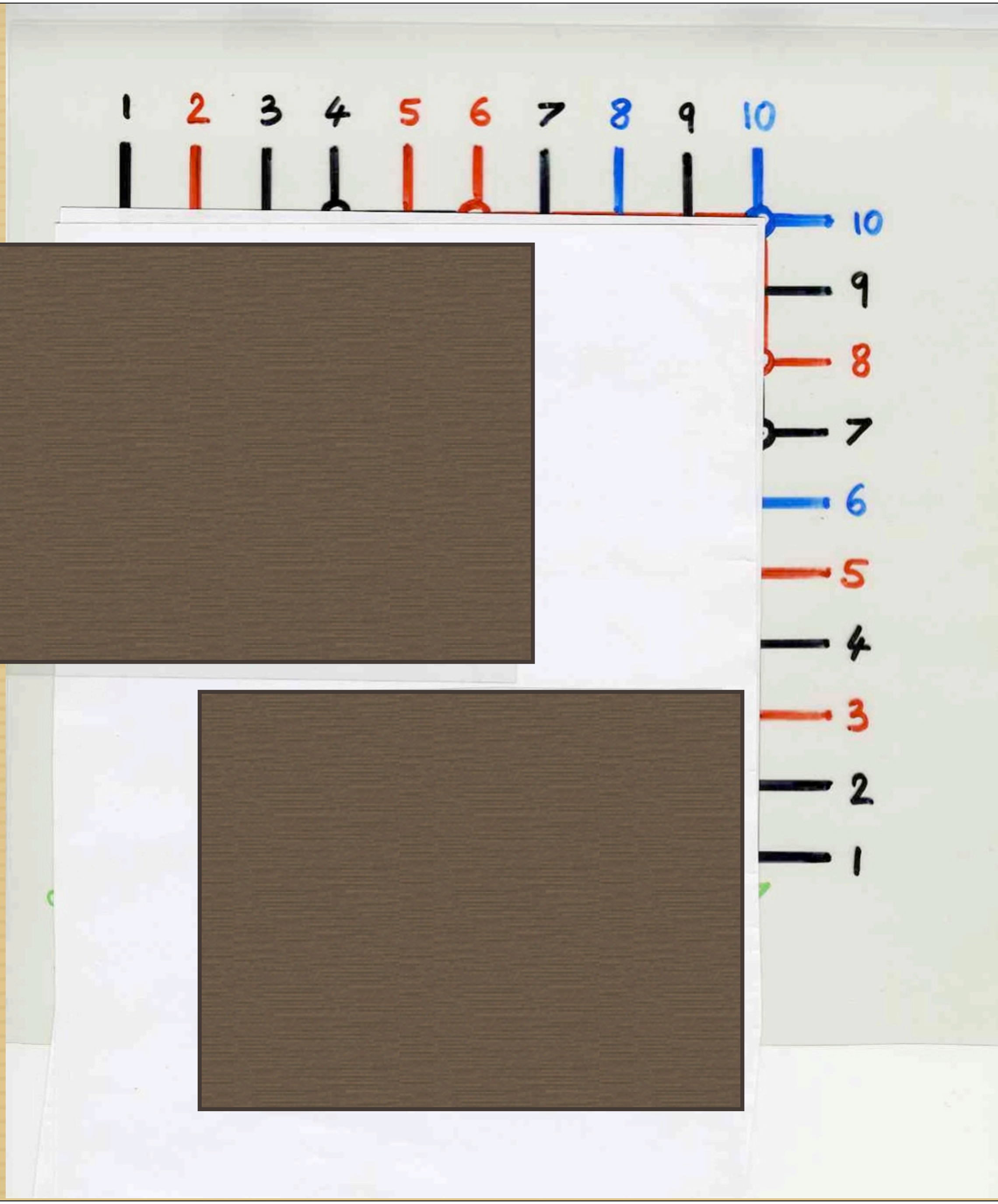


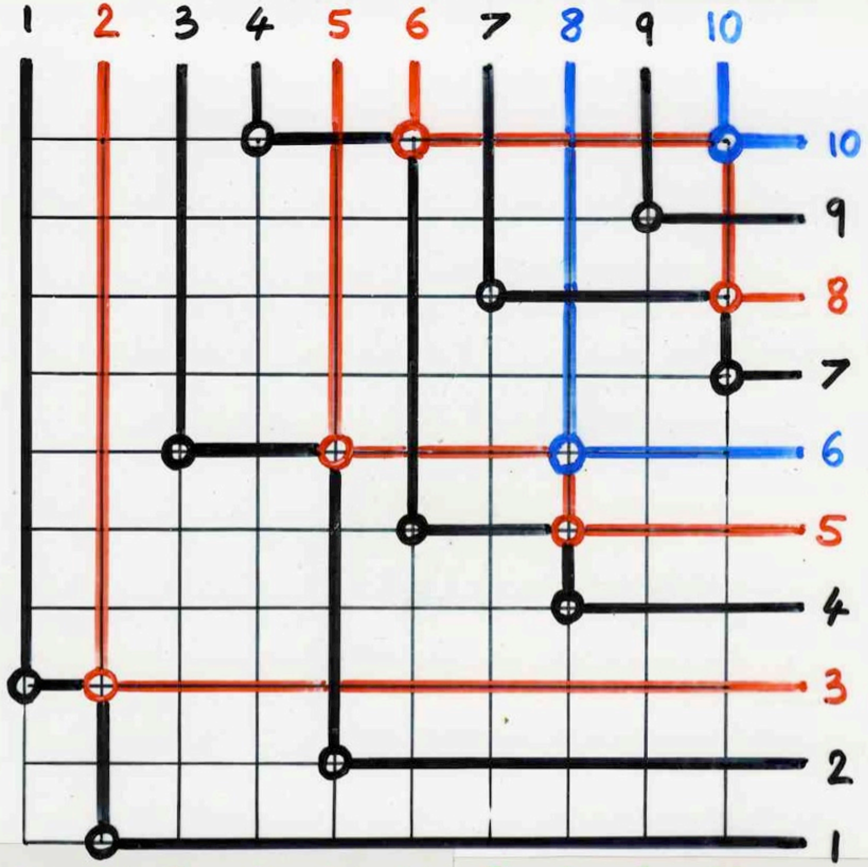
σ = 3 1 6 10 2 5 8 4 9 7



1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1
σ = 3 1 6 10 2 5 8 4 9 7


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1


1 2 3 4 5 6 7 8 9 10
10
9
8
7
6
5
4
3
2
1



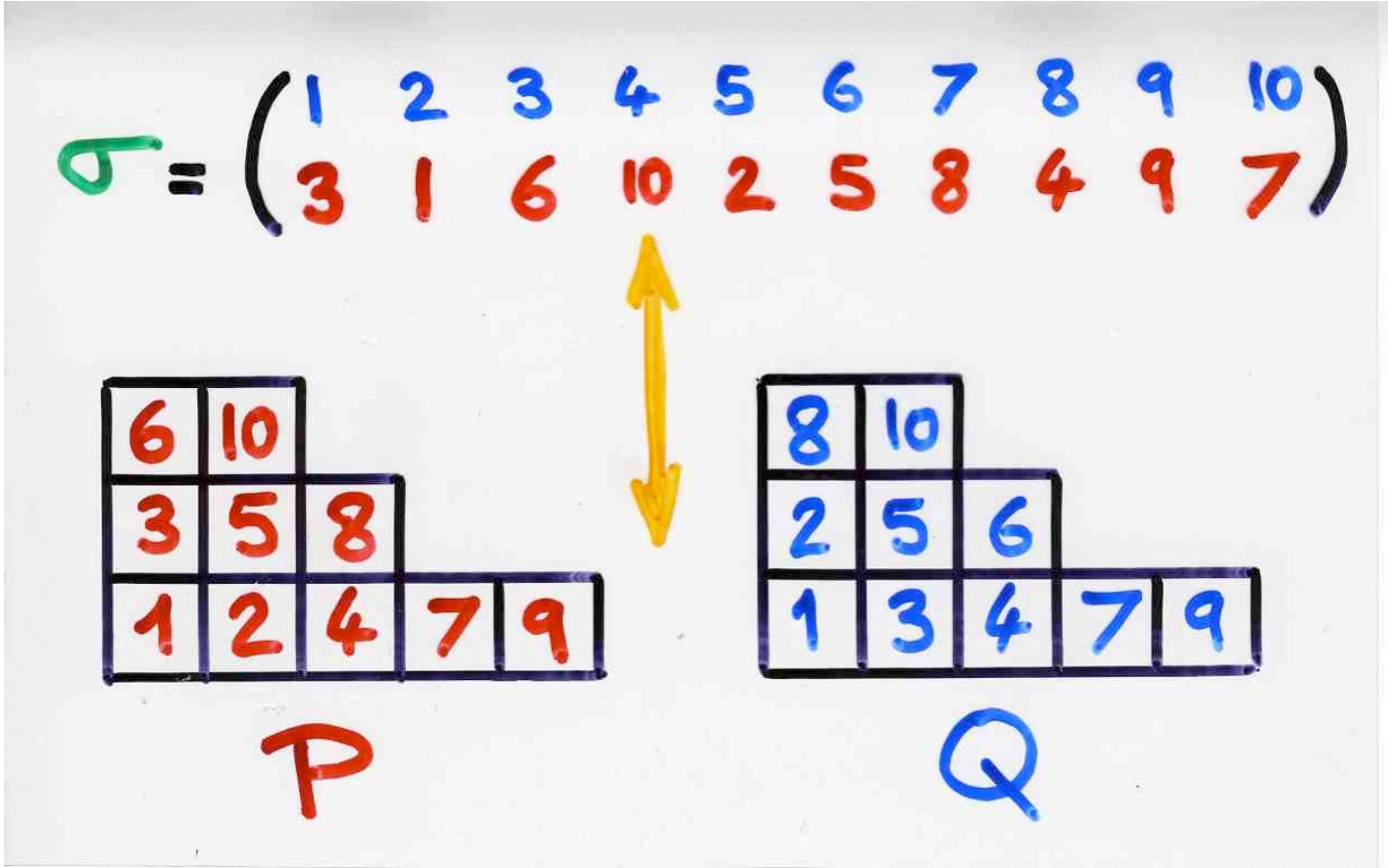
$\sigma = 3\ 1\ 6\ 10\ 2\ 5\ 8\ 4\ 9\ 7$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q



σ =
1 2 3 4 5 6 7 8 9 10
3 1 6 10 2 5 8 4 9 7
6 10
3 5 8
1 2 4 7 9
P
8 10
2 5 6
1 3 4 7 9
Q

Donald Knuth (1972)

" The unusual nature of these coincidences might lead us to suspect that some sort of withcraft is operating behind the scenes "

nombre de permutations

$$n! = \sum_{\lambda} \left(f_{\lambda} \right)^2$$

forme

n cases

nombre de tableaux de Young de forme λ

$=$

meilleure
compréhension

$\longleftrightarrow$

dessiner des calculs

...

calculer des dessins

G fini

$$|G| = \sum_{\varphi} \deg^2(\varphi)$$

représentation irréductible

$$n! = \sum_{\lambda} f_{\lambda}^2$$

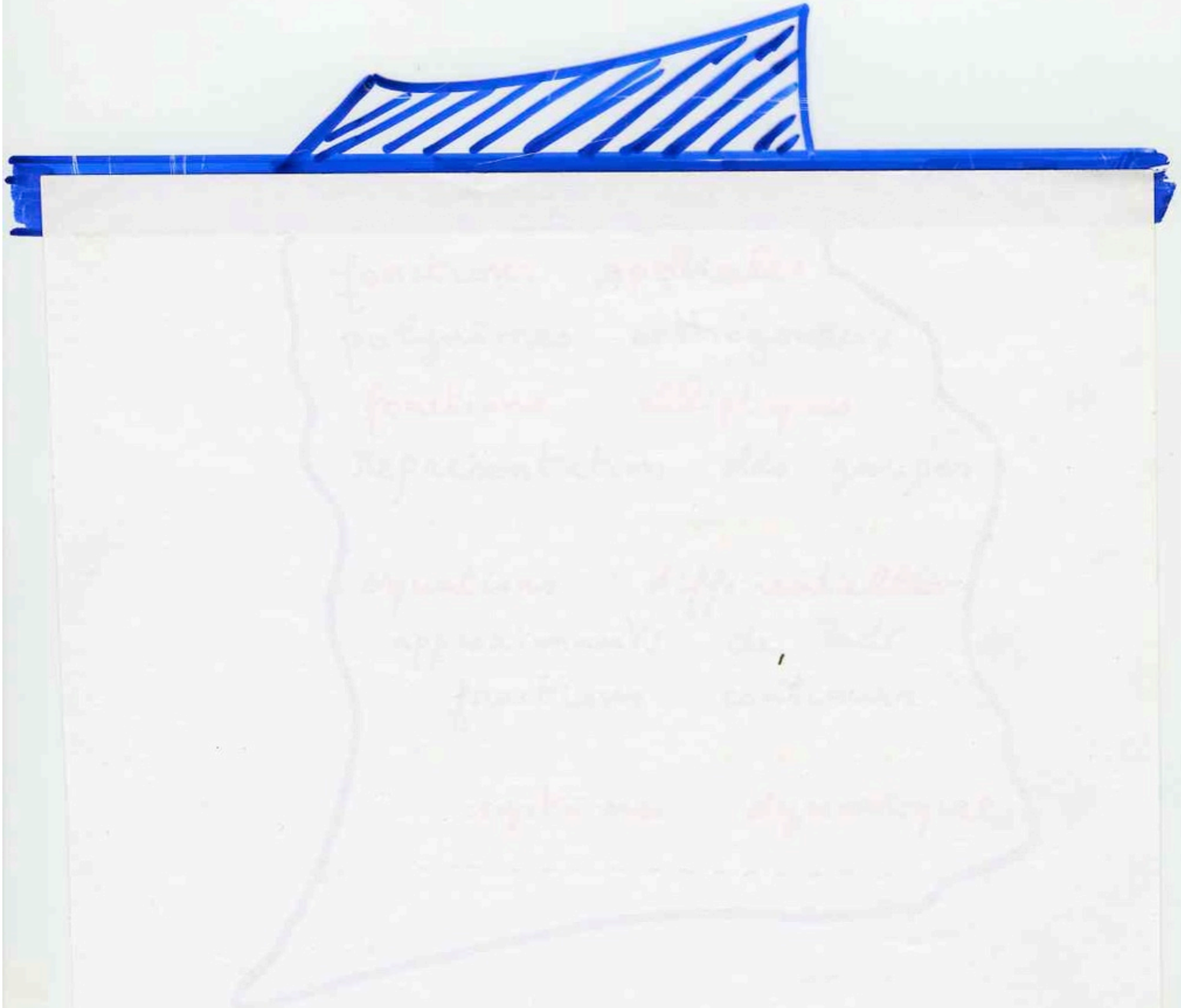
ordre

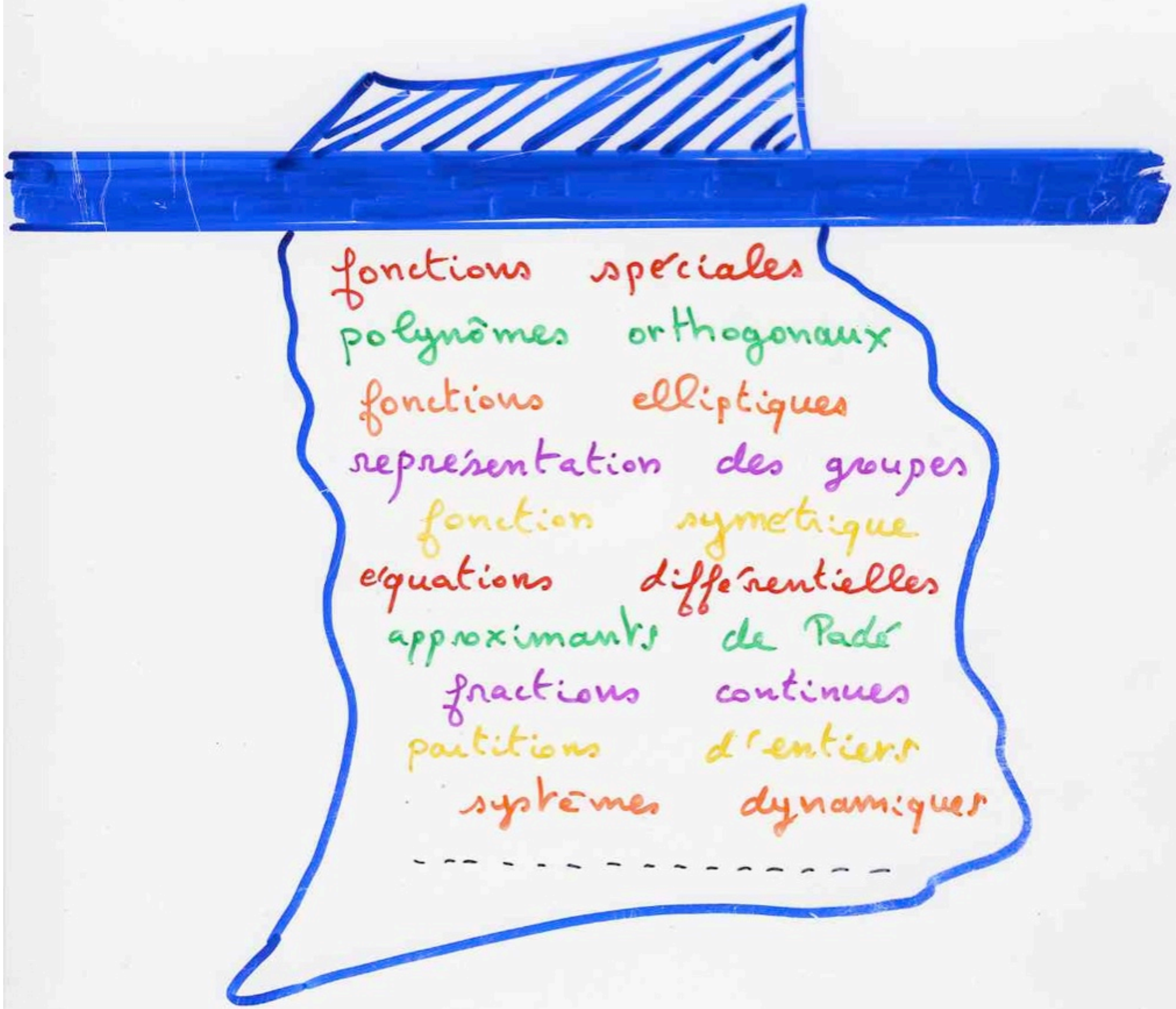
groupe fini

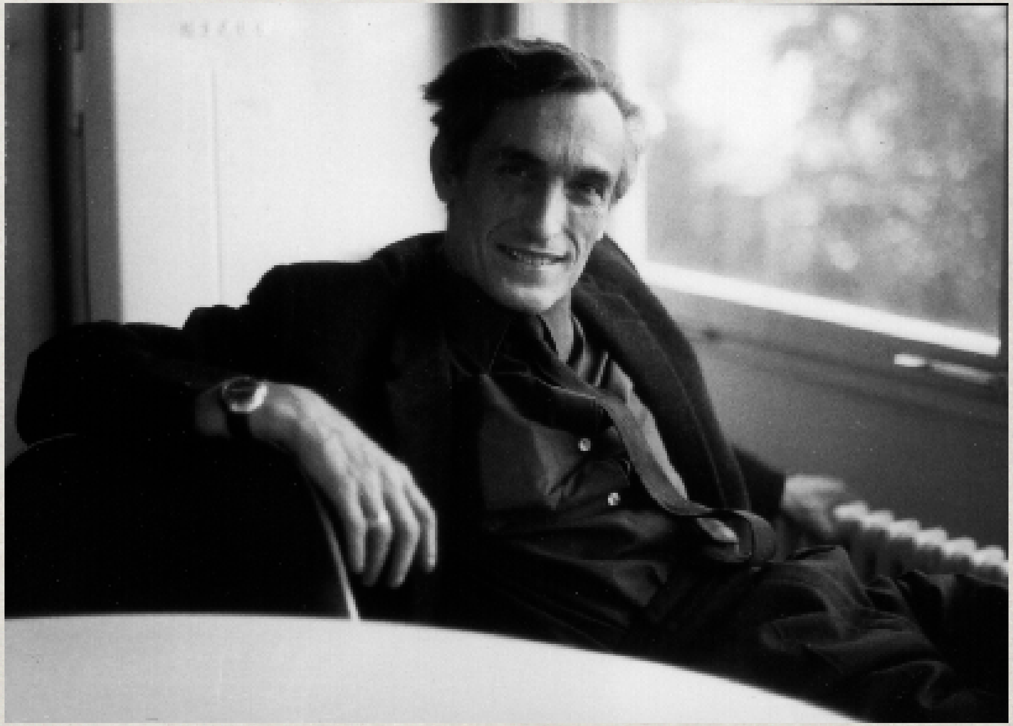
$\mathfrak{S}_n$

représentations irréductibles

degré


fonctions spéciales
polynômes orthogonaux
fonctions elliptiques
représentation des groupes
équations différentielles
approximations de toutes fonctions continues
systèmes dynamiques


fonctions spéciales
polynômes orthogonaux
fonctions elliptiques
représentation des groupes
fonction symetique
équations différentielles
approximants de Padé
fractions continues
partitions d'entiers
systèmes dynamiques
- - - - - - - - - - - - - -



Marcel Paul Schützenberger

Chaînes de spins quantiques ....

Spin chains and combinatorics

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The XXZ quantum spin chain model with periodic boundary conditions is one of the most popular integrable models which has been investigating by the Bethe Ansatz method during the last 35 years [3]. It is described by the Hamiltonian

$$H_{XXZ} = -\sum_{j=1}^{N} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right\}, \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{1}$$

The nonzero wave function components are

$$
\begin{aligned}
N = 3: &\quad \psi_{001} = 1; \\
N = 5: &\quad \psi_{00011} = 1, \quad \psi_{00101} = 2; \\
N = 7: &\quad \psi_{0000111} = 1, \quad \psi_{0001101} = \psi_{0001011} = 3, \quad \psi_{0010011} = 4 \quad \psi_{0010101} = 7.
\end{aligned}
$$

All components not included in the list can be obtained by shifting. Notice that the components of the ground state are positive in accordance with the Perron–Frobenius theorem.

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$
\begin{array}{llll}
\psi_{000001111} = 1, & \psi_{000010111} = 4, & \psi_{000011011} = 6, & \psi_{000100111} = 7, \\
\psi_{000101011} = 17, & \psi_{000101101} = 14, & \psi_{000110011} = 12, & \psi_{001001011} = 21, \\
 & \psi_{001010011} = 25, & \psi_{001010101} = 42. &
\end{array}
$$

Let us continue the list. For $N = 9$ the components of the eigenvector with the energy $-27/2$ and $S_z = -1/2$ are

$$
\begin{array}{llll}
\psi_{000001111} = 1, & \psi_{000010111} = 4, & \psi_{000011011} = 6, & \psi_{000100111} = 7, \\
\psi_{000101011} = 17, & \psi_{000101101} = 14, & \psi_{000110011} = 12, & \psi_{001001011} = 21, \\
 & \psi_{001010011} = 25, & \psi_{001010101} = 42. &
\end{array}
$$

We omit nonzero components which can be obtained by the reflection of the order of sites since this transformation is a symmetry of our state, as it is for the ground state. For example, we have

$$\psi_{000011101} = \psi_{000010111} = 4.$$

1, 2, 7, 42, 429, ...

?

M1791 0, 1, 2, 7, 32, 181, 1214, 9403, 82508, 808393, 8743994, 103459471, 1328953592, 18414450877, 273749755382, 4345634192131, 73362643649444, 1312349454922513
$a(n)=n.a(n-1)+(n-2)a(n-2)$. Ref R1 188. [0,3; A0153, N0706]

$$\text{E.g.f.: } (1-x)^{-3} e^{-x}.$$

M1792 1, 1, 2, 7, 32, 181, 1232, 9787, 88832, 907081, 10291712, 128445967, 1748805632, 25794366781, 409725396992, 6973071372547, 126585529106432
Expansion of $1/(1-\sinh x)$. Ref ARS 10 138 80. [0,3; A6154]

M1793 0, 1, 1, 2, 7, 32, 184, 1268, 10186, 93356, 960646, 10959452, 137221954, 1870087808, 27548231008, 436081302248, 7380628161076, 132975267434552
Stochastic matrices of integers. Ref DUMJ 35 659 68. [0,4; A0987, N0707]

M1794 1, 2, 7, 33, 192
Permutations of length n with n in second orbit. Ref C1 258. [2,2; A6595]

M1795 1, 2, 7, 34, 209, 1546, 13327, 130922, 1441729, 17572114, 234662231, 3405357682, 53334454417, 896324308634, 16083557845279, 306827170866106
$a(n)=2n.a(n-1)-(n-1)^2a(n-2)$. Ref SE33 78. [0,2; A2720, N0708]

M1796 1, 2, 7, 34, 257, 2606, 32300, 440564, 6384634
Polyhedra with n nodes. Ref GR67 424. UPG B15. Dil92. [4,2; A0944, N0709]

M1797 2, 7, 35, 219, 1594, 12935, 113945, 1070324, 10586856, 109259633, 1168384157, 12877168147, 145656436074, 1685157199175, 19886174611045
Two-rowed truncated monotone triangles. Ref JCT A42 277 86. Zei93. [1,1; A6947]

M1798 1, 1, 2, 7, 35, 228, 1834, 17382, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Coefficients of iterated exponentials. Ref SMA 11 353 45. [0,3; A0154, N0710]

M1799 1, 2, 7, 35, 228, 1834, 17582, 195866, 2487832, 35499576, 562356672, 9794156448, 186025364016, 3826961710272, 84775065603888, 2011929826983504
Expansion of $\ln(1+\ln(1+x))$. [0,2; A3713]

M1800 1, 0, 1, 2, 7, 36, 300, 3218, 42335, 644808
Circular diagrams with n chords. Ref BarN94. [0,4; A7474]

M1801 1, 2, 7, 36, 317, 5624, 251610, 33642660, 14685630688
$n \times n$ binary matrices. Ref CPM 89 217 64. SLC 19 79 88. [0,2; A2724, N0711]

M1802 2, 7, 37, 216, 1780, 32652
Semigroups of order n with 2 idempotents. Ref MAL 2 2 67. SGF 14 71 77. [2,1; A2787, N0712]

M1803 1, 2, 7, 37, 266, 2431, 27007, 353522, 5329837, 90960751, 1733584106, 36496226977, 841146804577, 21065166341402, 569600638022431
$a(n)=(2n-1)a(n-1)+a(n-2)$. Ref RCI 77. [0,2; A1515, N0713]

M1804 1, 1, 2, 7, 38, 291, 2932, 36961, 561948, 10026505, 205608536, 4767440679, 123373203208, 3525630110107, 110284283006640, 3748357699560961
Forests of labeled trees with n nodes. Ref JCT 5 96 68. SIAD 3 574 90. [0,3; A1858, N0714]

M1805 1, 1, 2, 7, 40, 357, 4824, 96428, 2800472, 116473461
n-element partial orders contained in linear order. Ref nbh. [0,3; A6455]

M1806 1, 2, 7, 41, 346, 3797, 51157, 816356, 15050581, 314726117, 7359554632, 190283748371, 5389914888541, 165983936096162, 5521346346543307
Planted binary phylogenetic trees with n labels. Ref LNM 884 196 81. [1,2; A6677]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \, . \, . \, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1810 0, 1, 2, 7, 44, 361, 3654, 44207, 622552, 10005041, 180713290, 3624270839, 79914671748, 1921576392793, 50040900884366, 1403066801155039
Modified Bessel function $K_n(1)$. Ref AS1 429. [0,3; A0155, N0716]

M1811 0, 1, 2, 7, 44, 447, 6749, 142176, 3987677, 143698548, 6470422337, 356016927083, 23503587609815, 1833635850492653, 166884365982441238
$a(n)=n(n-1)a(n-1)/2+a(n-2)$. [0,3; A1046, N0717]

M1812 1, 2, 7, 44, 529, 12278, 565723, 51409856, 9371059621, 3387887032202, 2463333456292207, 3557380311703796564, 10339081666350180289849
Sum of Gaussian binomial coefficients $[n,k]$ for $q=4$. Ref TU69 76. GJ83 99. ARS A17 328 84. [0,2; A6118]

M1813 2, 7, 52, 2133, 2590407, 3374951541062, 5695183504479116640376509, 16217557574922386301420514191523784895639577710480
Free binary trees of height n. Ref JCIS 17 180 92. [1,1; A5588]

M1814 1, 1, 2, 7, 56, 2212, 2595782, 3374959180831, 5695183504489239067484387, 16217557574922386301420531277071365103168734284282
Planted 3-trees of height n. Ref RSE 59(2) 159 39. CMB 11 87 68. JCIS 17 180 92. [0,3; A2658, N0718]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \, . \, . \, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 64539836938720749739356
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

M1807 1, 1, 2, 7, 41, 376, 5033, 92821, 2257166, 69981919, 2694447797, 126128146156, 7054258103921, 464584757637001, 35586641825705882, 3136942184333040727
Hammersley's polynomial $p_n(1)$. Ref MASC 14 4 89. [0,3; A6846]

M1808 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, 129534272700, 31095744852375, 12611311859677500, 8639383518297652500
Robbins numbers: $\Pi(3k+1)!/(n+k)!$, $k = 0 \, . \, . \, n-1$. Ref MINT 13(2) 13 91. JCT A66 17 94. [1,2; A5130]

M1809 1, 2, 7, 42, 582, 21480, 2142288, 575016219, 415939243032, 816007449011040, 4374406209970747314, 6453983693872074973935 6
Antisymmetric relations on n nodes. Ref PAMS 4 494 53. MIT 17 23 55. [1,2; A1174, N0715]

MSA

Matrices à signes alternants

"Matrices à signes alternés"

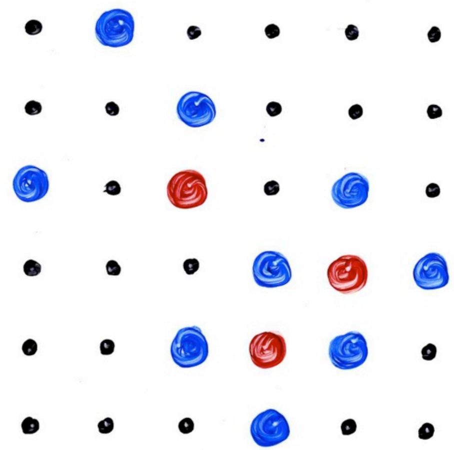
(alternating sign matrices)

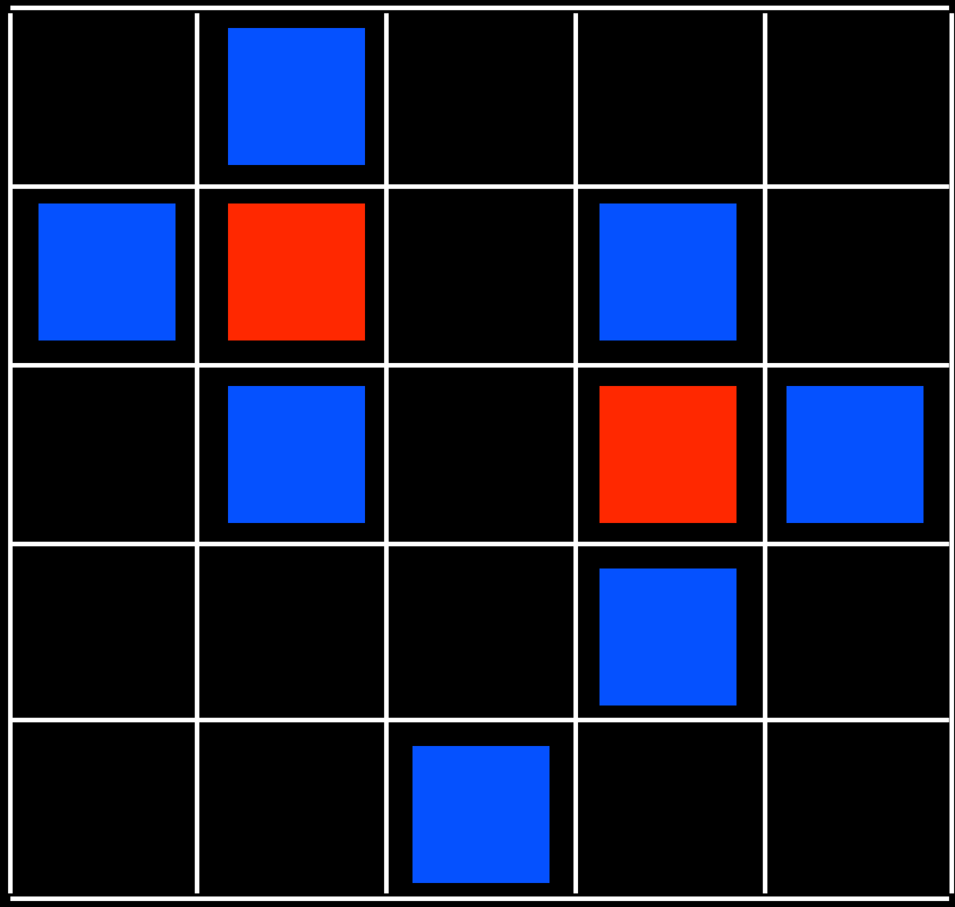
- entrées : 0, 1, −1
- somme des entrées, ligne, colonne = 1
- entrées ≠ 0 alternent en signe ligne, colonne

alternating sign matrices (ASM)

ex :

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$





Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \underset{\text{permutations}}{6}$$

1, 2, 7, 42, 429, ...

Enumérer les MSA

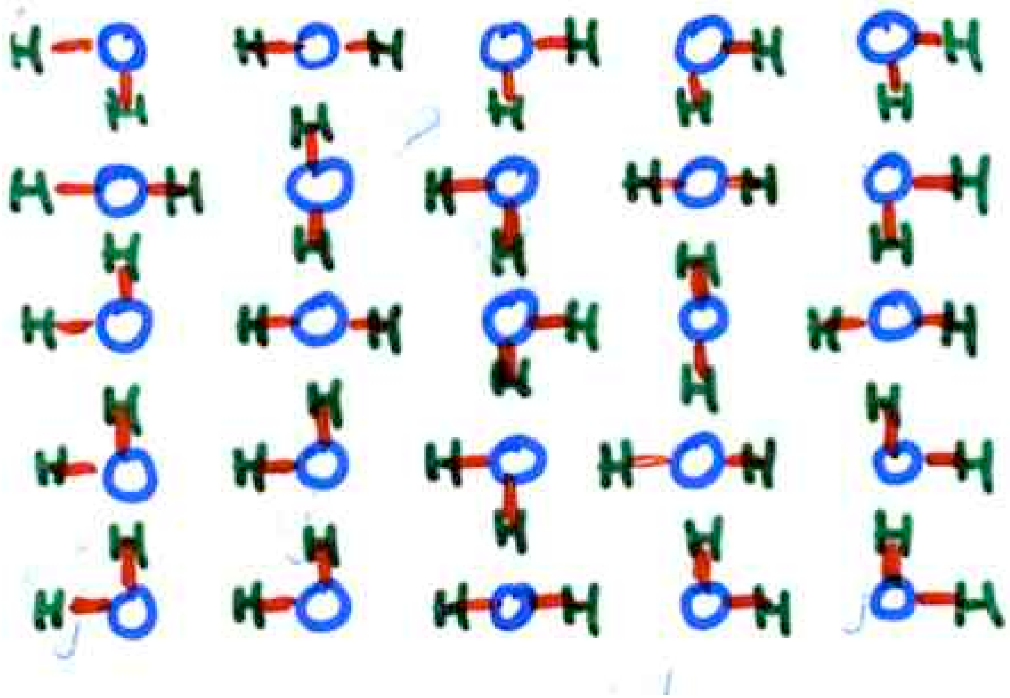
$$1\ ,\ 2\ ,\ 7\ ,\ 42\ ,\ 429\ ,\ \ldots$$

$$\frac{1!\ \ 4!}{n!\ (n+1)}$$

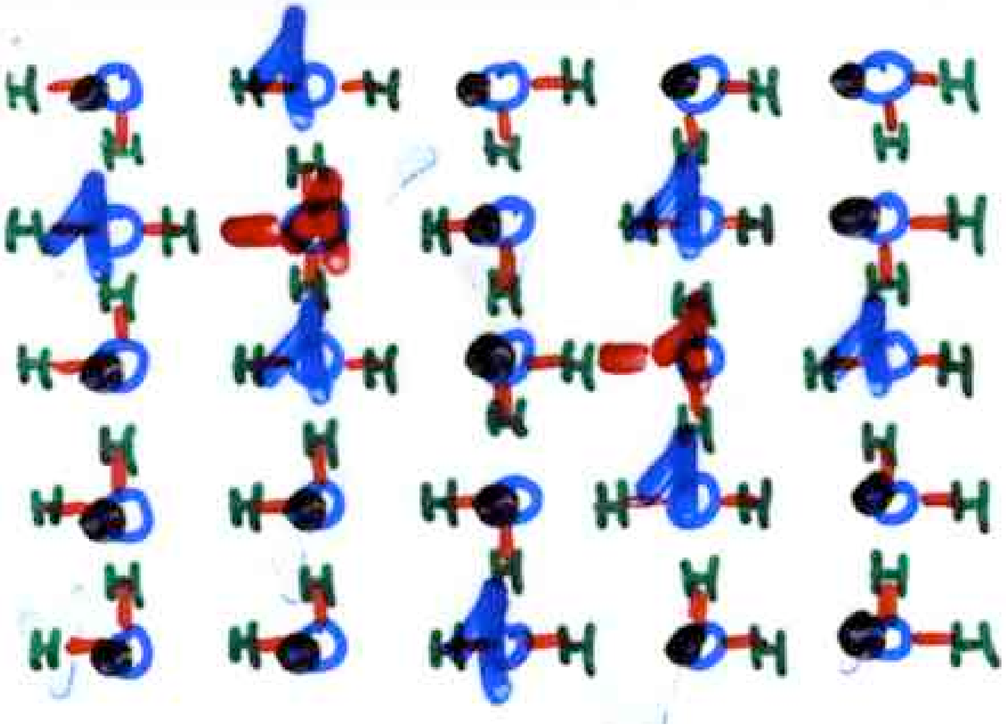
?

$$\frac{3n-2)!}{(n+n-1)!}$$

Modèle de la glace



$$\begin{array}{ccccc}
\bullet & 1 & \bullet & \bullet & \bullet \\
1 & -1 & \bullet & 1 & \bullet \\
\bullet & 1 & \bullet & -1 & 1 \\
\bullet & \bullet & \bullet & 1 & \bullet \\
\bullet & \bullet & 1 & \bullet & \bullet
\end{array}$$



MARS

"Physique combinatoire:
autour des matrices à signes alternants
et la conjecture de Razumov - Stroganov"

Projet Blanc ANR
LaBRI , IGM

" What else have you got in your pocket ?" he
went on, turning to A
" Only a thimble,"
" Hand it over her
Then they all crow
while the Dodo solen

Lewis Carroll

"Alice aux pays des merveilles"

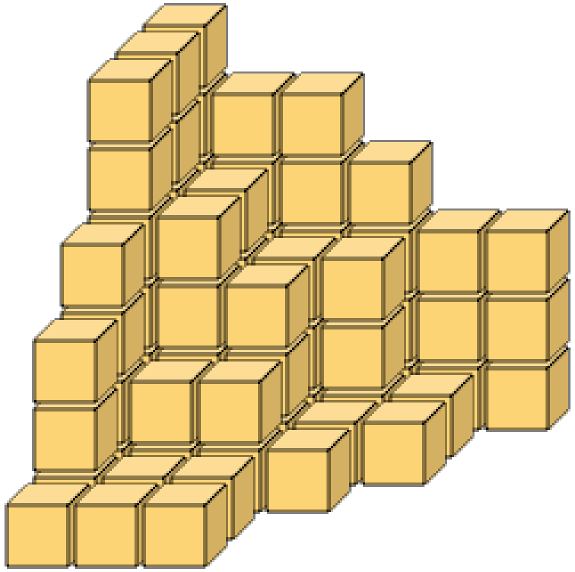
C. I. Dodgson (1866)

Condensation of determinants

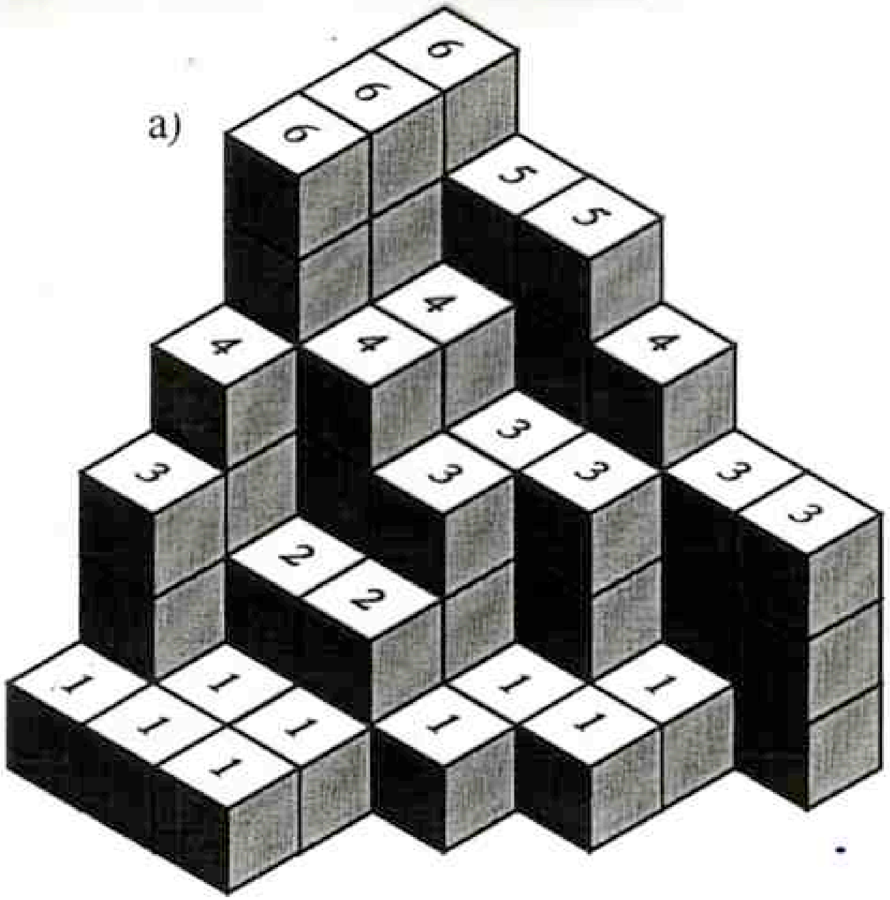
$$\textcolor{red}{det}(M) = \frac{M_{NO} M_{SE} - M_{NE} M_{SO}}{M_C}$$

Partitions planes

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

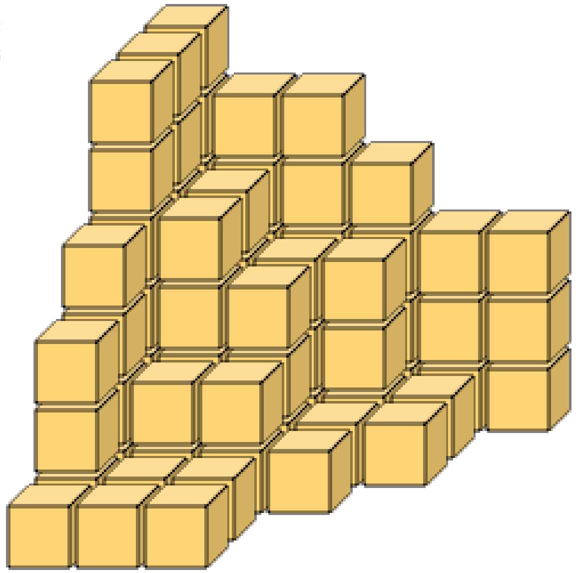


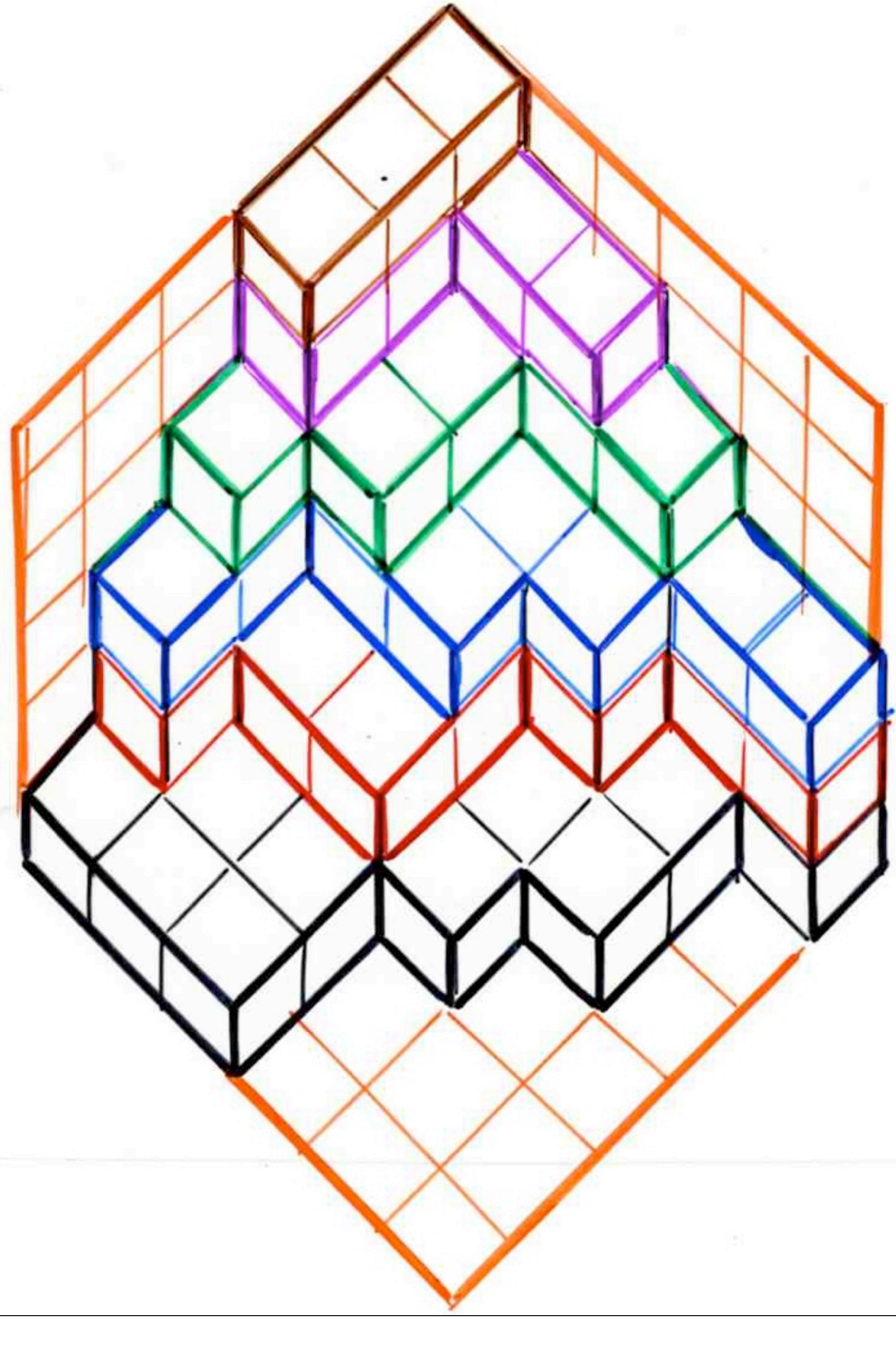
a)



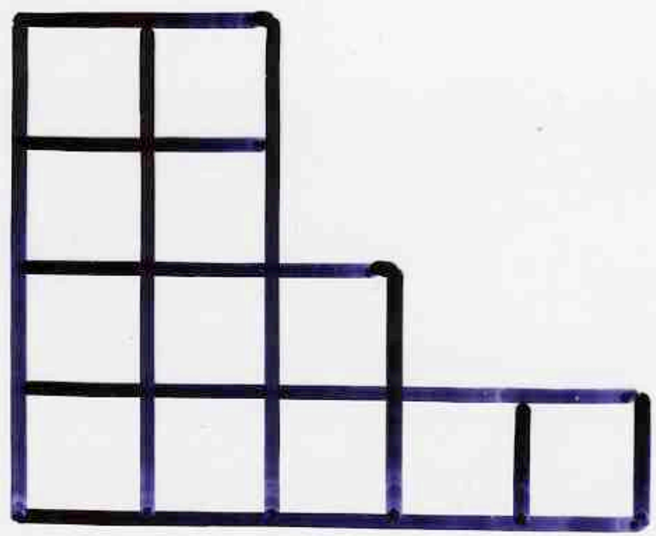
b)

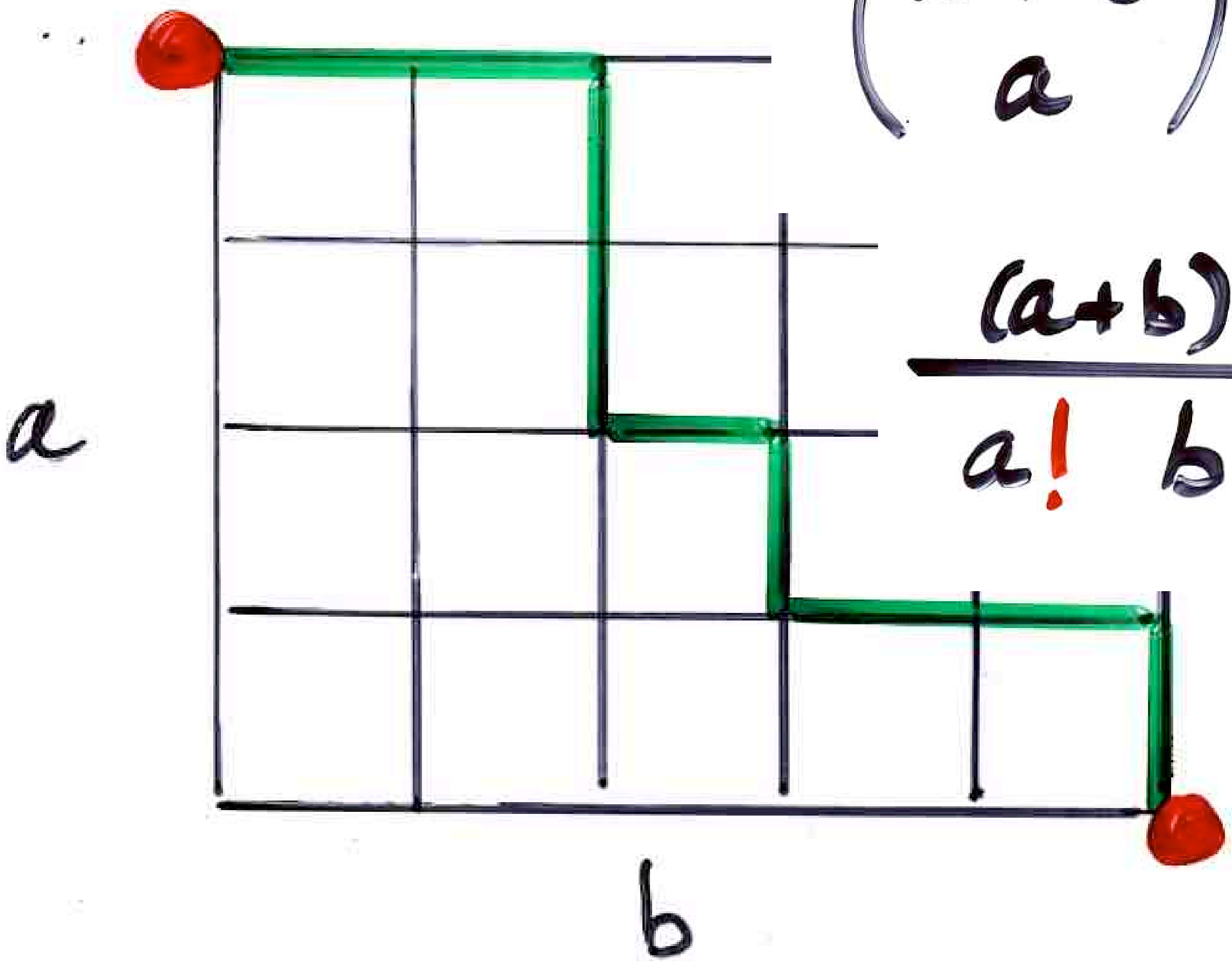
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			





$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

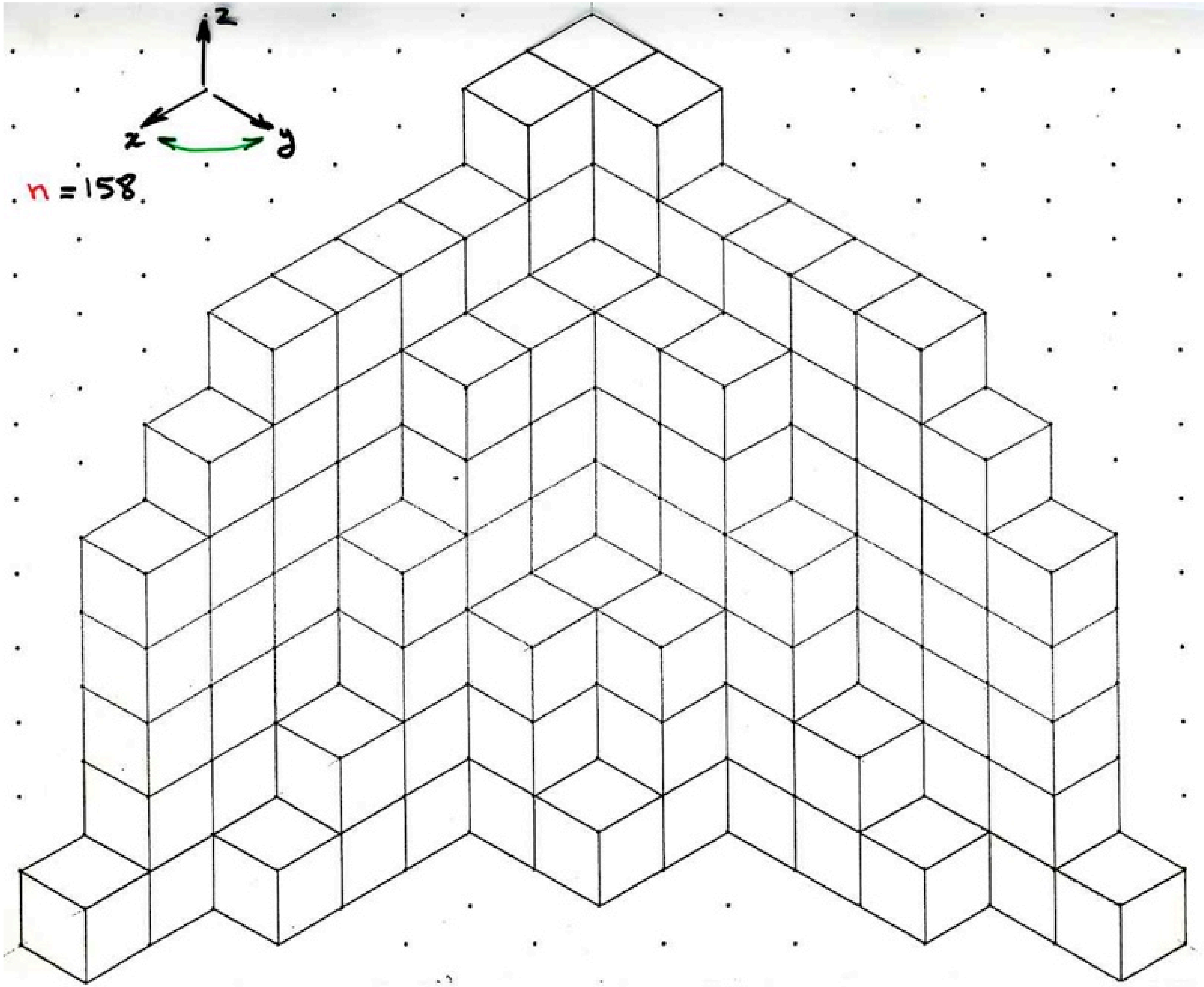


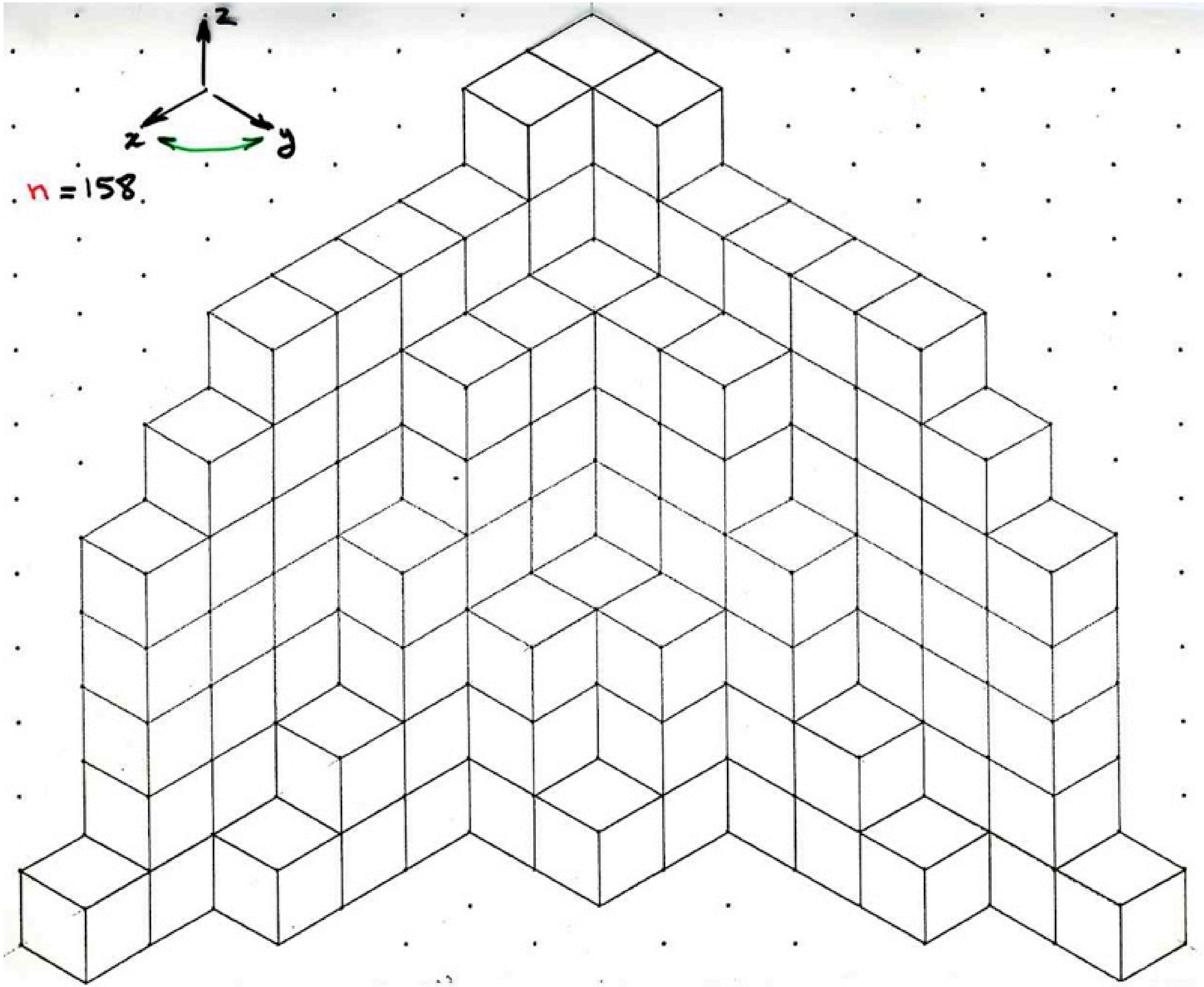


$$\binom{a+b}{a}$$

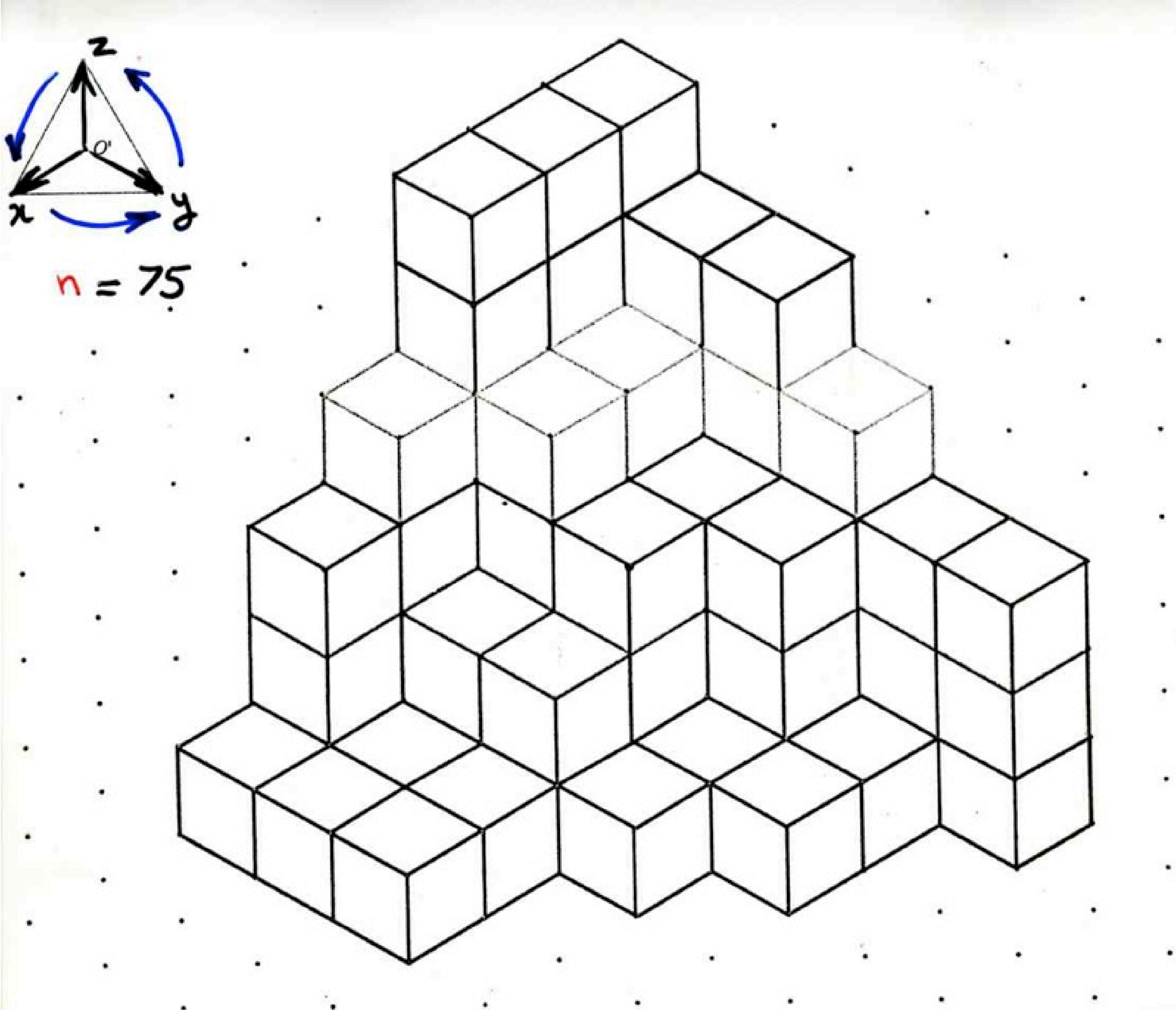
$$\frac{(a+b)!}{a!\,b!}$$

Partitions planes symétriques

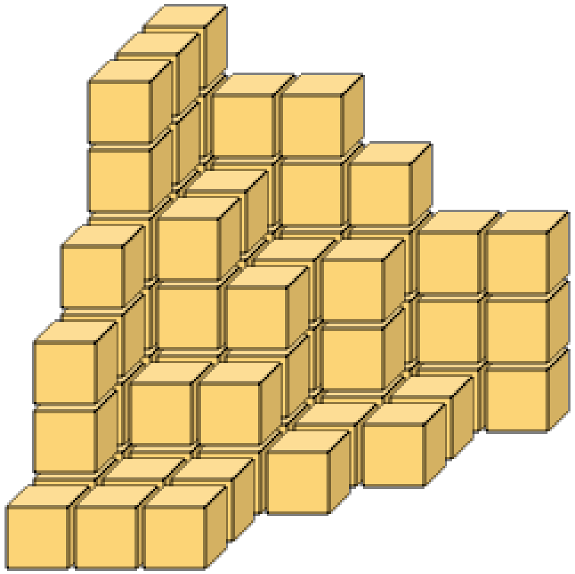

z
x
y
n = 158


z
z
y
n = 158

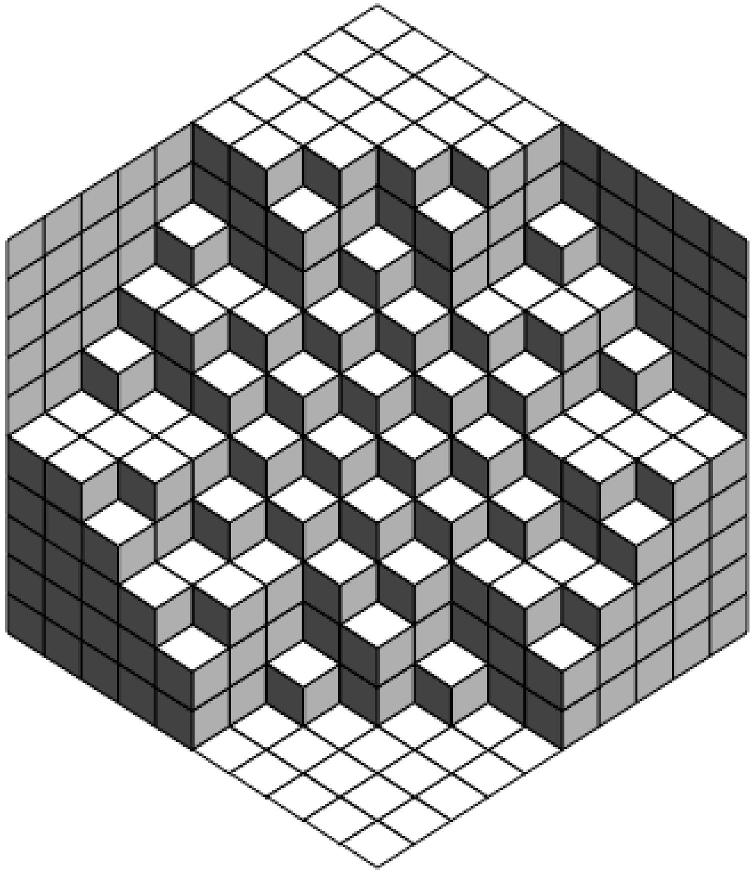
Partitions planes cycliquement symétriques


z
O'
x
y
n = 75

Partitions planes
cycliquement symétriques



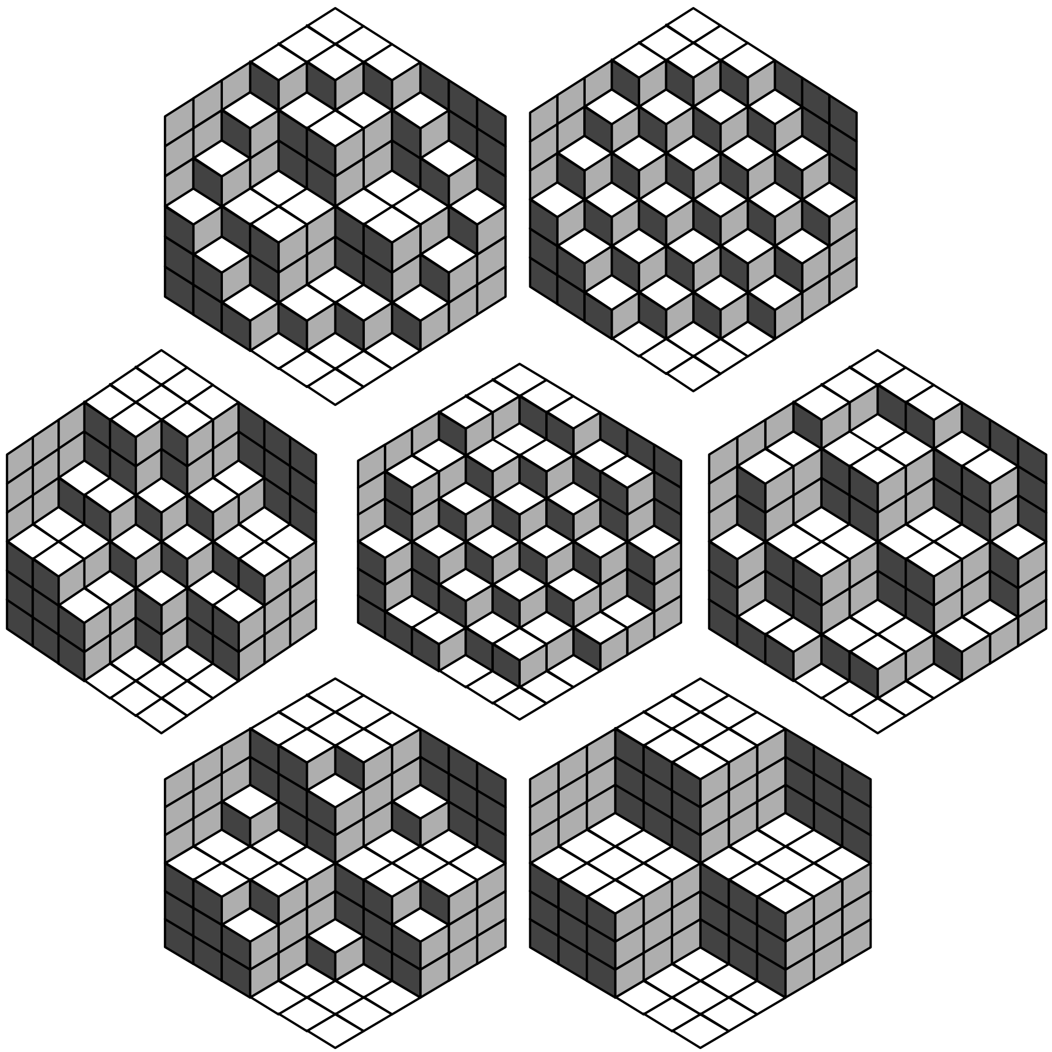
complémentaire
d'une partition plane

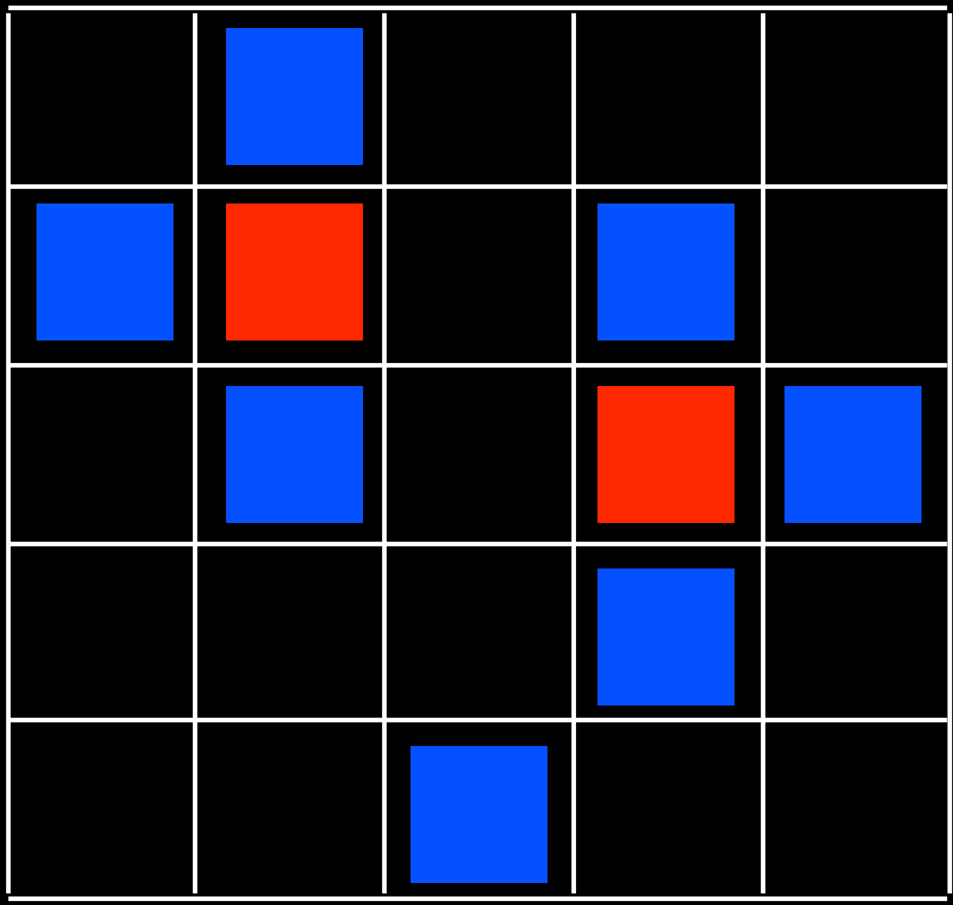


partitions planes totalement symétriques
et autocomplémentaires

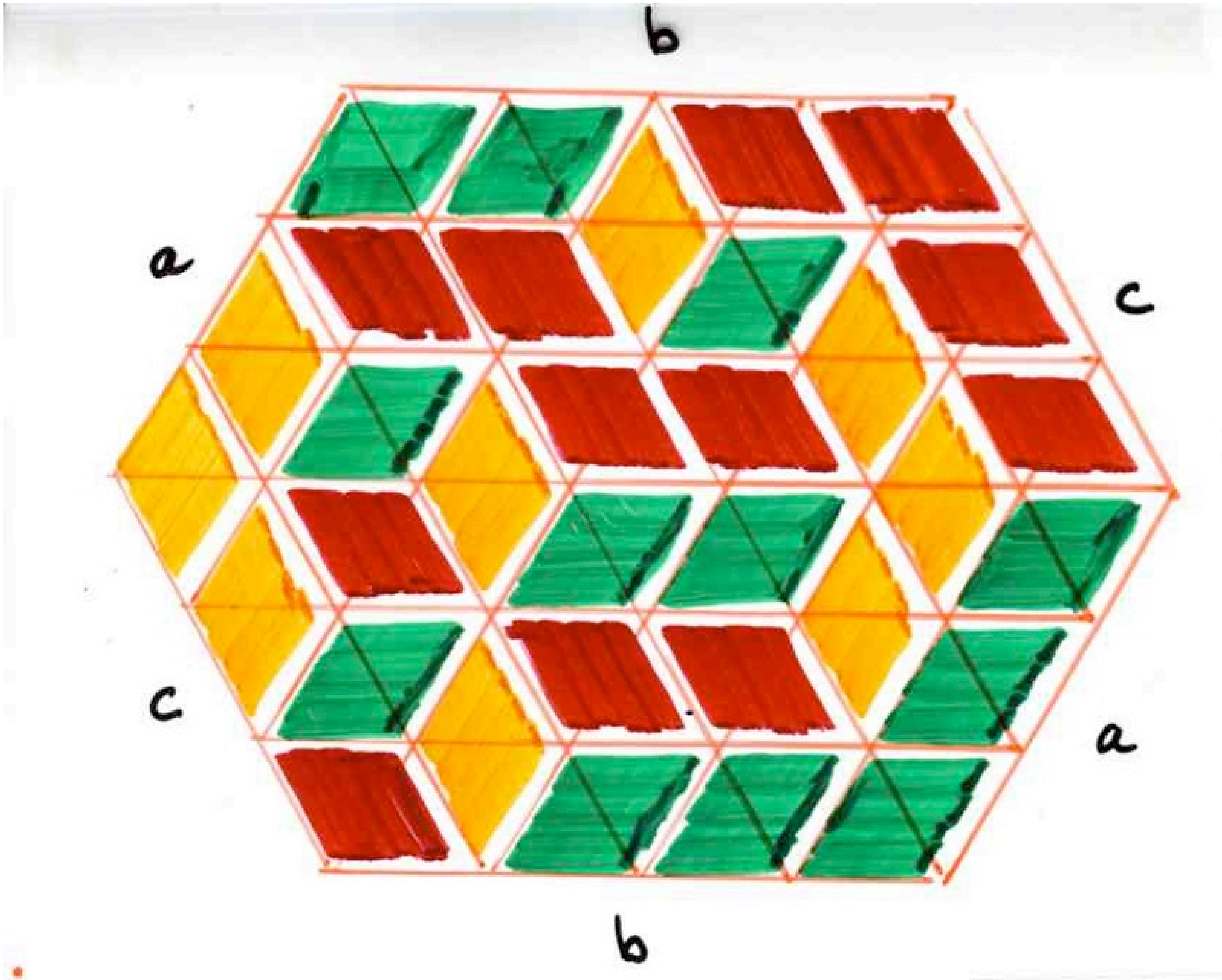
Le nombre de telles partitions
dans une boîte $2n \times 2n \times 2n$ est

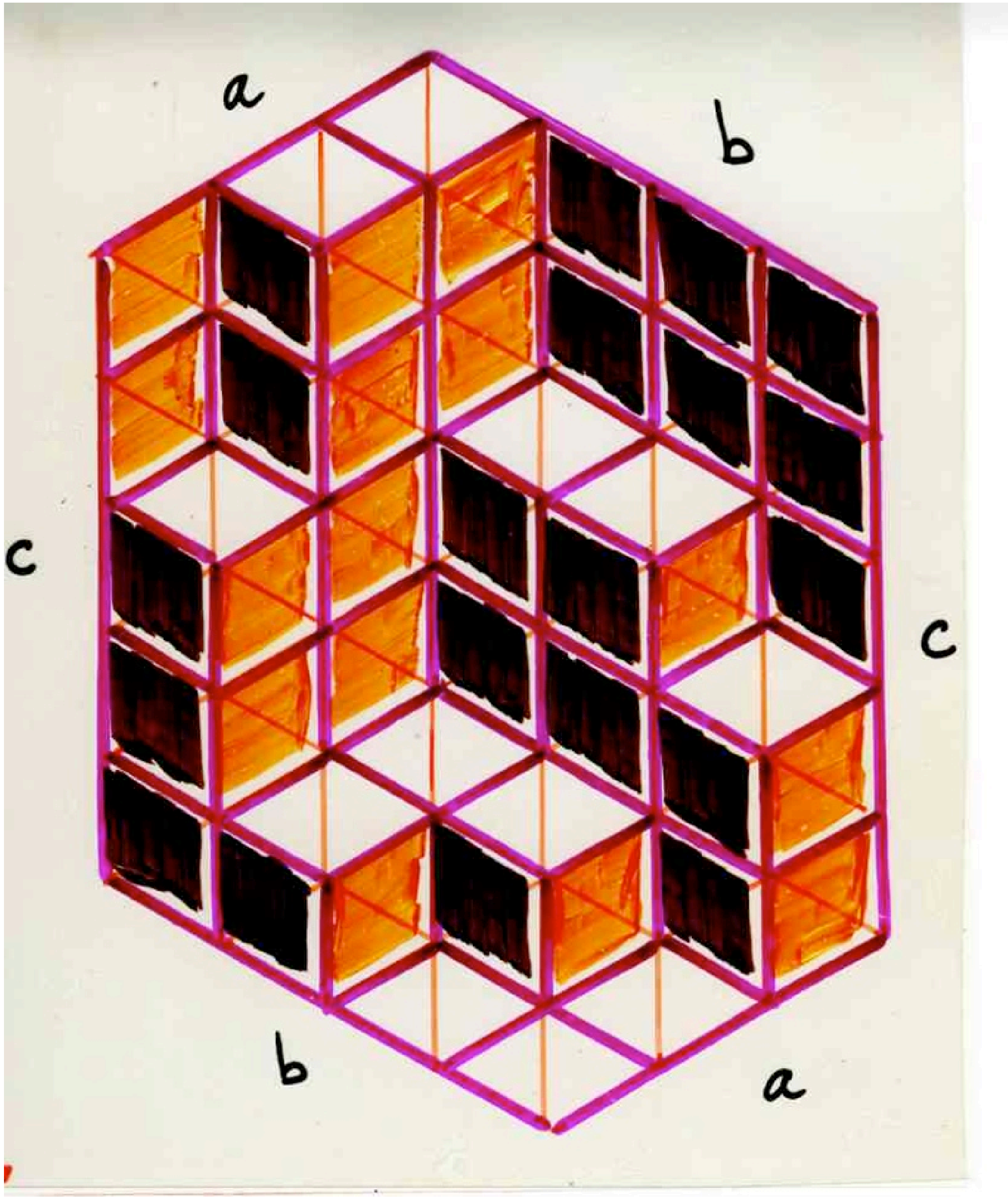
le même que celui de
matrices à signes alternants de
taille $n \times n$



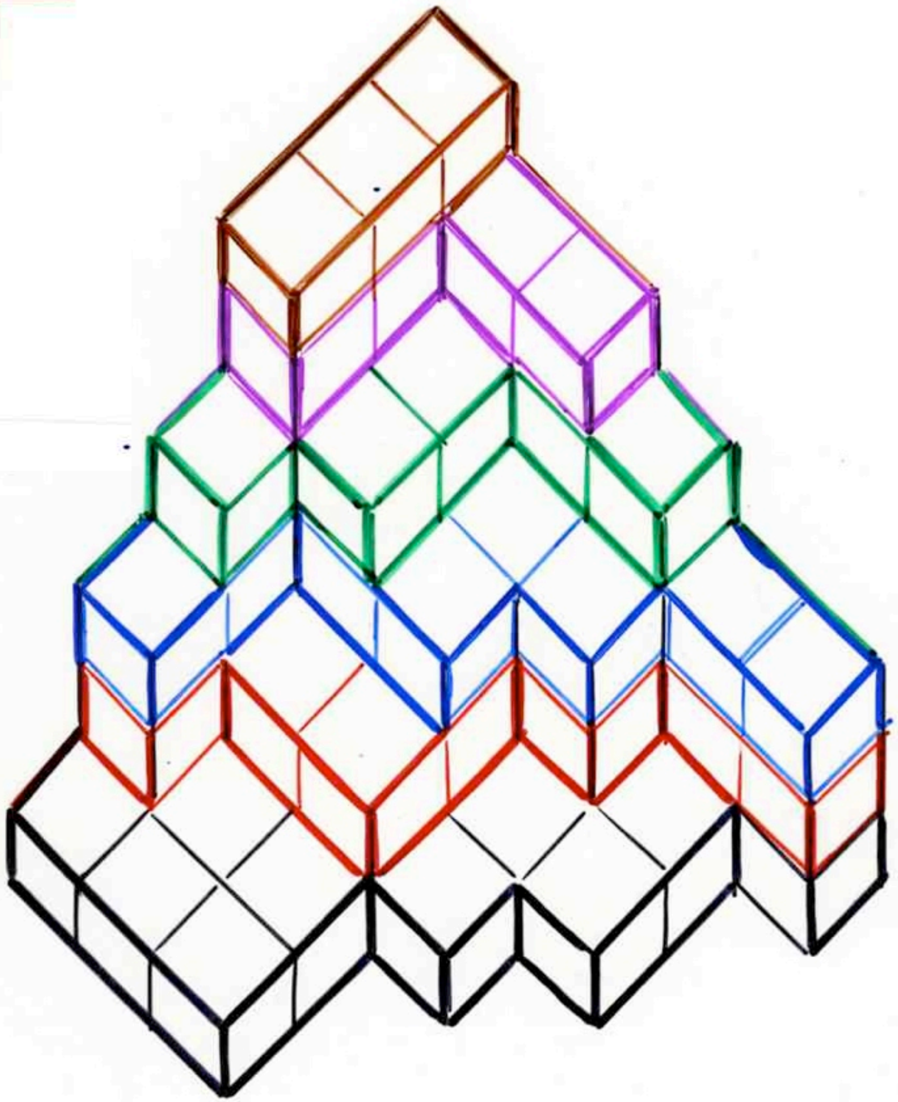


Pavages


b
a
c
c
a
b


a
b
c
c
b
a

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

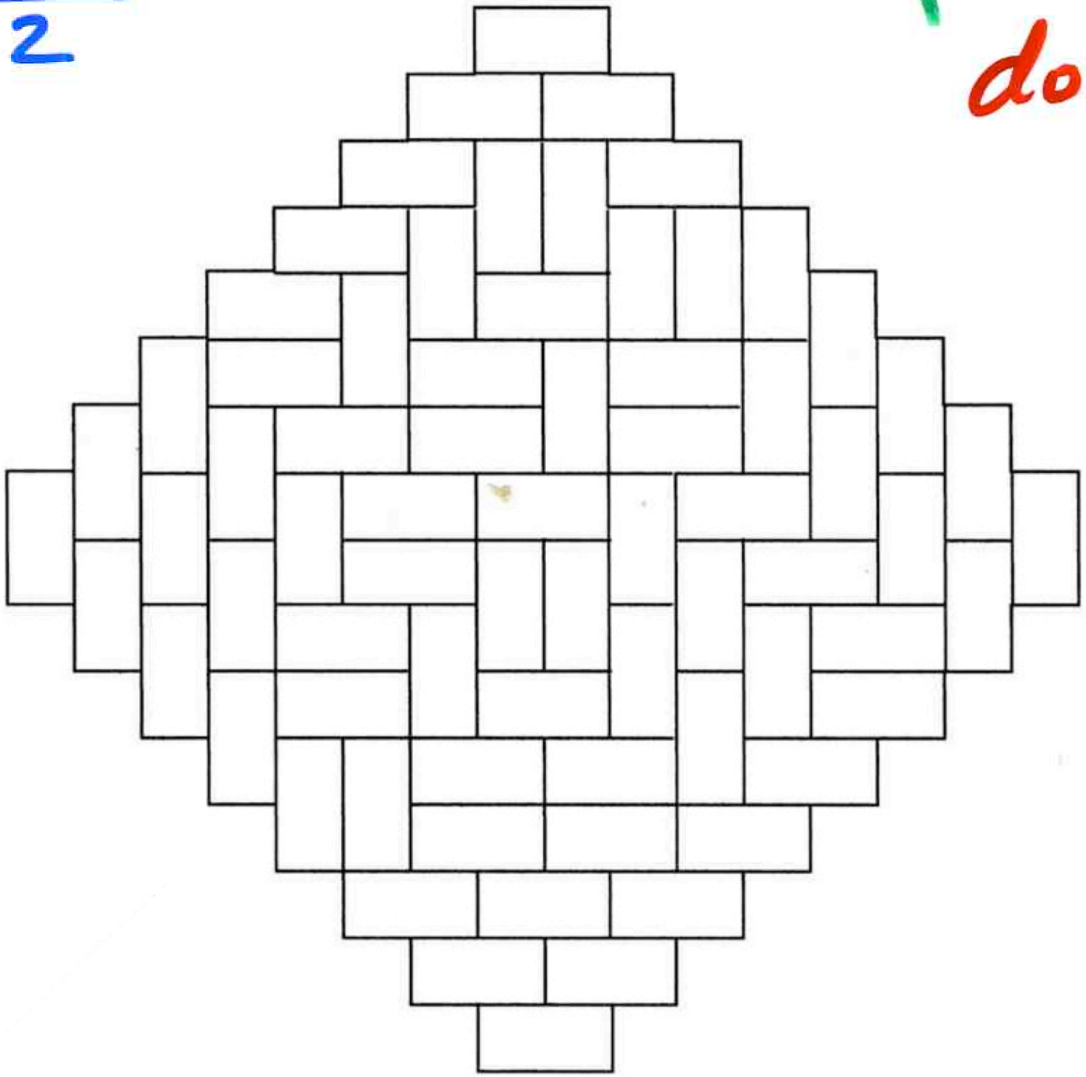


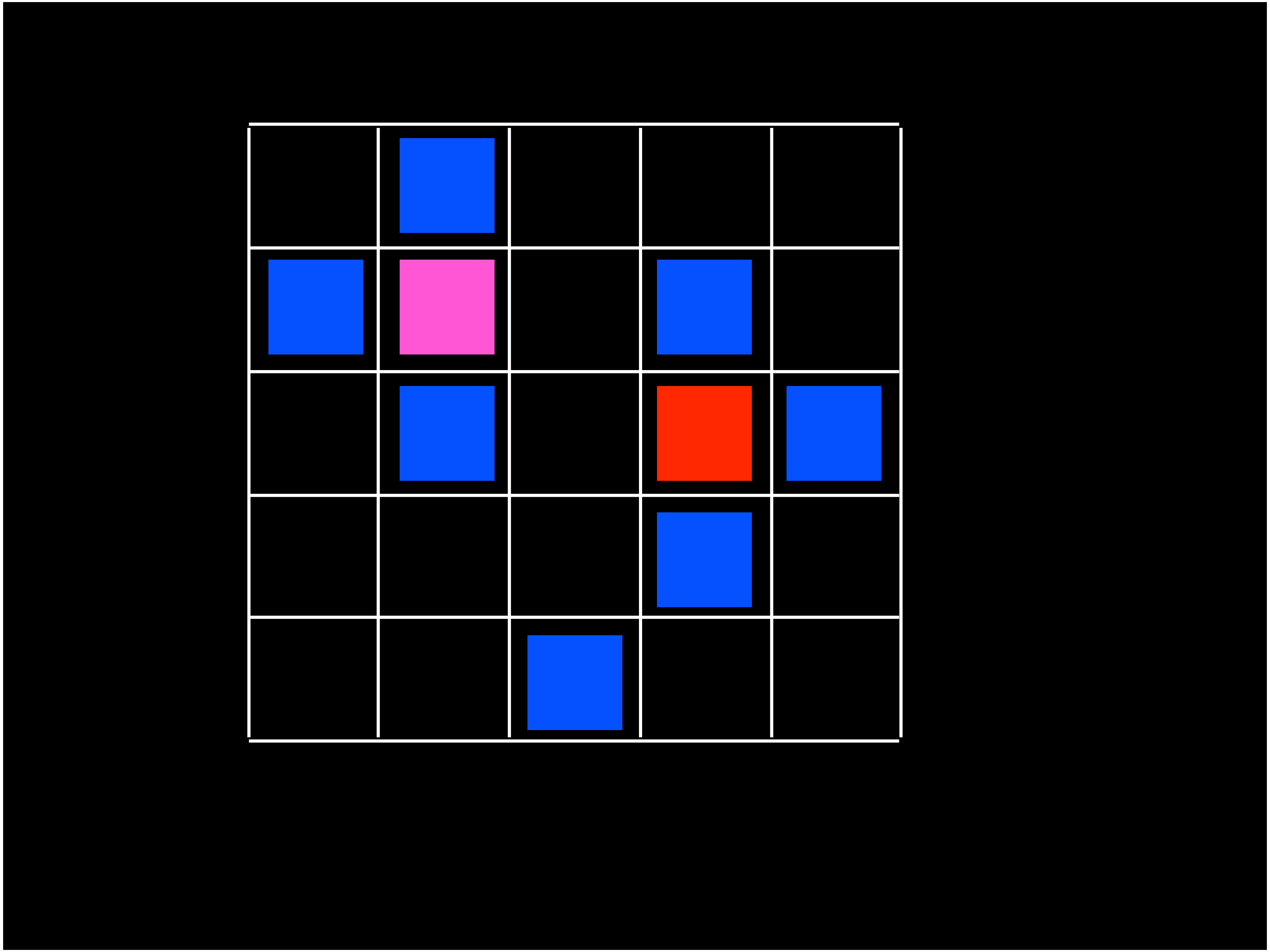
$$\prod_{\substack{1\le i\le a\\ 1\le j\le b\\ 1\le k\le c}} \frac{i+j+k-1}{i+j+k-2}$$

$2^{\frac{n(n-1)}{2}}$

pavage

dominos



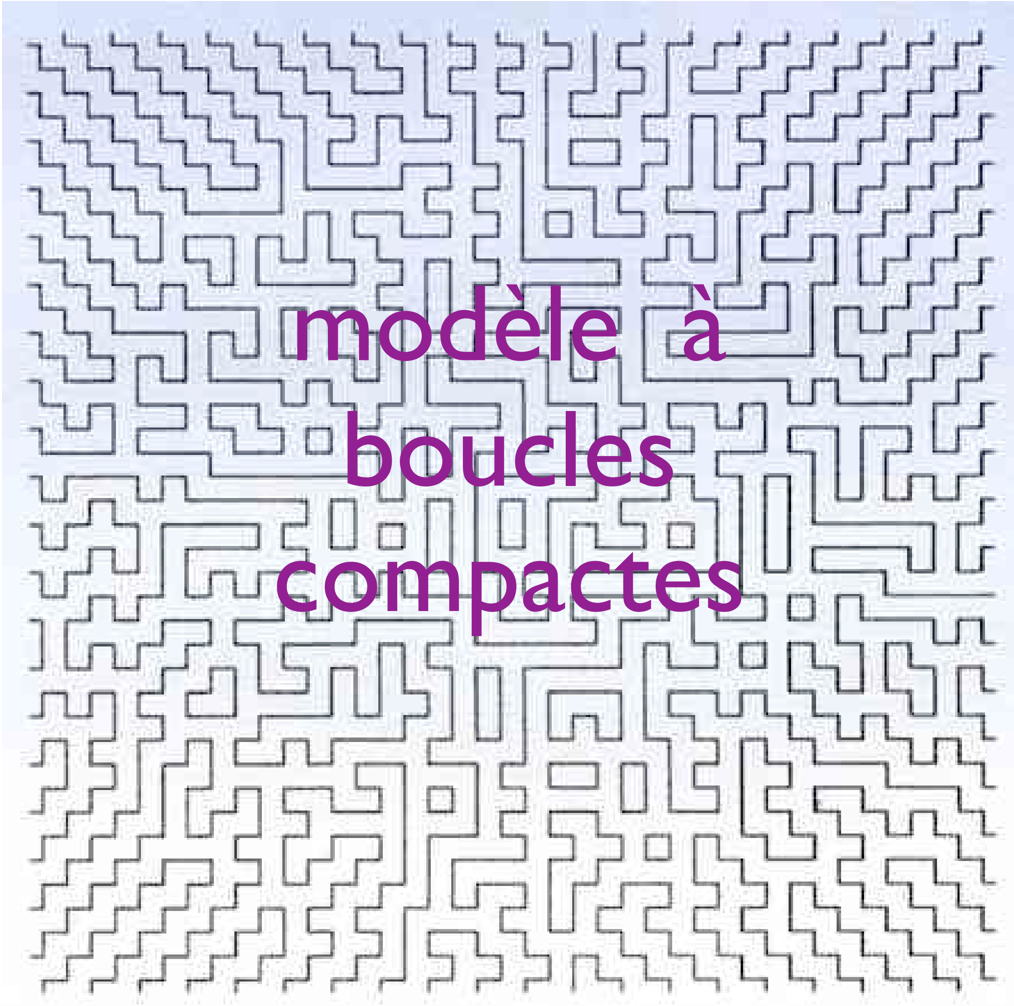


Structures discrètes aléatoires

SADA

Projet blanc ANR

LaBRI, INRIA, LIX, ...

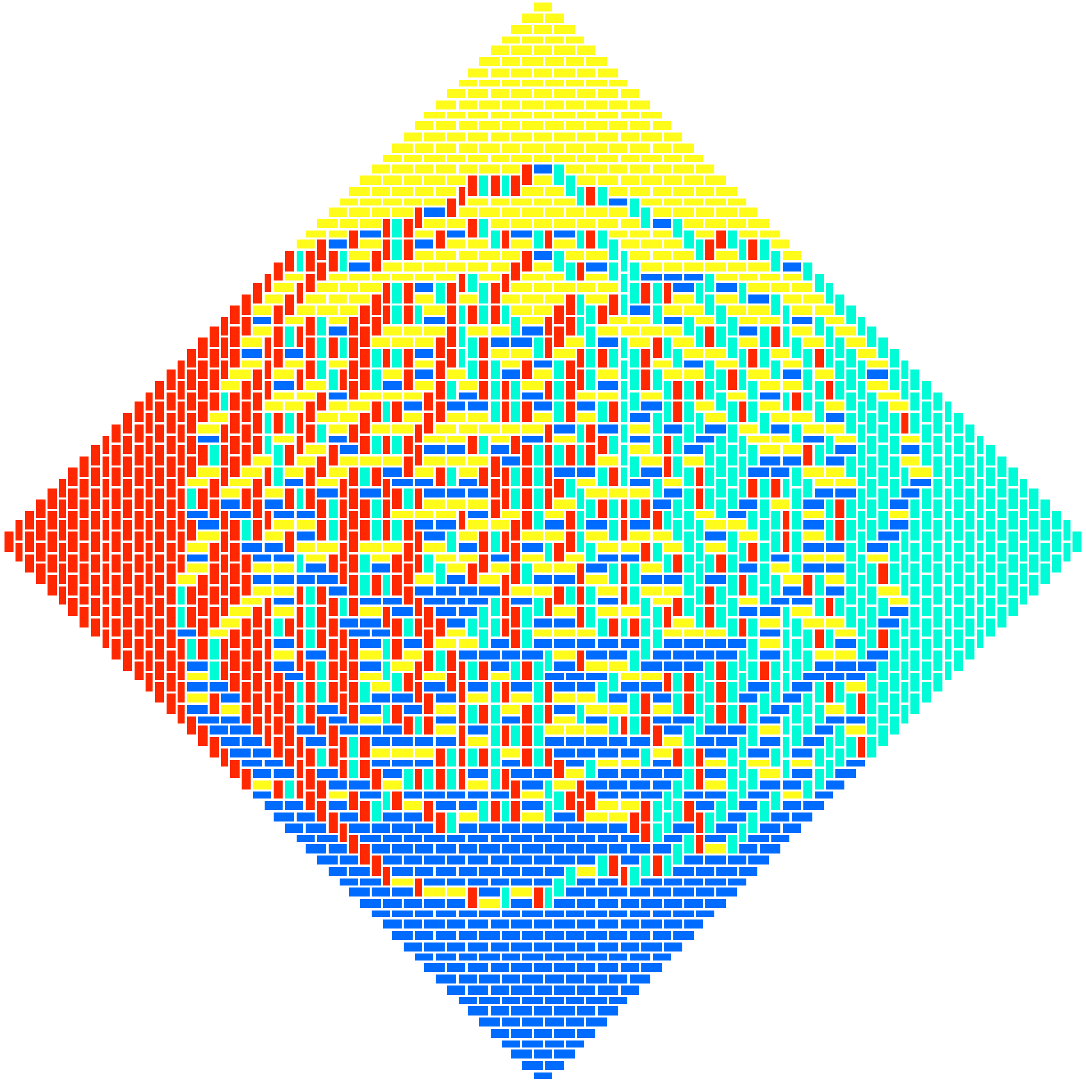

modèle à
boucles
compactes

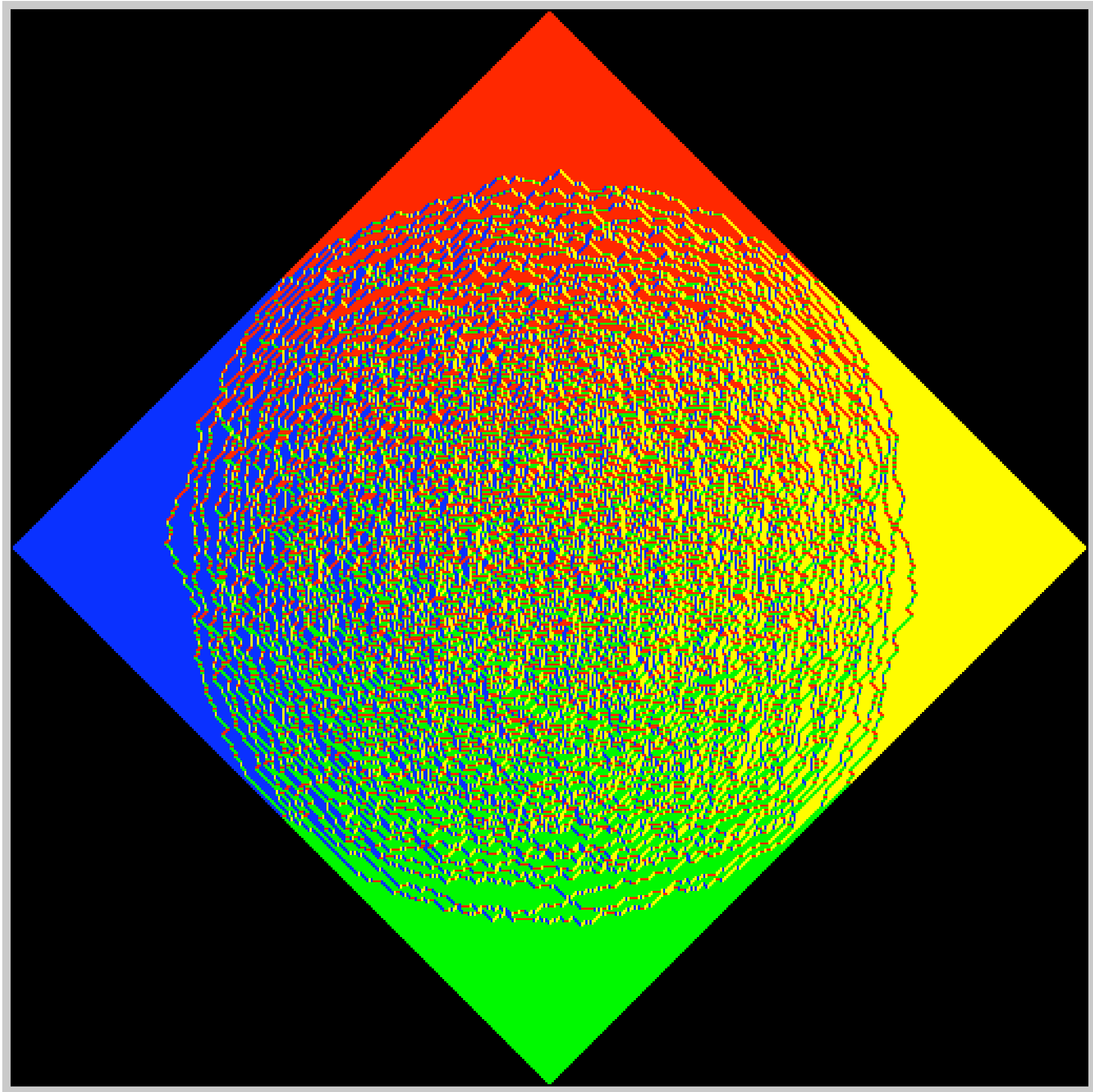
Partitions planes aléatoires

Andrei Okoukov

avec Richard Kenyon

le théorème
du
cercle antique





merci à tous !